Ruprecht-Karls-Universität Heidelberg
Institut für Mathematik und Informatik

Masterarbeit

Andreev's Theorem on 3-dimensional Hyperbolic Polyhedra

Name: Andrea Rincón
E-Mail-Adreesse: presipao@gmail.com
Betreuer: Prof. Dr. Anna Wienhard, Dr. Gye-Seon Lee
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Erklärung

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Heidelberg, July 6, 2017 ....

Abstract

Andreev's Theorem (1970) [3] [4] gives us a classification of polyhedra of finite volume, with non-obtuse dihedral angles, in the 3-dimensional hyperbolic space. The main purpose of this thesis is to give an introduction to such polyhedra and their combinatorics, and to study the tools used by Andreev to state and prove his theorem.

Zusammenfassung

Andreevs Theorem (1970) [3] [4] gibt uns eine Klassifikation von Polyedern endlichen Volumens mit nicht-stumpfen Diederwinkeln im dreidimensionalen hyperbolischen Raum. Der Hauptzweck dieser Arbeit ist es, eine Einführung in solche Polyeder und ihre Kombinatorik zu geben und die von Andreev verwendeten Instrumente um seinen Satz zu beweisen zu verstehen.

Acknowledgements

I am really thankful to all the people with whom I shared this journey. In particular, I would like to thank Prof. Wienhard for her nice lectures and the opportunity to work in this beautiful topic, and Dr. Lee for all his patient explanations, help and support during the development of this project. Finally, a special thanks to my familly for all its love and support.

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Contents

1. Hyperbolic Polyhedra

We begin by defining convex polyhedra on the 3-dimesional hyperbolic space. For this purpose we will use the hyperboloid model $\mathbf{H}^3$, which is the model that we get by taking the positive unit time-like vectors of the Minkowski space. On the other hand, let us remember that in the Euclidean space a convex polyhedron is the convex set that we get by intersecting finitely many half-spaces. We will define hyperbolic polyhedra in a similar way. Therefore, we have to see how a k-dimensional hyperbolic subspace in $\mathbf{H}^3$ looks and to understand the relation between them and the time-like vectors.

1.1. The Lorentzian n-space

The **Lorenz n-Space** $\mathbb{E}^{1,n}$ is the vector space $\mathbb{R}^{n+1}$ together with the Lorentzian product, which is given by

$$\langle x, y\rangle_L = -x_0y_0 + x_1y_1 + \cdots + x_ny_n. \tag{1.1}$$

where $x = [x_0, x_1, \ldots, x_n]^T$ and $y = [y_0, y_1, \ldots, y_n]^T \in \mathbb{R}^{n+1}$. Note that for all $x, y \in \mathbb{R}^{n+1}$ equation (1.1) is equivalent to

$$\langle x, y\rangle_L = x^T J y \tag{1.2}$$

where J is the $(n + 1) \times (n + 1)$ indefinite matrix given by

$$J = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & I_n & \\ 0 & & & \end{bmatrix}$$

Moreover, the Lorentzian product induces the pseudo-Riemannian metric defined by the form $-dx_0^2 + dx_1^2 + \cdots + dx_n^2$. When $n = 3$ we call $\mathbb{E}^{1,3}$ the Minkowski space.

Remark 1.1. We will denote the elements of the standard basis of $\mathbb{E}^{1,n}$ by e_i, for $i \in \{0, ..., n\}$, where e_i is the vector that has 1 in the i-th coordinate and zero in the others.

Vector Subspaces

We will study the vector subspaces of the general Lorentz n-space and their relation with the light cone. We begin by classifying the vectors in $\mathbb{E}^{1,n}$.

Definition 1.1. Take $x \in \mathbb{E}^{1,n}$, then

1. x is **space-like** if $\langle x, x\rangle_L > 0$,
2. x is **light-like** if $\langle x, x\rangle_L = 0$,
3. x is **time-like** if $\langle x, x\rangle_L < 0$.

4. We also say that x is **positive** if $x_0 > 0$.

The light cone $\mathbf{C}$ is the set of light-like vectors. Moreover, $\mathbf{C} - \{0\}$ is a n-dimensional manifold with two different connected components, one consists of the positive light-like vectors and the other contains the negative light-like vectors. The light-cone is the boundary of the two open sets containing the time-like and the space-like vectors, respectively.

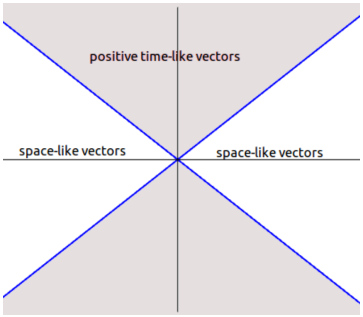

Figure 1.1.: $\mathbf{C}$ in $\mathbb{E}^{1,1}$

1. We denote by $\mathbf{C}^+$ the set of positive light-like vectors.
2. The **interior** of $\mathbf{C}$ is the open set in $\mathbb{E}^{1,n}$ containing all the time-like vectors.
3. The **exterior** of $\mathbf{C}$ is the open set in $\mathbb{E}^{1,n}$ containing all the space like vectors.

If V is a vector subspace of $\mathbb{E}^{1,n}$, then we have three possibilities for $V-\{0\}$: it intersects the interior of $\mathbf{C}$, it lies completely on the exterior of $\mathbf{C}$, or it intersects $\mathbf{C}$ but not its interior. This corresponds to the classification given in the following definition.

Definition 1.2. If V is a vector subspace of $\mathbb{E}^{1,n}$, it is called

- **time-like** if contains a time-like vector.
- **space-like** if all the vectors in $V - \{0\}$ are space-like vectors.
- **light-like** otherwise.

Now, let V be a k-dimensional vectors subspace of $\mathbb{E}^{1,n}$. The restriction of the Lorentzian product to V, denote it by $\langle , \rangle_L\big|_V$, is a scalar product on V. Given a basis $B = \{v_1, \dots, v_k\}$ of V, the matrix representation of the scalar product $\langle , \rangle_L\big|_V$ with respect to B is

$$G(B) = \begin{bmatrix} \langle v_1, v_1\rangle_L & \dots & \langle v_1, v_k\rangle_L \\ \vdots & \ddots & \vdots \\ \langle v_k, v_1\rangle_L & \dots & \langle v_k, v_k\rangle_L \end{bmatrix} \tag{1.3}$$

and for all $x, y \in V$ it holds that

$$\langle x, y\rangle_L\big|_V = \tilde{x}^T G(B)\tilde{y}, \tag{1.4}$$

where $\tilde{x}$ and $\tilde{y}$ are the column vectors in $\mathbb{R}^k$ whose entries are the coefficients of x and y with respect to the basis B.

Equation 1.4 tells us that:

- $G(B)$ is positive definite if and only if V is space-like.
- $G(B)$ is positive semi-definite if and only if V is light-like.
- $G(B)$ is indefinite if and only if V is time-like.

This is equivalent to the following statement.

Proposition 1.1. *If V is a k-dimensional vector subspace of $\mathbb{E}^{1,n}$ and $B = \{v_1, ..., v_k\}$ is a basis of V, then*

1. *det $G(B) > 0$ if and only if V is space-like.*
2. *det $G(B) = 0$ if and only if V is light-like.*
3. *det $G(B) < 0$ if and only if V is time-like.*

Remark 1.2. If $V{=}\mathbb{E}^{1,n}$ and $B = \{e_0, \dots, e_n\}$ is the standard basis, then $G(B) = J$.

The Lorentzian Complement

The **Lorentzian-Complement** of a vector subspace V of $\mathbb{E}^{1,n}$ is the set

$$V^L = \{x \in \mathbb{E}^{1,n} | \langle x, y\rangle_L = 0 \quad \text{for all} \quad y \in V\}.$$

Since the Lorentz product is a bilinear form, it is clear that V^L is also a vector subspace of $\mathbb{E}^{1,n}$.

Example 1.1. If $V = \mathbb{E}^{1,n}$, then $V^L = \{0\}$.

A vectors subspace V is space-like if and only if its Lorentzian complement is a time-like vector subspace and in this case the whole space is the direct sum of both vector subspaces. On the other hand, V is light-like if and only if its Lorentzian-complement is light-like.

Lemma 1.1. *If V is a vector subspace of $\mathbb{E}^{1,n}$, then*

1. *$J(V^\perp) = J(V)^\perp = V^L$, where $V^\perp$ denotes the Euclidean orthogonal complement of V.*
2. *$(V^L)^L = V$*

Proof.

1. From the equation (1.2) we know that

$$\langle x, y\rangle_L = x^T J y = y^T J x \quad \text{for all} \quad x, y \in \mathbb{E}^{1,n}. \tag{1.5}$$

and the statement follows from the definition.

2. Note that

$$(V^L)^L = J(V^L)^\perp = J(J(V^\perp))^\perp = J(J(V^\perp)^\perp) = J(J(V)) = V.$$

□

Proposition 1.2. *If V is a k-dimensional space-like subspace of $\mathbb{E}^{1,n}$, then*

$$\mathbb{E}^{1,n} = V \oplus V^L$$

Proof. We want to see that $\dim(V^L) = n + 1 - k$ and $V \cap V^L = \{0\}$.

1. V and $J(V)$ are vector subspaces of the same dimension and from Lemma 1.1 we know that $V^L = J(V)^{\perp}$. Therefore,

$$\dim(V^L) = \dim(J(V)^{\perp}) = n + 1 - k.$$

2. Since V is space-like, it holds that $\langle y, y\rangle_L > 0$ for all vectors $y \in V$. Let us assume that $V \cap V^L \neq \{0\}$, then there is a non-zero vector $x \in V$ such that $\langle x, y\rangle_L = 0$ for all vectors $y \in V$, specially $\langle x, x\rangle_L = 0$ and so we get a contradiction.

□

Proposition 1.3. *If V is k-dimensional space-like subspace of $\mathbb{E}^{1,n}$, then V^L is time-like.*

Proof. From the last proposition we know that $\mathbb{E}^{1,n} = V \oplus V^L$. Let us take a basis $B = \{v_0, \ldots, v_n\}$ of $\mathbb{E}^{1,n}$ such that

$$\begin{aligned} B_1 = &\ \{v_0, \ldots, v_{k-1}\} \quad \text{is a basis of V, and} \\ B_2 = &\ \{v_k, \ldots, v_n\} \qquad \text{is a basis of } V^L. \end{aligned}$$

Also let us assume that V^L is not a time-like vector subspace.

Take the representation matrices $G(B_1)$ and $G(B_2)$ of the scalar product on the respective vector subspaces. By proposition 1.1 it holds that that

$$\det G(B_1) > 0 \quad \text{and} \quad \det G(B_2) \geq 0. \tag{1.6}$$

Additionally, since V^L is the Lorentzian complement of V we have

$$\langle v_i, v_j\rangle_L = \langle v_j, v_i\rangle_L = 0 \qquad \text{for} \quad i = 0, \ldots, k-1 \quad \text{and} \quad j = k, \ldots, n.$$

From the last equation we can conclude that the matrix of representation of $\langle , \rangle_L$ with respect to the basis B is

$$G(B) = G(B_1) \oplus G(B_2) = \begin{bmatrix} G(B_1) & 0 \\ 0 & G(B_2) \end{bmatrix}$$

and so, by the last result and equation (1.6)

$$\det G(B) = \det G(B_1) \det G(B_2) \geq 0,$$

which contradicts the fact that the whole space is time-like. □

Proposition 1.4. *If V be a proper time-like subspace of $\mathbb{E}^{1,n}$, then V^L is space-like.*

Proof. Let $x \in V$ be a time-like vector, then

$$\begin{aligned} \langle x, x\rangle_L = & -x_0^2 + x_1^2 + \cdots + x_n^2 < 0 \\ \Rightarrow \quad & x_1^2 + \cdots + x_n^2 < x_0^2. \end{aligned} \tag{1.7}$$

Now, let us assume that there is a vector $y \in V^L - \{0\}$ that is not space-like, then

$$\begin{aligned} \langle y, y\rangle_L = & -y_0^2 + y_1^2 + \cdots + y_n^2 \leq 0 \\ \Rightarrow \quad & y_1^2 + \cdots + y_n^2 \leq y_0^2. \end{aligned} \tag{1.8}$$

Note that both x_0 and y_0 are different from zero. Otherwise, we would have a contradiction to the two last equations. In addition, since $\langle \lambda x, \lambda x\rangle_L = \lambda^2 \langle x, x\rangle_L$, we can assume without loss of generality that $x_0, y_0 > 0$.

Let $\hat{x} = [x_1, \ldots, x_n]^T$ and $\hat{y} = [y_1, \ldots, y_n]^T$, it holds

$$\begin{aligned} \langle \hat{x}, \hat{y}\rangle =& x_1 y_1 + \cdots + x_n y_n \\ \|\hat{x}\|^2 =& x_1^2 + \cdots + x_n^2 \\ \|\hat{y}\|^2 =& y_1^2 + \cdots + y_n^2 \end{aligned}$$

From equations (1.7), (1.8) and the Schwartz inequality it follows that

$$\begin{aligned} x_0 y_0 > & \|\hat{x}\| \|\hat{y}\| \geq \langle \hat{x}, \hat{y}\rangle \\ \Rightarrow \quad & 0 > -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n = \langle x, y\rangle_L, \end{aligned}$$

but $\langle x, y\rangle_L = 0$, since $y \in V^L$. □

Proposition 1.5. *If V a light-like vector subspace different from zero, then V^L is light-like.*

Proof. Let us assume that V^L is time-like, by Proposition 1.4 we know that $(V^L)^L = V$ is space-like. Analogously, if we assume that V^L is space-like, then we can conclude that V is time-like and in both cases we get a contradiction. □

Corollary 1.1. *Given a non-zero proper vector subspace V of $\mathbb{E}^{1,n}$. It holds that*

$$\begin{aligned} V^L \textit{ is time-like} \quad & \Longleftrightarrow \quad V \textit{ is space-like} \\ & \Longleftrightarrow \quad \textit{det } G(B) > 0 \textit{ for any basis } B \textit{ of } V \end{aligned}$$

$$\begin{aligned} V^L \textit{ is space-like} \quad & \Longleftrightarrow \quad V \textit{ is time-like} \\ & \Longleftrightarrow \quad \textit{det } G(B) < 0 \textit{ for any basis } B \textit{ of } V \end{aligned}$$

$$\begin{aligned} V^L \textit{ is light-like} \quad & \Longleftrightarrow \quad V \textit{ is light-like} \\ & \Longleftrightarrow \quad \textit{det } G(B) = 0 \textit{ for any basis } B \textit{ of } V \end{aligned}$$

Corollary 1.2. *If V is a time-like or space-like vector subspace of $\mathbb{E}^{1,n}$, then*

$$\mathbb{E}^{1,n} = V \oplus V^L$$

Example 1.2. Let $v \in \mathbb{E}^{1,n}$ be a space-like vector, and $\langle v\rangle$ be the 1-dimensional vector space generated by this vector. Since $\langle v, v\rangle_L > 0$, it holds that $\langle v\rangle$ is space-like. Therefore, $\langle v\rangle^L$ is time-like. In fact, $\langle v\rangle^L$ is the n-dimensional hyperplane with normal vector $Jv = [-v_0, v_1, \ldots, v_n]$.

Lorentz Transformations

The Lorentz transformations are the linear transformations from $\mathbb{E}^{1,n}$ into itself that preserve the Lorentzian product. They preserve the light cone, its interior and its exterior. Moreover, they are linear isomorphisms and the set of all a Lorentz transformations is a Lie group.

Definition 1.3. A **Lorentz transformation** is a linear map $\phi : \mathbb{E}^{1,n} \longrightarrow \mathbb{E}^{1,n}$ such that $\langle \phi(x), \phi(y) \rangle_L = \langle x, y \rangle_L \quad \text{for all} \quad x, y \in \mathbb{E}^{1,n}$.

Let $\phi : \mathbb{E}^{1,n} \longrightarrow \mathbb{E}^{1,n}$ be a Lorentz transformation:

- If A_ϕ is its representation matrix with respect to the standard basis, then $A_\phi^T J A_\phi = J$.
- Since det $(A_\phi)^2 = 1$, ϕ is a linear isomorphism.
- ϕ preserves the light cone $\mathbf{C}$, its interior and its exterior. However, it doesn't necessarily sends positive vectors into positive vectors.

As mentioned before, the set of all Lorentz transformations is a Lie group, we call it the **Lorentz Group** and denote it by $O(1, n)$.

Proposition 1.6. *$O(1, n)$ is a Lie group of dimension $\binom{n+1}{2}$*

Proof. Let $e = I_n$ be the identity matrix. We want to see that $dim(T_e O(1, n)) = \binom{n+1}{2}$.

Take $A_0 \in T_e O(1, n)$, there is a smooth curve $A(t) : (-\epsilon, \epsilon) \to O(1, n)$, for some $\epsilon > 0$, such that $A(0) = I_n$ and $\dot{A}(0) = A_0$.

Since $A(t) \in O(1, n)$ for all $t \in (-\epsilon, \epsilon)$,

$$A(t)^T J A(t) = J \quad \Rightarrow \quad \frac{d}{dt} A^T(t) J A(t) = 0$$
$$\Rightarrow \quad \dot{A}(t)^T J A(t) + A(t)^T J \dot{A}(t) = 0.$$

The last equation holds for $t = 0$. Hence,

$$A_0^T J + J A_0 = 0 \quad \Rightarrow \quad [J A_0]^T + J A_0 = 0.$$

Let $A_0 = [a_{ij}]_{i,j=1,\ldots,n}$, the last equation tells us that:

- $a_{ii} = 0$ for all $i = 0, \ldots, n$,
- $-a_{0i} + a_{i0} = 0$ for all $i = 1, \ldots, n$,
- $a_{ij} + a_{ji} = 0$ for all $i, j = 1, \ldots, n$.

Therefore, as each $A \in T_e O(1, n)$ is defined by the choice of the entries a_{ij} with indices $i, j = 0, \ldots, n$ and $i \neq j$, $\dim(T_e O(1, n)) = \dim(O(1, n)) = \binom{n+1}{2}$. $\square$

Positive Lorentz Transformations

We are interested in Lorentz transformations that preserve $\mathbf{C}^+$. In other words, we are interested in Lorentz transformations that send positive light-like vectors into positive light-like vectors. Such a Lorentz transformation is called a **positive Lorentz transformation** and the set of all positive Lorentz transformations is a subgroup of $O(1,n)$, we call it **the positive Lorentz group** and denote it by $O^+(1,n)$.

Remark 1.3. Here are some facts about $O^+(1,n)$

- A positive Lorentz transformation ϕ sends positive time-like vectors into positive time-like vectors.
- Let $A_\phi = [a_{ij}]_{i,j=0,\ldots,n}$ be the representation matrix of a positive Lorentz transformation ϕ. Since $\phi(e_0)$ is the first column of A, it holds that $a_{00} > 0$.
- As a result of the above, $O^+(1,n)$ is an open set in $O(1,n)$. Hence, it is a Lie subgroup of $O(1,n)$.
- $O^+(1,n)$ gives a transitive action on the set of k-dimensional time-like subspaces of $\mathbb{E}^{1,n}$. In other words, for any two k-dimensional time-like vector subspaces V_1 and V_2 there is a $\phi \in O^+(1,n)$ such that $\phi(V_1) = V_2$.

1.2. The Hyperboloid Model

Recall that the hyperbolic space is up to isometry the simply connected n-dimensional manifold with constant sectional curvature $K = -1$. We will build hyperbolic polyhedra in the hyperboloid model. Therefore, we will develop some basic notions as isometries, lines, hyperplanes etc... in this model. Note that these notions are equivalent in other models of the hyperbolic space. Also, we say that two Riemmanian manifolds are conformally equivalent, if the measured angles are the same.

Let us take the Lorentzian space $\mathbb{E}^{1,n}$. The hyperboloid model is the n-dimensional manifold given by

$$\mathbf{H}^n = \{x \in \mathbb{R}^{n+1} \quad | \quad \langle x, x \rangle_L = -1 \quad \text{and} \quad x_0 > 0\}$$

together with the Riemannian metric induced from the form $-dx_0^2 + dx_1^2 + \cdots + dx_n^2$. We identify the points lying on the boundary with the positive light-like rays, i.e., $\partial \mathbf{H}^n = \mathbf{C}^+ / x \sim \lambda x$.

Remark 1.4. Let us denote by $\mathring{\mathbf{C}}^+$ the positive vectors lying in the interior of the light cone. The image of the projection

$$\begin{aligned} \pi_{\mathbb{H}} : \mathbf{H}^n &\longrightarrow \mathring{\mathbf{C}}^+ \cap \{x_0 = 1\} \\ p = [p_0, ..., p_n] &\longmapsto [1, \frac{p_1}{p_0}, ..., \frac{p_n}{p_0}] \end{aligned}$$

corresponds to the Klein model $\mathbb{K}^n$ of the hyperbolic space. The points lying on the boundary $\partial \mathbb{K}^n$ correspond to the points in the sphere $\mathbb{S}_1^{n-1} = \mathbf{C}^+ \cap \{x_0 = 1\}$.

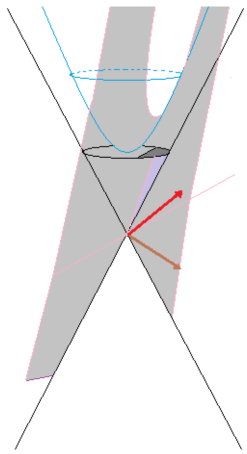

Figure 1.2.: $\mathbf{H}^2$, $\mathbb{K}^2$ and a hyperplane.

Isometries of $\mathbf{H}^n$

The group of isometries of $\mathbf{H}^n$ will be denoted by $\mathrm{Isom}(\mathbf{H}^n)$. It is the subgroup of diffeomorphisms of $\mathbf{H}^n$ that preserve the metric induced from the Lorentzian product. In other words, if $f \in \mathrm{diff}(\mathbf{H}^n) \quad \Rightarrow$

$$f \in \mathrm{Isom}(\mathbf{H}^n) \iff \quad \forall p \in \mathbf{H}^n \quad \text{and} \quad v, w \in T_p\mathbf{H}^n \quad \text{it holds that}$$
$$g_p(v,w) = g_{f(p)}(\mathrm{d}f_p v, \mathrm{d}f_p w)$$

Also remember that an element of $\mathrm{Isom}(\mathbf{H}^n)$ preserve lengths, distances and angles. Moreover, it sends geodesics, i.e, hyperbolic lines, into geodesics.

There is a close relation between the group $\mathrm{Isom}(\mathbf{H}^n)$ and the group of positive Lorentz transformations. The restriction of a positive Lorentz transformation to $\mathbf{H}^n$ is an isometry. On the other hand, any isometry can be extended in a unique way to a positive Lorentz transformation.

To see that the restriction of an element of $O^+(1,n)$ to $\mathbf{H}^n$ is an isometry, we need to take the following into account:

- The tangent space $T_p\mathbf{H}^n$ at a point $p \in \mathbf{H}^n$ is $\langle p\rangle^L$. In particular $T_{e_0}\mathbf{H}^n = \mathrm{span}\{e_1, \dots, e_n\}$.
- The Riemannian metric g evaluated at a point p is the Lorentzian product restricted to $\langle p\rangle^L$.

Proposition 1.7. *Given a positive Lorentz transformation $f \in O^+(1,n)$, its restriction to $\mathbf{H}^n$ is an element of Isom$(\mathbf{H}^n)$.*

Proof. It is clear that $\phi = f|_{\mathbf{H}^n}$ is an element of $\mathrm{diff}(\mathbf{H}^n)$.

Let A be the matrix representation of f with respect to the standard basis, then

$$f(x) = Ax \quad \text{and} \quad df_p v = Av, \quad \text{for all} \quad p \in \mathbb{E}^{1,n} \quad \text{and} \quad v \in T_p\mathbb{E}^{1,n}.$$

Take an element $p \in \mathbf{H}^n$. Since $T_p\mathbf{E}^{1,n} = \langle p\rangle \oplus \langle p\rangle^L = \langle p\rangle \oplus T_p\mathbf{H}^n$ (see Proposition 1.1), it holds that

$$d\phi_p = df_p\big|_{\langle p\rangle^L}$$

and for all $v, w \in T_p\mathbf{H}^n$

$$g_p(v,w) = \langle v,w\rangle_L = \langle Av, Aw\rangle_L = g_{\phi(p)}(d\phi_p v, d\phi_p w).$$

□

In fact, the restriction defines a group isomorphism between the positive Lorentz group and the group of isometries of $\mathbf{H}^n$ (see [16, Chap 2.] or [11, Chap.3]). Furthermore, together with Remark 1.3 and Proposition 1.6, this tells us that $\mathrm{Isom}(\mathbf{H}^n)$ is a Lie group of dimension $\binom{n+1}{2}$.

Some isometries that we will use are rotations, reflections through a given hyperbolic plane and translations on $\mathbf{H}^2$.

Hyperbolic Subspaces

To study the notion of hyperbolic subspaces, let us begin by studing hyperbolic lines. For this purpose take two different points $x_1, x_2 \in \mathbf{H}^n$. The vector space $V = \text{span}\{x_1, x_2\}$ is a 2-dimensional time-like vector subspace of $\mathbb{E}^{1,n}$. Therefore, Remark 1.3 tells us that there is an element $\phi \in O^+(1,n)$ such that $\phi(V) = \mathbb{E}^{1,1} = \text{span}\{e_0, e_1\}$. Also note that

$$\phi(V \cap \mathbf{H}^n) = \mathbb{E}^{1,1} \cap \mathbf{H}^n \tag{1.9}$$

is the hyperbolic line given by

$$l(t) = \cosh(t)e_0 + \sinh(t)e_1, \quad t \in \mathbb{R}. \tag{1.10}$$

Hence, since $\phi|_{\mathbf{H}^n} \in \text{Isom}(\mathbf{H}^n)$, the set $V \cap \mathbf{H}^n$ is a hyperbolic line. In fact, it is the unique hyperbolic line containing x_1, x_2 and it contains the hyperbolic line segment L_{x_1,x_2} between x_1 and x_2.

We can generalize the last result for higher dimensions. In other words, we will say that a **k-dimensional hyperbolic subspace** of $\mathbf{H}^n$ is the intersection of a $k+1$-dimensional time-like vector subspace V of $\mathbb{E}^{1,n}$ and $\mathbf{H}^n$.

Example 1.3.

- A 0-dimensional hyperbolic subspace is a point $p \in \mathbf{H}^n$.
- A 1-dimensional hyperbolic subspace is a hyperbolic line l.

Also, if $H_1 = V \cap \mathbf{H}^n$ is a k-dimensional hyperbolic subspace, then there is a positive Lorentz transformation such that $\phi(V) = \text{span}\{e_0, e_1, \ldots, e_k\} = \mathbb{E}^{1,k}$. Therefore, since $\phi(V \cap \mathbf{H}^n) = \mathbb{E}^{1,k} \cap \mathbf{H}^n = \mathbf{H}^k$ and $\phi|_{\mathbf{H}^n} \in \text{Isom}(\mathbf{H}^n)$, we can conclude that H_1 is isometric to $\mathbf{H}^k$.

Now, let $H_1 = V_1 \cap \mathbf{H}^n$ and $H_2 = V_2 \cap \mathbf{H}^n$ be two different hyperbolic subspaces of $\mathbf{H}^n$. Note that

$$H_1 \cap H_2 = V_1 \cap V_2 \cap \mathbf{H}^n$$

- $V_1 \cap V_2$ is a time-like vector subspace if and only if it intersects the interior of the light cone. In this case, $H_1 \cap H_2 \neq \emptyset$ and it is a hyperbolic subspace of $\mathbf{H}^n$.
- $V_1 \cap V_2$ is light-like if and only if it intersects the light cone but not its interior. The intersection of a non-zero light-like vector subspace with the light cone is a 1-dimensional light-like vectors subspace [11, Chap. 3]. For this reason, $V_1 \cap V_2 \cap \mathbf{C}$ corresponds to a single point in $\partial\mathbf{H}^n$ and we say that H_1, H_2 meet at a point at infinity.
- $V_1 \cap V_2$ is space-like if and only if this intersection is completely in the exterior of the light cone. In this case the intersection of the subspaces H_1, H_2 is empty and they don't meet at infinity.

