# Department of
# Mathematics & Computer Science

**Heidelberg University**


**Master Thesis**

in the degree program Mathematics (M. Sc.)
submitted by


**Tim Adler**

born in Sinsheim (Germany)


**2017**

Heidelberg University
Department of Mathematics & Computer Science
Im Neuenheimer Feld 205
69120 Heidelberg
Germany

**Master Thesis**

# A boundary map to the Roller boundary of a CAT(0) cube complex

Tim Adler

13.12.2017

Advised by
Prof. Dr. Anna Wienhard

Tim Adler
Eberlinweg 6
69121 Heidelberg
Germany
tim {at} emrys-merlin.de

**A boundary map to the Roller boundary of a CAT(0) cube complex**
Master Thesis in the degree program Mathematics (M. Sc.)

Heidelberg University
Department of Mathematics & Computer Science
Mathematical Institute

| | |
|---|---|
| Advised by | Prof. Dr. Anna Wienhard |
| Starting date | 13.06.2017 |
| Closing date | 13.12.2017 |

This work has been set using LaTeX and KOMA-Script.

| | |
|---|---|
| Main Font: | Linux Libertine |
| Sans Font: | Linux Biolinum |

## Abstract

In this thesis we will present the construction of a so-called *boundary map* between the strong $\Gamma$-boundary $B$ of a discrete, countable group $\Gamma$ and the Roller boundary $\partial X$ of a CAT(0) cube complex $X$ on which $\Gamma$ acts by automorphisms:

$$\varphi\colon B \to \partial X.$$

We will see that this boundary map is measurable and $\Gamma$-equivariant almost everywhere. The existence was first proven by Chatterji, Fernós, and Iozzi [CFI16] under the further assumption that $X$ is connected, locally countable and finite-dimensional and that $\Gamma$ acts non-elementary on $X$.

This thesis has an expository nature. We will give a brief introduction to CAT(0) cube complexes and then turn towards the Roller duality, which will lead us immediately to the Roller boundary. Additionally, we will explore group actions on CAT(0) cube complexes introducing the notions of non-elementarity and essentiality. Lastly, we will define ergodic group actions (with coefficients) and strong $\Gamma$-boundaries.

## Zusammenfassung

Diese Abschlussarbeit hat zum Ziel eine sogenannte *Randabbildung* von einem starken $\Gamma$-Rand $B$ einer diskreten, abzählbaren Gruppe $\Gamma$ in den Roller-Rand $\partial X$ eines CAT(0) Kubenkomplexes $X$ auf dem $\Gamma$ via Autmorphismen operiert zu konstruieren:

$$\varphi\colon B \to \partial X.$$

Wir werden sehen, dass diese Randabbildung messbar und fast überall $\Gamma$-äquivariant ist. Die Existenz dieser Abbildung wurde als erstes von Chatterji, Fernós und Iozzi [CFI16] bewiesen; unter den zusätzlichen Annahmen, dass $X$ zusammenhängend, lokal abzählbar und endlich dimensional und die Gruppenwirkung von $\Gamma$ auf $X$ nicht-elementar ist.

Diese Arbeit hat einen einführenden Charakter. Wir werden zunächst eine kurze Einführung in CAT(0) Kubenkomplexe geben und uns anschließend mit der Roller-Dualität auseinandersetzen, die direkt zum Roller-Rand führt. Zusätzlich werden wir Gruppenoperationen auf CAT(0) Kubenkomplexen untersuchen und dabei die Begriffe der Nicht-Elementarität und Essentialität einführen. Schlussendlich werden wir ergodische Gruppenoperationen (mit Koeffizienten) und starke $\Gamma$-Ränder einführen.

# Contents

# 1 Introduction

Geometric group theory is a relatively new field in mathematics. One starting point might be seen in Felix Klein's *Erlangen program* [Kle93]. There, he pointed out the deep connection between (abstract) groups and their realization as automorphism groups of topological and geometric spaces. In the following years this point of view was refined and graphs and trees (for example the Cayley graph) were identified as important geometrical objects in order to understand groups from a geometrical point of view. The 1980s and the 1990s were especially fruitful with Gromov's work [Gro87] and the progress in Thurston's geometrization program [Thu82]. In his treatise Gromov introduced (word-)hyperbolic groups. These groups extend the notion of an isometry group of a hyperbolic space by generalizing hyperbolic geometry to the geometry of non-positively curved spaces, captured in the notion of CAT(0) spaces and considering groups operating on these spaces.

In the 1990s, Sageev [Sag95] brought certain special CAT(0) spaces, the so-called *CAT(0) cube complexes*, to the attention of a wider audience. General CAT(0) spaces are often too complicated to facilitate the understanding of a group. Instead, cube complexes introduce a combinatorial structure which makes these spaces more rigid and hence, easier to handle. At the same time, they are still flexible enough to admit interesting group actions. In particular, every fundamental group of a surface of genus at least one admits an action on a CAT(0) cube complex (see Examples 2.30 and 2.31). Another reason why CAT(0) cube complexes were quickly adopted as a natural object of study is that every tree is a CAT(0) cube complex. Hence, this new object generalized the old workhorse of the field. Interestingly, Sageev was not the first to introduce the complexes. This had already been done by Gromov in his 1987 treatise [Gro87], but only as a particular example of a CAT(0) space.

Another old and very important object class in geometric group theory are the Lie groups. Also in the 1990s Margulis [Mar91] was able to prove an astounding result which was then coined as *Margulis superrigidity*. It states that under certain (rather weak) conditions a linear representation of a lattice in a Lie group can be extended to the whole group. This superrigidity result had a deep impact and it became an objective to find analogous results in slightly different settings.

In 2016, Chatterji, Fernós, and Iozzi [CFI16] proved a superrigidity theorem for groups acting *essential* and *non-elementary* on CAT(0) cube complexes:

**Theorem 1.1** ([CFI16, Theorem 1.5]). *Let $Y$ be an irreducible finite-dimensional CAT(0) cube complex and $\Gamma < G_1 \times \cdots \times G_l =: G$ an irreducible lattice in the product of $l \geq 2$ locally compact groups. Let $\Gamma \to \mathrm{Aut}(Y)$ be an essential and non-elementary action on $Y$. Then the action of $\Gamma$ on $Y$ extends continuously to an action of $G$ by factoring via one of the factors.*

The main ingredient in their result was the construction of a so-called *boundary map*. This map connects the group $\Gamma$ via a *strong $\Gamma$-boundary $B$* to the *Roller boundary $\partial X$* (a subset of the *Roller compactification $\bar{X}$*) of a CAT(0) cube complex $X$. More precisely, they proved the existence of a measurable map

$$\varphi\colon B \to \partial X$$

which is $\Gamma$-equivariant almost everywhere. The space $B$ is in fact a probability space. The set $\bar{X}$ is deeply intertwined with the combinatorial structure of $X$. Each CAT(0) cube complex has an associated set of hyperplanes $\hat{\mathcal{H}}$ dividing the complex in two convex components. The set of all these components $\mathcal{H}$ is called the set of all *halfspaces* of $X$. Certain subsets of these sets are denoted *ultrafilters* (see Section 3.1). The space $\bar{X}$ is then simply given as the set of all ultrafilters (the detailed construction can be found in Chapter 3).

The aim of this thesis is to understand and present the construction of this map (Theorem 6.20 and Corollary 6.21). The idea of the proof is as follows: A strong $\Gamma$-boundary $B$ is defined via two key properties. First, the $\Gamma$-action on $B$ must be amenable. Secondly, the $\Gamma$-action must be doubly ergodic with coefficients, which is a strengthening of the standard notion of (double) ergodicity. The group $\Gamma$ also acts on $X$ and this action can be extended to the Roller compactification $\bar{X}$, where $\Gamma$ acts via homeomorphisms. The amenability provides a measurable map

$$\psi\colon B \to \mathcal{P}(\bar{X}) \tag{1.1}$$

which is $\Gamma$-equivariant almost everywhere and where $\mathcal{P}(\bar{X})$ denotes the set of all regular probability measures on $\bar{X}$. The hard part of the proof is then that every probability measure in the image of $\psi$ identifies a point in $\bar{X}$, i. e. in some sense we would like the probability measures in the image to have a point mass. In order to make this statement precise, we need to introduce *weighted halfspaces* (see Section 4.2). Let $\mu$ be a probability measure then the associated weighted halfspaces give the following decomposition of the set of halfspaces

$$\mathcal{H} = H_\mu^- \sqcup H_\mu \sqcup H_\mu^+.$$

It turns out that if $H_\mu = \varnothing$ then $H_\mu^+$ is an ultrafilter. This would give the desired map from $\mathcal{P}(\bar{X})$ to $\bar{X}$. The main work then is to show that $H_\mu$ indeed vanishes for every $\mu$ in the image of $\psi$. For this part to work, we need to introduce further restrictions. We need our complex $X$ to be indecomposable (i. e. *irreducible*, see Section 2.2) and finite-dimensional. Furthermore, the group action on the complex needs to be well-behaved. This is encoded in two properties. First, the group needs to act *essential* which means that $\Gamma$ needs to be well-behaved with regard to the combinatorial structure of $X$. Secondly, the group needs to act *non-elementary* which means that $\Gamma$ needs to be well-behaved with regard to the CAT(0) structure of $X$. The details of both notions can be found in Section 5.2. With all these definitions in place, we will first be able to show that $H_\mu$ is always finite. Then, in a second step, we will see that finiteness always implies emptiness. This closes the main proof. As a last step, we will see that the image of $\varphi$ indeed is in the Roller *boundary* not only in the Roller *compactification*. In all the steps of the proof ergodicity (with coefficients) will play a crucial role.

## Chapter outline

In Chapter 2, we will introduce CAT(0) cube complexes. We will start with some metric preliminaries before introducing general CAT(0) spaces. Most important in this early part is the definition of the *visual boundary* (see Definition 2.12). Afterwards, we will introduce cube complexes and combinatorial maps, which are the structure-preserving mappings (i. e. morphisms) for our objects. We will give a combinatorial property (Gromov's link condition, see Theorem 2.27) to check the CAT(0) property for cube complexes. Afterwards, we will talk about hyperplanes and halfspaces and some of their important properties (convexity, non-empty intersections, countability).

In Chapter 3, we will first introduce *pocsets*. Pocsets are partially ordered sets admitting a fixed point free, order reversing involution. It turns out that the set of halfspaces of CAT(0) cube complex is always a (discrete) pocset. Roller [Rol99] proved the reverse, namely that every (discrete) pocset gives rise to a unique CAT(0) cube complex with this pocset as pocset of halfspaces. The main ingredient of this construction is the notion of an *ultrafilter* which we introduce next. The important observation was that there is a one-to-one correspondence between principal ultrafilters and the vertex set of the CAT(0) cube complex. However, Roller went further and noted that the set of all ultrafilters equipped with a natural topology is a compactification of the vertex set of every CAT(0) cube complex. This lead to the definition of one of our main objects of study: the *Roller compactification* of a CAT(0) cube complex. We will introduce some topological and metric results regarding this space. Afterwards, we will revisit ultrafilters and introduce a second (equivalent) viewpoint, which is more natural for the later arguments. Lastly, we turn towards applications and introduce *intervals* of ultrafilters. These are special (sub-)complexes which have strong restrictions when it comes to sets of halfspaces. In particular, we will see that sets of halfspaces can contain at most finitely many *terminal elements* (i. e. minimal or maximal elements with regard to the partial order up to going over to the opposite under involution). This property is the main reason we are interested in intervals. We will close the chapter by studying sub-pocsets of halfspaces. We will give conditions under which the associated CAT(0) cube complex can be embedded into the CAT(0) cube complex associated to the actual pocset (Lemma 3.37).

After these two chapters, we will shortly leave the realm of CAT(0) cube complexes and (in Chapter 4) turn towards measure theory and functional analysis. We will start with generalities concerning *measurable spaces, measurable maps* and *(probability) measures*. We will recall the connection between the vector space of continuous functions and the vector space of (signed) measures. However, with these generalities in place, we return to our special case and introduce *weighted halfspaces* (see Definition 4.15). Lastly, we will prove measurability for certain key maps.

Up to this point, we did not talk about group actions. This will be remedied in Chapter 5. First, we will talk about groups acting on CAT(0) cube complexes. We will see how the group action can be extended to the Roller compactification and we will introduce the notions of *essential* and *non-elementary* group actions citing some important results by Caprace and Sageev [CS11]. The second half of the chapter is concerned with the introduction of *strong* $\Gamma$-*boundaries* (where $\Gamma$ is a countable, discrete group). One essential ingredient for this boundary as well as for the proof in general is *ergodicity*. Hence, we have a whole section reserved for this topic. The most important results are:

- If we have a finite group acting ergodically on a space $B$, then $B$ is purely atomic (Lemma 5.33);
- Ergodicity is inherited by finite index subgroups (Lemma 5.36).

Afterwards, we will strengthen the notion of ergodicity to *ergodicity with coefficients*. Both notions can be defined via requiring certain $\Gamma$-equivariant, measurable maps to be constant. Whereas in the case of ergodicity these maps always have $\mathbb{R}$ as codomain (with the trivial action by $\Gamma$), in the case of ergodicity with coefficients we allow the dual of any separable Banach space that admits a unitary $\Gamma$-action. This stronger version of ergodicity leads to a condition for the essentiality of the $\Gamma$-action (see Corollary 5.42), which we will regularly use. Next, we turn towards *amenable group actions*, which guarantee the existence of certain measurable maps which are $\Gamma$-equivariant almost everywhere. With this notion in place, we can define strong $\Gamma$-boundaries which are special probability spaces on which $\Gamma$ acts amenably and ergodically with coefficients. We close the chapter with the result that (thanks to the amenability) we find the map $\psi$ in Equation (1.1) (Corollary 5.52).

Chapter 6 contains the statement and the proof of our main result (Theorem 6.20 and Corollary 6.21). First, we will construct the boundary map using the further assumption that $H_\mu$ vanishes (as described above). Afterwards, we will prove that if the map exists, it takes values in the Roller boundary. We will then see that $H_\mu$ being finite already implies it to be empty. The remainder of the chapter builds up the necessary tools to exclude $H_\mu$ being infinite. The most important results are Proposition 6.6, Proposition 6.13 and Proposition 6.14. Finally, we descend into the main proof plugging all the previous results together.

As a closing remark, we would like to point out that the material in this thesis by its very nature is very close to the exposition in Chatterji, Fernós, and Iozzi [CFI16].

# 2 CAT(0) cube complexes

This chapter is divided into four sections, which are all concerned with the introduction of our basic objects of study: CAT(0) cube complexes. The Section 2.1 will introduce general metric preliminaries and properties of CAT(0) spaces. In Section 2.2 we will be in a position to introduce cubes and cube complexes and give a combinatorial condition for them to be CAT(0). Section 2.3 is closely related to the former, as we will need to talk about maps between CAT(0) cube complexes. As it turns out, combinatorial maps are the right generalization of simplicial maps for us. In this section, we will also introduce the automorphism group of CAT(0) cube complex. In Section 2.4 we will talk a bit more about the geometry of the complexes and introduce hyperplanes and halfspaces. These objects will be heavily used throughout this thesis and will be especially important in Chapter 3, when they are used to construct the Roller compactification of our complexes.

## 2.1 Preliminaries on metric and CAT(0) spaces

This section is concerned with basic metric properties of spaces and their connection to CAT(0) spaces. First, we will recall some generalities about metric spaces. Afterwards, we will state some basic results about the geometry and topology of CAT(0) spaces. The section will close with the definition of the visual boundary of a CAT(0) space. The exposition follows closely [BH99] and [Rol12].

### Generalities on metric spaces

**Definition 2.1** (Geodesics)**.**

- Let $(X, d)$ be a metric space. A *geodesic* from $x$ to $y$ with $x, y \in X$ is a map $c\colon [a, b] \to X$ ($a, b \in \mathbb{R}$) such that $c(a) = x$, $c(b) = y$ and

  $$d(c(t), c(t')) = |t - t'|$$

  for all $t, t' \in [a, b]$.
- The pair $(X, d)$ is called *$r$-geodesic* ($r > 0$) if $d(x, y) < r$ implies that there is a geodesic joining $x$ and $y$. It is called *uniquely $r$-geodesic* if this geodesic is unique (up to reparametrization). The pair $(X, d)$ is called *(uniquely) geodesic* if it is (uniquely) $r$-geodesic for all $r > 0$.

**Example 2.2** (Euclidean space)**.** The pair $(\mathbb{R}^n, d_0)$, where $d_0(x, y) := \|x - y\|_2$, is uniquely geodesic. The space $\mathbb{R}^n \setminus \{0\}$ is no longer geodesic, since every pair of antipodal points can no longer be joined by a line segment.

**Example 2.3** (Riemannian manifolds:)**.** By the Hopf-Rinow theorem [HR31], every complete Riemannian manifold is geodesic, but not necessarily uniquely geodesic (consider the sphere).

**Definition 2.4** (Comparison triangles)**.** Let $(X, d)$ be a metric space.

- A *geodesic triangle* $\Delta \subset X$ consists of three points $p, q, r \in X$, its *vertices*, together with a choice of three *edges*, that is, geodesic segments $[p, q], [q, r], [r, p]$ joining them (recall that geodesics might not be unique). If necessary, the notation $\Delta = \Delta(p, q, r)$ or $\Delta = \Delta([p, q], [q, r], [r, p])$ will be used. The first case is a slight abuse of notation, as the three vertices might not determine the triangle.

- A *comparison triangle* in $\mathbb{R}^2$ for $\Delta = \Delta(p, q, r) \subset X$ is a choice of three points $\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}^2$ such that $\|\bar{p} - \bar{q}\| = d(p, q)$, $\|\bar{q} - \bar{r}\| = d(q, r)$ and $\|\bar{r} - \bar{p}\| = d(r, p)$. It will be denoted by $\bar{\Delta} = \Delta(\bar{p}, \bar{q}, \bar{r})$. Such a comparison triangle always exists [c. f. BH99, Sec. I.2].

**Definition 2.5** (CAT(0) and non-positive curvature spaces)**.** Let $(X, d)$ be a metric space.

- Let $\Delta = \Delta(p, q, r) \subset X$ be a triangle and $\bar{\Delta} = \Delta(\bar{p}, \bar{q}, \bar{r}) \subset \mathbb{R}^2$ its comparison triangle. Each point $x \in [p, q]$ has a unique associated point $\bar{x} \in [\bar{p}, \bar{q}]$, which has the same distance from $\bar{p}$ and $\bar{q}$ as $x$ from $p$ and $q$. The same is of course true for any point lying on any of the other edges.

  With this in mind we say that $\Delta$ satisfies the *CAT(0) inequality* if $d(x, y) \leq \|\bar{x} - \bar{y}\|$ for any $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta} \subset \mathbb{R}^2$ the two associated points defined above.

- $X$ is called a *CAT(0) space* if $X$ is geodesic and if each geodesic triangle $\Delta$ satisfies the CAT(0) inequality.

- $X$ is called *of curvature* $\leq 0$ or *non-positively curved* if it is locally a CAT(0) space, i. e. for each $x \in X$ there exists an $r > 0$ such that $B(x, r)$ together with the induced metric is a CAT(0) space.

**Example 2.6** (Euclidean space)**.** $\mathbb{R}^n$ is by definition a CAT(0) space. Furthermore, it is easy to see if we remove the interior $\Delta$ of a non-degenerate triangle from $\mathbb{R}^2$ and equip this space with the induced length metric (i. e. the distance between two points is given by the infimum of the length over all piecewise linear paths), then $\mathbb{R}^2 \setminus \Delta$ is no longer CAT(0). Indeed, the interior of $\Delta$ is missing and this lengthens all shortest paths which would normally go through the interior. However, this space is still non-positively curved.