Let us see that any hyperbolic subspace is the intersection of finitely many $(n-1)$-dimensional hyperbolic subspaces, and a criterion to see when we have one of the last three cases.

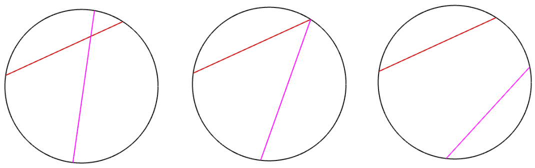

Figure 1.3.: Secant, parallel and ultra-parallel lines in $\mathbb{K}^2$

Hyperplanes

A $(n-1)$-dimensional hyperbolic subspace in $\mathbf{H}^n$ is called a **hyperbolic hyperplane**. The set of hyperbolic hyperplanes is parametrized by the set of Lorentz unit vectors, or, the **de Sitter sphere**. In other words, each Lorentz unit vector defines a hyperbolic hyperplane and for each hyperbolic hyperplane, we can find a Lorentz unit vector that defines it.

Example 1.4.

- In dimension 3 a hyperplane is a **hyperbolic plane**.
- In dimension 2 a hyperplane is a **hyperbolic line**.

The **de Sitter sphere** is the set of Lorentz unit vectors in $\mathbb{E}^{1,n}$, i.e., the set

$$\mathcal{H}_n = \{ \quad x \in \mathbb{E}^{1,n} \quad | \quad \langle x, x \rangle_L = 1 \quad \}$$

First, remember that for all $v \in \mathcal{H}_n$ the Lorentzian complement $\langle v \rangle^L$ of $\langle v \rangle$ is a time-like hyperplane (remember example 1.2). Therefore,

$$P_v = \langle v \rangle^L \cap \mathbf{H}^n \tag{1.11}$$

is a hyperbolic hyperplane.

On the other hand, if V is a n-dimensional time-like vector subspace, then

1. V^L is space-like.
2. $\mathbb{E}^{1,n} = V \oplus V^L$.
3. $(V^L)^L = V$

(see Corollaries 1.1, 1.2)

Therefore, we can find a space-like vector $v \in \mathcal{H}_n$ such that $V = \langle v \rangle^L$ and so, the hyperbolic hyperplane $V \cap \mathbf{H}^n$ can be written as in the equation (1.11).

Also note that:

- If $v \in \mathcal{H}_n$, then $-v \in \mathcal{H}_n$. Furthermore, $P_v = P_{-v}$.
- If $v, w \in \mathcal{H}_n$ are linear independent vectors. Then,

$$P_v = \langle v \rangle^L \cap \mathbf{H}^n \neq \langle w \rangle^L \cap \mathbf{H}^n = P_w.$$

Thus, v and $-v$ are the unique Lorentz unit vectors which define the hyperplane P_v.

Lorentz Product and Intersections

Lemma 1.2. *Let us take $v_1, \ldots, v_k \in \mathbb{E}^{1,k}$. If $V = Span\{v_1, \ldots, v_k\}$, then*

$$V^L = \bigcap_{i=1}^{k} \langle v_i \rangle^L.$$

Proof.

$$x \in V^L \iff \langle x, v_i \rangle_L = 0 \quad \text{for all i}$$

$$\iff x \in \bigcap_{i=1}^{k} \langle v_i \rangle^L$$

□

Proposition 1.8. *A k-dimensional hyperbolic subspace of $\mathbf{H}^n$ is the intersection of $(n-k)$ hyperplanes.*

Proof. Let $H = V \cap \mathbf{H}^n$ be a k-dimensional hyperbolic subspace. Since V is a $(k+1)$-dimensional time-like vector subspace, from the corollaries 1.1 and 1.2 we know that

- V^L is space-like.
- $\mathbb{E}^{1,n} = V \oplus V^L$.

Therefore, dim$V^L = n-k$ and we can find $n-k$ linear independent vectors $v_1, \ldots, v_{n-k} \in \mathcal{H}_n$ such that $V^L = \text{span}\{v_1, \ldots, v_{n-k}\}$. By Lemma 1.2

$$V = (V^L)^L = \bigcap_{i=1}^{n-k} \langle v_i \rangle^L$$

and so

$$H = \bigcap_{i=1}^{n-k} \langle v_i \rangle^L \cap \mathbf{H}^n = \bigcap_{i=1}^{n-k} P_{v_i}$$

□

Now, let $\bar{\mathbf{H}}^n = \mathbf{H}^n \cup \partial \mathbf{H}^n$ be the compactification of the hyperbolic space, and $\bar{P}_v$ be the closure of hyperbolic hyperplane defined by v in $\bar{\mathbf{H}}^n$. If we take $v, w \in \mathcal{H}_n$, the value of the Lorentzian inner product between the two vectors tells us if the intersection of the closures of the corresponding hyperplanes is empty or not.

Proposition 1.9. *Take $v, w \in \mathcal{H}_n$ such that $v \neq \pm w$, then*

1. $P_v \cap P_w \neq \emptyset \iff 1 - \langle v, w \rangle_L^2 > 0.$
2. $\emptyset \neq \bar{P}_v \cap \bar{P}_w \subset \partial \mathbf{H}^n \iff \langle v, w \rangle_L^2 = 1.$
3. $\bar{P}_v \cap \bar{P}_w = \emptyset \iff 1 - \langle v, w \rangle_L^2 < 0.$

Proof. Take $V = \text{span}\{v, w\}$ and note that by Lemma 1.2 it holds that

$$\begin{aligned} \bar{P}_v \cap \bar{P}_w &= \langle v \rangle^L \cap \langle w \rangle^L \cap \bar{\mathbf{H}}^n \\ &= V^L \cap \bar{\mathbf{H}}^n \end{aligned}$$

Thus, condition 1. holds if and only if V^L is time-like, condition 2. holds if and only if V^L is light-like and condition 3. holds if and only if V^L is space-like.

The representation matrix of the inner product on V with respect to the basis $B = \{v, w\}$ is

$$G(B) = \begin{bmatrix} 1 & \langle v, w \rangle_L \\ \langle w, v \rangle_L & 1 \end{bmatrix}$$

and $\det G(B) = 1 - \langle v, w \rangle_L^2$. Therefore, by Corollary 1.1

$$\begin{aligned} &\text{Condition 1. holds if and only if} \quad 1 - \langle v, w \rangle_L^2 > 0 \\ &\text{Condition 2. holds if and only if} \quad 1 - \langle v, w \rangle_L^2 = 0 \\ &\text{Condition 3. holds if and only if} \quad 1 - \langle v, w \rangle_L^2 < 0 \end{aligned}$$

□

- If the intersection of two hyperplanes P_v and P_w is not empty, then it holds that

  $$\langle v, w \rangle_L = -\cos(\measuredangle vw),$$

  where $\measuredangle vw$ is the interior angle between the hyperplanes P_v and P_w. We call it the **dihedral angle** at the intersection.

- If the hyperplanes P_v, P_w meet at a point at infinity, then the **dihedral angle** is equal to zero.

- If the closures don't intersect, then the Lorentzian inner product gives us the hyperbolic distance between the two hyperplanes

  $$\langle v, w \rangle_L = -\cosh(d_{\mathbb{H}}(P_v, P_w)).$$

Remark 1.5. If $\{v_1, \dots, v_k\}$, with $k < n$, is a set of linearly independent vectors in $\mathcal{H}_n$ we can use a similar argument to the one given in Proposition 1.9 to show that the intersection $\bigcap_{i=1}^k \bar{P}_{v_i} \neq \emptyset$ if and only if

$$\begin{vmatrix} \langle v_1, v_1 \rangle_L & \dots & \langle v_1, v_k \rangle_L \\ \vdots & \ddots & \vdots \\ \langle v_k, v_1 \rangle_L & \dots & \langle v_k, v_k \rangle_L \end{vmatrix} \geq 0. \tag{1.12}$$

Some Computations and Results in $\mathbf{H}^3$

Let us see some results in the 3-dimensional hyperbolic space.

Proposition 1.10. *Let P_{v_1} and P_{v_2} two hyperbolic planes in $\mathbf{H}^3$. If $v_1 \neq \pm v_2$ and $P_{v_1} \cap P_{v_2} \neq \emptyset$, then the intersection is a hyperbolic line and $\langle v_1 v_2 \rangle_L = -\cos(\measuredangle v_1 v_2)$.*

Proof. Since $P_{v_1} \cap P_{v_2} \neq \emptyset$ and $v_1 \neq \pm v_2$ we know that $\langle v_1, v_2 \rangle_L = -\cos(\measuredangle v_1, v_2)$. On the other hand, by lemma 1.2

$$P_{v_1} \cap P_{v_2} = \langle v_1 \rangle^L \cap \langle v_2 \rangle^L \cap \mathbf{H}^3 = V^L \cap \mathbf{H}^3,$$

where $V = \text{span}\{v_1, v_2\}$. It is clear that V^L is time-like and $\dim(V) = 2$, which tells us that $\dim(V^L) = 2$. Therefore, $P_{v_1} \cap P_{v_2}$ is a 1-dimensional hyperbolic subspace, i.e, a hyperbolic line. □

Note that by the Propositions 1.8 and 1.10, a hyperbolic line l in $\mathbf{H}^3$ is the intersection of two hyperbolic planes, i.e, there are $v_1, v_2 \in \mathcal{H}_3$ such that $l = P_{v_1} \cap P_{v_2}$. Moreover, the hyperbolic line l is also a hyperbolic line of both hyperbolic planes.

Proposition 1.11. *If $P_{v_1} P_{v_2} P_{v_3}$ is a sequence of hyperbolic planes such that:*

1. *The vectors v_1, v_2 and v_3 are linearly independent,*
2. *$l_1 = P_{v_1} \cap P_{v_2}, l_2 = P_{v_2} \cap P_{v_3}$ and $l_3 = P_{v_3} \cap P_{v_1}$ are hyperbolic lines.*
3. *The dihedral angles $\alpha_1 = \measuredangle v_1 v_2, \alpha_2 = \measuredangle v_2 v_3, \alpha_3 = \measuredangle v_3 v_1$ are less or equal to $\frac{\pi}{2}$.*

Then, $\bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} \neq \emptyset$ if and only if $\alpha_1 + \alpha_2 + \alpha_3 \geq \pi$. Moreover, the intersection is a single point p and

$$p \in \mathbf{H}^3 \quad \Longleftrightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 > \pi$$

$$p \in \partial \mathbf{H}^3 \quad \Longleftrightarrow \alpha_1 + \alpha_2 + \alpha_3 = \pi.$$

Proof. conditions 1. and 2. tells us that $\langle v_1, v_2 \rangle_L = -cos(\alpha_1)$, $\langle v_2, v_3 \rangle_L = -cos(\alpha_2)$ and $\langle v_3, v_1 \rangle_L = -cos(\alpha_3)$. On the other hand, by remark 1.5 $\bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} \neq \emptyset$ if and only if the determinant of the matrix

$$G(B) = [\langle v_i, v_j \rangle_L]_{i,j=1,2,3} = \begin{bmatrix} 1 & -\cos(\alpha_1) & -\cos(\alpha_3) \\ -\cos(\alpha_1) & 1 & -\cos(\alpha_2) \\ -\cos(\alpha_3) & -\cos(\alpha_2) & 1 \end{bmatrix}$$

is bigger or equal to zero.

We compute to get that

$$\det G(B) = 1 - \cos^2(\alpha_1) - \cos^2(\alpha_2) - \cos^2(\alpha_3) + 2\cos(\alpha_1)\cos(\alpha_2)\cos(\alpha_2)$$

and using the equality $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$, the last expression is equal to

$$-4\cos\left(\frac{\alpha_1 + \alpha_2 + \alpha}{2}\right)\cos\left(\frac{-\alpha_1 + \alpha_2 + \alpha_3}{2}\right)\cos\left(\frac{-\alpha_2 + \alpha_1 + \alpha_3}{2}\right)\cos\left(\frac{-\alpha_3 + \alpha_1 + \alpha_2}{2}\right)$$

(see [14, page.838]).
Since $0 < \alpha_1, \alpha_2, \alpha_2 \leq \frac{\pi}{2}$, it holds that $-\pi < -\alpha_i+\alpha_j+\alpha_k < \pi$ and so $\cos\left(\frac{-\alpha_i+\alpha_j+\alpha_k}{2}\right) > 0$. Thus,

$$\det G(B) \begin{cases} > 0 & \iff & \alpha_1 + \alpha_2 + \alpha_3 > \pi \\ = 0 & \iff & \alpha_1 + \alpha_2 + \alpha_3 = \pi \\ < 0 & \iff & \alpha_1 + \alpha_2 + \alpha_3 < \pi \end{cases} \tag{1.13}$$

and so the first part of the proposition is done.

For the second part, note that $P_{v_1} \cap P_{v_2} \cap P_{v_3}$ can be seen as the intersection of the hyperbolic lines $l_1 = P_{v_1} \cap P_{v_2}$ and $l_2 = P_{v_3} \cap P_{v_2}$ on the hyperbolic plane P_{v_2}. Therefore, $\bar{l}_1 \cap \bar{l}_2 = \bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3}$ is either the empty set, a point on the boundary of P_{v_2} or a point in P_{v_2} (see Figure 1.3). From the last argument, if $\bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} \neq \emptyset$, then the intersection corresponds to a single point $p \in \bar{\mathbf{H}}^3$.

In our case $p \in \mathbf{H}^3$ if and only if $\det\ G(B) > 0$, and, $p \in \partial\mathbf{H}^3$ if and only if $\det\ G(B) = 0$. Hence, by the equation (1.13) we are done. $\square$

Other Hyperbolic Models

In the course of the proof we will also use, the projective Klein model $\mathbb{K}^n$, the Poincare ball model $\mathbb{D}^n$ and the upper half-space model $\mathbb{H}^n$ of the hyperbolic space.

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- A brief description of the projective Klein model and its relation with the hyperboloid model was given at the beginning of this chapter (see Remark 1.5). Note that the set of points of projective Klein model is the intersection of the interior of the light cone and the Euclidean hyperplane $\{x \in \mathbb{E}^{1,n} | x_0 = 1\}$, i.e, $\mathbb{K}^n = \mathring{\mathbf{C}}^+ \cap \{x_0 = 1\}$. On the other hand, if $v \in \mathcal{H}_n$, the image of the hyperbolic hyperplane P_v in the projective Klein model is the set
$$E_v = \langle v \rangle^L \cap \mathbb{K}^n.$$
In general, hyperbolic subspaces correspond to the intersection of time-like vector subspaces and $\mathbb{K}^n$. Also this model is not conformally equivalent to the open unit ball in the Euclidean space.
- The Poincare ball model is given by the set
$$\mathbb{D}^n = \{ \quad x \in \mathbb{R}^n \quad | \quad \|x\| < 1\}$$
together with the metric
$$4\frac{dx_1^2 + \cdots + dx_n^2}{(1 - \|x\|^2)^2}.$$
The k-dimensional hyperbolic subspaces are the intersection of $\mathbb{D}^n$ with k-spheres and k-planes in $\mathbb{R}^n$ orthogonal to $\partial\mathbb{D}^n$. The projection $\pi_{\mathbb{D}} : \mathbf{H}^n \to \mathbb{D}^n$ given by
$$p(x) = \frac{x}{x_0 + 1}$$
is an isometry between both spaces. Moreover, This model is conformally equivalent to the open unit ball in the n-dimensional Euclidean space.

- The upper half-space model is given by the set

$$\mathbb{H}^n = \{ \quad x \in \mathbb{R}^n \quad | \quad x_n > 0 \quad \},$$

together with the metric

$$\frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

The k-dimensional hyperbolic spaces are k-spheres and k-planes that intersect $\partial\mathbb{H}^n$ perpendicularly. This model is conformally equivalent to the upper half-space in the Euclidean space.

1.3. Convex Polyhedra

In the euclidean space and the hyperbolic space a **convex set** is a set $\mathbf{D}$ with the property that it contains the unique line segment between any two points in $\mathbf{D}$.

It is well know that any vector subspace, a ball and any euclidean half-space

$$\tilde{h}_w = \{ \quad x \in \mathbb{R}^{1+n} \quad | \quad \langle x, w \rangle \leq 0 \quad \}, \quad w \in \mathbb{R}^{n+1}$$

are convex sets in the euclidean space.

An important property of convex sets is that the intersection of two convex sets is again a convex set. Note that in the Klein model a hyperbolic line segment L_{xy} between two points $x, y \in \mathbb{K}^n$ coincide with the euclidean line segment between them. Moreover, since $\mathbb{K}^n$ is the interior of a unit sphere, if W is a time-like vector subspace or an euclidean half-space, then

$$W \cap \mathbb{K}^n$$

is a convex set in both the euclidean space and the hyperbolic space.

The last remark tells us that any hyperbolic subspace is convex set. Moreover, since the interior of them with respect to $\mathbf{H}^n$ is empty, we say that they are **degenerate** convex sets.

As we already stated at the beginning of this chapter, a convex polyhedron is the intersection of finitely many half-spaces. We will begin by describing half-spaces in $\mathbf{H}^n$. After that we will study convex polyhedra in $\mathbf{H}^n$, specially polyhedra of finite volume, and some of their combinatorial, topological and geometrical properties.

Half-Spaces

Take $v \in \mathcal{H}_n$. The **(hyperbolic) closed half-space** defined by v is the closed set

$$H_v = \{ \quad x \in \mathbf{H}^n \quad | \quad \langle x, v \rangle_L \leq 0 \quad \}.$$

For a unit space-like vector $v \in \mathcal{H}_n$ we have:

1. $\partial H_v = P_v$.
2. $H_v \cap H_{-v} = P_v$ and $H_v \cup H_{-v} = \mathbf{H}^n$. Hence, P_v divides $\mathbf{H}^n$ in two closed regions or two sides.
3. With respect of the Lorentzian product, v is the outward pointing vector with respect of the Lorentz product of the half-space

   $$\begin{aligned} h_v &= \{ \quad x \in \mathbf{R}^{n+1} \quad | \quad \langle x, v \rangle_L \leq 0 \quad \} \\ &= \tilde{h}_{Jv} \end{aligned}$$

   and

   $$H_v = h_v \cap \mathbf{H}^n.$$

   Thus, by the equation (1.3) and the remark that follows it, H_v is a convex set in $\mathbf{H}^n$ and v is **the outward pointing vector** of H_v.

Although $P_v = P_{-v}$, the closed half spaces H_v and H_{-v} are different (see Figure 1.4). This tells us that there is a bijection between the elements of $\mathcal{H}_n$ and the set of closed half-spaces in $\mathbf{H}^n$. In this sense we can understand $\mathcal{H}_n$ as the space of closed half-spaces and give it the structure of a differentiable manifold of dimension n.

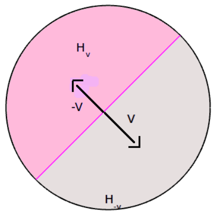


Figure 1.4.: Half-Planes in $\mathbb{K}^2$

Proposition 1.12. *The space of half-spaces $\mathcal{H}_n$ is a differentiable manifold of dimension n.*

Proof. Let us take the differentiable map

$$\begin{aligned} f : \mathbb{E}^{n,1} &\longrightarrow \mathbb{R} \\ x &\longmapsto -x_0^2 + x_1^2 + \cdots + x_n^2, \end{aligned}$$

note that $\mathcal{H}_n = f^{-1}(1)$ and $df = [-2x_0, \ldots, 2x_n]$. The differential is equal to zero if and only if $x = 0$, but zero doesn't belong to $\mathcal{H}_n$. Therefore, by the inverse function theorem $\mathcal{H}_n$ is a differentiable manifold of dimension $\quad dim\mathbb{E}^{n,1} - dim\mathbb{R} = n$. □

Convex Polyhedra

Definition 1.4. A **convex polyhedron** is the intersection of finitely many half-spaces (both in the euclidean and the hyperbolic space). We also ask that the interior of the polyhedron is not empty.

Take a convex polyhedron

$$\mathbf{P} = \bigcap_{i=1}^{N} H_{v_i} \quad \text{in} \quad \mathbf{H}^n.$$

To have a better understanding of $\mathbf{P}$ consider the $(n+1)$-dimensional polyhedral cone

$$\mathbf{C_P} = \bigcap_{i=1}^{N} h_{v_i} \quad \text{in} \quad \mathbb{E}^{1,n}.$$

We can assume without loss of generality that $\{v_1, \ldots, v_N\}$ is a minimal set of vectors specifying $\mathbf{C_P}$. Minimality means that there is no a proper subset $A \subset \{v_1, \ldots, v_N\}$ such that the intersection of the half-spaces defined by the vectors in A is equal to $\mathbf{C_p}$. Furthermore, since

$$\mathbf{P} = \mathbf{C_p} \cap \mathbf{H^n}$$

$\{v_1, \ldots, v_N\}$ is **the unique minimal subset of vectors** in $\mathcal{H}_n$ specifying $\mathbf{P}$.

Proposition 1.13. *If $\{v_1, ..., v_N\}$ is the minimal set of vectors specifying a polyhedron* $\mathbf{P}$ *in* $\mathbf{H}^n$*. Then,*

1. *Any two vectors are linear independent.*
2. *Any three different vectors v_i, v_j and v_k are linearly independent.*

Proof.

1. It is clear since we assumed that $\mathbf{P}$ is non-degenerate.
2. We can assume that $v_i = \lambda v_k + \beta v_j$. Since $\{v_1, \ldots, v_N\}$ is the minimal set of vectors specifying the polyhedron $\mathbf{P}$, none of the half-spaces defined by this vectors contain an intersections of the others. Take $x \in H_{v_k} \cap H_{v_j}$ and let us assume without loss of generality that λ, β are both bigger than zero, it holds that $\langle x, v_i \rangle_L = \lambda \langle x, v_k \rangle_L + \beta \langle x, v_j \rangle_L \leq 0$. This means that $x \in H_{v_i}$ and so, $H_{v_k} \cap H_{v_j} \subseteq H_{v_i}$, which contradicts the minimality of the set $\{v_1, \ldots, v_N\}$. □

Faces of $\mathbf{C_P}$

To understand the combinatorial and topological properties of the polyhedron $\mathbf{P}$, let us begin by taking the boundary of $\mathbf{C_P}$ with respect to $\mathbb{E}^{1,n}$

$$\partial \mathbf{C_P} = \{x \in \mathbf{C_P} \quad | \quad \langle x, v_i \rangle_L = 0 \quad \text{for some} \quad v_i\}.$$

A **supporting hyperplane** of $\mathbf{C_P}$ is a subspace of codimension 1 that intersects $\mathbf{C_P}$ along $\partial\mathbf{C_P}$, and a **face** of $\mathbf{C_P}$ is the intersection of $\mathbf{C_P}$ with a supporting hyperplane. We also consider that the whole polyhedron and the empty set are faces of $\mathbf{C_P}$.

If F is a proper non-empty face of $\mathbf{C_P}$, then F is a convex set lying on $\partial\mathbf{C_P}$. Furthermore, if k is the dimension of the vector space generated by F, we say that F is a **k-dimensional face**. It is clear that $\langle v_i \rangle^L$ is a supporting hyperplane of $\mathbf{C_P}$. In fact, the faces of the form

$$C_{v_i} = \langle v_i \rangle^L \cap \mathbf{C_P}$$

are all the n-dimensional faces of $\mathbf{C_P}$, $\partial\mathbf{C_P} = \bigcup_{i=1}^N C_{v_i}$ and any lower dimensional face is the intersection of some of these faces.

Faces of $\mathbf{P}$ and Volume

Now, let us consider the boundary of $\mathbf{P}$ with respect to $\mathbf{H}^n$, note that

$$\partial \mathbf{P} = \partial \mathbf{C_P} \cap \mathbf{H}^n.$$

A **k-dimensional face** of $\mathbf{P}$ is a non-empty intersection of a $(k+1)$-dimensional face of $\mathbf{C_P}$ and $\mathbf{H^n}$. We also consider that the empty set and the whole polyhedron are faces of $\mathbf{P}$, in this case $\mathbf{P}$ is the unique face of dimension n and the empty set is the unique face of dimension -1. Since the euclidean hyperplane $\langle v_i \rangle^L$ is time-like, it holds that

$$\begin{aligned} F_{v_i} &= C_{v_i} \cap \mathbf{H}^n \\ &= \langle v_i \rangle^L \cap \mathbf{P} \\ &= P_{v_i} \cap \mathbf{P} \end{aligned} \tag{1.14}$$

is a $(n-1)$-dimensional face of $\mathbf{P}$. Note that these are all the $(n-1)$-dimensional faces of $\mathbf{P}$. Thus, $\partial\mathbf{P} = \bigcup_{i=1}^{N} F_{v_i}$ and any face of lower dimension is the intersection of some of the $(n-1)$-dimensional faces.

We also say that P_{v_i} is the **supporting (hyperbolic) hyperplane** of the face F_{v_i} or the hyperplane carrying the face. Furthermore, $\mathbf{P}$ lies entirely in one of the sides of P_{v_i}, i.e, closed half-space H_{v_i}.

The polyhedral cone $\mathbf{C_P}$ intersects with the positive part of the interior of the light cone. Hence, it makes sense to consider the n-dimensional euclidean polyhedron

$$\tilde{P}_{\mathbb{K}} = \mathbf{C_P} \cap \{x_0 = 1\},$$

note that

$$\begin{aligned}\mathbf{P}_{\mathbb{K}} &= \tilde{P}_{\mathbb{K}} \cap \mathbb{K}^n \\ &= \mathbf{C_P} \cap \mathbb{K}^n\end{aligned}$$

is the image of the hyperbolic polyhedron $\mathbf{P}$ in the Klein model.

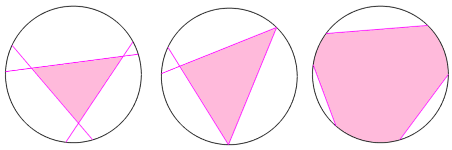


Figure 1.5.: $\tilde{P}_{\mathbb{K}} \cap \mathbb{K}^2$

For $n \geq 2$, $\mathbf{P}$ has **finite volume** if and only if $\tilde{P}_{\mathbb{K}} \subset \bar{\mathbb{K}}^n$, in this case $\tilde{P}_{\mathbb{K}}$ is a bounded polyhedron, i.e, a n-dimensional polytope. We have two possibilities:

- $\tilde{P}_{\mathbb{K}}$ has some of its vertices (0-dimensional faces) inscribed on $\partial\mathbb{K}^n = \mathbb{S}_1^{n-1}$. We call this vertices **ideal vertices** and in the case that all the vertices lie on $\partial\mathbb{K}^n$ we say that $\mathbf{P}$ is an ideal polyhedron.
  For the sake of completeness we also consider that the ideal vertices are 0-dimensional faces of $\mathbf{P}$ and we distinguish between ideal vertices and finite vertices, or vertices in $\mathbf{H}^n$.
- $\mathbf{P}_{\mathbb{K}} = \tilde{P}_{\mathbb{K}}$. In this case $\mathbf{P}$ is **compact** and all the vertices are finite.

Now, if $\mathbf{P}$ is a finite volume polyhedron in $\mathbf{H}^n$, the above tells us that the closure of $\partial\mathbf{P}$ in $\bar{\mathbf{H}}^n$ is isomorphic to a cellular decomposition of $\mathbb{S}^{n-1}$. There is also a bijection between the set of faces of $\mathbf{P}$ and the set of faces of $\tilde{P}_{\mathbb{K}}$. Hence, we have the following laws of incidence:

a. If F and G are faces of $\mathbf{P}$, then $F \cap G$ is a face of $\mathbf{P}$.

b. If F is a face of $\mathbf{P}$ and G is a face of F, then G is a face of $\mathbf{P}$.

This tells us that the inclusion " $\subseteq$ " defines a partial order on the set of faces of $\mathbf{P}$. We also call it the incidence relation on the faces of $\mathbf{P}$ (an ideal vertex is assumed to be included in the faces of $\mathbf{P}$ that contain it on their closures).

Definition 1.5. Let $\mathbf{F_P}$ be the set of faces of a polyhedron $\mathbf{P}$. Take $A, B \in \mathbf{F_P}$.

- If $A \subseteq B$, then A, B are incident faces.
- If A, B are k-dimensional faces such that $k > 0$ and $A \cap B$ is a $(k-1)$-dimensional face, then A, B are adjacent faces.
- If A, B intersect at a vertex, we can also say that they meet at a vertex.

Now, assume that F_{v_i}, F_{v_j} are adjacent faces of $\mathbf{P}$. Since the intersection of the faces lies in $\mathbf{H}^n$

$$\langle v_i, v_j \rangle_L = -cos(\measuredangle v_i v_j).$$

In this case $F_{ij} = F_{v_i} \cap F_{v_i}$ is a $(n-2)$-dimensional face, and the dihedral angle $\alpha = \measuredangle v_1 v_2$ is the interior angle between the faces F_{v_i} and F_{v_j}.

Convex Hull

In the euclidean and the hyperbolic geometry, the **convex hull** of a set $\mathbf{C}$ is the smallest convex set that contains $\mathbf{C}$. We denote it by

$$\mathrm{conv}(\mathbf{C}).$$

It is a well known result that a polytop is the convex hull of its set of vertices. Thus, if $\mathbf{P}$ is a polyhedron of finite volume in $\mathbf{H}^n$ and $V(\mathbf{P})$ its set of vertices, then

$$\mathbf{P} = \mathrm{conv}(V(\mathbf{P})).$$

Example 1.5. A compact polyhedron in $\mathbf{H}^1$ is a hyperbolic line segment $L_{x_1 x_2}$ and it is clear that

$$L_{x_1 x_2} = \mathrm{conv}(\{x_1, x_2\}).$$

1.4. Polyhedra of Finite Volume in $\mathbf{H}^2$ and $\mathbf{H}^3$

The goal of this section is to describe finite volume polyhedra in $\mathbf{H}^2, \mathbf{H}^3$ and to study some properties of non-obtuse compact polyhedra in this spaces. We will begin by taking a polyhedron of finite volume

$$\mathbf{P} = \bigcap_{i=1}^{N} H_{v_i} \tag{1.15}$$

in $\mathbf{H}^n$, where n=2 or 3, and assume that $\{v_1, \ldots, v_N\}$ is the minimal set of $\mathcal{H}_n$ specifying it. Note that in this case the maximal dimension that of a proper face is 2. Therefore, as usual we call 0-dimensional faces **vertices**, 1-dimensional faces **sides** or **edges** and 2-dimensional faces just **faces**.

Remark 1.6. In this section and further chapters we might assume that $\mathbf{P}$ lies in other model of the hyperbolic space, in this case we will replace the index v_i by just i.

Polyhedra of Finite Volume in $\mathbf{H}^2$

Assume that $\mathbf{P}$ lies in $\mathbf{H}^2$, as we saw in the last section, the closure of the image of $\mathbf{P}$ in the compactification of the Klein model is a convex euclidean polygon. Therefore, $\mathbf{P}$ is a convex hyperbolic polygon with N-sideas and N-vertices. Also, a side of $\mathbf{P}$ is either a hyperbolic line, a hyperbolic ray or a hyperbolic line segment, and the vertices of $\mathbf{P}$ are the endpoints of its sides (the limit points on $\partial\mathbf{H}^2$, in the case that the side is a hyperbolic line or ray).

The corresponding dihedral angles $\beta_1, \ldots, \beta_N$ are the interior angles at the vertices of $\mathbf{P}$ (note that $\beta_i = 0$ in the case that β_i is the interior angle at an ideal vertex). Moreover, by the Gauss-Bonnet Theorem (see Appendix A)

$$\beta_1 + \cdots + \beta_N < (n-2)\pi. \tag{1.16}$$

Remark 1.7. In fact, given a collection $\beta_1, \ldots, \beta_N$ of real numbers in the interval $[0, \pi)$, we can find a convex hyperbolic N-gon with interior angles $\beta_1, \ldots, \beta_N$ if and only if equation (1.16) holds.

Remark 1.8. Instead of F_{v_i} we will denote a side of $\mathbf{P}$ by s_{v_i} and by l_{v_i} the corresponding supporting line. We will also denote by $x_1, \ldots, x_N$ the vertices of $\mathbf{P}$ and assume that β_i is the interior angle at x_i.

Compact Polygons and Parallelograms

We will assume that $s_{v_i}, s_{v_{i+1}}$, for $i = 1, \ldots, N-1$ and s_{v_1}, s_{v_N} are adjacent sides, i.e, $s_{v_i}, s_{v_{i+1}}$ (or s_{v_1}, s_{v_N}) intersect at a finite vertex or meet at a ideal vertex. In fact, if $\mathbf{P}$ is a compact polygon, then its sides are hyperbolic line segments and we can assume without loss of generality that

$$s_{v_1} = L_{x_N x_1} \quad \text{and} \quad s_{v_i} = L_{x_{i-1} x_i} \quad \text{for} \quad i \in \{2, \ldots, N\}.$$

Note that

$$x_N = s_{v_N} \cap s_{v_1} \quad \text{and} \quad x_i = s_{v_i} \cap s_{v_{i+1}} \quad \text{for} \quad i \in \{1, \ldots, N-1\}.$$

Otherwise, $\quad s_{v_i} \cap s_{v_j} = \emptyset$.

Definition 1.6. A compact polygon $\mathbf{P}$ in $\mathbf{H}^2$ is called a **parallelogram**, if for any two sides s_{v_i}, s_{v_j} of $\mathbf{P}$ such that $s_{v_i} \cap s_{v_j} = \emptyset$, its corresponding supporting lines l_{v_i}, l_{v_j} don't intersect and don't meet at infinity.

Proposition 1.14. *If the interior angles β_i at the vertices of a compact polygon $\mathbf{P}$ are less or equal to $\frac{\pi}{2}$, then $\mathbf{P}$ is a parallelogram.*

Proof. Let us assume that $\mathbf{P}$ is not a parallelogram. Then, there are two non-adjacent sides s_{v_i}, s_{v_j} of $\mathbf{P}$ such that its corresponding supporting lines l_{v_i}, l_{v_j} intersect or meet at a point at infinity. Let $p \in \bar{H}^2$ be the point of intersection of $\bar{l}_{v_i}$ and $\bar{l}_{v_j}$. Note that the polygon $\mathbf{P}$ lies inside the closed region $H_{v_i} \cap H_{v_j}$. Moreover, $H_{v_i} \cap H_{v_j} - \mathbf{P}$ is a set divided in two disjoint regions inside of $H_{v_i} \cap H_{v_j}$. Let $\mathbf{D}$ be the closure of the region containing p. We have two possibilities:(see Figure 1.6)

a. $\mathbf{D}$ is a hyperbolic triangle,

b. $\mathbf{D}$ is a non-convex hyperbolic polygon.

(a) (b)

Figure 1.6.

We may assume that $i < j$ and that β is the interior angle of $\mathbf{D}$ at p.

a. If $\mathbf{D}$ is a triangle, the interior angles of $\mathbf{D}$ are $\beta, \pi - \beta_i$ and $\pi - \beta_j$. Furthermore, since $\beta_i, \beta_j \leq \frac{\pi}{2}$, we know that $\pi - \beta_i \geq \frac{\pi}{2}$, then

$$\beta + (\pi - \beta_i) + (\pi - \beta_j) \geq \beta + \pi \geq \pi$$

which contradicts equation (1.16).

b. Now let **D** be a non convex polygon. Using the Gauss-Bonnet-Theorem (see appendix A) we get

$$\begin{aligned} Area(D) =&(\pi-\beta)+\beta_i+\sum_{k=i+1}^{j-1}(\beta_k-\pi)+\beta_j-2\pi \\ \leq&(\pi-\beta)-\pi+\sum_{k=i+1}^{j-1}(\beta_i-\pi) \\ =&-\beta+\sum_{k=i+1}^{j-1}(\beta_i-\pi) \\ \leq&0. \end{aligned}$$

As a result of the above, we get a contradiction. □

Polyhedra of Finite Volume in $\mathbf{H}^3$

Now, if **P** is a polyhedron of finite volume in $\mathbf{H}^3$, we know that the closure of the image of **P** in the ompactification of the Klein model is a 3-dimensional polytope. Therefore, each face F_{v_i} of **P** is a convex hyperbolic polygon carried by the supporting plane P_{v_i}. Also. by the incidence laws, a non-empty intersection of two faces is a common side, or common vertex of them.

In fact, each edge e of **P** is the intersection of exactly two adjacent faces. Let us assume that

$$\begin{aligned} e &= F_{v_i}\cap F_{v_j} \\ &= \langle v_i\rangle^L\cap\langle v_J\rangle^L\cap\mathbf{P}, \end{aligned}$$

note that e is a commune side of both F_{v_i} and F_{v_j}, and that the hyperbolic line

$$l=\langle v_i\rangle^L\cap\langle v_j\rangle^L\cap\mathbf{H}^3=P_{v_i}\cap P_{v_j}$$

is the supporting line of e with respect of the supporting planes P_{v_i} and P_{v_j}. Also, the dihedral angle $\alpha=\measuredangle v_1v_2$ is strictly bigger than zero, otherwise e would be a point at infinity. We say that **$\boldsymbol{\alpha}$ is the dihedral angle** at e.

Finally, note that the vertices of **P** are the end points of its edges and

$$V(\mathbf{P})=\bigcup_{i=1}^{N}V(F_{v_i}).$$

Cellular Structure

To have a better understanding of the shape of the boundary of **P** and its combinatorial structure, let us denote by $E(\mathbf{P})$ set of edges of **P** and consider the graph $G(\mathbf{P})=(V(\mathbf{P}),E(\mathbf{P}))$ (in this case the 1-skeleton of $\partial\mathbf{P}$ as a CW-complex). As we will see in Chapter 2, $G(\mathbf{P})$ is a simple, 3-connected and planar graph (see Appendix C). This tells us that

- For all $y \in V(\mathbf{P})$ the number of edges incident to y is bigger or equal to 3.
- $G(\mathbf{P})$ has one connected component.
- There is an embedding $\tilde{\psi} : G(\mathbf{P}) \longrightarrow \mathbb{S}^2$. (If we assume that $\mathbf{P}$ lies in $\mathbb{K}^3$ a possible embedding is given by the radial projection.)

The graph $G = \tilde{\psi}(G(\mathbf{P}))$ is a plane graph. Hence, it divides $\mathbb{S}^2$ in N disjoint regions whose boundaries are sub-cycles of G, we denote by $\tilde{F}_1, \dots, \tilde{F}_N$ the closure of those regions. Using this, G defines a cellular decomposition $\mathcal{M}$ of the unit sphere $\mathbb{S}^2$, where

- G is the 1-skeleton of $\mathcal{M}$. Hence, the vertices of G are the closures of the 0-cells and the edges of G are the closures of the 1-cells. We also say that G is the graph of $\mathcal{M}$ and use the notation $G = G(\mathcal{M}) = (V(\mathcal{M}), E(\mathcal{M}))$.
- The closed regions $\tilde{F}_1, \dots, \tilde{F}_N$ are the closures of the 2-cells of $\mathcal{M}$, and we call them the faces of $\mathcal{M}$.

The graph isomorphism

$$\tilde{\psi} : G(\mathbf{P}) \longrightarrow G$$

can be extended to a **cellular homeomorphism**

$$\psi : \partial\mathbf{P} \longrightarrow \mathcal{M}$$

i.e, a homeomorphism that sends vertices to vertices, edges to edges and faces to faces. Moreover, since ψ is a homeomorphism, for all $A, B \in \mathbf{F_P}$ such that $A \subseteq B$ it holds that $\psi(A) \subseteq \psi(B)$. Therefore, if we consider the set $\mathbf{F}_{\mathcal{M}}$ that contains the empty set, the vertices, edges and faces of $\mathcal{M}$ and the hole unit ball, and order it by the inclusion, we can conclude that $\mathbf{F}_{\mathcal{M}}$ and $\mathbf{F_P}$ are isomorphic partially ordered sets.

Remark 1.9. Following Andreev [3], $\mathcal{M}$ is a cellular decomposition of $\mathbb{S}^2$ with the following properties:

M1. Every edge belongs to exactly two faces.

M2. Every face contains no fewer than three edges.

M3. A non empty intersection of two faces is either an edge or a vertex.

Remark 1.10. We can do the last construction for any simple, 3-connected, planar graph $\mathbf{G}$. Moreover, the cellular decomposition doesn't depend on the choice of the embedding. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two cellular decomposition of $\mathbb{S}^2$ that we get via two different embeddings of $\mathbf{G}$, by a theorem by Whitney on 3-connected simple planar graphs there is a cellular homeomorphism $\varphi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$. (see [6, Chap.4]).

Remark 1.11. Since the smallest planar simple, 3-connected, planar graph is the complete graph with four vertices we can conclude that $N \geq 4$. If $N = 4$, then $\mathbf{P}$ is a tetrahedron.

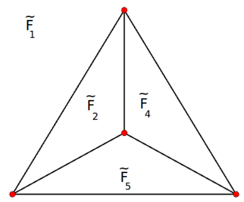


Figure 1.7.: Complete graph with 4-vertices

Congruent Polyhedra

Now, take an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ and let $\tilde{\phi}$ be its unique extension in $O^+(1,3)$. Since $\tilde{\phi}$ is a positive Lorentz transformation, it holds that

$$\langle x, v_i \rangle_L = \langle \tilde{\phi}(x), \tilde{\phi}(v_i) \rangle_L \text{ for all } x \in \mathbf{H}^3 \text{ and } i \in \{1, \dots, N\}.$$

Using this equation, we can conclude that $\phi(P_{v_i}) = P_{\tilde{\phi}(v_i)}$, and that

$$\phi(\mathbf{P}) = \bigcap_{i=1}^{N} H_{\tilde{\phi}(v_i)}.$$

As ϕ is an isometry, it holds that $\mathbf{P}$ and $\phi(\mathbf{P})$ have the same volume, hence $\phi(\mathbf{P})$ is a polyhedron of finite volume. Furthermore, note that $\{\tilde{\phi}(v_1), \dots, \tilde{\phi}(v_N)\}$ is the minimal set of vectors in $\mathcal{H}_3$ specifying $\phi(\mathbf{P})$ and that

$$\begin{aligned} \phi(F_{v_i}) &= \phi(P_{v_i} \cap \mathbf{P}) \\ &= P_{\tilde{\phi}(v_i)} \cap \phi(\mathbf{P}) \\ &= F_{\tilde{\phi}(v_i)}. \end{aligned} \tag{1.17}$$

As any element $A \in \mathbf{F_P} - \{\mathbf{P}\}$ is the intersection of some of the faces of $\mathbf{P}$, equation (1.17) tells us that ϕ sends faces to faces, edges to edges and vertices to vertices. Moreover, since any isometry is a homeomorphism, we can conclude that $\phi\big|_{\partial\mathbf{P}}$ is a cellular homeomorphism between $\partial\mathbf{P}$ and $\partial\phi(\mathbf{P})$.

The last consideration tells us that the sets of faces $F_\mathbf{P}$ and $F_{\phi(\mathbf{P})}$ are isomorphic as partially ordered sets. Moreover, since ϕ preserves lengths, dihedral angles etc... we can say that $\mathbf{P}$ and $\phi(\mathbf{P})$ are basically the same polyhedron but in different positions of the space. If $\mathbf{P}$ lies in $\mathbf{H}^2$ and $\phi \in \mathrm{Isom}(\mathbf{H}^2)$ we can use a similar argumentation to get an analogous result.

Definition 1.7. Two finite volume polyhedra $\mathbf{P_1}, \mathbf{P_2}$ in $\mathbf{H}^n$, $n = 2, 3$, are **congruent** if there is an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^n)$ such that $\phi(\mathbf{P_1}) = \mathbf{P_2}$.

Remark 1.12. Take two unit vectors $v, w \in \mathcal{H}_3$. If $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are convex polygons lying respectively on the hyperbolic planes P_v and P_w, we can say that $\mathbf{Q_1}$ and $\mathbf{Q_2}$ are congruent if their images under the isometries that take P_v and P_w to $\mathbf{H}^2$ are congruent polygons. This is equivalent to say that there is an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ such that $\phi(\mathbf{Q_1}) = \mathbf{Q_2}$. Moreover, if this is the case it must hold that $\phi(P_v) = P_w$

Non-obtuse compact Polyhedra in $\mathbf{H}^3$

Assume that a vertex y of $\mathbf{P}$ is incident to exactly k edges, each edge is the intersection of exactly two faces. Therefore, we can find k faces $F_{v_{i_1}}, \ldots, F_{v_{i_k}}$ of $\mathbf{P}$ such that

$$F_{v_{i_1}} \cap F_{v_{i_k}} \quad \text{and} \quad F_{v_{i_j}} \cap F_{v_{i_{j+1}}} \quad \text{for} \quad , j \in \{1, \ldots, k\},$$

are the edges incident to y and so, we can conclude that a vertex is the intersection of at least 3 faces. In fact, if $\mathbf{P}$ is a compact polyhedron with non-obtuse dihedral angles in $\mathbf{H}^3$, each vertex of $\mathbf{P}$ is the intersection of exactly 3 faces.

Proposition 1.15. *If $\mathbf{P}$ is a compact polyhedron in $\mathbf{H}^3$ with non-obtuse dihedral angles, then*

1. *Every vertex of $\mathbf{P}$ is the intercession of exactly three faces.*
2. *The interior angles of the faces of $\mathbf{P}$ are also less or equal to $\frac{\pi}{2}$*

Proof. (For this proof we will assume that $\mathbf{P}$ lies in $\mathbb{D}^3$).
Assume that $y \in V(P)$ is the intersection of k faces. It is clear that none of the vertices of $\mathbf{P}$ lies on $\partial\mathbb{D}^3$, hence by using an appropriate isometry we may assume that y lies at the origin and that the faces containing y lie on euclidean planes perpendicular to $\partial\mathbb{D}^3$. Take a small sphere S_y centred at y, note that $Q = \mathbf{P} \cap S_y$ is a spherical polygon with k-sides. In addition, the interior angles at the vertices of Q are the dihedral angles $\alpha_1, \ldots, \alpha_k$ at the edges that are incident to y.

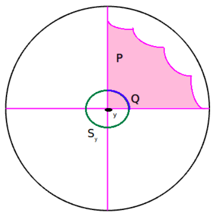


Figure 1.8.: $Q = S_y \cap P$

Rescale Q such that it lies on the sphere of radius 1.

1. We know that $k \geq 3$. On the other hand, by the Gauss-Bonneth Theorem, it holds that

$$\begin{aligned} (k-2)\pi &= \sum_{i=1}^{k} \alpha_i - \text{Area}(Q) \Rightarrow \\ \text{Area(Q)} &= \sum_{i=1}^{k} \alpha_i - (k-2)\pi \\ &\leq k\frac{\pi}{2} - (k-2)\pi \end{aligned} \tag{1.18}$$

Since $Area(Q)$ is strictly bigger than zero, from the equation (1.18) we can conclude that $k\pi < 4\pi$ and so $k = 3$.

2. By the last result, we know that Q is a spherical triangle, and that there are exactly three faces F_1, F_2, F_3 such that $y = F_1 \cap F_2 \cap F_3$. Assume that

$$e_1 = F_1 \cap F_2, \quad e_2 = F_2 \cap F_3 \quad \text{and} \quad e_3 = F_3 \cap F_1$$

are the edges incident to y and that α_i are their dihedral angels. Moreover, if we assume that β_j is the interior angle at y on the face F_j, then β_j is the length of corresponding side of Q.

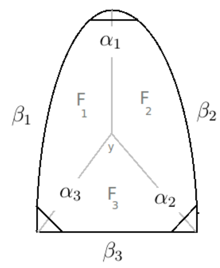


Figure 1.9.

By the law of cosines in the spherical geometry (see Appendix B) we get that

$$\cos(\beta_3) = \frac{\cos(\alpha_1) + \cos(\alpha_2)\cos(\alpha_3)}{\sin(\alpha_2)\sin(\alpha_3)}.$$

Therefore, as $0 < \alpha_i \leq \frac{\pi}{2}$, we get that $cos(\beta_3) \geq 0$. Furthermore, since we are considering convex polyhedra, the dihedral angels are less than π, and so we can conclude that

$$0 < \beta_3 \leq \frac{\pi}{2}.$$

In a equivalent way, we can see that β_1 and β_2 are non-obtuse angles. □

Corollary 1.3. *The faces of a compact hyperbolic polyhedron in* $\mathbf{H}^3$ *whose dihedral angles are less or equal to* $\frac{\pi}{2}$ *are parallelograms.*

Proof. It is a consequence of the Propositions 1.15 and 1.14. □

2. 3-dimensional Polyhedra and Andreev's Theorem

In his Papers [3] [4] Andreev gives us a classification of the non-obtuse polyhedra of finite volume, different to the tetrahedron, in the 3- dimensional hyperbolic space. In this chapter the basic combinatorial and topological tools used by Andreev to state and prove his theorem will be explained. We will also state the theorem for compact polyhedra and understand the strategy to prove it.

2.1. Combinatorics of Polyhedra and Andreev's Theorem for Compact Polyhedra

As Andreev did in his paper, we will begin by defining abstract polyhedra which are the basic combinatorial tools to classify polyhedra. An abstract polyhedron is basically a partially ordered set (in short poset) $(\mathfrak{m}, \leq)$ that looks like the poset of faces of a polyhedron (polytope) in the euclidean space. Before we give a precise definition, we must make some basic definitions about partially ordered sets.

Definition 2.1. Let $(\mathfrak{m}, \leq)$ be a poset,

- $\mathfrak{m}$ is bounded if it has a minimal element $\hat{0}$ and a maximal element $\hat{1}$.
- A chain is a totally ordered subset of $\mathfrak{m}$ and its length is the number of its elements minus 1.
- If $A, B \in \mathfrak{m}$, then the interval $[A, B]$ is the set $\{C \in \mathfrak{m} | \quad A \leq C \leq B\}$.
- If $A \in \mathfrak{m}$, the length of a maximal chain in the interval $[\hat{0}, A]$ is the rank of A.
- $\mathfrak{m}$ is graded if it is bounded, and every maximal chain has the same rank.
- $\mathfrak{m}$ is connected, if given any two elements $A, B \in \mathfrak{m}$, there is a sequence of elements $H_1, \dots, H_k \in \mathfrak{m}$ such that $A = H_1$, $B = H_k$ and either $H_i \leq H_{i+1}$ or $H_{i+1} \leq H_i$. Also, $\mathfrak{m}$ is strongly connected if it is connected and every interval is connected.

Definition 2.2. An **abstract polyhedron** is a poset $(\mathfrak{m}, \leq)$ whose elements will be called faces such that

1. $\mathfrak{m}$ is a graded poset.
2. $\mathfrak{m}$ is strongly connected.
3. Each interval of rank 2 is isomorphic the the poset of faces of an edge.

The partial order $\leq$ can be understand as the incidence relation and instead of $(\mathfrak{m}, \leq)$, we will denote an abstract polyhedron by $\mathfrak{m}$.

A **realization** of an abstract polyhedron $\mathfrak{m}$ is a geometrical or topological representation of the abstract polyhedron. For example

- A polyhedron $\mathbf{P}$ whose poset of faces $\mathbf{F_P}$ is isomorphic to $\mathfrak{m}$.
- A CW-complex $\mathcal{M}$ such that the poset $F_{\mathcal{M}}$ that we obtain by ordering their their closed cells by the inclusion (we also include the empty set) is isomorphic to $\mathfrak{m}$.

We are interested in abstract polyhedra that are realizable as polyhedra of finite volume in the 3-dimensional hyperbolic space. As we saw in the last chapter, if we consider the closure of a polyhedron of finite volume in $\mathbb{K}^3$, this is a 3-dimensional polytope. Hence, if an abstract polyhedron $\mathfrak{m}$ is realizable as a 3-dimensional hyperbolic polyhedron, then it is a graded poset of rank 4. We can denote by $V(\mathfrak{m})$ the set of faces of rank 1 (vertices), $E(\mathfrak{m})$ the set of faces of rank 2 (edges) and by $F(\mathfrak{m})$ the set of faces of rank 3 (faces). Moreover, from now on by an abstract polyhedron, we will mean an abstract polyhedron that is realizable as a 3-dimensional polytope.

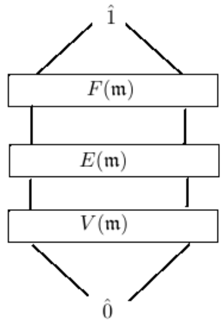


Figure 2.1.: Hasse diagram of $\mathfrak{m}$

Two polytopes are **combinatorial equivalent** if their face posets are isomorphic as partially ordered sets (in general we will say that two realizations of the same abstract polyhedron are combinatorial equivalent) and the equivalent classes of 3-dimensional polytopes under this equivalence relation are called **combinatorial types**. Note that up to isomorphism there is a unique abstract polyhedron $\mathfrak{m}$ representing each combinatorial type. Furthermore, the abstract polyhedra are classified by the Steinitz theorem and the theorem of Whitney that we mentioned in Remark 1.10.

Definition 2.3. The graph of a polytope is the graph whose set of vertices is the set of vertices of the polytope and whose set of edges is the set of edges of the polytope.

Theorem 2.1 (**Steinitz Theorem**)**.** *The graph of a 3-dimensional polytope is a simple, 3-connected, planar graph. On the other hand, any simple, 3-connected, planar graph is isomorphic to the graph of a 3-dimensional polytop.*

By the Steinitz Theorem, if $\mathfrak{m}$ is an abstract polyhedron, then the graph

$$G(\mathfrak{m}) = (V(\mathfrak{m}), E(\mathfrak{m})) \tag{2.1}$$

is a simple, 3-connected planar graph. Therefore, by Remark 1.10 we can find a cellular decomposition $\mathcal{M}$ of $\mathbb{S}^2$ that realizes $\mathfrak{m}$ (we just forget about the interior of the unit ball), also $\mathcal{M}$ is unique up to cellular homeomorphism. In less sophisticated words, to find an abstract polyhedron it is enough to draw a plane graph on $\mathbb{R}^2$ (or $\mathbb{S}^2$) following the properties (axioms) given in Remark 1.9. Moreover, two different drawings of the same polyhedron are combinatorial equivalent .

Compact Polyhedra in $\mathbf{H}^3$ and Abstract Polyhedra

Now, let us assume that an abstract polyhedron $\mathfrak{m}$ is realizable as a compact polyhedron $\mathbf{P}$ in $\mathbf{H}^3$ with non-obtuse dihedral angles, and choose a cellular decomposition $\mathcal{M}$ of $\mathbb{S}^2$ realizing $\mathfrak{m}$. The isomorphism of posets

$$\hat{\psi} : \mathfrak{m} \longrightarrow \mathbf{F_P}$$

can be translated into a cellular homeomorphism

$$\psi : \mathcal{M} \longrightarrow \partial\mathbf{P}$$

In fact, we can understand $\mathcal{M}$ as a schematic drawing of $\mathbf{P}$. Moreover, by proposition 1.15, it holds that $\mathfrak{m}$ and $\mathcal{M}$ are trivalent, i.e, each vertex of $\mathfrak{m}$ (or $\mathcal{M}$) is incident to exactly three edges.

Definition 2.4. The dual $\mathfrak{m}^*$ of $\mathfrak{m}$ is the poset with the same elements of $\mathfrak{m}$ but with the reversed order.