**Example 2.7** (Trees)**.** Trees equipped with the edge metric are another example of CAT(0) spaces. Every triangle in a tree takes the form of one midpoint $m$ with three edge paths connecting $m$ to the three vertices $p, q$ and $r$. The situation is depicted in Figure 2.1. We see that each point on the triangle always lies on at least two of its sides at the same time. Hence, we always find two associated points in the comparison triangle. In the figure, $x$ has the points $\bar{x}_i$ associated and $y$ the points $\bar{y}_i$. We see that two of the representatives (in our notation $\bar{x}_2$ and $\bar{y}_2$) lie on a common edge and hence satisfy the CAT(0) (in-)equality (blue line segment). The other three possible combinations of representatives all have a

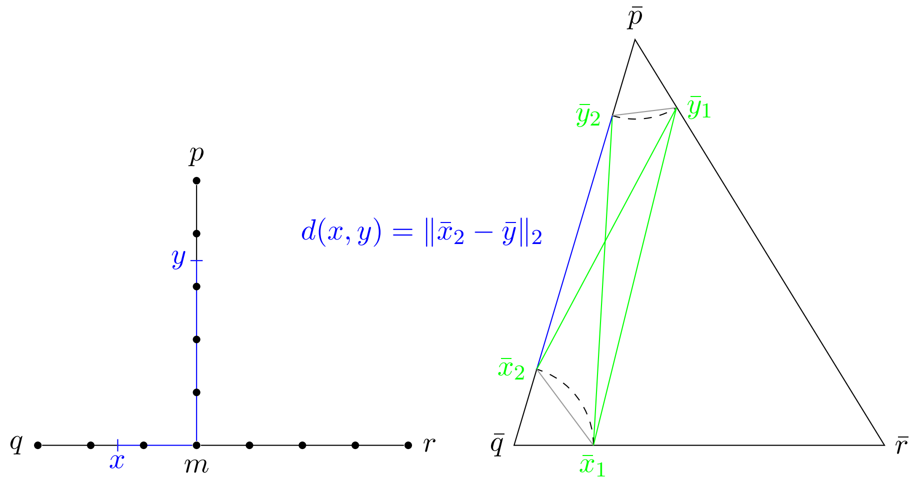


Figure 2.1: A triangle in a tree and the corresponding comparison triangle in $\mathbb{R}^2$. The different choices of comparison points for $x$ and $y$ are inscribed as $\bar{x}_i$ and $\bar{y}_i$. The blue segment in the comparison triangle corresponds to the only comparison pair with the same length as in the tree. The three green line segments are all longer.

longer distance (green line segments). This can be deduced from the fact, that the triangles $\Delta(\bar{q}, \bar{x}_1, \bar{x}_2)$ and $\Delta(\bar{p}, \bar{y}_1, \bar{y}_2)$ are isosceles. This implies that the angles $\angle(\bar{x}_2; \bar{x}_1, \bar{y}_2)$ and $\angle(\bar{y}_2; \bar{y}_1, \bar{x}_2)$ are larger than $\frac{\pi}{2}$. We see that the triangle satisfies the CAT(0) inequality. Since every tree is geodesic, we see that trees indeed are CAT(0) spaces.

**Example 2.8** (Hyperbolic space)**.** Every hyperbolic $n$-space $\mathbb{H}^n$ is CAT(0). The interested reader may find further information in Bridson and Haefliger [BH99, Section II.1].

### Properties of general CAT(0) spaces

This section contains some important facts about general CAT(0) spaces. For the omitted proofs, see [BH99].

**Proposition 2.9** ([BH99, Prop II.1.4])**.** *Let $X$ be a CAT(0) space. Then*

1. *$X$ is uniquely geodesic and*
2. *$X$ is contractible.*

**Definition 2.10.** Let $X, Y$ be metric spaces. The map $\varphi\colon X \to Y$ is called

- an *isometric embedding* if $d_X(x,y) = d_Y(\varphi(x), \varphi(y))$ for any two $x, y \in X$,

- an *isometry* if it is an isometric embedding and surjective (and hence bijective), and
- a *local isometry* if for each $x \in X$ there exists an open neighborhood $U \subset X$ containing $x$ such that $\varphi|_U : U \to \varphi(U)$ is an isometry.

**Proposition 2.11** ([Rol12, Propositions 1 & 2]). *Let $X, Y$ be geodesic spaces and let $Y$ be CAT(0). Then every local isometry $\varphi\colon X \to Y$ is an isometric embedding. In particular, every local geodesic is a geodesic.*

On CAT(0) spaces one regularly defines a boundary via identifying certain geodesic rays. This so called *visual boundary* will play a minor role in this thesis, but we will still need it.

**Definition 2.12** (Visual boundary, [BH99, Sec. II.8]). Let $\gamma_i\colon [0,\infty) \to X$ two geodesic rays into a CAT(0) space $X$. We say $\gamma_1 \sim \gamma_2$ if and only if there exists a constant $K > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) < K$$

for all $t \geq 0$. The set of equivalence classes $\partial_\triangleleft X$ is called the *visual boundary of $X$*.

Clearly, each group action on $X$ by isometries extends to an action on $\partial_\triangleleft X$.

**Remark 2.13.** $X \sqcup \partial_\triangleleft X$ can be topologized in a way that it agrees with the topology induced by the metric on $X$. If $X$ is locally compact, $X \sqcup \partial_\triangleleft X$ is also compact [c. f. BH99, Sec. II.8].

**Example 2.14** (Euclidean space). In $\mathbb{R}^n$ two geodesics (i. e. straight lines) are equivalent if and only if they are parallel. Hence, we can fix any point $x \in \mathbb{R}^n$ and see that there is a one-to-one correspondence between points at the visual boundary and (signed) directions. In other words, we attach an $(n-1)$-sphere at infinity and indeed it can be shown that $\mathbb{R}^n \sqcup \partial_\triangleleft \mathbb{R}^n$ is homeomorphic to the closed unit ball $D^n$, with $\mathbb{R}^n$ homeomorphic to the interior [c. f. BH99, Section II.8].

**Example 2.15** (Trees). In the case of trees, every edge path is a geodesic and two of them, $c_1$ and $c_2$, are equivalent if and only if there exist $n, m \in \mathbb{N}$ such that $c_1(t+m) = c_2(t+n)$ for all $t \geq 0$. This means that up to a finite starting interval the two geodesics have to coincide. In other words we are only interested in the tails of the geodesic rays.

**Lemma 2.16** ([CFI16, Lemma 2.9]). *Let $X = X_1 \times \cdots \times X_m$ be a product of CAT(0) spaces $X_j$ and let $G = G_1 \times \cdots \times G_m$, where $G_j \leq \mathrm{Isom}(X_j)$. Then any $G_j$-fixed point in $\partial_\triangleleft X_j$ defines a $G$-fixed point in $\partial_\triangleleft X$.*

*Proof.* Let $d_i$ and $d$ be the CAT(0) metrics on $X_i$ and $X$. Up to permuting the indices we can assume that we have a $G_1$-fixed point in $\partial_\triangleleft X_1$. Let the geodesic ray $l_1\colon [0,\infty) \to X_1$ represent this fixed point, i. e.

$$\sup_{t \geq 0} d_1(l_1(t), g_1 l_1(t)) < \infty$$

for every $g_1 \in G_1$. For each $i > 1$ we fix a point $x_i \in X_i$. Then

$$l\colon [0, \infty) \to X,\ t \mapsto (l_1(t), x_2, \dots, x_m)$$

is a geodesic in $X$ and for any $g = (g_1, \dots, g_m) \in G$ we have

$$\sup_{t \geq 0} d^2(l(t), gl(t)) = \sup_{t \geq 0} \left[ d_1^2(l_1(t), g_1 l_1(t)) + \sum_{i=2}^{m} d_i^2(x_i, g_i x_i) \right] < \infty.$$

Hence, $l$ defines a $G$-fixed point in $\partial_{\sphericalangle} X$. $\square$

## 2.2 Cube complexes

We are now able to define the central object of this thesis: CAT(0) cube complexes. First, we will introduce Euclidean cubes and some necessary notation (faces and links). Afterwards, we will define the gluing process that will lead to cube complexes. We will state some basic properties and define flag complexes. This allows us to connect the geometric property of being CAT(0) to a purely combinatorial notion which is stated in Gromov's link condition (Theorem 2.27). Lastly, we will turn towards locally countable complexes and prove some necessary lemmas concerning the countability of vertex sets.

**Definition 2.17** (Cubes)**.** A set $C = [0, 1]^n \subset \mathbb{R}^n$ is called a *cube*. A *face* is a subset of the form

$$F = C \cap \{x_{i_1} = e_1, \dots, x_{i_k} = e_k\},$$

where $0 \leq k \leq n$, $i_1, \dots, i_k$ are pairwise different elements in $\{1, \dots, n\}$ and $e_j \in \{0, 1\}$. $F$ is called a *proper face* if $F \neq C$. The notation $F \preceq C$ will be used for faces. The *dimension* of $F$ is $n - k$. The *interior* $\mathring{F}$, is the interior of $F$ equipped with its $\mathbb{R}^{n-k}$-structure. Any subset $C \cap \{x_i = 1/2\}$ is called a *midcube of $C$*. The *$m$-skeleton of $C$* is defined by

$$C^{(m)} := \bigcup \{F \mid F \preceq C \text{ and } \dim F \leq m\}.$$

For a fixed $x \in C$, the *support of $x$*, $\operatorname{supp}(x)$, is the unique face of $C$ containing $x$ in its interior or alternatively the unique face with minimal dimension containing $x$.

The *link of $x$ in $C$* is given by

$$\operatorname{Lk}(x, C) := \{u \in U_x\mathbb{R}^n \mid \exists t > 0\colon \exp_x(tu) \in C\} \subset U_x\mathbb{R}^n \cong \mathbb{S}^{n-1},$$

where $U_x\mathbb{R}^n$ is the unit tangent space at $x$ in $\mathbb{R}^n$ considered as a Riemannian manifold. It can be isometrically identified with $\mathbb{S}^{n-1}$.

**Remark 2.18.** The link $\operatorname{Lk}(v, C) \subset \mathbb{S}^{n-1}$ is a simplex for all vertices $v \in C$ (unless $\dim C = 0$). Its edges have length $\frac{\pi}{2}$.

Now, that we have introduced the vocabulary concerning Euclidean cubes, we turn towards cube complexes. These are obtained by gluing disjoint cubes along their faces via *isometries*. The restriction on the gluing maps ascertain that many of the combinatorial properties we know from simplicial complexes will transfer to cube complexes.

**Definition 2.19** (Cube complexes)**.**

- Let $(C_\lambda)_{\lambda\in\Lambda}$ be a family of cubes and $\mathcal{C} := \bigsqcup_{\lambda\in\Lambda} C_\lambda$ its disjoint union. Furthermore, let $\sim$ denote an equivalence relation on $\mathcal{C}$ and denote by $X$ the space of equivalence classes with natural projection $p\colon \mathcal{C} \to X$. Lastly, let $p_\lambda\colon C_\lambda \to X$ be the embedding of $C_\lambda$ into $\mathcal{C}$ concatenated with the projection.

  $X$ is called a *cube complex* if

  1. $p_\lambda$ is injective and
  2. for arbitrary $\lambda_1, \lambda_2 \in \Lambda$ and $x_i \in C_{\lambda_i}$ such that $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$, there exists an isometry $h\colon \operatorname{supp}(x_1) \to \operatorname{supp}(x_2)$ such that $p_{\lambda_1}|_{\operatorname{supp}(x_1)} = p_{\lambda_2} \circ h$.

Now, let $X$ be a cube complex.

- $C \subset X$ is called an *$n$-dimensional cube*, if it is the image of an $n$-dimensional face $F \preceq C_\lambda$ under $p_\lambda$. The interior of $C$ is given by $\mathring{C} := p_\lambda(\mathring{F})$. A *midcube of* $C$ is the image of a midcube of $F$ under $p_\lambda$.