It is well known that $\mathfrak{m}^*$ is also an abstract polyhedron. In fact, if we consider the cellular decomposition $\mathcal{M}^*$ of $\mathbb{S}^2$, that we get by taking the dual of the plane graph $G(\mathcal{M})$, then $\mathcal{M}^*$ is a realization of $\mathfrak{m}^*$. It holds that,

- A face $\tilde{F}_i$ of $\mathcal{M}$ corresponds to dual vertex v_i^*.
- The edge $e = \tilde{F}_i \cap \tilde{F}_j$ corresponds to the dual edge $e^* = (v_i^* v_j^*)$.
- A vertex v_i corresponds to the dual face $\tilde{F}_i^*$.

Proposition 2.1. *$\mathcal{M}^*$ is a triangulation of $\mathbb{S}^2$.*

Proof. Let us take $y \in V(\mathcal{M})$ and let e_1, e_2, e_3 be the edges of $\mathcal{M}$ incident to y. The vertex y is the intersection of exactly three faces $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ such that $e_1 = \tilde{F}_1 \cap \tilde{F}_2$, $e_2 = \tilde{F}_2 \cap \tilde{F}_3$ and $e_3 = \tilde{F}_1 \cap \tilde{F}_3$. Therefore, the edges e_1, e_2, e_3 correspond to the dual edges $e_1^* = (v_1^* v_2^*)$, $e_2^* = (v_2^* v_3^*)$ and $e_3^* = (v_1^* v_3^*)$. Note that $\gamma = e_1^* e_2^* e_3^*$ is a cycle in $\mathcal{M}^*$ that surrounds the vertex v, and that no other vertices or edges of $\mathcal{M}^*$ lie in the interior region defined by the cycle γ. Therefore, $F_v^* = int(\gamma) \cup \gamma$ is a face of $\mathcal{M}^*$ and it is the dual face corresponding to v. □

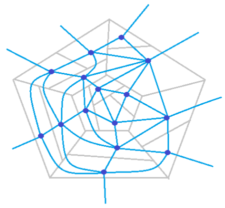


Figure 2.2.: $\mathcal{M}^*$

We will use use $\mathcal{M}$ and $\mathcal{M}^*$ to learn more about the about the combinatorial properties of $\mathbf{P}$.

Definition 2.5. Let $\tilde{F}_{i_1}, ..., \tilde{F}_{i_k}$ be a sequence of faces of $\mathcal{M}$ (or $\mathbf{P}$) such that $\tilde{F}_{i_l}, \tilde{F}_{i_{l+1}}$ for $l \in \{1, \ldots, k-1\}$, and $\tilde{F}_1, \tilde{F}_k$ are adjacent faces, and none of them is adjacent to other faces in the sequence. If no three faces meet at a vertex, then we say that the sequence of faces $\tilde{F}_{i_1}, ..., \tilde{F}_{i_k}$ is a ***k*-angled prismatic element**.

Definition 2.6. If $\gamma = e^*_{i_1} \ldots e^*_{i_k}$ is a cycle in $\mathcal{M}^*$ such that no two edges in $\{e_{i_1}, \ldots, e_{i_k}\}$ share a commune vertex, then γ is a k-prismatic cycle.

The last two definitions are equivalent. If $\tilde{F}_{i_1}, ..., \tilde{F}_{i_k}$ is a k-angled prismatic element and

$$e_{i_k} = \tilde{F}_{i_1} \cap \tilde{F}_{i_k} \quad \text{and} \quad e_{i_j} = \tilde{F}_{i_j} \cap \tilde{F}_{i_{j+1}},$$

then $\gamma = e^*_{i_1} \ldots e^*_{i_K}$ is k-prismatic cycle. On the other hand, if $\gamma = e^*_{i_1} \ldots e^*_{i_K}$ is a k-prismatic cycle, then the sequence of faces whose intersections are the edges in $\{e_{i_1}, \ldots e_{i_k}\}$ is a k-angled prismatic element.

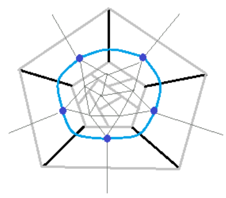


Figure 2.3.: 5-angled prismatic element

In general a n-cycle $\gamma = e^*_{i_1} e^*_{i_2} \ldots e^*_{i_n}$ in $\mathcal{M}^*$ can be translated into a sequence of adjacent faces in $\mathcal{M}$ (or $\mathbf{P}$). However, not every n-cycle is prismatic.

Proposition 2.2. *If $\gamma = e^*_{i_1} e^*_{i_2} e^*_{i_3}$ is a cycle that is not prismatic, then the edges e_{i_1}, e_{i_2} and e_{i_3} meet at a vertex.*

Proof. If the cycle $\gamma = e^*_{i_1} e^*_{i_2} e^*_{i_3}$ is not prismatic, at least two of the corresponding edges in $\mathcal{M}$ meet at a vertex y. Assume that e_{i_1} and e_{i_2} meet at the vertex y. Since $\mathcal{M}$ is trivalent, there is a third edge e of $\mathcal{M}$ that is incident to y. If $e \neq e_{i_3}$, then $e^* e^*_{i_3}$ is a cycle in $\mathcal{M}^*$, which contradicts the fact that the dual graph is simple. (see Figure 2.4) $\square$

Now, note that in the case that $\mathcal{M}$ has a 3-angled prismatic element, a 4-cycle in $\mathcal{M}^*$ may looks as in the following figure.

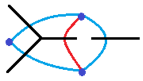
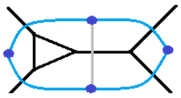


Figure 2.4.



Proposition 2.3. *If $\mathcal{M}$ (or* **P***) doesn't have a 3-angled prismatic element, then each 4-cycle that is not prismatic surrounds exactly one edge.*

Proof. If $\gamma = e^*_{i_1}e^*_{i_2}e^*_{i_3}e^*_{i_4}$ is a 4-cycle that is not prismatic, then at least two of the corresponding edges in $\mathcal{M}$ meet at a vertex. Assume that e_{i_1}, e_{i_2} meet at a vertex y, since $\mathcal{M}$ is trivalent, there is a third edge e of $\mathcal{M}$ incident to y. Note that $\gamma_1 = e^*e^*_{i_3}e^*_{i_4}$ is a 3-cycle. Furthermore, since $\mathcal{M}$ doesn't have 3-angled prismatic elements, γ_1 is not a prismatic 3-cycle and so, by Proposition 2.2 the edges e, e_{i_3} and e_{e_4} meet at a vertex. Therefore, γ surrounds the edge e and no other one. (see Figure 2.5)

□

Figure 2.5.

We are ready to state the Andreev's Theorem for compact polyhedra.

Theorem 2.2 (**Andreev's Theorem**). *Let $\mathfrak{m}$ be a trivalent abstract polyhedron with more than 4 faces, fix a cellular decomposition $\mathcal{M}$ of $\mathbb{S}^2$ that realizes $\mathfrak{m}$ and consider a weight function*

$$\alpha : E(\mathcal{M}) \longrightarrow (0, \frac{\pi}{2}].$$

Up to isometry, there is a unique 3-dimensional compact hyperbolic polyhedron that realizes $\mathfrak{m}$, and whose dihedral angles are the corresponding weights on the edges of $\mathcal{M}$ if and only if

[a1] For all $e \in E(\mathcal{M})$, then $\alpha(e) \in (0, \frac{\pi}{2}]$.

[a2] If the edges e_{i_1}, e_{i_2} and e_{i_3} meet at a vertex, then

$$\alpha(e_{i_1}) + \alpha(e_{i_2}) + \alpha(e_{i_3}) > \pi$$

[a3] If $\gamma = e^*_{i_1} e^*_{i_2} e^*_{i_3}$ is a prismatic 3-cycle, then

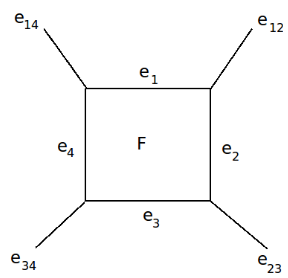
$$\alpha(e_{i_1}) + \alpha(e_{i_2}) + \alpha(e_{i_3}) < \pi$$

[a4] If $\gamma = e^*_{i_1} e^*_{i_2} e^*_{i_3} e^*_{i_4}$ is a prismatic 4-cycle, then

$$\alpha(e_{i_1}) + \alpha(e_{i_2}) + \alpha(e_{i_3}) + \alpha(e_{i_4}) < 2\pi$$

[a5] If $\tilde{F}$ is a quadrilateral face with boundary $\partial\tilde{F} = e_1e_2e_3e_4$ and e_{ij} is the third edge meeting the adjacent edges e_i and e_j, then

$$\alpha(e_1) + \alpha(e_3) + \alpha(e_{12}) + \alpha(e_{23}) + \alpha(e_{34}) + \alpha(e_{14}) < 3\pi$$
$$\alpha(e_2) + \alpha(e_4) + \alpha(e_{12}) + \alpha(e_{23}) + \alpha(e_{34}) + \alpha(e_{14}) < 3\pi$$



2.2. The Space of Marked Polyhedra

To prove Theorem 2.2 we will consider the set of compact polyhedra with non-obtuse dihedral angles in $\mathbf{H}^3$, realizing $\mathfrak{m}$ and with a enumeration of its faces induced by $\mathfrak{m}$, modulo isometries. We will give a topology to this set by seen it as a subset of a manifold of dimension $3N-6$, where N is the number of faces of $\mathfrak{m}$.

To define the manifold, first let us consider a compact polyhedron $\mathbf{P}$ in $\mathbf{H}^3$ realizing $\mathfrak{m}$ and take a pair $(\mathbf{P}, [\psi]_\sim)$ where $[\psi]_\sim$ is an equivalence class of cellular homeomorphims

$$\psi : \mathcal{M} \longrightarrow \partial \mathbf{P}$$

under the equivalence relation $\psi_1 \sim \psi_2$ if and only if there is a cellular isotopy between them. The equivalence class $[\psi]_\sim$ is called a **marking** and it holds that ψ_1 and ψ_2 are cellular isotopic if and only if $\psi_1(A) = \psi_2(A)$ for all $A \in \mathbf{F}_{\mathcal{M}}$. Therefore, by choosing a fix enumeration $\{\tilde{F}_1, \dots, \tilde{F}_N\}$ of the faces of $\mathcal{M}$, it induces an enumeration, or marking, of the faces of $\mathbf{P}$ given by

$$\psi(\tilde{F}_i) = F_{v_\psi(i)},$$

where $v_\psi(i)$ is a unit vector specifying $\mathbf{P}$. The N-tuple $[v_\psi(1), \dots, v_\psi(N)] \in \mathcal{H}_3^N$ contains the whole information about the marking $[\psi]_\sim$ and it gives us a coordinate for the pair $(\mathbf{P}, [\psi]_\sim)$.

Now, let us take the set of **marked polyhedra**

$$\mathcal{O}_{\mathfrak{m}} = \Big\{ (\mathbf{P}, [\psi]_\sim) | \quad \mathbf{P} \text{ is a compact polyhedron in } \mathbf{H}^3 \text{ realizing } \mathfrak{m} \text{ and } [\psi]_\sim \text{ is a marking} \Big\}$$

and consider the injective map

$$\begin{array}{rccc} C: & \mathcal{O}_{\mathfrak{m}} & \longrightarrow & \mathcal{H}_3^N \\ & (\mathbf{P}, [\psi]_\sim) & \mapsto & [v_\psi(1), \dots, v_\psi(N)]. \end{array}$$

Proposition 2.4. *$C(\mathcal{O}_{\mathfrak{m}})$ is an open set in $\mathcal{H}_3^N$.*

Proof. Let $\hat{V} = [V_\psi(1), \dots, V_\psi(N)]$ be the coordinate of a pair $(\mathbf{P}, [\psi]_\sim)$ in $\mathcal{O}_{\mathfrak{m}}$. Since $\mathbf{P}$ is a compact polyhedron, by moving the unit vectors $V_\psi(i)$ a little bit we get a polyhedron that is combinatorially equivalent to $\mathbf{P}$ and we don't change the enumeration of the faces.

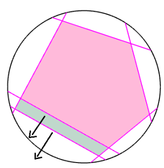


Figure 2.6.

Therefore, we can find an open neighbourhood $U_{\hat{V}}$ of $\hat{V}$ in $C(\mathcal{O}_{\mathfrak{m}})$. $\square$

From the Proposition 1.12 we can deduce that $\mathcal{H}_3^N$ is a manifold of dimension $3N$. Therefore, by giving $\mathcal{O}_{\mathfrak{m}}$ the topology induced by C, the last proposition tells us that $\mathcal{O}_{\mathfrak{m}}$ is a manifold of dimension $3N$.

Now, consider the subset $\mathcal{O}_{\mathfrak{m}}^1$ of $\mathcal{O}_{\mathfrak{m}}$ containing the marked polyhedra $(\mathbf{P}, [\psi]_\sim)$ where $\mathbf{P}$ is a polyhedron whose faces are parallelograms.

Proposition 2.5. *$\mathcal{O}_{\mathfrak{m}}^1$ is an open subset of $\mathcal{O}_{\mathfrak{m}}$.*

Proof. Let us take a face $\tilde{F}_i$ of $\mathcal{M}$, two edges e_{i_1}, e_{i_2} on the boundary of $\tilde{F}_i$ such that $e_{i_1} \cap e_{i_2} = \emptyset$, and a marked polyhedron $[(\mathbf{P}, [\psi]_\sim)] \in \mathcal{O}_{\mathfrak{m}}^1$. Assume without loss of generality that

$$\begin{aligned}
\psi(\tilde{F}_j) &= F_{v_j} \\
\psi(e_{i_1}) &= P_{v_1} \cap P_{v_j} \cap \mathbf{P} = s_{i_1} \\
\psi(e_{i_2}) &= P_{v_2} \cap P_{v_j} \cap \mathbf{P} = s_{i_2}.
\end{aligned}$$

Since ψ is a cellular homeomorphism, it holds that $s_{i_1} \cap s_{i_2} = \emptyset$. Moreover, as F_{v_j} is a parallelogram the supporting lines

$$l_{i_1} = \langle v_1 \rangle^L \cap \langle v_j \rangle^L \cap \mathbf{H}^3$$

and

$$l_{i_2} = \langle v_2 \rangle^L \cap \langle v_j \rangle^L \cap \mathbf{H}^3$$

don't intersect and don't meet at infinity. On the other hand, by Remark 1.5 and Proposition 1.13

$$\bar{l}_{i_1} \cap \bar{l}_{i_2} = \langle v_1 \rangle^L \cap \langle v_2 \rangle^L \cap \langle v_j \rangle^L \cap \bar{\mathbf{H}}^3 = \emptyset$$

if and only if

$$\begin{vmatrix} 1 & \langle v_1, v_2 \rangle_L & \langle v_1, v_j \rangle_L \\ \langle v_1, v_2 \rangle_L & 1 & \langle v_2, v_j \rangle_L \\ \langle v_1, v_j \rangle_L & \langle v_2, v_j \rangle_L & 1 \end{vmatrix} < 0. \tag{2.2}$$

The last paragraph tells us that a marked polyhedron $(\mathbf{P}, [\psi]_\sim) \in \mathcal{O}_{\mathfrak{m}}$ is an element of $\mathcal{O}_{\mathfrak{m}}^1$ if and only if a set of open conditions as the one given in equation (2.2) are satisfied. Hence, $\mathcal{O}_{\mathfrak{m}}^1$ is an open set in $\mathcal{O}_{\mathfrak{m}}$. $\square$

The last proposition tells us that $\mathcal{O}_{\mathfrak{m}}^1$ is also a manifold of dimension $3N$. On the other hand, an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ preserves distances and angles. Therefore, if the faces of $\mathbf{P}$ are parallelograms, then the faces pf $\phi(\mathbf{P})$ are parallelograms as well. Moreover, as we saw in section 1.4 $\phi\big|_{\partial \mathbf{P}}$ is a cellular homeomorphism from $\partial \mathbf{P}$ to $\partial \phi(\mathbf{P})$. Thus, $\phi\big|_{\partial \mathbf{P}} \circ \psi$ is a cellular homeomorphism from $\mathcal{M}$ to $\partial \mathbf{P}$ and we can consider the marking $[\phi\big|_{\partial \mathbf{P}} \circ \psi]_\sim$. Using this we can define an action of $\mathrm{Isom}(\mathbf{H}^3)$ on $\mathcal{O}_{\mathfrak{m}}^1$ by

$$\phi(\mathbf{P}, [\psi]_\sim) = (\phi(\mathbf{P}), [\phi\big|_{\partial \mathbf{P}} \circ \psi]_\sim) \quad \text{where} \quad \phi \in \mathrm{Isom}(\mathbf{H}^3).$$

Let us consider the set of orbits or equivalent classes of $\mathcal{O}_{\mathfrak{m}}^1$ under this action

$$\mathsf{P}_{\mathfrak{m}}^1 = \mathcal{O}_{\mathfrak{m}}^1 / \mathrm{Isom}(\mathbf{H}^3)$$

Proposition 2.6. $\mathsf{P}^1_{\mathfrak{m}}$ *is a manifold of dimension* $3N-6$.

Proof. Take $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ and assume that

$$\begin{aligned}\phi(\mathbf{P}, [\psi]_\sim) &= (\phi(\mathbf{P}), [\phi\big|_{\partial\mathbf{P}} \circ \psi]_\sim)\\ &= (\mathbf{P}, [\psi]_\sim).\end{aligned}$$

Since $\phi|_{\partial\mathbf{P}} \circ \psi$ and ψ are cellular isotopic, it holds that $\phi(\psi(A)) = \psi(A)$ for all $A \in \mathbf{F_P}$. Therefore, ϕ fixes the vertices, edges and faces of $\mathbf{P}$ and so, if we consider the unique extension $\tilde{\phi} \in O^+(1,3)$ of ϕ, we can conclude that $\tilde{\phi}$ is a linear isomorphism that fixes more that 4 linear independent vectors of $\mathbb{E}^{1,3}$. Hence, we get that $\tilde{\phi} = id_{\mathbb{E}^{1,3}}$ and so $\phi = id_{\mathbf{H}^3}$, which tells us that the action of $\mathrm{Isom}(\mathbf{H}^3)$ on $\mathcal{O}^1_{\mathfrak{m}}$ is free. On the other hand, if $[v_\psi(1), \ldots, v_\psi(N)]$ is the coordinate of $(\mathbf{P}, [\psi]_\sim)$ in $C(\mathcal{O}^1_{\mathfrak{m}})$, then its image $(\phi(\mathbf{P}), [\phi\big|_{\partial\mathbf{P}} \circ \psi]_\sim)$ has coordinate $[\tilde{\phi}(v_\psi(1)), \ldots, \tilde{\phi}(v_\psi(N))]$. Thus, as $\tilde{\phi}$ is a linear isomorphism, it defines a homeomorphism from $C(\mathcal{O}^1_{\mathfrak{m}})$ into itself and we can conclude that the action of $\mathrm{Isom}(\mathbf{H}^3)$ on $\mathcal{O}^1_{\mathfrak{m}}$ is smooth. Finally, let E be the number of edges of $\mathfrak{m}$, fix a enumeration $e_1, \ldots, e_E$ of the edges of $\mathcal{M}$ and consider the continuous map

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}: \quad & \mathcal{O}^1_{\mathfrak{m}} \longrightarrow \mathbb{R}^E\\ & (\mathbf{P}, [\psi]_\sim) \mapsto [\alpha(\psi(e_1)), \ldots, \alpha(\psi(e_N))]^T\end{aligned}$$

where $\alpha(\psi(e_i))$ is the dihedral angle of $\mathbf{P}$ at the edge $\psi(e_i)$. Note that the continuity of $\tilde{\boldsymbol{\alpha}}$ comes from the way that we defined the topology of $\mathcal{O}_{\mathfrak{m}}$ and the continuity of $\langle , \rangle_L$. Also, it is clear that

$$\tilde{\boldsymbol{\alpha}}(\phi\mathbf{M}) = \tilde{\boldsymbol{\alpha}}(\mathbf{M})$$

for all $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ and $\mathbf{M} \in \mathcal{O}^1_{\mathfrak{m}}$. Moreover, in Section 3.2 we will see that $\tilde{\boldsymbol{\alpha}}(\mathbf{M}_1) = \tilde{\boldsymbol{\alpha}}(\mathbf{M}_2)$ if and only if there is an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ such that $\phi\mathbf{M}_1 = \mathbf{M}_2$. Therefore, if $\mathbf{M}_1$ and $\mathbf{M}_2$ don't lie on the same orbit under the action of $\mathrm{Isom}(\mathbf{H}^3)$, it holds that $\tilde{\boldsymbol{\alpha}}(\mathbf{M}_1) = \mathbf{a}_{\mathbf{M}_1}$ and $\tilde{\boldsymbol{\alpha}}(\mathbf{M}_2) = \mathbf{a}_{\mathbf{M}_2}$ are different vectors in $\mathbb{R}^E$. Let $\epsilon > 0$ be the distance between $\mathbf{a}_{\mathbf{M}_1}$ and $\mathbf{a}_{\mathbf{M}_2}$, and take the open balls $\mathbb{B}_{\epsilon/2}(\mathbf{a}_{\mathbf{M}_1}), \mathbb{B}_{\epsilon/2}(\mathbf{a}_{\mathbf{M}_2})$ of radius $\frac{\epsilon}{2}$ around them. Since the open balls are disjoint open sets, it holds that

$$\mathbf{U}_{\mathbf{M}_1} = \tilde{\boldsymbol{\alpha}}^{-1}(\mathbb{B}_{\epsilon/2}(\mathbf{a}_{\mathbf{M}_1})) \quad \text{and} \quad \mathbf{U}_{\mathbf{M}_2} = \tilde{\boldsymbol{\alpha}}^{-1}(\mathbb{B}_{\epsilon/2}(\mathbf{a}_{\mathbf{M}_2}))$$

are disjoint neighbourhoods in $\mathcal{O}^1_{\mathfrak{m}}$ of $\mathbf{M}_1$, and $\mathbf{M}_2$ respectively. Moreover, both sets are invariant under the action of $\mathrm{Isom}(\mathbf{H}^3)$, i.e,

$$\phi\mathbf{M} \in \mathbf{U}_{\mathbf{M}_i} \quad \text{for all} \quad \mathbf{M} \in \mathbf{U}_{\mathbf{M}_i}, i = 1, 2, \quad \text{and} \quad \phi \in \mathrm{Isom}(\mathbf{H}^3).$$

The above tells us that we can define a smooth manifold structure on $\mathsf{P}^1_{\mathfrak{m}}$ in such a way that the canonical projection

$$\pi : \mathcal{O}^1_{\mathfrak{m}} \longrightarrow \mathbf{P}^1_{\mathfrak{m}}$$

is a smooth submersion. Using this smooth structure it holds that

$$\dim(\mathsf{P}^1_{\mathfrak{m}}) = \dim(\mathcal{O}^1_{\mathfrak{m}}) - \dim(\mathrm{Isom}(\mathbf{H}^3)).$$

By proposition 1.6 we know that $\mathrm{Isom}(\mathbf{H}^3)$ is a Lie-group of dimension 6 and so, $\dim(\mathsf{P}^1_{\mathfrak{m}}) = 3N - 6$. □

Finally, take the subset $\mathcal{O}^0_{\mathfrak{m}}$ of $\mathcal{O}_{\mathfrak{m}}$ of pairs containing compact polyhedra whose dihedral angles are non-obtuse, by Corollary 1.3 we know that $\mathcal{O}^0_{\mathfrak{m}} \subseteq \mathcal{O}^1_{\mathfrak{m}}$. Moreover, it is clear that $\mathcal{O}^0_{\mathfrak{m}}$ is invariant under the action of $\mathrm{Isom}(\mathbf{H}^3)$. The set that we are going to consider for the proof of Theorem 2.2 is $\mathsf{P}^0_{\mathfrak{m}}$ the subset of $\mathsf{P}^1_{\mathfrak{m}}$ containing equivalent classes of marked compact polyhedra whit non-obtuse dihedral angles or $\mathcal{O}^0_{\mathfrak{m}}/\mathrm{Isom}(\mathbf{H}^3)$.

2.3. The Proof of Andreev's Theorem

As we mentioned in the last section, the set $\mathsf{P}^0_{\mathfrak{m}}$ plays a central role in the proof of Theorem 2.2. To see how this works, let E be the number of edges of $\mathfrak{m}$, take the fix enumeration $e_1, \dots, e_E$ the edges $\mathcal{M}$ that we used in the proof of Proposition 2.6 and consider the map

$$\begin{aligned} \boldsymbol{\alpha} : \qquad \mathsf{P}^1_{\mathfrak{m}} &\longrightarrow \mathbb{R}^E \\ [(\mathbf{P}, [\psi]_\sim)] &\mapsto [\alpha(\psi(e_1)), \dots, \alpha(\psi(e_E))]^T \end{aligned}$$

where $\alpha(\psi(e_i))$ is the dihedral angle of $\mathbf{P}$ at the edge $\psi(e_1)$. As we discussed in the proof of Proposition 2.6, $\boldsymbol{\alpha}$ is a well defined continuous map.

Now, note that an equivalent class of marked polyhedra $[(\mathbf{P}, [\psi]_\sim)] \in \mathsf{P}^1_{\mathfrak{m}}$ contains the following information,

- A compact polyhedron $\mathbf{P}$ realizing $\mathfrak{m}$, up to isometry.
- A marking on the faces of $\mathbf{P}$ induced by $\mathfrak{m}$.
- A weight function, or weights on the edges of $\mathcal{M}$ given by the vector

$$[\alpha(\psi(e_1)), \dots, \alpha(\psi(e_E))]^T$$

.

Example 2.1. Consider an abstract polyhedron $\mathfrak{m}$ corresponding to a cube, and the plane graph $\mathcal{M}$ realizing $\mathfrak{m}$ given in Figure 2.7 with the given enumeration of its faces and edges.

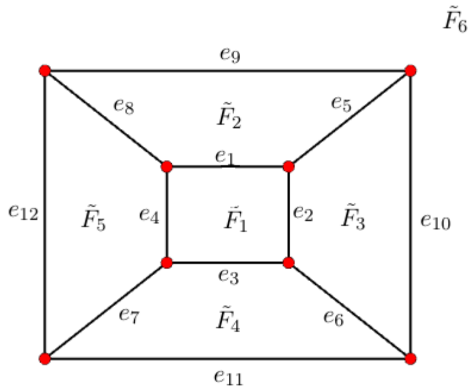


Figure 2.7.: $\mathcal{M}$

Also, assume that the polyhedron $\mathbf{P}$ from Figure 2.8 is a compact polyhedron in $\mathbb{K}^3$ and that $\text{Area}(F_{v_6}) > \text{Area}(F_{v_1})$.

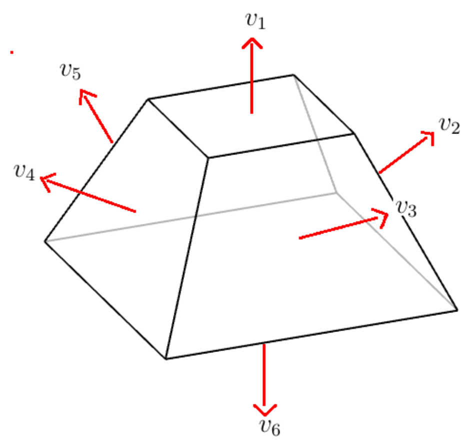


Figure 2.8.: **P**

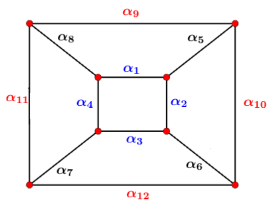
The dihedral angles of **P** are

$$\begin{array}{lll} \alpha_1 = \measuredangle v_1v_2 & \alpha_5 = \measuredangle v_2v_3 & \alpha_9 = \measuredangle v_6v_2 \\ \alpha_2 = \measuredangle v_1v_3 & \alpha_6 = \measuredangle v_3v_4 & \alpha_{10} = \measuredangle v_6v_3 \\ \alpha_3 = \measuredangle v_1v_4 & \alpha_7 = \measuredangle v_4v_5 & \alpha_{11} = \measuredangle v_6v_4 \\ \alpha_4 = \measuredangle v_1v_5 & \alpha_8 = \measuredangle v_5v_2 & \alpha_{12} = \measuredangle v_6v_5 \end{array}.$$

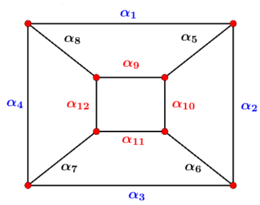
We consider the marked polyhedra $\mathbf{M}_1 = (\mathbf{P}, [\psi_1]_\sim)$ with coordinate $[v_1, v_2, v_3, v_4, v_5, v_6]$ and $\mathbf{M}_2 = (\mathbf{P}, [\psi_2]_\sim)$ with coordinate $[v_6, v_2, v_3, v_4, v_1]$, i.e,

$$\begin{aligned} \psi_1(\tilde{F}_i) &= \psi_2(\tilde{F}_i) = F_{v_i} \quad \text{for} \quad i = 2,3,4,5, \\ \psi_1(\tilde{F}_1) &= \psi_2(\tilde{F}_6) = F_{v_1} \\ \psi_1(\tilde{F}_6) &= \psi_2(\tilde{F}_1) = F_{v_6}. \end{aligned}$$

Since, $\text{Area}(F_{v_6}) > \text{Area}(F_{v_1})$, there is not an isometry $\phi \in \text{Isom}(\mathbf{H}^3)$ such that $\psi_2 = \phi\big|_{\partial \mathbf{P}} \circ \psi_1$ and so $[\mathbf{M}_1] \neq [\mathbf{M}_2]$. However, $\mathbf{M}_1$ gives us the weight function from Figure 2.9a and $\mathbf{M}_2$ gives us the weight function from Figure 2.9b, which by our initial assumption and the discussion that we will held in Section 3.2 are different.



(a)



(b)

Figure 2.9.

We want to find the image of $\mathsf{P}^0_{\mathfrak{m}}$ under $\boldsymbol{\alpha}$. Let us denote by $A_{\mathfrak{m}}$ the convex set defined by the inequalities in Theorem 2.2, i.e, the set of vectors $[\alpha_1, \ldots, \alpha_E]^T \in \mathbb{R}^E$ such that

[a1] $0 < \alpha_i \leq \frac{\pi}{2}$ for all i.

[a2] $\alpha_i + \alpha_j + \alpha_k > \pi$ whenever the edges e_i, e_j and e_k meet at a vertex.

[a3] $\alpha_i + \alpha_j + \alpha_k < \pi$ whenever $\gamma = e_i^* e_j^* e_k^*$ is a prismatic 3-cycle.

[a4] $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$ whenever $\gamma = e_i^* e_j^* e_k^* e_l^*$ is a prismatic 4-cycle.

[a5] $\alpha_{i_1} + \alpha_{i_3} + \alpha_{i_1 i_2} + \alpha_{i_2 i_3} + \alpha_{i_3 i_4} + \alpha_{i_4 i_5} < 3\pi$ and $\alpha_{i_2} + \alpha_{i_4} + \alpha_{i_1 i_2} + \alpha_{i_2 i_3} + \alpha_{i_3 i_4} + \alpha_{i_4 i_5} < 3\pi$ whenever $e_{i_1} e_{i_2} e_{i_3} e_{i_4}$ is the boundary of a quadrilateral face and each edge $e_{i_k i_j}$ is the third edge that meets the adjacent edges e_{i_k}, e_{i_j}.

Our goal is to show that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) = A_{\mathfrak{m}}$. We will give a proof for **simple abstract polyhedra** or trivalent abstract polyhedra with more than 4 faces and without prismatic 3-cycles. For this proof, we need the following statements,

[i.] $E = \dim(\mathsf{P}^1_{\mathfrak{m}})$.

[ii.] $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \to \mathbb{R}^E$ is injective.

[iii.] $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) \subseteq A_{\mathfrak{m}}$.

[iv.] $\boldsymbol{\alpha} : \mathsf{P}^0_{\mathfrak{m}} \to A_{\mathfrak{m}}$ is proper.

[v.] If $\mathfrak{m}$ is simple, then $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) \neq \emptyset$.

Statement [i.] is a straight forward argument that we will prove in the next lemma. In chapter 3, we will give a proof for [ii.],[iii.] and [iv.] following the ideas in [14] and [3]. The proof of statement [v.] is slightly complicated. However, in section 3.4 we will give an explanation of the basic tools to prove it.

Lemma 2.1. *$E = dim(\mathsf{P}^1_{\mathfrak{m}})$*

Proof. Let V be the number in vertices in $\mathfrak{m}$. Since $\mathfrak{m}$ is realizable as a cellular decomposition of $\mathbb{S}^2$, it holds that $\chi(\mathbb{S}^2) = V - E + N = 2$. We also know that the number of edges incident to each vertex of $\mathfrak{m}$ is equal to 3, hence $2E = 3V$. Combining these two facts together, we get that

$$2 = \frac{2}{3}E - E + N \quad \text{and so} \quad E = 3N - 6.$$

Hence, by Proposition 2.6 we get that $E = \dim(\mathsf{P}^1_{\mathfrak{m}})$.

□

Proposition 2.7. *If* $\mathfrak{m}$ *is simple, then* $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) = A_{\mathfrak{m}}$

Proof. (**Proof of Andreev's Theorem for simple polyhedra**)
Statements [i.] and [ii.] tell us that $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \longrightarrow \mathbb{R}^E$ is an injective continuous map between two manifolds of the same dimension. Therefore, by the invariance of domain principle (see [9]), it is a local homeomorphism and so, for each vector $\hat{\mathbf{a}} \in \boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}})$ there is an open set $V_{\hat{\mathbf{a}}} \subseteq \mathsf{P}^1_{\mathfrak{m}}$ and an open neighbourhood $\tilde{U}_{\hat{\mathbf{a}}} \subseteq \mathbb{R}^E$ of $\hat{\mathbf{a}}$ such that $\boldsymbol{\alpha} : V_{\hat{\mathbf{a}}} \longrightarrow \tilde{U}_{\hat{\mathbf{a}}}$ is a homeomorphism. By statement [iii.], $\hat{\mathbf{a}} \in A_{\mathfrak{m}}$. Moreover, if there is a vector $\tilde{\mathbf{a}} \in A_{\mathfrak{m}}$ such that $\boldsymbol{\alpha}^{-1}(\tilde{\mathbf{a}}) \neq \emptyset$, the condition [a1] tells us that $\tilde{\mathbf{a}} \in \boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}})$. Hence, $U_{\hat{a}} = \tilde{U}_{\hat{a}} \cap A_{\mathfrak{m}}$ is an open neighbourhood of $\hat{\mathbf{a}}$ in $A_{\mathfrak{m}}$ that is completely contain in $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}})$ and so, $\alpha(\mathsf{P}^0_{\mathfrak{m}})$ is an open set in $A_{\mathfrak{m}}$.

On the other hand, statement [iii.] also tells us that $\boldsymbol{\alpha} : \mathsf{P}^0_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ is a well defined continuous map. Furthermore, since we are using the subspace topology on $A_{\mathfrak{m}}$, it is hausdorff and locally compact. Therefore, statement [iv.] tells us that $\boldsymbol{\alpha} : \mathsf{P}^0_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ is a closed map, and we can conclude that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}})$ is a closed set in $A_{\mathfrak{m}}$.

From the last paragraphs, we know that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}})$ is both a open and closed set in $A_{\mathfrak{m}}$. In addition, since $A_{\mathfrak{m}}$ is a convex set in $\mathbb{R}^E$, it is a connected topological space. Thus, in the case that $A_{\mathfrak{m}} \neq \emptyset$, we must have that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) = \emptyset$ or $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) = A_{\mathfrak{m}}$. By the statements [iii.] and [iv.], we know that $A_{\mathfrak{m}} \neq \emptyset$. Moreover, since $\boldsymbol{\alpha}(\mathbf{P}^0_{\mathfrak{m}}) \neq \emptyset$ it is clear that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) = A_{\mathfrak{m}}$ □

For a complete proof, we need the more general statement

vi. If $A_{\mathfrak{m}} \neq \emptyset$, then $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) \neq \emptyset$.

A proof of this can be found in [14] or [13].

2.4. Andreev's Polytope and Abstract Polyhedra

The convex set that satisfies the set of linear inequalities in the Andreev's Theorem for compact polyhedra is called the **Andreev's Polytope**. In this section, we will see an explicit example of a combinatorial type (the triangular prism), its abstract polyhedron and how to compute its Andreev's Polytope. We will show that condition [a5] is a necessary condition only for this combinatorial type and we will introduce the prism with N-faces and the split-prim, which are the basic combinatorial types used for the proof of Andreev's Theorem.

The Triangular Prism

The triangular prism is the combinatorial type corresponding to the polytope that we get by taking the product of a triangle $\Delta \subset \mathbb{R}^2$ and the interval $I = [-1, 1] \subset \mathbb{R}^1$.

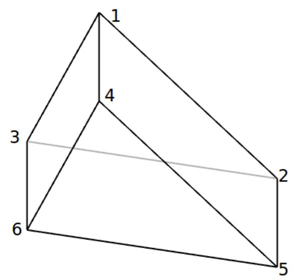


Figure 2.10.: $\text{prism}(\Delta) = \Delta \times I$

Note that prism(Δ) has 6 vertices, 9 edges and 5 faces. We denote by $\mathfrak{p}_5$ the face lattice (abstract polyhedron) corresponding to this combinatorial type. Now, by enumerating the vertices of prism(Δ) as in figure 2.10 we get that (by ijk we mean the set $\{i \quad j \quad k\}$)

$$V(\mathfrak{p}_5) = \{1, 2, 3, 4, 5, 6\}$$

$$E(\mathfrak{p}_5) = \{12, 23, 13, 45, 56, 46, 14, 36, 25\}$$

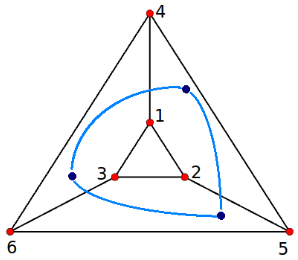
$$F(\mathfrak{p}_5) = \{123, 456, 1346, 2356, 1245\}$$

and so $\mathfrak{p}_5$ is the set $\emptyset \cup V(\mathfrak{p}_5) \cup E(\mathfrak{p}_5) \cup F(\mathfrak{p}_5) \cup \{123456\}$ ordered by the inclusion " $\subseteq$ ".

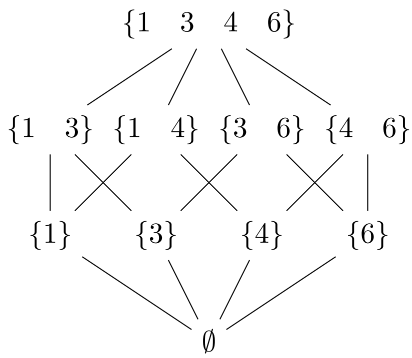
Let

$$\begin{array}{lll} e_1 = 12 & e_4 = 45 & e_7 = 14 \\ e_2 = 13 & e_5 = 46 & e_8 = 36 \\ e_3 = 23 & e_6 = 56 & e_9 = 25 \end{array}$$

and set $\alpha(e_i) = \alpha_i$.



(a) Cellular Realization of $\mathfrak{p}_5$



(b) Hasse diagram of the face 1346

Figure 2.11.

The Andreev's Polytope $A_{\mathfrak{p}_5}$ is the set of vectors $[\alpha_1, \ldots, \alpha_9]^T \in \mathbb{R}^9$ that satisfy the following linear inequalities,

[a1] $$0 < \alpha_i \leq \tfrac{\pi}{2}.$$

[a2]

$$\begin{array}{ll} \alpha_1 + \alpha_2 + \alpha_7 > \pi & (\text{for} \quad 1) \\ \alpha_1 + \alpha_3 + \alpha_9 > \pi & (\text{for} \quad 2) \\ \alpha_2 + \alpha_3 + \alpha_8 > \pi & (\text{for} \quad 3) \\ \alpha_7 + \alpha_4 + \alpha_5 > \pi & (\text{for} \quad 4) \\ \alpha_9 + \alpha_4 + \alpha_6 > \pi & (\text{for} \quad 5) \\ \alpha_8 + \alpha_6 + \alpha_5 > \pi & (\text{for} \quad 6) \end{array}$$

[a3] It has one prismatic 3-circuit, the one given by $e_7^* e_9^* e_8^*$. Hence,

$$\alpha_7 + \alpha_8 + \alpha_9 < \pi.$$

[a4] $\mathfrak{p}_5$ has no prismatic 4-circuits.

[a5] The quadrilateral faces are 1254, 2536 and 1436 and we have

For the face 1254 we have the inequalities

$$\alpha_2 + \alpha_5 + \alpha_6 + \alpha_3 + \alpha_7 + \alpha_9 < 3\pi.$$
$$\alpha_2 + \alpha_5 + \alpha_6 + \alpha_3 + \alpha_1 + \alpha_4 < 3\pi.$$

For the face 2356 we have the inequalities

$$\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9 + \alpha_8 < 3\pi.$$
$$\alpha_1 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_3 + \alpha_6 < 3\pi.$$

And finally, for the face 1346 we have the inequalities

$$\alpha_1 + \alpha_4 + \alpha_6 + \alpha_3 + \alpha_7 + \alpha_8 < 3\pi.$$
$$\alpha_2 + \alpha_4 + \alpha_6 + \alpha_3 + \alpha_5 + \alpha_2 < 3\pi.$$

Proposition 2.8. $A_{\mathfrak{p}_5} \neq \emptyset$

Proof. 1. First note that the inequalities for the quadrilateral faces in [a5] hold if $\alpha_i < \frac{\pi}{2}$ for all i.

2. If $\alpha_7 = \alpha_8 = \alpha_9 = \frac{\pi}{5}$, then the inequality in [a3] is satisfied.

3. For $i = 1, \dots, 6$ take a value for α_i such that

$$\frac{2\pi}{5} < \alpha_i < \frac{\pi}{2}.$$

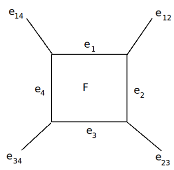
Using this values and the one that we choose in the second item, the inequalities in [a2] are satisfied. Additionally, by the first item we are done.

□

The triangular prism condition and simple abstract polyhedra

Proposition 2.9. *If $\mathfrak{m} \not\cong \mathfrak{p}_5$ is a trivalent abstract polyhedron, then condition* [a5] *is a consequence of the conditions* [a3] *and* [a4].

Proof. Let $\mathcal{M}$ be a realization of $\mathfrak{m}$ as a cell complex, $\tilde{F}$ a quadrilateral face of $\mathcal{M}$, $e_1e_2e_3e_4$ the boundary of $\tilde{F}$ and e_{ij} the edge adjacent to the edges e_i and e_j.



Take a map $\alpha : E(\mathcal{M}) \longrightarrow (0, \frac{\pi}{2}]$ such that conditions $[a1] - [a4]$ hold. We will see that $[a4] \Rightarrow [a5]$, $[a3] \Rightarrow [a5]$, and, if $[a4]$ or $[a3]$ are empty conditions for the edges surrounding $\tilde{F}$, then $\mathfrak{m}$ is isomorphic to $\mathfrak{p}_5$.

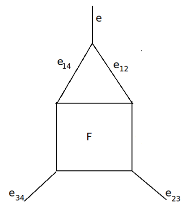
First, note that condition $[a5]$ holds for F if and only if there is an i such that $\alpha(e_i) < \frac{\pi}{2}$ or a pair ij such that $\alpha(e_{ij}) < \frac{\pi}{2}$.

Case 1. If $\gamma = e_{12}^*e_{23}^*e_{34}^*e_{14}^*$ is a prismatic 4-cycle, then

$$\alpha(e_{12}) + \alpha(e_{23}) + \alpha(e_{34}) + \alpha(e_{14}) < 2\pi.$$

Therefore, $\alpha(e_{ij}) < \frac{\pi}{2}$ for all ij an so it's clear that $[a5]$ follows from $[a4]$.

Case 2. Assume that e_{12}, e_{14} meet at a vertex v. Since $\mathfrak{m}$ is trivalent, there is an edge $e \in E(\mathcal{M})$ such that e_{12}, e_{14}, e are the edges incident to the vertex v.

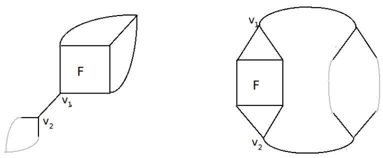


If $e^* e_{23}^* e_{34}^*$ is a prismatic 3-circuit, then

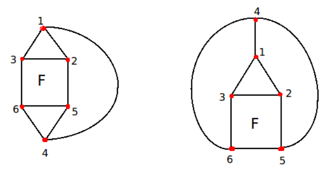
$$\alpha(e) + \alpha(e_{12}) + \alpha(e_{14}) < \pi.$$

Therefore, $\alpha(e_{12}), \alpha(e_{14}) < \frac{\pi}{2}$ and so it's clear that condition $[a5]$ follows from condition $[a3]$.

Case 3. Finally, if $[a4]$ and $[a3]$ are empty conditions, we can argue that $\mathcal{M}$ looks like one of the following cellular complexes,



(a)



(b)

In the case (a), $G(\mathcal{M}) - \{v_1, v_2\}$ is a disconnected graph, which contradicts the fact that $G(\mathcal{M})$ is 3-connected. In the case (b) $\mathcal{M}$ is combinatorial equivalent to the cellular complex of Figure 2.11a. Therefore, $\mathfrak{m}$ is isomorphic to $\mathfrak{p}_5$. $\square$

The last proposition tells us that condition $[a5]$ is only necessary for the triangular prism. We can use this result to see that the Andreev's Polytope of a simple abstract polyhedron is not empty.

Corollary 2.1. *If $\mathfrak{m}$ is a simple abstract polyhedron, then $A_{\mathfrak{m}} \neq \emptyset$.*

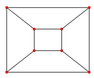
Proof. We want to see that there is a $[\alpha_1, \ldots, \alpha_E]^T \in \mathbb{R}^E$ that satisfies the set of inequalities given in Andreev's Theorem. Since $[a5]$ is not a necessary condition, we only have to check conditions $[a1] - [a4]$. Take

$$\alpha_i = \frac{2\pi}{5} \quad \text{for all} \quad i,$$

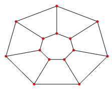
it's easy to see that conditions $[a1], [a2]$ and $[a4]$ are satisfied. Also, since $\mathfrak{m}$ has not prismatic 3-circuits, condition $[a3]$ is empty and we are done. $\square$

The prism with N-faces and The Split Prism

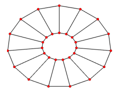
The **prism with N-faces** is the combinatorial type corresponding to the polytope that we get by taking the product of a $(N-2)$-gon $Q_{N-2} \subset \mathbb{R}^2$ and the interval $I = [-1, 1]$. It has N-faces, $3(N-2)$ edges and $2(N-2)$ vertices.



(a) Prism with 6 faces

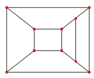


(b) Prism with 9 faces

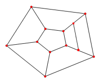


(c) Prism with 17 faces

The **split prism with N-faces** is the combinatorial type that we get by dividing in two a quadrilateral face of the prism with $(N-1)$-faces.



(a) Split prism with 7 faces



(b) Split prism with 8 faces

Note that in both cases the Andreev's Polytope is not empty.

3. Andreev's Theorem

In this chapter we will see that the map $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \longrightarrow \mathbb{R}^E$ satisfies the necessary conditions to prove Theorem 2.2.

3.1. $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) \subset A_{\mathfrak{m}}$

We wan to see that $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathbf{C}_{\mathfrak{m}}}) \subseteq \mathbf{A}_{\mathfrak{m}}$.

Proof. Take and equivalence class $[(\mathbf{P}, [\phi]_\sim)] \in \mathsf{P}^0_{\mathfrak{m}}$. We want to see that the conditions [a1], [a2], [a3], [a4] and [a5] from Theorem 2.2 hold for the dihedral angels at the edges of $\mathbf{P}$.

[a1] By our assumption, if we take an edge $\tilde{e} \in E(\mathbf{P})$, its corresponding dihedral angle $\alpha(\tilde{e})$ is less or equal to $\frac{\pi}{2}$. Also, it is clear that $\alpha(\tilde{e}) > 0$.

[a2] Let us assume that the edges $\tilde{e}_1, \tilde{e}_2$ and $\tilde{e}_3 \in E(\mathbf{P})$ meet at a vertex $v \in V(\mathbf{P})$, also let us assume that F_{v_1}, F_{v_2} and F_{v_3} are the faces of $\mathbf{P}$ such that

$$\tilde{e}_1 = F_{v_1} \cap F_{v_2}, \quad \tilde{e}_2 = F_{v_2} \cap F_{v_3} \quad \text{and} \quad \tilde{e}_3 = F_{v_3} \cap F_{v_1}.$$

By proposition 1.11 we can conclude that $F_{v_1} \cap F_{v_2} \cap F_{v_3} = P_{v_1} \cap P_{v_2} \cap P_{v_3} = \{v\}$ and that

$$\alpha(\tilde{e}_1) + \alpha(\tilde{e}_2) + \alpha(\tilde{e}_3) > \pi.$$

[a3] Now, let us assume that the faces $F_{v_1} F_{v_2} F_{v_3}$ of $\mathbf{P}$ form a 3-angled prismatic element and that

$$\tilde{e}_1 = F_{v_1} \cap F_{v_2}, \quad \tilde{e}_2 = F_{v_2} \cap F_{v_3} \quad \text{and} \quad \tilde{e}_3 = F_{v_3} \cap F_{v_1}.$$

Since the sequence of faces is a 3-angled prismatic element, it holds that

$$e_i \cap e_j = \emptyset \quad \text{for} \quad i \neq j. \tag{3.1}$$

On the other hand, let l_i be the corresponding supporting line of the edge $\tilde{e}_i$, i.e,

$$l_1 =< v_1 >^L \cap < v_2 >^L \cap \mathbf{H}^3, \quad l_2 =< v_2 >^L \cap < v_3 >^L \cap \mathbf{H}^3$$

$$\text{and} \quad l_3 =< v_3 >^L \cap < v_1 >^L \cap \mathbf{H}^3.$$

The faces of $\mathbf{P}$ are parallelograms. Therefore, from condition (3.1) we can deduce that

$$\bar{l}_i \cap \bar{l}_j = \emptyset \text{ for } i \neq j$$

and so

$$\bar{l}_1 \cap \bar{l}_2 \cap \bar{l}_3 = \bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} = \emptyset.$$

The last result and proposition 1.11 tell us that

$$\alpha(e_1) + \alpha(e_2) + \alpha(e_3) < \pi.$$

[a4] Let us assume that the sequence of faces $F_{v_1}F_{v_2}F_{v_3}F_{v_4}$ is a 4-angled prismatic element and that

$$\tilde{e}_1 = F_{v_1} \cap F_{v_2}, \quad \tilde{e}_2 = F_{v_2} \cap F_{v_3}$$
$$\tilde{e}_3 = F_{v_3} \cap F_{v_4} \quad \text{and} \quad \tilde{e}_4 = F_{v_4} \cap F_{v_1}.$$

Note that $\alpha(\tilde{e}_1)+\alpha(\tilde{e}_2)+\alpha(\tilde{e}_3)+\alpha(\tilde{e}_4) = 2\pi$ if and only if $\alpha(\tilde{e}_i) = \frac{\pi}{2}$ for all i. Otherwise, the condition holds and we are done. Assume that $\alpha(\tilde{e}_i) = \frac{\pi}{2}$ for $i = 1, 2, 3$ and 4, and, note that

$$\langle v_1, v_2\rangle_L = -\cos(\alpha(\tilde{e}_1)) = 0, \quad \langle v_2, v_3\rangle_L = -\cos(\alpha(\tilde{e}_2)) = 0$$
$$\langle v_3, v_4\rangle_L = -\cos(\alpha(\tilde{e}_3)) = 0 \qquad \text{and} \quad \langle v_4, v_1\rangle_L = -\cos(\alpha(\tilde{e}_4)) = 0.$$

also take the matrix

$$G = \left[\langle v_i, v_j\rangle_L\right]_{i,j=1,2,3,4} = \begin{bmatrix} 1 & 0 & \langle v_1, v_3\rangle_L & 0 \\ 0 & 1 & 0 & \langle v_2, v_4\rangle_L \\ \langle v_1, v_3\rangle_L & 0 & 1 & 0 \\ 0 & \langle v_2, v_4\rangle_L & 0 & 1 \end{bmatrix}.$$

We have the two following cases:

Case a. If v_1, v_2, v_3 and v_4 are linear independent vectors, then the G is the representation matrix of the Lorentzian inner product on $\mathbb{E}^{1,3}$ with respect to the basis $B = \{v_1, v_2, v_3, v_4\}$. Therefore, in this case we have

$$\det\,(G) < 0.$$

Case b. If $\{v_1, v_2, v_3, v_4\}$ is not a set of linear independent vectors, at least one of the columns of G can be written as a linear combination of the other three columns. Therefore, in this case we have

$$\det\,(G) = 0.$$

From the last two cases we get that $\det\,(G) \leq 0$. Moreover, if we compute we get that

$$\det(G) = (1 - \langle v_1, v_3\rangle_L^2)(1 - \langle v_2, v_4\rangle_L^2).$$

The last computation tells us that either $1 - \langle v_1, v_3\rangle_L^2 \geq 0$ or $1 - \langle v_2, v_4\rangle_L^2 \geq 0$. We can assume without loss of generality that

$$1 - \langle v_1, v_3\rangle_L^2 \geq 0$$

by Proposition 1.9

$$\bar{P}_{v_1} \cap \bar{P}_{v_3} \neq \emptyset,$$

this tells us that $\langle v_1, v_3\rangle_L = -\cos(\alpha)$, where α is the dihedral angle at $\bar{P}_{v_1} \cap \bar{P}_{v_3}$.

Now, take the matrix $A = [\langle v_i, v_j\rangle_L]_{i,j=1,2,3}$, since $\langle v_1, v_2\rangle_L = \langle v_2, v_3\rangle_L = 0$, it holds that $\det(A) = 1 - cos^2(\alpha)$. The last expression is always bigger or equal to zero, hence by remark 1.5 we get that

$$\bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} \neq \emptyset.$$

On the other hand, since $F_{v_1}F_{v_2}F_{v_3}F_{v_4}$ is a four angled prismatic element, we know that $\tilde{e}_1 \cap \tilde{e}_2 = \emptyset$. Also, $\tilde{e}_1$ and $\tilde{e}_2$ are both edges of the face F_{v_2}, which as a parallelogram. Hence, if l_1 and l_2 are the corresponding supporting lines of the edges $\tilde{e}_1$ and $\tilde{e}_2$, we get that

$$\bar{l}_1 \cap \bar{l}_2 = \bar{P}_{v_1} \cap \bar{P}_{v_2} \cap \bar{P}_{v_3} = \emptyset,$$

which leads to a contradiction .

[a5] Assume that the face F_{v_i} is a quadrilateral, whose boundary is $\partial F_{v_i} = \tilde{e}_1\tilde{e}_2\tilde{e}_3\tilde{e}_4$ and let $\tilde{e}_{ij}$ be the edge adjacent to the edges $\tilde{e}_i$ and $\tilde{e}_j$.

Since **P** has non-obtuse dihedral angels, the condition [a5] is violated if we get equality in one of the linear equations defining this condition. Assume that

$$\alpha(\tilde{e}_1) + \alpha(\tilde{e}_3) + \alpha(\tilde{e}_{12}) + \alpha(\tilde{e}_{23}) + \alpha(\tilde{e}_{34}) + \alpha(\tilde{e}_{14}) = 3\pi.$$

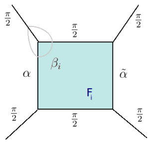
Then,

$$\alpha(\tilde{e}_1) = \alpha(\tilde{e}_3) = \alpha(\tilde{e}_{ij}) = \frac{\pi}{2}.$$

By the law of cosines in the spherical geometry (see equation B.2), if β_i is an interior angel of F_{v_i}, we get that

$$\cos(\beta_i) = \frac{\cos(\frac{\pi}{2}) + \cos(\alpha)\cos(\frac{\pi}{2})}{\sin(\alpha)\sin(\frac{\pi}{2})} = 0$$

and so $\beta_i = \frac{\pi}{2}$.