- The *$m$-skeleton* of $X$ is given by

$$X^{(m)} := \bigsqcup_{\lambda\in\Lambda} C_\lambda^{(m)} \Big/ \sim \, ,$$

  where $\sim$ is given by the restriction of the equivalence relation on $\mathcal{C}$ to the disjoint union of the $m$-skeleta of the cubes.

- Let $x \in X$ and $(x_i)_{i\in I}$ be the family of all the points $x_i \in C_{\lambda(i)}$ lying in the fiber over $X$. Consider the disjoint union $\bigsqcup_{i\in I} \operatorname{Lk}(x_i, C_{\lambda(i)})$. We define an equivalence relation: $u_i \sim u_j$ if and only if there exist $t_i, t_j > 0$ such that $\exp_{x_i}(t_i u_i) \in C_{\lambda(i)}$, $\exp_{x_j}(t_j u_j) \in C_{\lambda(j)}$ and $p_{\lambda(i)}(\exp_{x_i}(t_i u_i)) = p_{\lambda(j)}(\exp_{x_j}(t_j u_j))$. Then the *link of $x$ in $X$* is given by

$$\operatorname{Lk}(x, X) := \bigsqcup_{i\in I} \operatorname{Lk}(x_{\lambda(i)}, C_{\lambda(i)}) \Big/ \sim \, .$$

**Remark 2.20.**

- In the language of $M_\kappa$-polyhedral complexes the link $\operatorname{Lk}(x, X)$ is a $M_1$-polyhedral complex whenever $x$ is a vertex of $X$. For more details see Bridson and Haefliger [BH99, Section I.7]. Although all the cells of $\operatorname{Lk}(x, X)$ consist of simplices, it might happen that $\operatorname{Lk}(x, X)$ is not a simplicial complex. An example is given in Example 2.29.

- Since the cubes are glued isometrically, one can think of $\mathrm{Lk}(x, X)$ as being inscribed in $X$. The key observation is that one can fix $t_i$ respectively $t_j$ to any (common) value smaller than 1 (one common choice is ⅓). Then the maps $p_{\lambda(i)}(\exp_{x_i}(t_i\cdot))$ induce the embedding. Vertices correspond to the intersection of the 1-skeleton with the image of this embedding and edges must lie in the 2-skeleton. For an example see Figure 2.2.

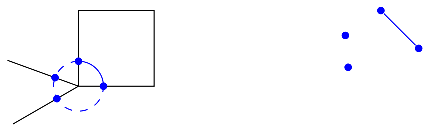

Figure 2.2: The left-hand side depicts a CAT(0) cube complex (black). One vertex link is inscribed into the complex via the intersection of a small sphere (blue). The right-hand side depicts the vertex link without the CAT(0) cube complex.

- Our definition of a cube complex is not standard. Usually, the above defined object is called a cubical complex [c. f. BH99, Def. I.7.37]. The difference between the two concepts lies solely in the fact that in the cubical case we need the maps $p_\lambda$ to be injective on the whole cube, whereas in the cube case they are only assumed to be injective on the *interior* of each cube. However, as Leary [Lea13, Thm. C.4] has shown, in the case of CAT(0) cube complexes the two definitions are equivalent, hence we will adopt it from the start.

- By definition, two cubes either intersect in a common face or have an empty intersection. In this sense, they are completely analogous to simplicial complexes (c. f. Definition 2.26).

In the following we will list some useful results about cube complexes. For the proofs see [Lea13, Appendices A, B] or [BH99, Sec. I.7, II.5].

**Theorem 2.21** ([BH99, p. I.7.10])**.** *Every cube complex $X$ is a metric space, when equipped with the metric $d_p$ induced by the piecewise linear paths in $X$.*

With the above definition we can define:

**Definition 2.22** (CAT(0) cube complex)**.** A *CAT(0) cube complex* is a cube complex $X$ such that the pair $(X, d_p)$ is a CAT(0) space, where $d_p$ is the metric induced by the piecewise linear paths on $X$.

**Remark 2.23.** Although the path metric $d_p$ gives $X$ its CAT(0) structure, this metric is often of no great importance in the theory of CAT(0) cube complexes. The reason for this is described in Remark 2.45.

**Theorem 2.24** ([Lea13, Theorem A.6], [BH99, Theorem I.7.50]). *A cube complex is complete if and only if every chain of ascending cubes is finite.*

**Notation 2.25.** Let $S$ be a set. We will denote its *power set* by

$$\mathrm{Pot}(S) = \{A \subset S\}.$$

**Definition 2.26** (Flag complexes and joins).

- Let $S$ be a set and $P \subset \mathrm{Pot}(S)$. A pair $K = (S, P)$ is called a *simplicial complex* if $\{s\} \in P$ for every $s \in S$ and for every $X \in P$ and $Y \subset X$ with $Y \neq \varnothing$ we have $Y \in P$.

  An $n$-*simplex* is an element $X \in P$ such that $|X| = n + 1$.

  A 0-simplex is called a *vertex* and a 1-simplex is called an *edge*.

- A simplicial complex $K = (S, P)$ is *flag*, if every finite subset of $S$ that is pairwise joined by edges spans a simplex (see [BH99, Definition II.5.15]).

- Let $K_1 = (S_1, P_1)$ and $K_2 = (S_2, P_2)$ be two simplicial complexes. Their *join* $K$ is the simplicial complex $(S, P)$ such that $S := S_1 \sqcup S_2$ and $X \in P$ if and only if one of the following is true:

  1. $X \in P_1$,
  2. $X \in P_2$, or
  3. there exist $X_i \in P_i$ such that $X = X_1 \sqcup X_2$.

We write $K = K_1 * K_2$ (See [BH99, Definition I.7A.2]).

**Theorem 2.27** (Gromov's link condition, [Lea13, Theorem B.8], [BH99, Theorem II.5.20]). *A cube complex $X$ is* non-positively curved *if and only if* $\mathrm{Lk}(v, X)$ *is a flag complex for each vertex $v \in X$.*

*A cube complex $X$ is* CAT(0) *if and only if* $\mathrm{Lk}(v, X)$ *is a flag complex for each vertex $v \in X$ and $X$ is simply connected.*

The following examples were taken from Sageev [Sag12]:

**Example 2.28** (Graphs). The link of every vertex in a graph is a set of disconnected vertices, which is, by the non-existence of any higher dimensional simplices, a flag complex. Hence, every graph is non-positively curved. In the case of graphs, simply connectedness is equivalent to the graph being a tree such that we see that CAT(0) cube complexes are in a natural way a generalization of trees.

**Example 2.29** (Sphere)**.** Figure 2.3 contains three representations of the two-dimensional sphere as a quotient of a disjoint union of cubes. None of these is a CAT(0) cube complex. In case 2.3a the cube is not embedded into the quotient. Nonetheless, one can define links for the vertices and we see that these are not simplicial complexes, since one contains a loop and the other contains parallel edges. In the case 2.3b, we see that we can indeed realize the sphere as a cube complex. However, we still have a non-simplicial vertex link. In the case 2.3c, we can even find a realization as a cube complex such that the vertex links are simplicial. However, even then the link is not flag.

Naturally, this is as it should be as we know that a sphere is an example of a positively curved space and our definition agrees with the one in the case of manifolds.

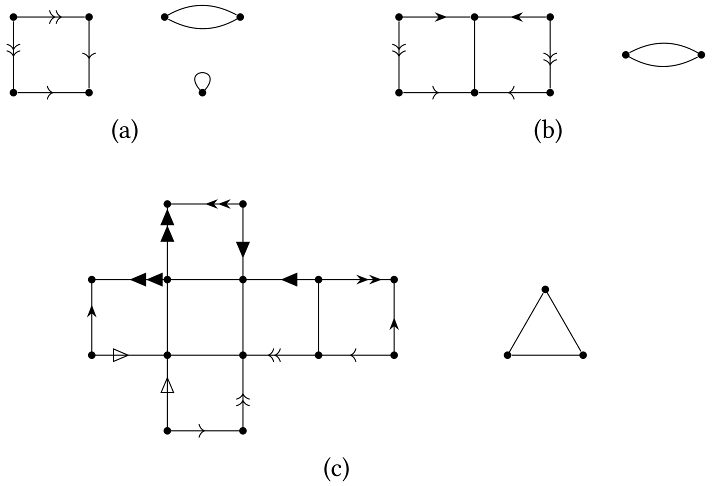


Figure 2.3: Three topological realizations of a sphere and their vertex links. Figure a is not a cube complex, whereas Figures b and c are. In Figure b the link is not a simplicial complex (parallel edges). In Figure c the link is simplicial, but not flag.

**Example 2.30** (Torus)**.** Figure 2.4 contains a realization of a two-dimensional torus as a cube complex. The vertex links show that the torus is indeed non-positively curved. This is what we would expect, since a torus is an example of a flat space. However, it is not a CAT(0) cube complex, as it is not simply connected. A generalization of the Cartan-Hadamard theorem shows that the universal cover of a non-positively curved cube complex is always CAT(0). In the case of a torus the universal cover can be chosen to be $\mathbb{R}^2$ with its standard cubulation via $\mathbb{Z}^2$.

The above considerations generalize to tori in arbitrary dimensions.

**Example 2.31** (Higher genus surfaces)**.** In order to complete our discussion of surfaces, we consider in Figure 2.5 the example of a genus 2 surface. Inscribed in the standard octagon, there are four more geodesics creating eight cubes embedded in the quotient. The vertex

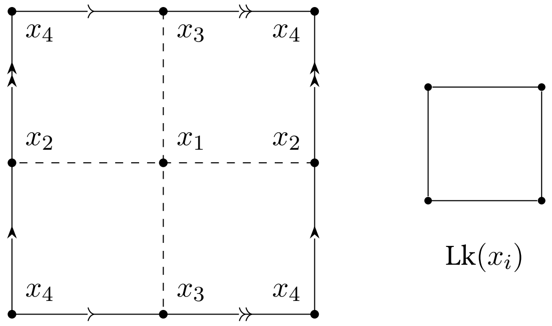


Figure 2.4: Realization of a 2-dimensional torus as a cube complex and its vertex link. We see that the torus is non-positively curved.

links are clearly flag and so we see that the surface can be realized as a non-positively curved cube complex. The construction can be generalized to all higher genus surfaces. This is expected, as all these higher genus surfaces are negatively curved.

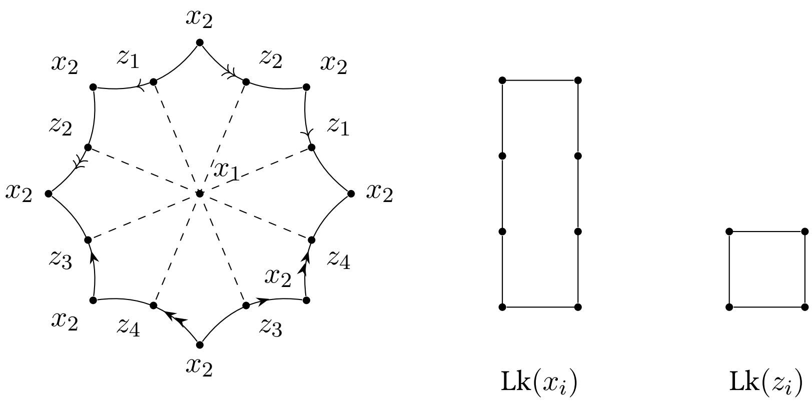


Figure 2.5: Realization of a genus 2 surface as a cube complex and the two non-isomorphic vertex links. The links indicate that the surface is non-positively curved.

As in the previous example, these surfaces lead to a cube complex structure on the universal cover, which can be chosen to be $\mathbb{H}^2$.

**Lemma 2.32.** *The join $K = (S, P)$ of two simplicial complexes $K_i = (S_i, P_i)$ ($i = 1, 2$) is a simplicial complex. If $K_1$ and $K_2$ are flag, so is $K$.*

*Proof.* Let $X \in P$ and $\varnothing \neq Y \subset X$. The set $X$ can be decomposed as $X = X_1 \sqcup X_2$, where $X_i \in P_i$ or $X_i = \varnothing$. With this we have $Y = Y_1 \sqcup Y_2$ where $Y_i := X_i \cap Y$. If both $Y_1$ and $Y_2$ are non-empty, then $Y_i \in P_i$ and, by construction, we have $Y \in P$. If $Y_2 = \varnothing$, then $Y_1 \neq \varnothing$ and hence $Y = Y_1 \in P_1 \subset P$. Likewise for $Y_1 = \varnothing$. We conclude that $K$ is a simplicial complex.

We will now show that $K$ is flag if the $K_i$ are flag. Let $v_1, \dots, v_n \in P$ be distinct vertices that are pairwise connected in the 1-skeleton, i. e. $v_i \cup v_j \in P$ for all $i \neq j$. We have to show that $X \coloneqq \bigcup_i v_i \in P$. After renaming the vertices we may assume that $v_1, \dots, v_k \in P_1$ and $v_{k+1}, \dots v_n \in P_2$. If $k = 0$ or $k = n$, we are done as $K_i$ is flag and we have $P_i \subset P$. Otherwise $X_1 \coloneqq \bigcup_{i=1}^{k} v_i \in P_1$ and $X_2 \coloneqq \bigcup_{i=k+1}^{n} v_i \in P_2$, again since the $K_i$ are flag. However, then $X = X_1 \sqcup X_2 \in P$ by definition. Hence, $K$ is flag. $\square$

**Proposition 2.33.** *Let $X_1$ and $X_2$ be two cube complexes, then $X \coloneqq X_1 \times X_2$ is a cube complex. If $X_1$ and $X_2$ are both CAT(0), so is $X$.*

*Proof.* We will first prove that $X_1 \times X_2$ is a cube complex. If $X_1$ and $X_2$ are cube complexes, we have the following maps:

$$p_i \colon \bigsqcup_{\lambda \in \Lambda_i} C_\lambda \to X_i.$$

Hence, we have the map

$$(p_1 \times p_2) \colon \bigsqcup_{\lambda' \in \Lambda_1} \bigsqcup_{\lambda \in \Lambda_2} C_{\lambda'} \times C_\lambda \to X_1 \times X_2$$

and via the embedding of the cubes the maps

$$p_{\lambda',\lambda} \colon C_{\lambda'} \times C_\lambda \to X_1 \times X_2.$$

These maps are injective because each $p_i$ is injective on every cube. The fact that the cubes are glued by isometries can also be seen on each factor separately.

We turn towards the proof that $X$ is CAT(0), if the $X_i$ are CAT(0). We will show that for any vertex $(v, w) \in X$ the link $\mathrm{Lk}((v, w), X)$ is the join of $\mathrm{Lk}(v, X_1)$ and $\mathrm{Lk}(w, X_2)$. We use Remark 2.20 and think of the link as inscribed into the complex. Let us first consider two Euclidean cubes $C_{\lambda'}$ and $C_\lambda$ and the origin as the vertex. Let $n = \dim C_{\lambda'}$ and $k = \dim C_\lambda$. Then a vertex $v \in \mathrm{Lk}(0, C_{\lambda'} \times C_\lambda)$ is uniquely defined by the unique coordinate $v_i$ which is non-zero. If $i \leq n$, then $v$ corresponds to a vertex in $\mathrm{Lk}(0, C_{\lambda'})$, otherwise to a vertex in $\mathrm{Lk}(0, C_\lambda)$. Now, the three links are indeed a $(n-1)$-, $(k-1)$- and $(n+k-1)$-simplex respectively and we have $\mathrm{Lk}(0, C_{\lambda'} \times C_\lambda) = \mathrm{Lk}(0, C_{\lambda'}) * \mathrm{Lk}(0, C_\lambda)$.

In the case of a general $X$, we can accomplish the above decomposition on each cube separately. Since the isometric gluing is also factor wise, it preserves the decomposition and we have $\mathrm{Lk}((v, w), X) = \mathrm{Lk}(v, X_1) * \mathrm{Lk}(v, X_2)$. With this assertion in place Lemma 2.32 finishes the proof. $\square$

The above proposition leads to the following definition:

**Definition 2.34.** A CAT(0) cube complex $X$ is called *reducible*, if it can be decomposed as a (proper) product of CAT(0) cube complexes. Otherwise, it is called *irreducible*.

**Definition 2.35.** A cube complex $X$ is called *locally finite* if every $x \in X$ is contained in only finitely many cubes. It is called *locally compact* if for every $x \in X$ there exists a compact neighborhood $K \subset X$. It is called *locally countable* if every $x \in X$ is contained in at most countably many cubes.

**Proposition 2.36** ([Rol12, Prop. 14])**.** *Let $X$ be a cube complex. Then the following are equivalent:*

1. *$X$ is locally finite,*
2. *each bounded subset of $X$ meets only finitely many cubes,*
3. *$X$ is proper (i. e. closed balls are compact), and*
4. *$X$ is locally compact.*

**Definition 2.37** (Edge metric)**.** The *edge metric* on $V(X) := X^{(0)}$, the vertex set of $X$, is the graph metric of the 1-skeleton $X^{(1)}$ considered as a simplicial graph with all edges assigned unit length.

**Lemma 2.38.** *Let $X$ be a locally countable CAT(0) cube complex. Let $V(X)$ be its vertex set equipped with the edge metric $d$. Then for every $x_0 \in V(X)$ and $n \in \mathbb{N}_0$ the set $X_n := \{x \in V(X) \mid d(x_0, x) = n\}$ is countable.*

*Proof.* We fix $x_0 \in V(X)$ and proceed by induction. Since $d$ is a metric we have $X_0 = \{x_0\}$. Now, assume $X_n$ to be countable and let for each $x \in V(X)$ be $N(x)$ the set of its neighboring vertices (i. e. all vertices connected by an edge to $x$ or equivalently all vertices with distance 1 from $x$). Because of the local countability $N(x)$ is countable. Thus

$$X_{n+1} \subset \bigcup_{x \in X_n} N(x)$$

is a countable set. $\square$

**Remark 2.39.** In this thesis we will be mostly concerned with connected, locally countable, finite-dimensional CAT(0) cube complexes. We will recall in each section which assumptions are necessary on our space.

## 2.3 Combinatorial maps

This is a short section concerned with maps between cube complexes. We will deal with these maps in order to understand group actions on CAT(0) cube complexes and the decomposition of the complexes in factors.

**Definition 2.40** (Combinatorial maps)**.** Let $X, Y$ be cube complexes. The map $f \colon X \to Y$ is called a *morphism of cube complexes* or a *combinatorial map*, if

1. each vertex $v \in X^{(0)}$ is mapped to a vertex $f(v) \in Y^{(0)}$,
2. each cube $C \subset X$ is mapped to a cube $f(C) \subset Y$ and
3. the induced map given by

$$f_{\lambda,\omega} \colon C_\lambda \xrightarrow{p_{X,\lambda}} C \xrightarrow{f} f(C) \xrightarrow{p_{Y,\omega}^{-1}} C_\omega$$

can be represented as $f_{\lambda,\omega}(x) = \sum_{i=1}^{n} a_i f(v_i)$, where $v_1, \ldots, v_n$ are the vertices of $C_\lambda$ and $x = \sum_{i=1}^{n} a_i v_i$ is an arbitrary element of $C_\lambda$ in its convex representation.

The automorphism group of a cube complex $X$ will be denoted by $\mathrm{Aut}(X)$.

**Remark 2.41.** The above definition of a combinatorial map is completely analogous to the one of a simplicial map of simplicial complexes [c. f. ST76].

We have two direct observations stemming from the above definition:

**Lemma 2.42.** *After possibly rotating $C_\lambda \subset \mathbb{R}^n$ and $C_\omega \subset \mathbb{R}^m$, the map $f_{\lambda,\omega}$ is induced by the restriction of the natural projection from $\mathbb{R}^n$ to $\mathbb{R}^m$. In particular, we have $n \geq m$.*

**Corollary 2.43.** *Let $f \colon X \to Y$ be a combinatorial map. The restriction $f|_C \colon C \to Y$ is distance non-increasing for each cube $C \subset X$, i. e. $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in C$.*

**Proposition 2.44.** *Let $f \colon X \to Y$ be a combinatorial map. Then $f$ is distance non-increasing with regard to the metric $d_p$. In particular, a combinatorial isomorphism is an isometry.*

*Proof.* By the combinatorial structure of $f$, each piecewise linear path $c$ in $X$ is mapped to a piecewise linear path in $Y$. For each $x, y \in X$ we denote by $\mathrm{PL}(x, y)$ the set of piecewise linear paths joining $x$ to $y$. Furthermore, each segment of $c$ lying in a cube is shortened by Corollary 2.43. Hence, $l(f \circ c) \leq l(c)$ and thus

$$\begin{aligned} d_X(x, y) &= \inf\{l(c) \mid c \in \mathrm{PL}(x, y)\} \\ &\geq \inf\{l(f \circ c) \mid c \in \mathrm{PL}(x, y)\} \\ &\geq \inf\{l(c) \mid c \in \mathrm{PL}(f(x), f(y))\} \\ &= d_Y(x, y), \end{aligned}$$

which is the desired result. □

**Remark 2.45.** In the last two sections we have seen that there are two closely intertwined aspects to CAT(0) cube complexes. First, there is the combinatorial nature of their construction, which is also mirrored in the definition of combinatorial maps. Second, there is the geometric structure as CAT(0) spaces. As stated in Theorem 2.21, $X$ is a metric space with regard to the path metric and it is with regard to this metric, that it is a CAT(0) space. However, often (and in particular in our case) one prefers to work on the combinatorial side and one introduces a second metric on the vertex set $V(X) := X^{(0)}$ of $X$. This so-called *edge metric*, as seen in Definition 2.37, is given by the infimum over the length of all paths between vertices along edges, i. e. the infimum of the length over all paths in the 1-skeleton $X^{(1)}$. Indeed, because of Gromov's link condition (Theorem 2.27), we are in the special situation that all the geometric information of our space is already encoded in its 1-skeleton (or equivalently in its vertex set equipped with the edge metric). This is also the reason why in the following chapters the length metric of CAT(0) cube complexes will not appear any longer and we will mostly be concerned with its vertex set and the equipped edge metric.