Therefore, by the Gauss-Bonnet-theorem (see example A.1) we get that

$$2\pi = \frac{4\pi}{2} + \text{Area}(F_{v_i}) \quad \Rightarrow \quad \text{Area}(F_{v_i}) = 0$$

and so we get a contradiction. $\square$

3.2. $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \to \mathbb{R}^E$ is injective.

To see that $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \longrightarrow \mathbb{R}^E$ is injective we will show that two marked polyhedra in $\mathcal{O}_{\mathfrak{m}}$ whose correspondent dihedral angles coincide have congruent faces. It is well know that two n-gons are congruent if and only if their interior angles and their lengths coincide, similarly we will show that two marked polyhedra in $\mathcal{O}^1_{\mathfrak{m}}$ whose faces are congruent and whose corresponding dihedral angles coincide are in the same orbit under the action of $\mathrm{Isom}(\mathbf{H}^3)$. In addition, to be able to prove our main statement, we will state and prove an important lemma about parallelograms in $\mathbf{H}^2$, and state a lemma about simple plane graphs.

Proposition 3.1. *Let us take $(\mathbf{P_1}, [\psi_1]_\sim), (\mathbf{P_2}, [\psi_2]_\sim) \in \mathcal{O}^1_{\mathfrak{m}}$. If the faces are congruent and the corresponding dihedral angles coincide, then there is a $\phi \in$ Isom$(\mathbf{H}^3)$ such that $\phi(\mathbf{P_1}, [\psi_1]_\sim) = (\mathbf{P_2}, [\psi_2]_\sim)$.*

Proof. Assume that $(\mathbf{P_1}, [\psi_1]_\sim)$ has coordinate $[v_1, \dots, v_N] \in \mathcal{H}^N_3$ and that $(\mathbf{P_2}, [\psi_2]_\sim)$ has coordinate $[w_1, \dots, w_N] \in \mathcal{H}^N_3$. By our assumption F_{v_i} and F_{w_i} are congruent polygons for $i = 1, ..., N$. Since F_{v_N} and F_{w_N} are congruent polygons, we can find an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ such that $\phi(F_{v_N}) = F_{w_N}$, also note that $\phi(P_{v_N}) = P_{w_N}$. Let us assume that $a_1, \dots, a_k$ are the sides of F_{v_N}, and that $\bar{a}_1, \dots, \bar{a}_k$ are the corresponding sides of F_{w_N}. Moreover, we can assume that F_{v_i} is the face adjacent to F_{v_N} at a_i, and respectively F_{w_i} is the face adjacent to F_{w_N} at $\bar{a}_i$. If it is necessary we can compose ϕ with the reflection through the hyperbolic plane P_{w_i} and (or) a rotation that preserves F_{w_N} to get that $\phi(\mathbf{P_1})$ and $\mathbf{P_2}$ lie in the half-space H_{w_N}, and that $\phi(a_i) = \bar{a}_i$. Since ϕ is an isometry and the dihedral angles coincide, the dihedral angles of $\phi(\mathbf{P_1})$ and $\mathbf{P}_2$ at $\bar{a}_i$ are the same and we can conclude that $\phi(P_{v_i}) = P_{w_i}$ for $i = 1, \dots, k$. This tells us that $\phi(F_{v_i})$ and F_{w_i} are congruent polygons lying on the hyperbolic plane P_{w_i}. Moreover, since they share the commune edge $\bar{a}_i$, the supporting line $l_i = \langle w_i \rangle^L \cap \langle w_N \rangle^L \cap \mathbf{H}^3$, the supporting line $\tilde{l}_i = \langle w_i \rangle^L \cap \langle w_{i+1} \rangle^L \cap \mathbf{H}^3$ ($\langle w_k \rangle^L \cap \langle w_1 \rangle^L \cap \mathbf{H}^3$ for $i = k$) and lie on the same side of the supporting line l_i with respect of the supporting plane P_{w_i}, we can conclude that $\phi(F_{v_i}) = F_{w_i}$. In a recursive way, we can use a similar argument to see that $\phi(F_{v_i}) = F_{w_i}$ for all i, which implies that $V(\phi(\mathbf{P_1})) = V(\mathbf{P_2})$. Therefore, we can conclude that $\phi(\mathbf{P_1}) = \mathrm{conv}(V(\phi(\mathbf{P_1}))) = \mathrm{conv}(V(\mathbf{P_2})) = \mathbf{P_2}$. Finally, note that $\phi \circ \psi_1(\tilde{F}_i) = \phi(F_{v_i}) = F_{w_i}$ for all i, and so $\psi_2 = \phi|_{\partial \mathbf{P_1}} \circ \psi_1$. ∎

Lemma 3.1 (**Andreev's Auxiliary lemma on parallelograms**).
Let

$$\mathbf{Q_1} = \bigcap_{i=1}^{m} H_{v_i} \quad \textit{and} \quad \mathbf{Q_2} = \bigcap_{i=1}^{m} H_{w_i}$$

be polygons in $\mathbf{H}^2$, not necessary of finite volume such that

$$x_i = s_{v_i} \cap s_{v_{i+1}} \quad \textit{for} \quad i = 1, \dots, m-1,$$
$$x'_i = s_{w_i} \cap s_{w_{i+1}} \quad \textit{for} \quad i = 1, \dots, m-1,$$

are finite vertices. We are also going to ask that the condition for parallelograms holds for both polygons, i.e, if $s_{v_i} \cap s_{v_j} = \emptyset$ (or $s_{w_i} \cap s_{w_j} = \emptyset$), then $\bar{l}_{v_i} \cap \bar{l}_{v_j} = \emptyset$ (resp. $\bar{l}_{w_i} \cap \bar{l}_{w_j} = \emptyset$). Assume that

$$\measuredangle v_i v_{i+1} = \measuredangle w_i w_{i+1} \quad \textit{for all} \quad i = 1, \dots, m-1$$

and

$$|s_{v_i}| \leq |s_{w_i}| \quad \textit{for} \quad i = 2, \ldots, m-1$$

where at least one of the inequalities is strict, then it holds that

$$\langle w_1, w_m \rangle_L < \langle v_1, v_m \rangle_L$$

Proof. We will first assume that just one inequality is strict, let us say that $|s_{v_k}| < |s_{w_k}|$. We can overlap both polyhedra in such a way that $v_1 = w_1$ and $v_k = w_k$ (see Figure 3.1) and use a parallel translation $\pi_k(t)$ along the hyperbolic line l_{v_k} to go from $\mathbf{Q_1}$ to $\mathbf{Q_2}$, i.e, we apply $\pi_k(t)$ on the appropriate half-spaces to get the polyhedron

$$\mathbf{Q}(t) = \bigcap_{i=1}^{k-1} H_{v_i} \cap \bigcap_{i=k}^{m} \pi_k(t) H_{v_i}$$

where $\mathbf{Q}(0) = \mathbf{Q_1}$ and $\mathbf{Q}(t_f) = \mathbf{Q_2}$ for some $t_f > 0$, our goal is to see that $\langle v_1, v_m(t) \rangle_L$ is a decreasing function, for that purpose we will show that $\frac{d}{dt}\langle v_1, v_m(t) \rangle_L < 0$ for all $t \geq 0$.

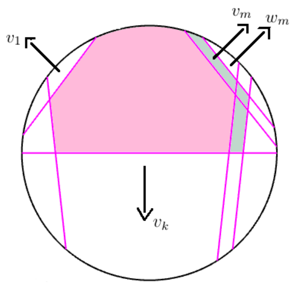


Figure 3.1.

Using an appropriate isometry we can assume that $v_k = [0, 0, -1]^T$ and that the middle point of s_{v_k} is $O = [1, 0, 0]^T$. Note that $l_{v_k} = \{x_2 = 0\} \cap \mathbf{H^2}$. Moreover, we can also assume that the sides of $\mathbf{Q}(t)$ are enumerated in counter-clockwise direction with respect to l_{v_k}.

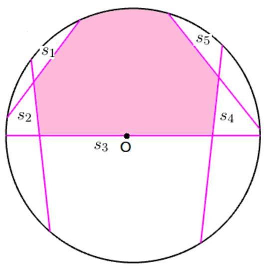


Figure 3.2.: k=3

The respective parallel translation is given by

$$\tilde{\pi}_k(t) = \begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we assume that $v_1 = [v_{01}, v_{11}, v_{21}]^T$ and that $v_m = [v_{0m}, v_{1m}, v_{2m}]^T$, then we have that $v_m(t) = [\cosh(t)v_{0m} + \sinh(t)v_{1m}, \sinh(t)v_{0m} + \cosh(t)v_{1m}, v_{2m}]^T$ and so,

$$\frac{d}{dt}\langle v_1, v_m(t)\rangle_L = \sinh(t)(v_{11}v_{1m} - v_{01}v_{0m}) - \cosh(t)(v_{1m}v_{01} - v_{11}v_{0m}). \tag{3.2}$$

We want to see that the expression in the Equation 3.2 is < 0 for $t \geq 0$. Since O is the middle point of s_{v_k}, it holds that $O \in \mathbf{Q_1}$ and so,

$$\langle [1,0,0]^T, [v_{01}, v_{11}, v_{21}]^T\rangle_L = -v_{01} < 0,$$
$$\langle [1,0,0]^T, [v_{0m}, v_{1m}, v_{2m}]^T\rangle_L = -v_{0m} < 0.$$

Therefore, v_1 and v_m are positive space-like vectors. Moreover, by the way that we chose the enumeration of the sides of $\mathbf{Q_1}$ we can see that $v_1, v_k = [0, 0, -1]^T, v_m$ in that order form a positive oriented basis of $\mathbb{E}^{1,2}$.

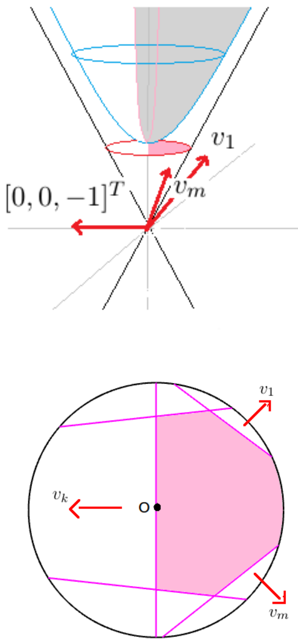


Figure 3.3.

Thus, it holds that

$$\begin{vmatrix} v_{01} & 0 & v_{0m} \\ v_{11} & 0 & v_{1m} \\ v_{21} & -1 & v_{2m} \end{vmatrix} = [v_{01}v_{1m} - v_{0m}v_{11}] > 0.$$

This tells us that $\frac{d}{dt}\big|_{t=0}\langle v_1, v_m(t)\rangle_L < 0$, and that for $t > 0$ equation (3.2) is less than zero if and only if

$$\frac{v_{11}v_{1m} - v_{01}v_{0m}}{v_{01}v_{1m} - v_{0m}v_{11}} < \frac{\cosh(t)}{\sinh(t)}. \tag{3.3}$$

Since $\frac{\cosh(t)}{\sinh(t)} > 1$ for $t > 0$, to see that last equation holds, it is enough to show that

$$\frac{v_{11}v_{1m} - v_{01}v_{0m}}{v_{01}v_{1m} - v_{0m}v_{11}} < 1. \tag{3.4}$$

We know that the parallelogram condition holds for $\mathbf{Q_1}$, this means that $\bar{l}_{v_1}$ intersects $\bar{l}_{v_k}$ if and only if s_1 and s_k are adjacent sides. Since we chose a counter-clockwise enumeration, s_1 and s_k are adjacent if and only if $k = 2$. Moreover, if the intersection is non-empty, it happens to the left of O. Let us consider $\mathbf{Q_1}$ in $\mathbb{K}^2$ as in Figure 3.4, the hole line segment that goes from $O = [1, 0, 0]^T$ to $[1, 1, 0]^T$ lies on $\bar{l}_k = \{x_2 = 0\} \cap \mathbb{K}^2$ and is completely situated to the right of O. Therefore, by our last discussion it never intersect the hyperplane $\langle v_1 \rangle^L$ and we can conclude that $[1, 1, 0]^T \in h_{v_i}$ but not in $\langle v_1 \rangle^L$. Thus,

$$\langle [1, 1, 0]^T, v_1 \rangle_L = -v_{01} + v_{11} < 0. \tag{3.5}$$

Using the same argument, we see that $\bar{l}_{v_m}$ and $\bar{l}_{v_k}$ intersect if and only if $k = m - 1$, and if the intersection is not empty, then it happens to the right of O. Therefore, $[1, -1, 0]^T \in h_{v_m}$ but not in $\langle v_m \rangle^L$. Thus,

$$\langle [1, -1, 0]^T, v_1 \rangle_L = -v_{0m} - v_{1m} < 0. \tag{3.6}$$

From the equations (3.5) and (3.6), it holds that $(-v_{01} + v_{11})(-v_{0m} - v_{1m}) > 0$, this implies that $v_{01}v_{1m} - v_{11}v_{0m} - v_{11}v_{1m} + v_{01}v_{0m} > 0$ and so we get equation (3.4).

Figure 3.4.

When we apply the translation $\pi_k(t)$ to get the polyhedron $\mathbf{Q}(t)$, the last discussion tells us that the cosine of the dihedral angle, or the distance, between the hyperbolic lines l_{v_1} and $l_{v_m(t)}$ get bigger as t increases. On the other hand, since $\pi_k(t)$ is an isometry, the lengths of the sides different from s_k and the dihedral angles different from $\measuredangle v_1 v_m$ are preserved. Also note that the parallelogram condition is preserved. Therefore, if there is more than one side where the inequality on the lengths is strict, we can do sequentially the last construction on each of those sides. As we always get an increase in the cosine of the dihedral angle, or the distance, between the first and the last supporting line, we get our desired result. $\square$

Lemma 3.2 (**Cauchy**). *Let* $\mathbf{G}$ *be a simple plane graph with more than two vertices. If the edges of* $\mathbf{G}$ *are two coloured, then there is a vertex* $\mathbf{G}$ *with at most two color changes in the cyclic order of the edges around the vertex.*

(for a proof see [1])

Proposition 3.2. *The map* $\boldsymbol{\alpha} : \mathsf{P}^1_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ *is injective.*

Proof. Assume that $(\mathbf{P_1}, [\psi_1]_\sim)$ and $(\mathbf{P_2}, [\psi_2]_\sim)$ are two marked polyhedra in $\mathcal{O}^1_{\mathfrak{m}}$ whose dihedral angles coincide. We want to see that there is a $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ such that $\phi(\mathbf{P_1}, [\psi_1]_\sim) = (\mathbf{P_2}, [\psi_2]_\sim)$. To do this we will show that $\mathbf{P_1}$ and $\mathbf{P_2}$ have congruent faces. Therefore, we have to see that the interior angles of the faces and the lengths of the edges of both polyhedra are the same. First, remember that using the spherical law of cosines, as we did in Proposition 1.15, we can compute the interior angles of the faces of $\mathbf{P_1}$ and $\mathbf{P_2}$ from the dihedral angles of the polyhedra. Therefore, as the dihedral angles of $\mathbf{P_1}$ and $\mathbf{P_2}$ are the same, we can conclude that the interior angles of the faces $\mathbf{P_1}$ and $\mathbf{P_2}$ coincide. To see that the length of the corresponding edges are the same, assume that this is not the case and consider the following marking (colouring) on the edges of the dual decomposition $\mathcal{M}^*$, take a dual edge e^* in $\mathcal{M}^*$,

- If $|\psi_1(e)| < |\psi_2(e)|$, then we will mark e^* with $-$.
- If $|\psi_1(e)| = |\psi_2(e)|$, then we will mark e^* with 0.
- If $|\psi_1(e)| > |\psi_2(e)|$, then we will mark e^* with $+$.

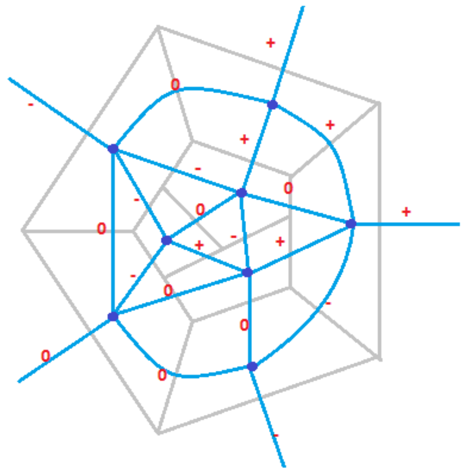


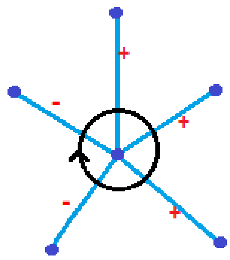
Figure 3.5.: Marking on $\mathcal{M}^*$

Consider the graph $\hat{\mathbf{G}}$ that we get from deleting (eliminating) the edges marked with zero from the dual graph and note that it is a simple plane graph whose edges are marked with $+$ and $-$, or two coloured. Therefore, by Lemma 3.2 there is a dual edge v^* such that when following the cyclic order of the edges in $\hat{\mathbf{G}}$ that are incident to v^*, there are at most two sign changes. Since the number of sign changes is even, this tells us that there are either zero or two sign changes.

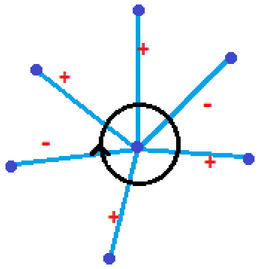
The dual vertex v^* corresponds to a face $\tilde{F}$ of $\mathcal{M}$ with boundary $\partial\tilde{F} = e_1 \dots e_n$ and from the last consideration we can conclude that there is a $m \leq n$ such that

- The dual edges $e_1^*, \dots, e_m^*$ are marked with $+$ or 0, and
- The dual edges $e_{m+1}^*, \dots, e_n^*$ are marked with $-$ or 0.

By our initial assumption, we can assume that there is at least one dual edge e_i^* marked with $+$ or $-$, and if all the edges that are not marked with 0 have the same sing, by interchanging the roles of $\mathbf{P_1}$ and $\mathbf{P_2}$, we can assume without lost of generality that those edges are all marked with $+$, which is the case when $m = n$.

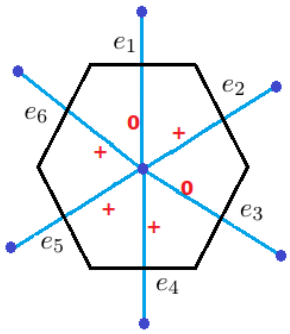


(a) 2 color changes in the cyclic order

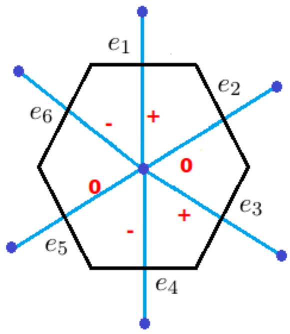


(b) 4 color changes in the cyclic order

Figure 3.6.



(a) m=n



(b) m<n

Figure 3.7.

Now, let $\mathbf{Q_1}$ and $\mathbf{Q_2}$ be the faces of $\mathbf{P_1}$ and $\mathbf{P_2}$ such that $\psi_1(\tilde{F}) = \mathbf{Q_1}$ and $\psi_2(\tilde{F}) = \mathbf{Q_2}$. We can assume that $\mathbf{H}^2$ is the supporting plane of both faces and that

$$\mathbf{Q_1} = \bigcap_{i=1}^{n} H_{w_i} \quad \text{and} \quad \mathbf{Q_2} = \bigcap_{i=1}^{n} H_{v_i}$$

where $\{w_1, \ldots, w_n\}$, $\{v_1, \ldots, v_n\}$ are the minimal set of vectors in $\mathcal{H}_2$ specifying this faces or convex n-gons. Also assume that $\psi_1(e_i) = s_{w_i}$ and that $\psi_2(e_i) = s_{v_i}$ for all i. We have the following possibilities,

Case 1 If $m = n$, then $\mathbf{Q_1}$ and $\mathbf{Q_2}$ satisfy the conditions from the Lemma 3.1 and it holds that

$$\langle w_1, w_n \rangle_L < \langle v_1, v_m \rangle_L$$

which contradicts the fact that all the interior angles are the same.

Case 2 If $m < n$, we can consider the two pair of polyhedra

[a]

$$\tilde{\mathbf{Q}}_{\mathbf{1}} = \bigcap_{i=1}^{m} H_{w_i} \quad \text{and} \quad \tilde{\mathbf{Q}}_{\mathbf{2}} = \bigcap_{i=1}^{m} H_{v_i}, \quad \text{and}$$

[b]

$$\hat{\mathbf{Q}}_{\mathbf{1}} = \bigcap_{i=m}^{n} H_{w_i} \cap H_{w_1} \quad \text{and} \quad \hat{\mathbf{Q}}_{\mathbf{2}} = \bigcap_{i=m}^{n} H_{v_i} \cap H_{v_1}.$$

Both pair of polyhedra satisfy the conditions from Lemma 3.1, from the first pair we get that

$$\langle w_1, w_m \rangle_L < \langle v_1, v_m \rangle_L$$

and from the second pair of polyhedra we get that

$$\langle v_1, v_m \rangle_L < \langle w_1, w_m \rangle_L$$

and we get a contradiction.

The last discussion tells us that that the lengths of the edges of $\mathbf{P_1}$ and $\mathbf{P_2}$ are equal, and so we can conclude that the corresponding faces of both polyhedra are congruent. In addition, since the dihedral angles of $(\mathbf{P_1}, [\psi_1]_\sim)$ and $(\mathbf{P_2}, [\psi_1]_\sim)$ coincide, from Proposition 3.1 we can conclude that there is an isometry $\phi \in \mathrm{Isom}(\mathbf{H}^3)$ such that $\phi(\mathbf{P_1}, [\psi_1]_\sim) = (\mathbf{P_2}, [\psi_2]_\sim)$. $\square$

3.3. $\boldsymbol{\alpha} : \mathsf{P}^0_{\mathfrak{m}} \to A_{\mathfrak{m}}$ is proper

To begin, let us consider a sequence of compact polyhedra $\{\mathbf{P}_j\}_{j\in\mathbb{N}}$ realizing $\mathfrak{m}$ in $\mathbf{H}^3$ such that

$$\mathbf{P}_j = \bigcap_{i=1}^{N} H_{v_i^j}$$

and that each polyhedron has a marking $[\psi_j]_\sim$ where $\psi_j(\tilde{F}_i) = F_{v_i^j} = F_i^j$. In the same way that the hyperbolic half-spaces are parametrized by the unit vectors in the de Sitter-Sphere, the euclidean half-spaces are parametrized by the vectors in the unit sphere $\mathbb{S}^3$ which is a compact set. Therefore, if we have an infinite sequence of closed half-spaces in the euclidean space, it has an infinite convergent subsequence. This tell us that each sequence of closed half-spaces $\{h_{v_i^j}\}_{j\in\mathbb{N}}$ has a convergent subsequence whose limit is a closed half-space h_j of $\mathbb{E}^{1,3}$. Now, transfer the polyhedra $\mathbf{P}_j$ to the Poincaré ball model $\mathbb{D}^3$ and consider the set their sets vertices

$$V(\mathbf{P}_j) = \{x_1^j, \ldots, x_V^j\},$$

where each x_l^j corresponds to a vertex $\tilde{x}_l$ of $\mathcal{M}$. Since $\bar{\mathbb{D}}^3$ is a compact set in $\mathbb{R}^3$, each sequence of vertices $\{x_l^j\}_{j\in\mathbb{N}}$ has a convergent subsequence with limit point $\hat{x}_l \in \bar{\mathbb{D}}^3$. The finite volume polyhedron, or **degenerate polyhedron** (i.e, a finite intersection of closed half-spaces with empty interior, in this case a finite or infinite point, a line segment, a hyperbolic ray, a hyperbolic line or a convex polygon) given by

$$\mathbf{P}_{\mathbb{D}^3} = \mathrm{conv}(\{\hat{x}_1, \ldots \hat{x}_V\}).$$

corresponds to the limit polyhedron $\mathbf{P}$ in $\mathbf{H}^3$ of a convergent subsequence of the compact polyhedra $\mathbf{P}_j$, we will maintain the indices and say that

$$\mathbf{P} = \lim_{j\to\infty} \mathbf{P}_j.$$

Note that in this case each sequence of half-space $h_{v_i^j}$ converges to an euclidean half-space whose boundary is a time-like plane, i.e, there is a unit vector $\hat{v}_i \in \mathcal{H}_3$ such that

$$h_{\hat{v}_i} = \lim_{j\to\infty} h_{v_i^j},$$

and so

$$\lim_{j\to\infty} [v_1^j, \ldots, v_N^j] = [\hat{v}_1, \ldots, \hat{v}_N] \in \mathcal{H}_3^N. \tag{3.7}$$

The last equation and the continuity of the Lorentzian-inner product tell us that

$$\lim_{j\to\infty} \langle v_k^j, v_l^j \rangle_L = \langle \hat{v}_k, \hat{v}_l \rangle_L \quad \text{for all} \quad k, l.$$

Hence, if e_j^i is the edge of $\mathbf{P}_j$ corresponding to the edge e_i in $\mathcal{M}$, it holds that $\alpha(e_j^i)$ is a convergent sequence. Also, if $\mathbf{P}$ is a polyhedron of finite volume realizing $\mathfrak{m}$, then

$$\lim_{j\to\infty} \alpha(e_i^j) = \alpha_i$$

is the dihedral angle at the corresponding edge and the limit N-tuple from equation (3.7) defines a marking on $\mathbf{P}$.

Lemma 3.3. *Let $\tilde{\mathbf{P}}$ be a compact polyhedron with non-obtuse dihedral angles in $\mathbf{H}^3$ and let F be one of its faces. If the interior angle at a vertex of F is equal to $\frac{\pi}{2}$, then the dihedral angle at the edge incident to the vertex and opposite to the face, and at least one of the dihedral angles at the edges of F entering the vertex are equal to $\frac{\pi}{2}$.*

Proof. Assume that $\tilde{e}_1$ is the edge opposite to the face and that $\tilde{e}_1, \tilde{e}_2$ are the edges entering the vertex.

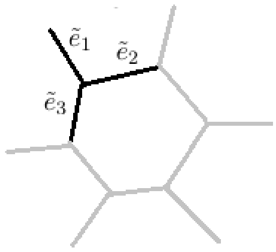


Figure 3.8.

By the spherical law of cosines it holds that

$$0 = \frac{\cos(\alpha(\tilde{e}_1)) + \cos(\alpha(\tilde{e}_2))\cos(\alpha(\tilde{e}_3))}{\sin(\alpha(\tilde{e}_2))\sin(\alpha(\tilde{e}_3))}.$$

Since we are considering non-obtuse dihedral angles, it holds that $\cos(\alpha(\tilde{e}_i)) \geq 0$ for all i. Therefore, the last equation tells us that $\cos(\alpha(\tilde{e}_1)) = 0$ and that at least one $\cos(\alpha(\tilde{e}_2))$ or $\cos(\alpha(\tilde{e}_3))$ is equal to zero. Hence, again as we consider non-obtuse dihedral angles, $\alpha(\tilde{e}_1) = \frac{\pi}{2}$, and at least one $\alpha(\tilde{e}_2)$ or $\alpha(\tilde{e}_3)$ is equal to $\frac{\pi}{2}$. □

Proposition 3.3. *Take a sequence of equivalent classes $\{[(\mathbf{P}_j, [\psi_j]_\sim)]\}_{j\in\mathbb{N}} \subseteq \mathsf{P}^0_{\mathfrak{m}}$. Assume that $\{\mathbf{P}_j\}_{j\in\mathbb{N}}$ is a convergent sequence, and that the sequence given by the vectors $\boldsymbol{\alpha}([(\mathbf{P}_j, [\psi_j]_\sim)]) = \mathbf{a}_j$ converges to a vector $\mathbf{a} \in \bar{A}_{\mathfrak{m}}$ with conditions* [a1], [a3]−[a4]*. Then, the limit polyhedron $\mathbf{P}$ is a polyhedron of finite volume realizing $\mathfrak{m}$.*

Proof. As in the last paragraph we transfer the problem to the Poincaré ball model $\mathbb{D}^3$. Since each $\mathbf{P}_j$ is a compact polyhedron realizing $\mathfrak{m}$, it is enough to see that all the limit vertices $\hat{x}_1, \ldots, \hat{x}_V$ are different.