## 2.4 Halfspaces

We will introduce hyperplanes, halfspaces and talk about their geometric properties. We will see that they are convex and have rather strong intersection properties. Later, we will introduce strongly separated halfspaces, which will be linked directly to the irreducibility of a CAT(0) cube complex (see Proposition 5.20). Lastly, we will be concerned with some combinatorial properties of halfspaces. The first part of this section follows the lecture notes by Rolli [Rol12].

**Definition 2.46** (Hyperplanes)**.** Let $X$ be a cube complex.

- The 0-, 1- and 2-dimensional cubes are called *vertices, edges* and *squares* respectively.
- We say that two edges $e$ and $e'$ are equivalent ($e \sim e'$) if and only if either $e' = e$ or there is a sequence of edges $e_1, \dots, e_n$ such that $e_1 = e$ and $e_n = e'$ and any two edges $e_{i-1}, e_i$ are opposite edges in a common square in $X$. Note that this is an equivalence relation and we will call it *square relation*.
- A midcube $M \subset X$ is *transverse* to a square relation class $E = [e]_\sim$ (write $M \pitchfork E$) if $M \cap X^{(1)}$ contains only midpoints of edges in $E$.
- The *hyperplane* defined by $E$ is given by
$$\hat{\mathfrak{h}}(E) := \bigcup_{M \pitchfork E} M \subset X.$$
We will often write $\hat{\mathfrak{h}}$ instead of $\hat{\mathfrak{h}}(E)$.

**Example 2.47.** Figure 2.6 contains an example of a CAT(0) cube complex with an equivalence class of edges (dark blue) and associated hyperplane (light blue).

**Proposition 2.48** (Convexity of halfspaces, [Rol12, Propositions 18 & 19])**.** *Let $X$ be a CAT(0) cube complex and $\hat{\mathfrak{h}} \subset X$ a hyperplane. Then $\hat{\mathfrak{h}}$ is closed and convex. Furthermore, if $\hat{\mathfrak{h}}$ contains at least two points of the image of any geodesic $\gamma$, then the whole image of $\gamma$ is contained in $\hat{\mathfrak{h}}$.*

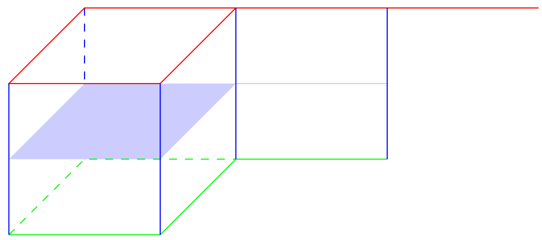

Figure 2.6: Example of a CAT(0) cube complex with a hyperplane inscribed. The dark blue edges form an edge equivalence class which defines the blue hyperplane. The red and green parts indicate the two halfspaces associated to the hyperplane. The figure follows closely the example in Sageev [Sag12].

**Corollary 2.49.** *Let $X$ be a CAT(0) cube complex. Every $\hat{\mathfrak{h}} \subset X$ is itself a CAT(0) cube complex.*

*Sketch.* By construction, it is easy to verify that the gluing of midcubes inherited from $X$ gives $\hat{\mathfrak{h}}$ a cube complex structure. Additionally, every convex closed subspace of a CAT(0) space is CAT(0) itself. □

**Theorem 2.50** (Separation, [Rol12, Proposition 21])**.** *Any hyperplane $\hat{\mathfrak{h}}$ separates $X$ in exactly two convex connected components.*

**Definition 2.51** (Halfspaces)**.** The two connected components of $X \setminus \hat{\mathfrak{h}}$ are called *halfspaces*. If $\mathfrak{h} \subset X \setminus \hat{\mathfrak{h}}$ is one of these halfspaces, then $\mathfrak{h}^*$ denotes the opposite halfspace leading to $X = \mathfrak{h} \sqcup \hat{\mathfrak{h}} \sqcup \mathfrak{h}^*$.

**Example 2.52.** Figure 2.6 also indicates the two halfspaces (red and green color).

**Theorem 2.53** (Intersection, [Rol12, Proposition 22 & 24])**.**

1. *Let $\hat{\mathfrak{h}}_1, \ldots, \hat{\mathfrak{h}}_n$ be hyperplanes with pairwise non-trivial intersection. Then*

$$\bigcap_{i=1}^{n} \hat{\mathfrak{h}}_i \neq \varnothing.$$

2. *Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ be halfspaces with pairwise non-trivial intersection. Then*

$$\bigcap_{i=1}^{n} \mathfrak{h}_i \neq \varnothing.$$

*In particular, the intersection contains a vertex of $X$.*

This concludes our discussion of geometric properties. The following definition will play a central role in Section 5 (see for example Proposition 5.20).

**Definition 2.54** (Strongly separated halfspaces)**.** Two hyperplanes are *strongly separated* if they are parallel (i. e. they do not intersect) and there is no hyperplane transverse to both. Two halfspaces are *strongly separated* if the same is true for their associated hyperplanes.

**Example 2.55** (Euclidean space)**.** The space $\mathbb{R}^d$ with its standard cubulation is one example of a CAT(0) cube complex *without* any strongly separated hyperplanes. Indeed, any pair of parallel hyperplanes is orthogonal to some coordinate axis. Any hyperplane parallel to this axis is transverse to both.

**Example 2.56** (A tree)**.** As an example *with* strongly separated halfspaces, consider the tree depicted in Figure 2.7. There we have three pairwise parallel hyperplanes $\hat{\mathfrak{h}}, \hat{\mathfrak{k}}$ and $\hat{\mathfrak{l}}$. Hence, any pair is strongly separated.

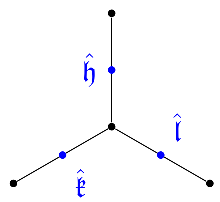


Figure 2.7: A tree with three pairwise parallel hyperplanes $\hat{\mathfrak{h}}, \hat{\mathfrak{k}}$ and $\hat{\mathfrak{l}}$.

We close this section with two technical results. The first one will be employed in the metrizability of the Roller compactification (see Corollary 3.8). The significance of the second result will become clear in Proposition 3.5 and Remark 3.6.

**Corollary 2.57.** *If $X$ is a locally countable CAT(0) cube complex, then its set of hyperplanes $\hat{\mathcal{H}}$ and its set of halfspaces $\mathcal{H}$ are countable.*

*Proof.* We fix a vertex $x_0 \in V(X)$ and consider the sets

$$Y_n := \{(x, y) \in X_{n-1} \times X_n \mid y \in N(x)\} \subset X_{n-1} \times X_n \quad \forall n \in \mathbb{N}.$$

By Lemma 2.38, the sets $X_n$ are countable and we have that

$$\hat{\mathcal{H}} = \bigcup_{n=1}^{\infty} \bigcup_{e \in Y_n} \hat{\mathfrak{h}}([e])$$

is countable. Since every hyperplane has exactly two halfspaces associated to it, the same is true for $\mathcal{H}$. $\square$

**Lemma 2.58.** *Let $X$ be a connected CAT(0) cube complex. Then for any two $\mathfrak{h}, \mathfrak{k} \in \mathcal{H}(X)$ such that $\mathfrak{h} \subset \mathfrak{k}$ we have*

$$|\{\mathfrak{l} \in \mathcal{H}(X) \mid \mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{k}\}| < \infty.$$

*Proof.* Let $M$ be the set as in the statement above and $\hat{M}$ the set of corresponding hyperplanes. Clearly, the two sets are one-to-one. We take any vertex $v \in \mathfrak{h}$ and $w \in \mathfrak{k}^*$. Then there exists a finite edge path $c$ joining the two. We claim that $\hat{M}$ is a subset of all the hyperplanes defined by the edges in $c$. Indeed, let $\mathfrak{l} \in M$. Then $v \in \mathfrak{l}$ and $w \in \mathfrak{l}^*$. Hence, $c$ has to cross $\hat{\mathfrak{l}}$. So $\hat{\mathfrak{l}}$ is one of the hyperplanes defined by an edge in $c$. □

# 3 Pocsets and the Roller compactification

This whole chapter is dedicated to the interplay between CAT(0) cube complexes and discrete pocsets. Roller [Rol99] showed that there is a one-to-one-correspondence between these two points of view, which is an important tool in this field of study.

This chapter is divided into five sections. In Section 3.1, we will introduce *pocsets* and *ultrafilters*, which are the main ingredients in the so-called *Roller compactification* of the vertex set of a CAT(0) cube complex. This compactification is described in detail in Section 3.2. Section 3.3 is again concerned with ultrafilters. There are two equivalent ways to define them and we will need both. The first viewpoint (in Section 3.1) has advantages when it comes to topological and metrical properties of the Roller compactification, whereas the second viewpoint (in Section 3.3) is more natural when it comes to measurability. The main result in Section 3.3 is Theorem 3.21, showing that the two viewpoints are equivalent. Section 3.4 is only loosely related to the previous sections. We will introduce *intervals* of ultrafilters, which are special subcomplexes of CAT(0) cube complexes. The most interesting property of these intervals is that they are always embeddable into some $\mathbb{R}^d$ (considered as a CAT(0) cube complex, see Example 2.30). The second topic of the section consists in so-called *terminal elements*. These are all the elements in a set of halfspaces that are minimal or maximal with regard to inclusion. The existence or non-existence of these elements is one cornerstone of the main proof. Intervals have at most finitely many terminal elements (c. f. Example 3.34) which is why they are so interesting for us. In Section 3.5, we are interested in one special case of the above mentioned correspondence between pocsets and CAT(0) cube complexes: If $\mathcal{H}' \subset \mathcal{H}$ are both pocsets, under which circumstances will the associated complex $X(\mathcal{H}')$ be a subcomplex of $X(\mathcal{H})$? We will be able to give a partial answer by providing a sufficient condition (which, however, might not be necessary).