[Case 1.] *If $\hat{x}_m$ is an ideal vertex for some $m \in \{1, \ldots, V\}$, then $\{x^j_m\}_{j\in\mathbb{N}}$ is the only sequence of vertices that converges to $\hat{x}_m$*

We can choose a face of $\mathcal{M}$ and use an isometry (see [14]) to situate the polyhedra $\mathbf{P}_i$ in such a way that three vertices x^j_a, x^j_b and x^j_c of the corresponding face lie respectively on the positive part of the x-axis, y-axis and z-axis of $\mathbb{R}^3$. This tells us that the limit vertices $\hat{x}_a, \hat{x}_b$ and $\hat{x}_c$ are all different.

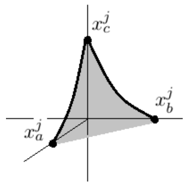


Figure 3.9.

Now, let us assume that there is more than one sequence of vertices that converges to $\hat{x}_m$ and let $\{x_1^j\}_{j\in\mathbb{N}}, \ldots, \{x_k^j\}_{j\in\mathbb{N}}$ be the sequences of vertices that converge to this vertex. Also, let us assume that $x_m^i = x_c^i$, i.e, $\hat{x}_m$ is the north pole, and note that by the way we situated the polyhedra, there are at least two sequences of vertices that don't converge to $\hat{x}_m$. For a large enough $l \in \mathbb{N}$ we can find a hyperbolic plane Q, that is perpendicular to the z-axis and such that for all $j \geq l$, all the vertices $x_1^j, \ldots, x_k^j$ lie in the interior of the half-space defined by Q looking to the north pole, and the other vertices lie in the interior of the other side of Q.

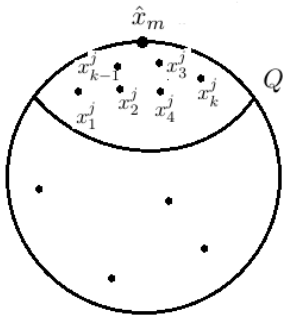


Figure 3.10.

For all $j \geq l$, we will take the hyperplane R_j that is perpendicular to the z-axis and intersects the vertex from the set $\{x_1^j, \ldots, x_k^j\}$ nearest to the origin. Moreover, we will also take the hyperplane S_i that is perpendicular to the z-axis and lies halfway with respect to the z-axis between R_i and Q. Let us denote by $\mathcal{N}$ the half-space of R_i looking to the north pole and by $\mathcal{S}$ the half-space defined by Q looking to the south pole, it is clear that $\{x_1^j, \ldots, x_k^j\} \subseteq \mathcal{N}$ and $\{x_{k+1}^j, \ldots, x_V^j\} \subseteq \mathcal{S}$. Also, note that each half-space contains at least two vertices.

Figure 3.11.

Consider the convex polygon $\mathbf{T}_j = S_j \cap P_j$, its interior angles are the dihedral angles at the edges $e_1^j, \ldots, e_n^j$ connecting the vertices in the half-space $\mathcal{N}$ with the vertices in the half-space $\mathcal{S}$. We normalize the polyhedra $\mathbf{P}_j$ by moving the half-space under S_j in such a way that S_j goes to the hyperbolic plane $H = \{z = 0\} \cap \mathbb{D}^3$ and the whole half-space goes to the half space defined by H looking to the south pole.

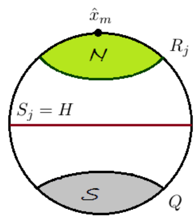


Figure 3.12.

By doing this, we obtain polyhedra $\tilde{\mathbf{P}}_j$, not necessary convex, realizing $\mathfrak{m}$ (,i.e, $\partial\tilde{\mathbf{P}}_j$ is cellular isomorphic to $\mathcal{M}$). Note that after the normalization, the combinatorics and the geometry of the vertices and edges in $\mathcal{N}$, and $\mathcal{S}$ are preserved. Let us maintain the names $e_1^j, \ldots, e_n^j$ for the edges connecting both half-spaces, as $j \to \infty$, the edges $e_1^j, \ldots, e_n^j$ tend to straight lines. Therefore, $H \cap \tilde{\mathbf{P}}_j$ is almost an euclidean polyhedron, it also bounds the hyperbolic polygon T_j and its interior angles are slightly bigger than the interior angles of T_j, i.e, the dihedral angles $\alpha(e_1^j), \ldots, \alpha(e_n^j)$.

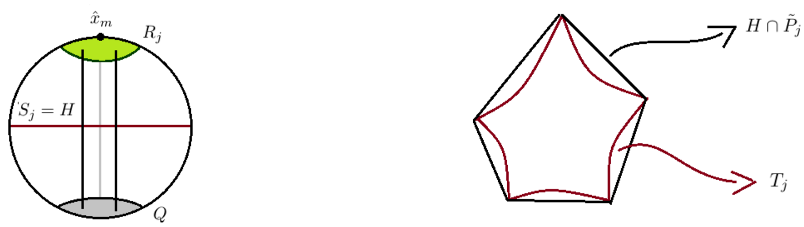


Figure 3.13.

Taking the above into consideration, the Gauss Bonnet theorem tells us that

$$\pi(n-2) \approx \sum_{i=n} \alpha(e_i^j). \tag{3.8}$$

Thus, since $\alpha(e_j^i) \leq \frac{\pi}{2}$, it holds that $\pi(n-2) = n\frac{\pi}{2} + \epsilon$ for a small $\epsilon > 0$, and we can conclude that $n \leq 4$. It is clear that $n \geq 3$ and so there are either 3 or 4 edges connecting $\mathcal{S}$ and $\mathcal{N}$.

First let us assume that $n = 3$ and that e_1^j, e_2^j, e_3^j are the edges connecting $\mathcal{N}$ and $\mathcal{S}$. Note that the corresponding dual edges in $\mathcal{M}^*$ form a 3-cycle. Moreover, it is a prismatic 3-cycle, otherwise Proposition 2.2 tells us that e_1^j, e_2^j, e_3^j meet at a vertex lying in $\mathcal{N}$ (or $\mathcal{S}$), this implies that the other vertices of $\tilde{\mathbf{P}}_j$ lying in $\mathcal{N}$ are disconnected from the vertices in $\mathcal{S}$, a contradiction the fact that $G(\tilde{\mathbf{P}}_j) \cong G(\mathbf{P}_j)$ is a connected graph. Since condition [a3] holds for $\mathbf{a} = [\alpha_1, \ldots, \alpha_E]^T$, the above tells us that

$$\alpha_1 + \alpha_2 + \alpha_3 < \pi.$$

However, by equation (3.8), we can conclude that

$$\lim_{j\to\infty} \alpha(e_1^j) + \alpha(e_2^j) + \alpha(e_3^j) = \alpha_1 + \alpha_2 + \alpha_3 = \pi$$

and we get a contradiction. Now, let us assume that $n = 4$ and that $e_1^j, e_2^j, e_3^j, e_4^j$ are the edges connecting $\mathcal{N}$ and $\mathcal{S}$. The corresponding dual edges form a 4-cycle. Furthermore, if the 4-cycle is prismatic, analogue to the last case, we get that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$$

which contradicts condition [a4]. On the other hand, if the 4-circuit is not prismatic, at least two edges from $\{e_1^j, e_2^j, e_3^j, e_4^j\}$ must meet at a vertex, and by equation (3.8),

$$\lim_{j\to\infty} \alpha(e_1^j) = \lim_{j\to\infty} \alpha(e_2^j) = \lim_{j\to\infty} \alpha(e_3^j) = \lim_{j\to\infty} \alpha(e_4^j) = \frac{\pi}{2}.$$

Assume that e_1^j and e_2^j meet at the vertex x^j, that e_l^j is the third edge incident to x^j and that F^j is the face of $\tilde{\mathbf{P}}_j$ containing e_1^j, e_2^j and x^j on its boundary.

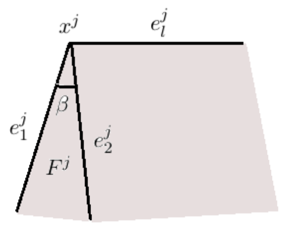


Figure 3.14.

Let β^j be the interior angle of F^j at x^j, by the spherical laws of cosines it holds that

$$\begin{aligned}\lim_{j\to\infty} \cos(\beta^j) &= \lim_{j\to\infty} \frac{\cos(\alpha(e_l^j)) + \cos(\alpha(e_1^j))\cos(\alpha(e_2^j))}{\sin(\alpha(e_1^j))\sin(\alpha(e_2^j))} \\ &= \lim_{j\to\infty} \cos(\alpha(e_l^j)).\end{aligned}$$

Therefore, as we are considering non-obtuse dihedral angles and all the vertices in $\mathcal{N}$ converge to the north pole, if x^j lies in $\mathcal{N}$, we must have that

$$\lim_{j\to\infty} \beta^j = \lim_{j\to\infty} \alpha(e_l^j) = \alpha_l = 0$$

which contradicts condition [a1]. Analogue, if x^j lies in $\mathcal{S}$, remember that the distance between $H = S_j$ and Q with respect to the z-axis is half the distance between Q and R_j before the normalization, call it d_j. Since $\frac{d_j}{2} \to \infty$ as $j \to \infty$, all the vertices in $\mathcal{S}$ converge to the south pole and we get a contradiction as the above.

Case 2. *If $\hat{x}_m$ is a finite point for some $m \in \{1, \ldots, V\}$, then $\{x^j\}_{j\in\mathbb{N}}$ is the only sequence of faces that converges to $\hat{x}_m$.*

First we will see that the faces of the polyhedra $\mathbf{P}_j$ don't degenerate, i.e, their limits are not polyhedra of less dimension.

Assume that a sequence of faces F_h^j degenerates, we already know that there is exactly one sequence of vertices converging to each ideal vertex. Therefore, F_h^j degenerates to a finite point, a line segment or a hyperbolic ray and we can conclude that

$$\lim_{j\to\infty} \text{Area}(F_h^j) = 0.$$

If $\beta_1^j, \dots, \beta_n^j$ are the interior angles of F_h^j, then the last equation and the Gauss-Bonnet theorem tell us that

$$\pi(n-2) = \lim_{j\to\infty} \sum_{i=1}^{n} \beta_i^j \tag{3.9}$$

Since the interior angles β_i^j are also less or equal to $\frac{\pi}{2}$, we can deduce from equation (3.9) that $\pi(n-2) \le n\frac{\pi}{2}$ and so $n = 3$ or $n = 4$. Therefore, F_h^j is either a triangle or a quadrilateral.

Assume that F_h^j is a triangle and let e_1^j, e_2^j and e_3^j be the edges leaving the vertices of F_h^j. If the corresponding dual edges don't form a prismatic 3-cycle, by Proposition 2.2 they meet at a vertex outside $\tilde{F}_h$, and we get that $N = 4$, which contradicts our assumption that the number of faces of $\mathfrak{m}$ is strictly bigger than 4.

Figure 3.15.

Now, if F_h^j degenerates to a finite point, it holds that

$$\lim_{j\to\infty} \alpha(e_1^j) + \alpha(e_2^j) + \alpha(e_3^j) = \alpha_1 + \alpha_2 + \alpha_3 > \pi$$

which contradicts condition $[a3]$. On the other hand, if F_h^j degenerates to a line segment or a ray, two of its interior angles tend to $\frac{\pi}{2}$, and using Lemma 3.3 we can conclude that two of the dihedral angles of the edges leaving F^j converge to $\frac{\pi}{2}$. Thus, we get that

$$\alpha_1 + \alpha_2 + \alpha_3 \ge \pi$$

which is again a contradiction to condition $[a3]$.

Now assume that F_h^j is a quadrilateral, by the equation (3.9)

$$\lim_{j\to\infty} \beta_1^j + \beta_2^j + \beta_3^j + \beta_4^j = 2\pi.$$

Therefore, since the interior angles of F_h^j are non-obtuse, we can conclude that the interior angles converge to $\frac{\pi}{2}$. Let $e_1^j, e_2^j, e_3^j, e_4^j$ be the edges leaving the vertices of F_h^j, by Lemma 3.3 and our last discussion, their dihedral angles converge to $\frac{\pi}{2}$ and the dihedral angles of at least two opposite edges on the boundary of F_h^j, let us say $e_{k_1}^j, e_{k_2}^j$, also converge to $\frac{\pi}{2}$.

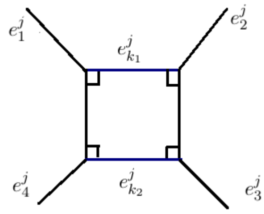


Figure 3.16.

We finally get that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_{k_1} + \alpha_{k_2} = 3\pi$$

which contradicts condition [a5].

Now, as no face of $\mathbf{P}_j$ degenerates, we can conclude that $\mathbf{P}_j$ don't degenerate as well. Therefore, we can find at least 3 sequences of vertices $\{x_a^j\}_{j\in\mathbb{N}}, \{x_b^j\}_{j\in\mathbb{N}}$ and $\{x_c^j\}_{j\in\mathbb{N}}$ whose limit is not $\hat{x}_m$. Assume that $x_1^j, \ldots, x_k^j$ are the vertices of that converge to $\hat{x}_m$ and use an isometry to situate the polyhedra in such a way that $\hat{x}_m$ lies at the origin, we want to see that $k = 1$. For a large enough $l \in \mathbb{N}$, we can find a small sphere $\mathbf{S}$ centred at the origin, such that for all $j > l$, it separates the vertices $y_1^i, \ldots, y_k^i$ from the vertices y_a^j, y_b^j and y_c^j, analogously as we did in Proposition 1.15 we can see that there are exactly three edges e_1^j, e_2^j, e_3^j going out of $\mathbf{S}$. If $k > 1$, then the corresponding dual edges form a prismatic 3-cycle. However, since $y_1^j, \ldots, y_k^j$ tend to the finite vertex $\hat{y}_m$, it holds that

$$\alpha_1 + \alpha_2 + \alpha_3 > \pi$$

which is a contradiction to condition $[a3]$. Thus, we can conclude that $k = 1$ and we are done. □

Proposition 3.4. *The map* $\boldsymbol{\alpha} : \mathsf{P}^0_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}}$ *is proper.*

Proof. Let $\mathbf{K} \subseteq A_{\mathfrak{m}}$ be a compact set, we want to see that $\boldsymbol{\alpha}^{-1}(\mathbf{K})$ is compact in $\mathsf{P}^0_{\mathfrak{m}}$. Take a infinite sequence of equivalence classes of marked polyhedra $\{[\mathbf{M}_l]\}_{l\in\mathbb{N}}$ in $\alpha(\mathbf{K})^{-1}$. Since $\mathbf{K}$ is compact, the sequence of vectors given by $\boldsymbol{\alpha}([\mathbf{M}_l]) = \mathbf{a}_l$ has a infinite convergent subsequence with limit point $\mathbf{a} \in \mathbf{K} \subseteq A_{\mathfrak{m}}$. Take the subsequence of marked polyhedra $\{[(\mathbf{P}_j, [\psi_j]_\sim)]\}_{j\in\mathbb{N}}$ whose corresponding dihedral angles are given by the vectors in the subsequence. It is clear that the conditions [a1],[a3]-[a5] hold for $\mathbf{a}$. Therefore, by the initial discussion and Proposition 3.3, the sequence of compact polyhedra $\{\mathbf{P}_j\}_{j\in\mathbb{N}}$, has infinite convergent subsequence whose limit is a polyhedron of finite volume $\mathbf{P}$ in $\mathbf{H}^3$ that realizes $\mathfrak{m}$, with a marking $[\hat{\psi}]_\sim$ as in equation 3.7, and whose corresponding dihedral angles are given by the entries of the vector $\mathbf{a}$. Moreover, since condition [a2] holds for $\mathbf{a}$, we can conclude that all the vertices of $\mathbf{P}$ are finite. Hence, $\mathbf{P}$ is a non-obtuse compact polyhedron realizing $\mathfrak{m}$ and $\boldsymbol{\alpha}([(\mathbf{P}, [\hat{\psi}]_\sim)]) = \mathbf{a}$, this tells us that $[(\mathbf{P}, [\hat{\psi}]_\sim)] \in \boldsymbol{\alpha}^{-1}(\mathbf{K})$ and we can conclude that $\boldsymbol{\alpha}^{-1}(\mathbf{K})$ is compact. □

3.4. If $\mathfrak{m}$ is simple, then $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{m}}) \neq \emptyset$

In this section an explanation of some of the tools that Andreev used in his paper [3] [14] to prove that Therorem 2.2 holds for simple abstract polyhedra will be given. Remember that by a simple abstract polyhedron we mean a trivalent abstract polyhedron that is different from the tetrahedron and doesn't have prismatic 3-cycles. The main idea of this proof is to give an explicit proof for a base case, in this case the split-prism, and then show that we can obtain any simple abstract polyhedron, that is not a prism, via a sequence of operations called Whitehead moves. We will show that Andreev's Theoream for compact polyhedra holds for the prism and the split prism, and then, show that the Whithead moves preserve realizability.

Proposition 3.5. *Let us take $N \geq 3$. The prism with $N+2$ faces is realizable as a compact non-obtuse hyperbolyc polyhedron.*

Proof. We will build the polyhedron in the Poincare ball model $\mathbb{D}^3$. First, let us take a collection of N real numbers $\beta_1, ..., \beta_N$ in the interval $(0, \frac{\pi}{2}]$, by Remark 1.7 there is an hyperbolic N-gon $\mathbf{Q}$ in $\mathbb{D}^2$ such that its interior angels are $\beta_1, ..., \beta_N$, also note that $\mathbf{Q}$ has no ideal vertices.

Let us see $\mathbb{D}^2$ as the equatorial plane of $\mathbb{D}^3$ and take the supporting lines $l_1, ..., l_N$ of the sides of $\mathbf{Q}$. By proposition 1.10 and the remark that follows it, we can find a hyperbolic plane P_i in $\mathbb{D}^3$ such that

$$l_i = P_i \cap \mathbb{D}^2.$$

Each plane P_i defines two half-spaces, let us take the half-spaces $H_1, ..., H_N$ such that $\mathbf{Q}$ lies in the interior of the 3-dimensional polyhedron

$$\hat{\mathbf{P}} = \bigcap_{i=1}^{N} H_i \quad \text{and} \quad \partial H_i = P_i.$$

Note that each face of $\hat{\mathbf{P}}$ contains exactly one side of $\mathbf{Q}$, and that the hyperbolic plane P_i is the supporting plane of the face containing the side s_i. Moreover, the dihedral angels of $\hat{\mathbf{P}}$ are $\beta_1, ..., \beta_N$.

Now take two hyperbolic planes P_{N+1} and P_{N+2} such that:

1. Both planes are perpendicular to the z-axis.
2. P_{N+1} lies over the equatorial plane and is very close to it
3. P_{N+2} lies under the equatorial plane an is very close to it.

The intersection of P_{N+k}, $k = 1, 2$, with the other planes has an interior angel smaller than $\frac{\pi}{2}$. Hence, by taking the appropriate half-spaces H_{N+1} and H_{N+2} the hyperbolic polyhedron

$$\mathbf{P} = \bigcap_{i=1}^{N+2} H_i$$

is a compact hyperbolic polyhedron, whose dihedral angels are non-obtuse and whose combinatorial type is the prism with $N+2$ faces. $\square$

The last result tells us that the Andreev's theorem for compact polyhedra holds for the combinatorial type corresponding to the prim with $N+2$ faces.

Corollary 3.1. *Let $\mathfrak{p}_{N+2}$ be an abstract polyhedron corresponding to the prism with $N+2$ faces. Then, the Theorem 2.2 holds with $\mathfrak{m} = \mathfrak{p}_{N+2}$*

Proof. By the last proposition $\boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{p}_{N+2}}) \neq \emptyset$. Since we already proved that the map $\boldsymbol{\alpha}$ is well define, injective and proper, Proposition 2.7 tells us that $A_{\mathfrak{p}_{N+2}} = \boldsymbol{\alpha}(\mathsf{P}^0_{\mathfrak{p}_{N+2}})$ (the same proof for the triangular prism), which tells us that Theorem 2.2 holds with $\mathfrak{m} = \mathfrak{p}_{N+2}$. $\square$

Corollary 3.2. *Let us take $N \geq 5$. The split-prism with $N+2$ faces is realizable as a compact non-obtuse hyperbolic polyhedron.*

Proof. First, let us take $N = 5$, note that the prism with 7 faces and the split-prism with 7 faces are combinatorial equivalent.

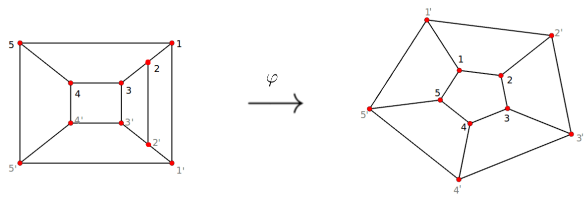


Figure 3.17.: Cellular isomorphism between $\mathfrak{d}_7$ and $\mathfrak{p}_7$

Now, for $N > 6$ by Corollary 3.1, there is a compact hyperbolic polyhedron $\mathbf{P}$ whose combinatorial type is the prism with $(N-1)+2$ faces and whose dihedral angels are given as in the Figure 3.18.

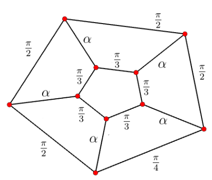


Figure 3.18.: $\alpha = \frac{\pi}{2}$.

Without loss of generality, we can assume that $\mathbf{P}$ lies in $\mathbb{D}^3$ and that F_1 is the face of $\mathbf{P}$ which corresponds to the exterior face of the cell complex of Figure 3.18.

Now, if P_1 is the supporting plane of the face F_1 and $\phi_1 \in \text{Isom}(\mathbb{H}^3)$ is the reflection through P_1, then $\phi_1(\mathbf{P})$ is compact hyperbolic polyhedron congruent to $\mathbf{P}$. Moreover, F_1 is a common face of both polyhedra. Gluing $\mathbf{P}$ and $\phi_1(\mathbf{P})$ together by the face F_1, we obtain a new compact polyhedron $\tilde{\mathbf{P}}$. Note that:(see Figure 3.19)

1. After gluing, the dihedral angle of an edge of F_1 is added to the dihedral angle of its mirror image. Therefore, the edges on F_1 with dihedral angle equal to $\frac{\pi}{2}$ disappear, and the remaining edge becomes an edge of $\tilde{\mathbf{P}}$ with dihedral angle $\frac{\pi}{2}$.
2. If e is an edge with dihedral angel α, it either glues together with its mirror image into a new edge of $\tilde{\mathbf{P}}$, or both the edge and its mirror image become edges of $\tilde{\mathbf{P}}$. In both cases the dihedral angle remains the same.
3. The other edges in $\mathbf{P}$ and $\phi_1(\mathbf{P})$ become edges of $\tilde{\mathbf{P}}$, and their dihedral angles don't change.

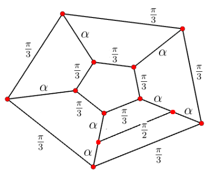


Figure 3.19.: Cell complex corresponding to the polyhedron $\tilde{\mathbf{P}}$

Note that $\tilde{\mathbf{P}}$ is a compact hyperbolic polyhedron whose combinatorial type corresponds to the split-prism with $N+2$ faces and whose dihedral angels are less or equal to $\frac{\pi}{2}$. □

In a similar way as we proved Corollary 3.1, the last result tells us that the Andreev's theorem for compact hyperbolic polyhedra holds for the split-prism with $N+2$ faces, where $N \geq 5$. Now, let us define the Whitehead movements and see why they preserve the property of being realizable as a compact polyhedron with non-obtuse dihedral angles.

Definition 3.1. Let $\mathcal{D}$ be a triangulation of $\mathbb{S}^2$ and let us take two triangles $\Delta(ADC)$ and $\Delta(ABC)$ in $\mathcal{D}$. A **Whitehead move** $\mathbf{W}(ABCD)$ sends the triangulation $\mathcal{D}$ to a new triangulation $\mathcal{D}'$ by deleting the edge $e = AC$ and adding the edge $e' = DB$, i.e, we switch the triangles $\Delta(ADC)$ and $\Delta(ABC)$ by the triangles $\Delta(BAD)$ and $\Delta(BCD)$.

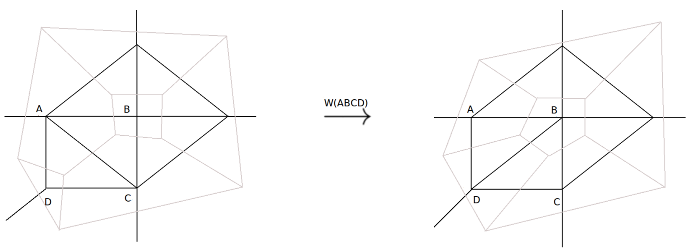


Figure 3.20.: Whitehead Move

Now assume that $\mathfrak{m}$ is a trivalent abstract polyhedron, and assume that $\mathcal{M}$ is a realization of $\mathfrak{m}$ as a cellular decomposition of $\mathbb{S}^2$. As we saw in the last chapter, $\mathcal{M}^*$ is a triangulation of $\mathbb{S}^2$, hence by applying a Whitehead move to $\mathcal{M}^*$ we obtain a new trivalent cellular decomposition $\mathcal{M}'$ of $\mathbb{S}^2$ whose dual decomposition is the triangulation that we get under the Whitehead move.

In terms of a the cellular decomposition $\mathcal{M}$ a Withehead move $\mathbf{W}(ABCD)$ can be see as follows (see Figure 3.21):

1. The vertices A, B, C and D of $\mathcal{M}^*$ corresponds to faces F_A, F_B, F_C and F_D of $\mathcal{M}$, and, the dual edge $e^* = AC$ corresponds to an edge e in $\mathcal{M}$. Moreover, it holds that
$$F_A \cap F_C = e \quad \text{and} \quad F_B \cap F_D = \emptyset.$$
2. Deleting the edge $e^* = AC$ of $\mathcal{M}^*$ corresponds to contracting the edge e to a vertex v' and so, we get a new cellular complex $\tilde{\mathcal{M}}$ where
$$F_A \cap F_B \cap F_C \cap F_D = \{\hat{v}\}$$
3. Adding the edge $e'^* = DB$ can be seen as stretching out the vertex v' so that it becomes the edge e' of the complex $\mathcal{M}'$. Also note that in $\mathcal{M}'$ we have that
$$F_B \cap F_D = e' \quad \text{and} \quad F_A \cap F_C = \emptyset.$$

Therefore, we can say that the Whitehead move is defined by the edge e and write $\mathbf{W}(e)$ instead of $\mathbf{W}(ABCD)$. Moreover, note that the face lattice $\mathfrak{m}'$ of $\mathcal{M}'$ is also a trivalent abstract polyhedron.