## 3.1 Pocsets and ultrafilters

In this section we will introduce a special type of partially ordered sets (or short: *posets*). These sets are special because they have a certain fixed point free involution. They are called *pocsets* (a poset with complementation). After having established this notion, we will introduce *ultrafilters* which are the necessary objects to construct a compactification of the vertex set of a CAT(0) cube complex. In this section, we will mostly be interested in topological and metric properties of the set of all ultrafilters. The connection to CAT(0) cube complexes will be established at the end of this section.

**Definition 3.1** (Pocset, [Rol99])**.**

- A *pocset* is a triple $(A, \prec, *)$ consisting of a set $A$, a partial ordering $\prec$ on $A$ and a fixed point free, order reversing involution

$$\begin{aligned} *\colon A &\to A, \\ a &\mapsto a^* \end{aligned}$$

such that $a$ and $a^*$ are incomparable for every $a \in A$. If there is no danger of ambiguity, we will often drop the triple and we will write $A$ for short.

Let $A, B$ be pocsets. The map $f\colon A \to B$ is called a *pocset morphism* if the following conditions hold:

1. $a \prec b$ implies that $f(a) \prec f(a)$ for every $a, b \in A$, and
2. $f(a^*) = f(a)^*$ for every $a \in A$.

- A pocset $A$ is called *discrete* if for any two $a, b \in A$ the *interval*

$$[a, b] \coloneqq \{c \in A \mid a \prec c \prec b\}$$

is finite.

- Two elements $a, b$ of a pocset $A$ are called *nested* if they satisfy $a \prec b$, $a^* \prec b$, $a \prec b^*$ or $a^* \prec b^*$. Otherwise, they are called *transverse*.

- A pocset $A$ has *finite width* if there exists a constant $N \in \mathbb{N}$ such that the cardinality of any subset of transverse elements of $A$ is bounded from above by $N$.

**Definition 3.2** (Ultrafilter)**.** Let $(A, \prec, *)$ be a pocset. Let $\tilde{A}$ be the set of equivalence classes via: $a \sim b$ if and only if $a^* = b$ or $a = b$. We define

$$P(A) \coloneqq \prod_{\tilde{a} \in \tilde{A}} \tilde{a},$$

i. e. $P(A)$ is the product over all the two element sets containing an element of $A$ and its opposite under involution. Let $\alpha \in P(A)$. The notation $a \in \alpha$ for some $a \in A$ means that the natural projection $P(A) \to \tilde{a}$ maps $\alpha$ to $a$ (instead of to $a^*$). With this notation introduced, we define:

An element $\alpha \in P(A)$ is called an *ultrafilter* if it satisfies the so called *consistency condition*, namely: If $a \in \alpha$ and $b \in A$ such that $a \prec b$, then $b \in \alpha$.

We denote by $\mathcal{U}(A) \subset P(A)$ the subset of all ultrafilters. We put on every $\tilde{a}$ the discrete topology. By Tychonoff's theorem [c. f. Jän94, Chapter 10], $P(A)$ is compact. A basis of the topology is given by the sets of the form

$$\mathcal{U}(a_1, \ldots, a_n) \coloneqq \{\alpha \in P(A) \mid a_1, \ldots, a_n \in \alpha\},$$

where $a_1, \ldots, a_n \in A$ are arbitrary elements.

We will postpone an example until we have introduce the pocset of halfspaces of a CAT(0) cube complex in Proposition 3.5.

**Proposition 3.3.** *Let $A$ be a pocset. The set of all ultrafilters $\mathcal{U}(A)$ is a compact space.*

*Proof.* Since $P(A)$ is already compact it suffices to show that $\mathcal{U}(A)$ is closed in $P(A)$. However, the complement

$$\begin{aligned}
P(A) \setminus \mathcal{U}(A) &= \{\alpha \in P(A) \mid \exists a, b \in A \colon a \in \alpha,\ b \notin \alpha,\ a \prec b\} \\
&= \bigcup_{a \in A} \{\alpha \in P(A) \mid a \in \alpha,\ \exists b \in A \colon a \prec b,\ b \notin \alpha\} \\
&= \bigcup_{a \in A} \{\alpha \in P(A) \mid a \in \alpha,\ \exists b \in A \colon a \prec b,\ b^* \in \alpha\} \\
&= \bigcup_{a \in A} \bigcup_{a \prec b} \{\alpha \in P(A) \mid a \in \alpha,\ b^* \in \alpha\} \\
&= \bigcup_{a \in A} \bigcup_{a \prec b} \mathcal{U}(a, b^*)
\end{aligned}$$

is open, which proves the claim. □

**Corollary 3.4.** *If $A$ is countable, then $\mathcal{U}(A)$ is a compact metrizable space.*

*Proof.* By Engelking [Eng89, Theorem 4.2.2], it holds that every countable product of metrizable spaces leads to a metrizable space. Thus, $P(A)$ is compact and metrizable. Since $\mathcal{U}(A)$ is a closed subset the same is true for this space. □

### The pocset of halfspaces of a CAT(0) cube complex

**Proposition 3.5.** *Let $X$ be a connected CAT(0) cube complex and $\mathcal{H}$ its set of halfspaces. Furthermore, let*

$$\begin{aligned}
*\colon \mathcal{H} &\to \mathcal{H}, \\
\mathfrak{h} &\mapsto \mathfrak{h}^*.
\end{aligned}$$

*Then $(\mathcal{H}, \subset, *)$ is a discrete pocset. If $X$ is finite-dimensional then $\mathcal{H}$ has finite width.*

*Proof.* Clearly, $(\mathcal{H}, \subset)$ is a partially ordered set and, by definition, $*$ has no fixed points and is order reversing. Hence, $\mathcal{H}$ is a pocset. By Lemma 2.58, it is discrete.

Assume that $\mathfrak{h}_1, \dots, \mathfrak{h}_n \in \mathcal{H}$ are transverse. Then they intersect pairwise and the same is true for the associated hyperplanes $\hat{\mathfrak{h}}_i$. By Theorem 2.53, the hyperplanes contain a common point. This point must lie in some cube $C$ of $X$ and in this cube all the hyperplanes are given by (transverse) midcubes. Hence, $\dim C \geq n$. We see that if $X$ is finite-dimensional with $\dim X = n$, then any subset of transverse elements of $\mathcal{H}$ can have at most $n$ elements. This proves that $\mathcal{H}$ has finite width. □

**Remark 3.6.** In his habilitation Roller [Rol99] showed that there is a one-to-one-correspondence between discrete pocsets and CAT(0) cube complexes (the so-called *Roller duality*).

The direction from the complexes $X$ to the pocsets has been realized in Proposition 3.5. The other way around is far more involved and makes heavy use of *ultrafilters* as defined above. This *Roller construction* is what lead to the *Roller compactification* of every finite-dimensional CAT(0) cube complex.

**Example 3.7** (Trees)**.** We describe the ultrafilters in the case of a tree. There, each edge defines a unique hyperplane, and hence the choice of a halfspace is equivalent to assigning a direction to an edge. We agree that the arrow we assign to an edge will point towards the chosen halfspace. Since each ultrafilter contains either a halfspace or its complement, we have to assign an arrow to each edge. The consistency condition implies that if we have an outgoing edge at a vertex, then all other edges must be incoming. In other words, an ultrafilter converts our undirected tree into a directed one such that each vertex has at most one outgoing edge. An example of this can be found in Figure 3.1.

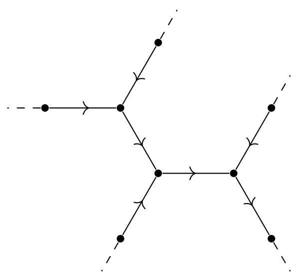

Figure 3.1: A tree with inscribed ultrafilter

This reformulation leads to another interesting observation. If all vertices have one outgoing edge, then we can start a geodesic ray from any vertex following the ultrafilter. Each of these geodesic rays will merge after finitely many steps and emanate on. Hence, each of these ultrafilters defines one element at the visual boundary of the tree (see Definition 2.12). Additionally, one can convince oneself that the only other possibility is that there is exactly one vertex with no outgoing edge [c. f. Sag12, p. 14]. Hence, these ultrafilters define a unique vertex in the tree. We see that for trees ultrafilters are a reformulation of the visual compactification of the CAT(0) space.

**Corollary 3.8.** *If $X$ is a locally countable CAT(0) cube complex and $\mathcal{H}$ is its pocset of halfspaces, then $\mathcal{U}(\mathcal{H})$ is a compact metrizable space.*

*Proof.* If $X$ is locally countable, then the pocset $\mathcal{H}$ is countable by Corollary 2.57 and we can apply Corollary 3.4. □

**Remark 3.9.** We would like to point out that the countability of $\mathcal{H}$ is the only place where the local countability of our CAT(0) cube complex $X$ comes into play. However, it is essential at this place. Indeed, the metrizability of $\mathcal{U}(\mathcal{H})$ is central in order for certain vector spaces of continuous functions to be separable. Otherwise, we could not construct the first part of our boundary map (see Theorem 5.51 and Corollary 5.52).

We will close this section with two results giving a criterion for reducibility of a CAT(0) cube complex $X$ using a decomposition of the pocset of halfspaces $\mathcal{H}(X)$.

**Proposition 3.10** ([CS11, Lemma 2.5])**.** *A CAT(0) cube complex $X$ is reducible, i. e. $X$ splits as a (proper) product if and only if there exists a partition $\mathcal{H}(X) = \mathcal{H}_1 \sqcup \mathcal{H}_2$ such that each halfspace in $\mathcal{H}_1$ is transverse to each halfspace in $\mathcal{H}_2$.*

*Sketch.* The key observation is that if we have two CAT(0) cube complexes $X_1$ and $X_2$ and we consider the cube complex $X_1 \times X_2$, then the halfspaces take the form $\mathfrak{h} \times X_2$ for any $\mathfrak{h} \in \mathcal{H}(X_1)$ or $X_1 \times \mathfrak{k}$ for any $\mathfrak{k} \in \mathcal{H}(X_2)$. Any two of them will always be transverse. Thus, if $X$ splits as a product, then this argument shows that we find the desired partition. If we have the partition, then the $\mathcal{H}_i$ are pocsets. Hence, up to isomorphism, we find two unique CAT(0) cube complexes corresponding to the two pocsets and their product has the same set of halfspaces as $X$. By the aforementioned uniqueness, they have to be isomorphic. $\square$

**Proposition 3.11** ([CS11, Proposition 2.6])**.** *Every finite-dimensional CAT(0) cube complex $X$ admits a decomposition*

$$X = X_1 \times \cdots \times X_m$$

*into a product of irreducible cube complexes $X_i$. This decomposition is canonical up to permutation. Every automorphism of $X$ preserves that decomposition, up to a permutation of possibly isomorphic factors. In particular, the image of the canonical embedding*

$$\mathrm{Aut}(X_1) \times \cdots \times \mathrm{Aut}(X_m) \hookrightarrow \mathrm{Aut}(X)$$

*has finite index in* $\mathrm{Aut}(X)$.