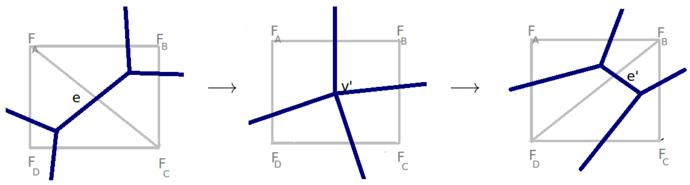


Figure 3.21.: Whitehead Move on the Complex

Proposition 3.6. *Let $\mathfrak{m}$ be an simple abstract polyhedron that is realizable as a non-obtuse compact hyperbolic polyhedron, and let $\mathfrak{m}'$ be the abstract polyhedron that we obtain from $\mathfrak{m}$ via a Whitehead move. Then, $\mathfrak{m}'$ is realizable as a non-obtuse compact hyperbolic polyhedron.*

Proof. (**Sketch of the proof**) Let us assume the $\mathcal{M}$ is a cellular realization of $\mathfrak{m}$, and let us assume e_0 is the edge of $\mathcal{M}$ in which we apply the Whitehead move. Also, let us assume that e_1, e_2, e_3 and e_4 are the edges adjacent to e_0 and take the values

$$\begin{aligned} \alpha(e_0) =& \epsilon \quad \text{for some} \quad \epsilon \in (0, \frac{\pi}{2}] \\ \alpha(e_1) =& \alpha(e_2) = \alpha(e_3) = \alpha(e_4) = \frac{\pi}{2} \\ \alpha(e) = \frac{2\pi}{5} & \quad \text{for a different edge} \quad e \quad \text{of} \quad \mathcal{M}. \end{aligned} \tag{3.10}$$

Since $\mathfrak{m}$ is realizable as compact hyperbolic polyhedron, Andreev's theorems holds, and since $\mathfrak{m}$ is simple, by Proposition 2.3 there is a compact hyperbolic polyhedron $\mathbf{P}_\epsilon$ whose dihedral angels are the ones given in the equation (3.10).

We can take a sequence $\{\epsilon_i\}_{i\in\mathbb{N}} \subseteq (0, \frac{\pi}{2}]$ such that

$$\lim_{n\to\infty} \epsilon_i = 0.$$

Since $\bar{\mathbb{D}}^3$ is a compact topological space, we can assume without loss of generality that there is a polyhedron $\tilde{\mathbf{P}}$ such that

$$\lim_{i\to\infty} \mathbf{P}_{\epsilon_i} = \tilde{\mathbf{P}}$$

We wan to see that $\tilde{\mathbf{P}}$ is a hyperbolyc polyhedron of finite volume with one ideal vertex and the other vertices lying inside of the hyperbolic space. Note that this corresponds to the situation of contracting the edge e to a vertex v' in $\mathcal{M}$.

To see the later us assume that $\{x_1^i, ..., x_V^i\} = V(P_{\epsilon_i})$ is the set of vertices of $\mathbf{P}_{\epsilon_i}$, e_0^i the edge corresponding to e_0 in $\mathbf{P}_{\epsilon_i}$ and $F_A^i F_B^i F_C^i F_D^i$ the faces of $\mathbf{P}_{\epsilon_i}$ such that

$$F_A^i \cap F_C^i = e_0^i \quad \text{and} \quad F_D^i \cap F_B^i = \emptyset.$$

Also, assume that x_1^i and x_2^i are the vertices adjacent to e_0^i.

1) Let P_A^i be the supporting plane of the face F_A^i, and P_C^i be the supporting plane of the face F_C^i. The dihedral angle $\alpha(e_0^i) = \epsilon_i$ corresponds to the dihedral angle of the intersection $P_A^i \cap P_C^i$. Since

$$\lim_{i\to\infty} \alpha(e_0^i) = 0,$$

it follows that the sequence $P_A^i \cap P_C^i$ converges to a single point $\hat{y}$ lying on the boundary of the hyperbolic space. Therefore,

$$\lim_{i\to\infty} x_1^i = \lim_{i\to\infty} x_2^i = \lim_{i\to\infty} e_0^i = \hat{y}.$$

2) Let $\hat{x}_1, ..., \hat{x}_V$ be the limit points of the sequences $x_1^i, ..., x_V^i$. We already know that $\hat{x}_1 = \hat{x}_2 = \hat{y}$. As in the last section, we can use a normalization to separate the vertices that converge to $\hat{y}$ from the others that converge to other vertices in two sets, one near to the north pole of $\mathbb{D}^3$ and the other on one near to the south pole. We use the last procedure to see that there are exactly four edges separating those two sets and that those edges do not form a prismatic four circuit. Moreover, this prismatic circuit separates exactly two vertices,in this case x_1^i and x_2^i ,from the others, hence x_1^i and x_2^i are the only vertices that converge to $\hat{y}$.

3) If F^i is a face of $\mathbf{P}_{\epsilon_i}$, using the law of cosines in the spherical geometry and the Gauss Bonnet theorem, we can show that $\{\text{Area}(F^i)\}_{i\in\mathbb{N}}$ is a constant sequence, hence no sequence of faces converges to a vertex or an edge. On the other hand, if $e^i \in E(\mathbf{P}_{\epsilon_i}) - \{e_0^i\}$, the sequence $\{\alpha(e^i)\}_{i\in\mathbb{N}}$ is also a constant sequence, either $\alpha(e^i) = \frac{\pi}{2}$ or $\alpha(e^i) = \frac{2\pi}{5}$. Therefore, since no face degenerates to an edge or a vertex, we can conclude that $\{e^i\}_{i\in\mathbb{N}}$

doesn't converge to a vertex.

The last result also tells us that for $j > 2$ the sum of the dihedral angels of the edges incident to $\hat{x}_j$ is strictly bigger that π. Consequently, by Proposition 1.11 we can conclude that $\hat{x}_j$ is not an ideal vertex.

Now, let us assume that $\tilde{\mathbf{P}}$ lies on upper-half space model $\mathbb{H}^3$, and that $P_A P_B P_C P_D$ are the hyperbolic planes such that

$$\bar{P}_A \cap \bar{P}_B \cap \bar{P}_C \cap \bar{P}_D = \{\hat{y}\}.$$

Also, let us assume that the hyperbolic planes $P_A P_B P_C P_D$ are half-spheres perpendicular to $\partial\mathbb{H}^3 = \mathbb{C}$.

We can modify the hyperbolic planes by a translating the boundaries of them on $\partial\mathbb{H}^3$ (See figure 3.22). Now, by a slight modification of the hyperbolic planes $P_A P_B P_C P_D$, we can find a polyhedron $\mathbf{P}'$ such that

$$F_B \cap F_D = e' \quad \text{and} \quad F_A \cap F_C = \emptyset$$

and that inherits the rest of the face structure from $\tilde{\mathbf{P}}$. Note that $\mathbf{P}'$ is a compact hyperbolic polyhedron. Moreover, this modification corresponds to stretching out the vertex v' of $\tilde{\mathcal{M}}$ to obtain the cell complex $\mathcal{M}'$. Hence, $\mathbf{P}'$ is a realization of $\mathfrak{m}'$ as a compact hyperbolic polyhedron.

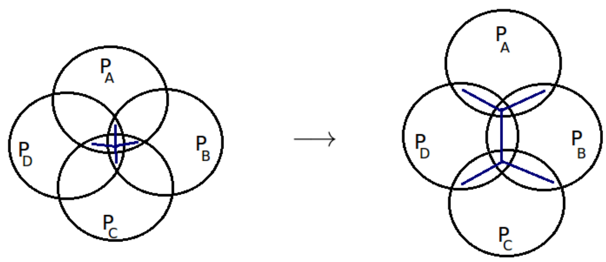
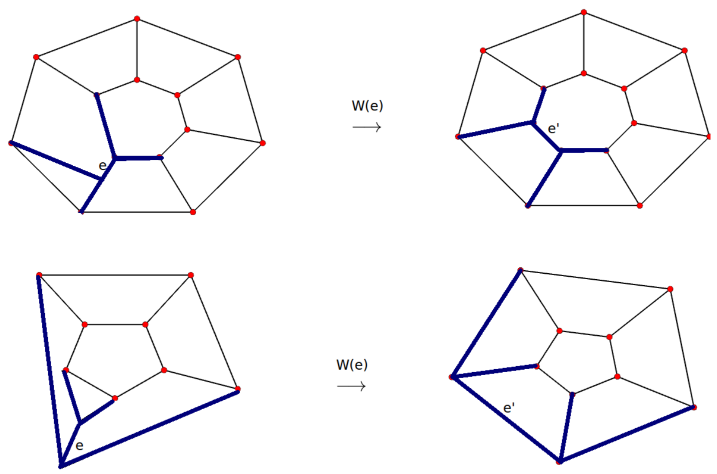


Figure 3.22.: Modification of the hyperbolic planes

Finally, since we obtained $\mathbf{P}'$ by a small modification of the hyperbolic planes $P_A P_B P_C P_D$, and by the way that we chose the dihedral angels in the original hyperbolic polyhedra, we can argue that the dihedral angels of $\mathbf{P}'$ are less or equal to $\frac{\pi}{2}$. $\square$

Example 3.1. Some polyhedra that can be obtained via Whitehead moves.



The algorithm proposed by Andreev in [3] shows that any trivalent simple abstract polyhedron (i.e its cellular representation) different from the prism, can be reduced to a split-prism via a sequence of Whitehead moves. Hence, from Proposition 3.6 we can deduce that the Andreev's theorem for compact hyperbolic polyhedra holds for this class of abstract polyhedra.

4. Polyhedra of Finite Volume

Let $\mathbf{P}$ be a non-obtuse polyhedron in $\mathbf{H}^3$ of finite volume and let $\{v_1, \ldots, v_N\}$ be the minimal set of vectors in $\mathcal{H}_3$ defining $\mathbf{P}$. Assume that a vertex $\hat{y} \in V(\mathbf{P})$ is the intersection of exactly three faces without lost of generality

$$\hat{y} = F_{v_1} \cap F_{v_2} \cap F_{v_3}$$

and that $\tilde{e}_1 = F_{v_1} \cap F_{v_2}$, $\tilde{e}_2 = F_{v_2} \cap F_{v_3}$ and $\tilde{e}_3 = F_{v_3} \cap F_{v_1}$. By Proposition 1.11, $\hat{y}$ is an ideal vertex if and only if

$$\alpha(\tilde{e}_1) + \alpha(\tilde{e}_2) + \alpha(\tilde{e}_3) = \pi. \tag{4.1}$$

On the other hand, remember that to prove proposition 3.6 we used a convergent sequence of non-obtuse compact polyhedra $\{\mathbf{P}_{\epsilon_i}\}_{i\in\mathbb{N}}$ in $\mathbb{D}^3$, where $\{\epsilon_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers that converges to zero, ϵ_i is the dihedral angle at an edge e_0^i of $\mathbf{P}_{\epsilon_i}$ and $\frac{\pi}{2}$ is the value of the dihedral angles of the edges incident to e_0^i. The sequence $\{\mathbf{P}_{\epsilon_i}\}_{i\in\mathbb{N}}$ converges to a non-obtuse polyhedron $\tilde{\mathbf{P}}$ of finite volume and the sequence $\{e_0^i\}_{i\in\mathbb{N}}$ converges to an ideal vertex $\hat{y}$ of $\tilde{P}$. Furthermore, if we assume that $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4$ are the edges of $\tilde{\mathbf{P}}$ that meet at the ideal vertex $\hat{y}$, it holds that $\alpha(\tilde{e}_i) = \frac{\pi}{2}$ for all $i \in \{1, 2, 3, 4\}$ and we get that

$$\alpha(\tilde{e}_1) + \alpha(\tilde{e}_2) + \alpha(\tilde{e}_3) + \alpha(\tilde{e}_4) = 2\pi. \tag{4.2}$$

In fact, if a non-obtuse polyhedron $\mathbf{P}$ of finite volume in $\mathbf{H}^3$ has an ideal vertex, we have one of the last two possibilities, i.e, if $\hat{y} \in V(\mathbf{P})$ is an ideal vertex, then the number of edges incident to $\hat{y}$ is either 3 or 4, and respectively either equation (4.1) or equation (4.2) holds for the dihedral angles at those edges.

The above tells us that a necessary condition, for a abstract polyhedron $\mathfrak{m}$ to be realizable as a non-obtuse polyhedron of finite volume in $\mathbf{H}^3$, is that the number of edges incident to a vertex of $\mathfrak{m}$ is 3 or 4. Moreover, if we assume that $\mathfrak{m}$ has more than 4 faces, we can extend Theorem 2.2 to tells us if there is a polyhedron $\mathbf{P}$ of finite volume in $\mathbf{H}^3$, that realizes $\mathfrak{m}$ and whose corresponding dihedral angles are the given values $\alpha(e_i) \in (0, \frac{\pi}{2}]$ for $e_i \in E(\mathfrak{m})$ (or $E(\mathcal{M})$) by adding the equation (4.1) to the condition [a2] and the following extra conditions to the theorem (see [4]),

[a6] If the edges e_i, e_j, e_k, e_l meet at a vertex, then

$$\alpha(e_i) + \alpha(e_j) + \alpha(e_k) + \alpha(e_l) = 2\pi.$$

[a7] If we have three faces $\tilde{F}_i, \tilde{F}_j, \tilde{F}_k$ such that, $\tilde{F}_i$ and $\tilde{F}_j$ are adjacent, $\tilde{F}_j$ and $\tilde{F}_k$ are adjacent, $\tilde{F}_i$ and $\tilde{F}_k$ are not adjacent but meet at a vertex, and not all the faces meet at a vertex. Then, if $e_{ij} = \tilde{F}_i \cap \tilde{F}_j$ and $e_{jk} = \tilde{F}_j \cap \tilde{F}_k$, it holds that

$$\alpha(e_{ij}) + \alpha(e_{jk}) < \pi$$

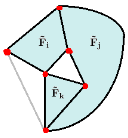


Figure 4.1.

Some Applications of Andreev's Theorem

- A 3-dimensional polytope is inscribed in the unit sphere $\mathbb{S}^2$ if all its vertices lie on $\mathbb{S}^2$. Note that an ideal polyhedron in the Klein model is a 3-dimensional polytope inscribed in $\mathbb{S}^2_1$. Therefore, Andreev's Theorem provides tools to find 3-dimensional polytopes inscribed in $\mathbb{S}^2$.

- The circle packing theorem, which we will discuss in the next chapter.

5. The Circle Packing Theorem

The first version of the circle packing theorem was given by Koebe in 1936 and later rediscovered by Thurston in his notes [15]. This theorem relates finite collections of adjacent circles on the plane, which we call a circle packings, with simple connected planar graphs. In the proof given by Thurston, the circle mapping theorem is a direct consequence of the Andreev's Theorem for finite volume hyperbolic polyhedra. To prove this theorem, we will first find a cellular decomposition of $\mathbb{S}^2$ that is combinatorial equivalent to an ideal polyhedron $\mathbf{P}$, and then we build a circle packing by taking some of the circles that we get when we intersect the boundary of the hyperbolic space with the closure of the supporting planes carrying the faces of $\mathbf{P}$.

Definition 5.1. A circle packing $\mathsf{C} = \{C_1, ..., C_n\}$ is a collection of circles with disjoint interior on a surface $\mathtt{D}$.

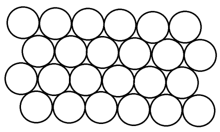


Figure 5.1.: A circle packing

Definition 5.2. The **nerve** $G(\mathsf{C})$ of the packing C is the graph whose vertices are the circles in the packing and whose edges correspond to tangent circles in the packing.

Note that the nerve $G(\mathsf{C})$ is a simple graph. It is clear that $G(\mathsf{C})$ has no loops, since no circle is tangent to itself, and there is only one possible edge between two circles. Moreover, if C is a circle packing on the plane, by sending a circle C_i to its center in $\mathbb{R}^2$ and the edge $e = (C_iC_j)$ to the line segment connecting the centres of both circles, we get an embedding of $G(\mathrm{C})$ in $\mathbb{R}^2$.

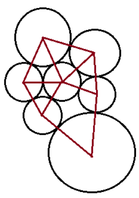


Figure 5.2.: Circle packing with its nerve

The last paragraph tells us that the nerve of a circle packing on the plane is a simple planar graph. Also, note that if the circle packing is connected, then the nerve is also a connected graph. On the other hand, The circle packing theorem tells us that any simple planar connected graph is the nerve of a connected circle packing on the plane.

Theorem 5.1. ***(The Koebe-Andreev-Thurston Circle Packing Theorem)*** *Every simple connected planar graph* $\mathbf{G}$ *is isomorphic to the nerve of a connected circle packing on the plane.*

Proof. Let G be the image of an embedding of $\mathbf{G}$ on $\mathbb{R}^2$. Extend G to a triangulation T by adding an extra vertex to the interior of each face of G that is not a triangle and connecting this vertex to its neighbour vertices.

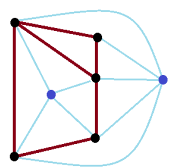


Figure 5.3.: $\mathbf{T}$

We will surround the vertices of T by a collection of simple closed curves $\{\gamma_v\}_{v\in V(T)}$ such that:

1. The vertex v lies on the interior of γ_v. Also, γ_v goes through the middle point of each edge incident to v.

2. $$\gamma_u \cap \gamma_v \neq \emptyset \quad \iff \quad (uv) \in E(T). \tag{5.1}$$
   Moreover, in this case the intersection is the middle point m_{uv} of the edge (uv).

3. The interiors of the curves don't intersect.

In this way we define a plane graph $\tilde{G}$ whose vertices are the middle points of the edges of T and whose edges are the segments of the curves between two middle points, containing no other middle points in their relative interiors.

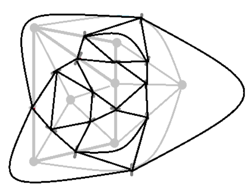


Figure 5.4.: $\tilde{\mathbf{G}}$

Note that:

- Since T is connected, $\tilde{G}$ is connected.
- Each γ_v is simple, hence $\tilde{G}$ is simple.
- Each γ_v defines exactly two edges incident to the middle point m_{uv} of the edge $(uv) \in E(T)$. Additionally, since there are exactly two curves passing through m_{uv}, the number of edges incident to m_{uv} is 4.

Therefore, $\tilde{G}$ is a 3-connected simple plane graph.

Let $\tilde{C}$ be the cellular decomposition of $\mathbb{S}^2$ defined by $\tilde{G}$ and for all $e \in E(\tilde{C})$ take the value $\alpha(e) = \frac{\pi}{2}$. Using this information, Andreev's Theorem tells us that there is an ideal polyhedron $\mathbf{P}$ that is combinatorially equivalent to $\tilde{C}$.

Assume that $\mathbf{P}$ lies in the Poincare ball model $\mathbb{D}^3$. In this model the intersection of the closure of a hyperbolic plane and $\partial\mathbb{D}^3 = \mathbb{S}^2 \cong \hat{\mathbb{C}}$ is a circle. Furthermore, as we already know the intersection of the closures of two hyperbolic planes P_1P_2 is either a point at infinity or the closure of a hyperbolic line. In the first case, the circles

$$C_1 = \bar{P}_1 \cap \partial\mathbb{D}^3 \quad \text{and} \quad C_2 = \bar{P}_2 \cap \partial\mathbb{D}^3$$

are tangent circles. In fact, C_1, C_2 are tangent circles if and only if P_1, P_2 meet at a point at infinity.

Since $\mathbf{P}$ is an ideal polyhedron, two faces F_1F_2 of $\mathbf{P}$ meet at vertex, if and only if, the supporting planes of these faces meet at a point at infinity (we may refer to figure 3.22). Therefore, the faces meet at a vertex, if and only if, the corresponding circles on $\partial\mathbb{D}^3$ are tangent circles.

Now, each curve γ_v is the boundary of a face $\tilde{F}_v$ of $\tilde{C}$ (see Figure 5.5). Let F_v be the corresponding face in $\mathbf{P}$ and C_v be the circle that we get by intersecting the supporting plane of F_v and $\partial\mathbb{D}^3$. By equation 5.1, the faces $\tilde{F}_v\tilde{F}_w$ intersect at a vertex if and only if $(uv) \in E(T)$. Therefore, $(uv) \in E(T)$ if and only if F_vF_w meet at a vertex, which as we saw in the last paragraph is equivalent to say that C_v and C_w are tangent circles. Also note these faces are not adjacent.

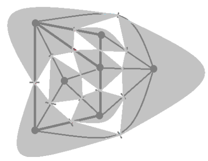


Figure 5.5.: $\tilde{\mathbf{C}}$

The latter tells us that T is the nerve of the circle packing $\mathsf{C} = \{C_v\}_{v\in V(T)}$ on $\partial\mathbb{D}^3 = \hat{\mathbb{C}}$. Moreover, since G is a sub-graph of T, the collection of circles $\{C_v\}_{v\in V(G)}$ is a circle packing whose nerve is G. We move the circle packing to the plane via the stereographic projection, which sends circles to circles, and so we are done. $\square$

In his proof Thurston also used the ideal polyhedron $\mathbf{P}$ to show that the circle packing of a graph G is unique up to Möbius transformation.

A. The Gauss Boneth Theorem

Let us take a unit-speed (piecewise) smooth simple closed curve $\gamma(t) : [0, s] \to S$ on a surface S of constant sectional curvature K. Moreover, let us assume that the orientation of γ is positive and that $s = l(\gamma)$ is its length. The **Gauss Bonnet Theorem** tells us that (see Presley [10, Chap. 5]):

$$\int_0^{l(\gamma)} k_g \mathrm{d}s = \int_0^{l(\gamma)} \dot{\theta} \mathrm{d}s - \int_{int(l(\gamma))} K \mathrm{d}A_S, \tag{A.1}$$

where k_g is the geodesical curvature.
If γ is a smooth curve the last equation tells us that

$$\int_0^{l(\gamma)} k_g \mathrm{d}s = 2\pi - \int_{int(l(\gamma))} K \mathrm{d}A_S$$

Otherwise, let $\gamma(t_0), \ldots, \gamma(t_n)$ be the point where the curve is not smooth, i.e, its vertices, and take the tangent vectors given by

$$\dot{\gamma}^-(t_i) = \lim_{t^- \to t_i} \frac{\gamma(t) - \gamma(t_i)}{t - t_i},$$

$$\dot{\gamma}^+(t_i) = \lim_{t^+ \to t_i} \frac{\gamma(t) - \gamma(t_i)}{t - t_i}.$$

Let θ_i^-, θ_j^+ be the angles between the vectors $\dot{\gamma}^-(t_i), \dot{\gamma}^+(t_i)$ and the x-coordinate of the tangent space. If $\delta_i = \theta_i^+ - \theta_i^-$, equation A.1 becomes

$$\int_0^{l(\gamma)} k_g \mathrm{d}s = 2\pi - \sum_{i=1}^{n} \delta_i - \int_{int(l(\gamma))} K \mathrm{d}A_S. \tag{A.2}$$

Example A.1.

1. Let $\mathbf{Q}$ be a n-sided polygon whose sides are geodesic segments. If $\beta_1, \ldots, \beta_n$ are the interior angels at the vertices of $\mathbf{Q}$, then $\delta_i = \pi - \beta_i$ and so, from equation A.2 it holds that

$$(n-2)\pi = \sum_{i=1}^{n} \beta_i - K \mathrm{Area}(\mathbf{Q}). \tag{A.3}$$

2. Let $\mathbf{Q}$ be a 4-sided non-convex polygon as the one from figure A.1. Moreover, let us assume that the sides of $\mathbf{Q}$ are geodesic segments. The angle at the fourth vertex is $\beta_4 = \pi + \delta_4$. Thus, equation A.2 tells us that

$$K\,\mathbf{Area}(\mathbf{Q}) = 2\pi - (\beta_4 - \pi) + \sum_{i=1}^{3}(\pi - \beta_i). \tag{A.4}$$

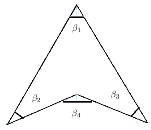


Figure A.1.: Non-convex polygon.

B. The Law of cosins in the spehrical geometry

We are going to consider spherical triangle Δ on $\mathbb{S}^2$, here we consider the 2-dimensional sphere of radius 1 in $\mathbb{R}^3$, defined by a set of linear independent unit vector $\{v_1, v_2, v_3\} \subset \mathbb{S}^2$ as follows:

- v_1, v_2 and v_3 are the vertices of Δ.
- B_1 is the side between v_2 and v_3, B_2 is the side between v_1 and v_3 , and B_3 is the side between v_1 and v_2. Also, they are are the spherical arcs between the respective vertices such that

$$|B_i| = d(v_j, v_k) < \pi.$$

- β_i is the interior angle at the vertex v_i.

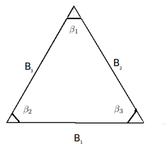


Figure B.1.: Non-convex polygon.

The Low of Cosines in the spherical geometry tells us that (see [16, chap.2]) the relation between the lengths of the sides and the angles of Δ is given by

$$\cos(\beta_i) = \frac{\cos(|B_i|) - \cos(|B_j|)\cos(|B_k|)}{\sin(|B_j|)\sin(|B_k|)}. \tag{B.1}$$

The dual law tells us that

$$\cos(|B_i|) = \frac{\cos(\beta_i) + \cos(\beta_j)\cos(\beta_k)}{\sin(\beta_j)\sin(\beta_k)} \tag{B.2}$$

C. Some Basics on Graph Theory and CW-Complexes

Some definitions from graph theory

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be an undirected graph.

Definition 1. A **path** in $\mathbf{G}$ is a sequence of edges, that connects a sequence of vertices such that all the vertices (except possible the first and the last one) are different. If the first and the last vertices are the same, a path is called **cycle** or a **circuit**.

Definition 2. The graph $\mathbf{G}$ is **connected** if for each pair of vertices in $\mathbf{G}$ there is a path connecting them.

Definition 3. The graph $\mathbf{G}$ is **simple** if there is no more than one edge between each pair of vertices and there are no edges that start and end at the same edge. If $\mathbf{G}$ is a simple graph and e is the edge connecting the vertices v and w, then we can use the notation $e = (vw)$.

Definition 4. If $\mathbf{G}$ is connected, has at least $d+1$ vertices, and after removing $d-1$ or fewer vertices from $\mathbf{G}$ we still have a connected graph, then G is d**-connected**.

Definition 5. The graph $\mathbf{G}$ is **planar** if it can be embedded in the plane (or $\mathbb{S}^2$), i.e, it can be drawn in the plane in such a way that its edges intersect only at its end points. A graph drawn in the plane (or $\mathbb{S}^2$) in such a way that its edges intersect only at its end points is called a **plane graph**.

If $\mathbf{G}$ is a planar graph, then there is an embedding or an injective map

$$\psi : G \longrightarrow \mathbb{R}^2 \tag{C.1}$$

such that $\psi(\mathbf{G})$ is a plane graph and $\psi : G \longrightarrow \psi(\mathbf{G})$ is a graph isomorphism.

We can also assume that the embedding goes from $\mathbf{G}$ to $\mathbb{S}^2$ and then consider the stereographic projection.

CW-complex

Definition 1. A d-cell is a topological space that is homeomorphic to the d-dimensional open ball $\mathbb{B}^d$ in $\mathbb{R}^d$.

Definition 2. A topological space $\mathcal{M}$ is called a cell complex, if there is a collection of disjoint cells

$$\mathcal{M}^\bullet = \{\sigma_i | i \in \mathbf{I}\}$$

such that

$$\mathcal{M} = \bigcup_{i \in \mathbf{I}} \sigma_i$$

and the following properties are satisfied,

1. $\mathcal{M}$ is Hausdorff.
2. For each d-cell $\sigma \in \mathcal{M}^\bullet$, there is a continuous map
$$f_\sigma : \bar{\mathbb{B}}^d \to \mathcal{M}$$
from the corresponding closed ball $\bar{\mathbb{B}}^d$ such that $f_\sigma^0 = f_\sigma|_{\mathbb{B}^d}$ is a homeomorphism
$$f_\sigma^0 : \mathbb{B}^d \to \mathcal{M}$$
and such that $f_\sigma(\mathbb{S}^{d-1})$ intersects only finitely many cell non-trivially, all which have dimension at most $d-1$.
3. A subset $A \subseteq \mathcal{M}$ is closed if and only if $A \cap \bar{\sigma}$ is closed in $\bar{\sigma}$ for all $\sigma \in \mathcal{M}^\bullet$.

(see [7])

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