## 3.2 The Roller compactification

This section contains the connection between CAT(0) cube complexes, pocsets and ultrafilters leading directly to the *Roller compactification.* This connection was first discovered by Roller [Rol99]. Hence, all constructions in this direction inherited his name. We will first define two special kinds of ultrafilters, the *principal ultrafilters* and the *non-terminating ultrafilters.* Afterwards, we will show how to embed the vertex set of a CAT(0) cube complex into its set of ultrafilters over the pocset of halfspaces.

**Definition 3.12** (Descending Chain Condition, non-terminating and principal)**.**

- An ultrafilter $\alpha$ satisfies the *descending chain condition* if all descending chains in $\alpha$ become stationary.
- An ultrafilter is *non-terminating* if every finite descending chain can be extended.

- Let $X$ be a finite-dimensional CAT(0) cube complex and $v \in X$ a vertex. Then

$$\alpha_v := \{\mathfrak{h} \in \mathcal{H}(X) \mid v \in \mathfrak{h}\}$$

is called a *principal ultrafilter* (see the next lemma).

**Lemma 3.13.** *Let $X$ be a finite-dimensional CAT(0) cube complex and $v \in X$ a vertex. Then $\alpha_v$ is an ultrafilter. Furthermore, it satisfies the descending chain condition and every ultrafilter satisfying the descending chain condition arises in this way.*

*Proof.* Clearly, either $\mathfrak{h}$ or $\mathfrak{h}^*$ contains $v$ such that $\alpha_v$ lies in $P(A)$. Furthermore, if $\mathfrak{h} \in \alpha_v$ and $\mathfrak{k} \in \mathcal{H}$ are such that $\mathfrak{h} \subset \mathfrak{k}$, we have $v \in \mathfrak{h} \subset \mathfrak{k}$. Hence, $\mathfrak{k} \in \alpha_v$ and $\alpha_v$ satisfies the consistency condition. This shows that $\alpha_v$ is an ultrafilter.

If $\alpha$ satisfies the descending chain condition, then each halfspace contains a minimal halfspace of $\alpha$. If we take the set of minimal halfspaces of $\alpha$, then all the elements must be pairwise transverse. Furthermore, since $X$ is finite-dimensional, every set of pairwise transverse elements must be finite. By Theorem 2.53, we find that the intersection over all minimal elements is non-empty and contains a vertex $v$. We claim that $\alpha_v = \alpha$. Indeed, if $\mathfrak{h} \in \alpha$, then there exists a minimal element $\mathfrak{k} \in \alpha$ and $v \in \mathfrak{k} \subset \mathfrak{h}$. Hence, $\mathfrak{h} \in \alpha_v$. Conversely, if $\mathfrak{h} \notin \alpha$ then $\mathfrak{h}^* \in \alpha$ and as before $v \in \mathfrak{h}^*$ and $\mathfrak{h} \notin \alpha_v$. □

**Theorem 3.14** (The Roller compactification)**.** *Let $X$ be a finite-dimensional CAT(0) cube complex with associated pocset $(\mathcal{H}, \subset, *)$. Let $V(X)$ be the vertex set of $X$. Then the map*

$$\iota\colon V(X) \hookrightarrow \mathcal{U}(\mathcal{H}),$$
$$v \mapsto \alpha_v$$

*is injective, continuous and the image is dense in $\mathcal{U}(\mathcal{H})$.*

**Definition 3.15.** The *Roller compactification* of a CAT(0) cube complex is $\bar{X} := \mathcal{U}(\mathcal{H})$. The *Roller boundary* $\partial X$ is the set of all ultrafilters which have at least one infinite descending chain (by abuse of notation one often writes $\partial X := \bar{X} \setminus X$).

*Proof of Theorem 3.14.* The map is well-defined by Lemma 3.13. In order to see the injectivity, consider two vertices $v \neq w$. There exists a halfspace $\mathfrak{h}$ separating the two, i. e. $v \in \mathfrak{h}$ and $w \in \mathfrak{h}^*$. Hence, $\mathfrak{h} \in \alpha_v$ and $\mathfrak{h}^* \in \alpha_w$ and $\alpha_v \neq \alpha_w$. The continuity is clear, since we have the discrete topology on $V(X)$. Lastly, we have to show that the image is dense. We consider a basic open set $\mathcal{U} := \mathcal{U}(\mathfrak{h}_1, \dots, \mathfrak{h}_n)$ in $\mathcal{U}(X)$. If $\mathcal{U} \neq \varnothing$, then the $\mathfrak{h}_i$ intersect pairwise. Otherwise, assume that $\mathfrak{h}_i \cap \mathfrak{h}_j = \varnothing$. We have $V(X) = \mathfrak{h}_i \sqcup \mathfrak{h}_i^*$. Hence, for any $v \in \mathfrak{h}_j$ we have $v \in \mathfrak{h}_i^*$. This yields $\mathfrak{h}_j \subset \mathfrak{h}_i^*$. However, no ultrafilter can contain both $\mathfrak{h}_i$ and $\mathfrak{h}_i^*$. Now, we know by Theorem 2.53 that $\bigcap_{i=1}^n \mathfrak{h}_i$ contains a vertex $v$ and hence $\alpha_v \in \mathcal{U}$. □

**Remark 3.16.** Many compactifications one encounters have an additional interesting property: The space itself is embedded as an *open* subset of the compactification. In our case this can fail. Here is an example: Consider an infinite family of copies of the non-negative reals $\mathbb{R}_{\geq 0}$ with their standard cubulation. We glue these lines together at their respective origins (and call this vertex $*$). This construction leads to a tree and hence to a CAT(0) cube complex. We claim that any open neighborhood of the ultrafilter $\alpha_*$ contains an ultrafilter which does not satisfy the descending chain condition. The construction is depicted in Figure 3.2

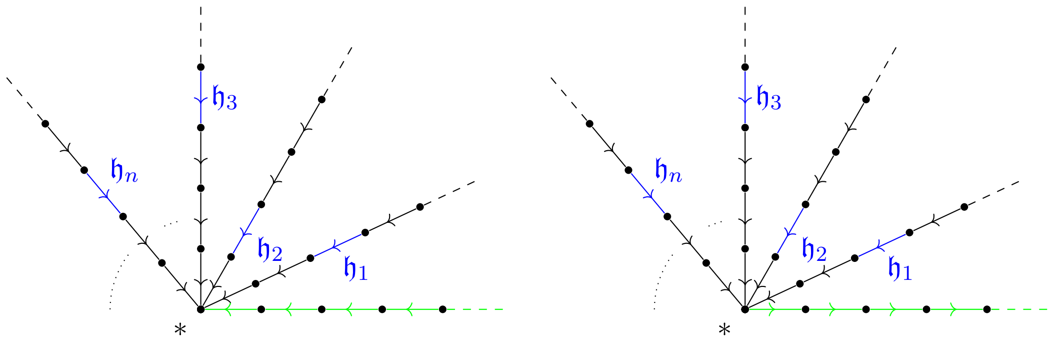


Figure 3.2: The left-hand side depicts the ultrafilter $\alpha_*$. The blue edges depict the halfspaces, determined by $\mathcal{U}(\mathfrak{h}_1, \dots, \mathfrak{h}_n)$. The green branch contains no $\mathfrak{h}_i$ and its halfspaces can therefore be reversed. This leads to the ultrafilter $\alpha$ on the right-hand side.

Let $\mathcal{U} := \mathcal{U}(\mathfrak{h}_1, \dots, \mathfrak{h}_n)$ be any basic open set containing $\alpha_*$. Then $\mathfrak{h}_i \in \alpha_*$. We recall from Example 2.28 that in the case of trees, halfspaces correspond to the choice of a direction at an edge and in the case of $\alpha_*$ all arrows have to point towards $*$. Since we have infinitely many branches with $*$ as their root, but only finitely many arrows prescribed by the $\mathfrak{h}_i$'s, we can construct the following ultrafilter: We take $\alpha_*$ and choose a branch which does not contain one of the edges with prescribed direction (the green branch in the figure). On this branch we reverse all arrows. This leads to a new ultrafilter $\alpha$ (each vertex has at most one outgoing edge), which does not satisfy the descending chain condition, but nonetheless lies in $\mathcal{U}$.

## 3.3 (Set-)Ultrafilters

Now that we have defined the Roller compactification, we have to consider a second viewpoint. The advantage of the above construction was that the topological and metric properties were easy to establish. However, the disadvantage of the construction is that the ultrafilters are not simply sets or more precisely special subsets of $\mathcal{H}(X)$, but elements in a product space. The new form has its advantages when it comes to measurability of certain maps. Therefore, we will establish this second viewpoint as well and prove the equivalence of the two.

**Definition 3.17.** Let $A$ be a set. Let $F_1, F_2 \subset A$ be finite subsets. The set

$$\mathcal{C}(F_1, F_2) \coloneqq \{H \subset A \mid F_1 \subset H \text{ and } F_2 \subset A \setminus H\} \subset \operatorname{Pot}(A)$$

is called a *cylinder set*. We set $\mathcal{C}(a) \coloneqq \mathcal{C}(\{a\}, \varnothing)$ for arbitrary $a \in A$.

**Proposition 3.18.** *The set of all cylinder sets is a basis for a topology on* $\operatorname{Pot}(A)$ *for any set* $A$.

*Proof.* There are two properties we have to establish:

1. We have to show that the union of all cylinder sets is all of the power set. However, $\mathcal{C}(\varnothing, \varnothing) = \operatorname{Pot}(A)$.

2. We have to show that the intersection of two cylinder sets is a union of arbitrarily many cylinder sets. Thus, let $F_1, F_2, G_1, G_2 \subset A$ be finite and consider

$$\begin{aligned}\mathcal{C}(F_1, F_2) \cap \mathcal{C}(G_1, G_2) &= \{H \subset A \mid F_1 \cap G_1 \subset H \text{ and } F_2 \cap G_2 \subset A \setminus H\} \\ &= \mathcal{C}(F_1 \cap G_1, F_2 \cap G_2).\end{aligned}$$

   Since $F_1 \cap G_1$ and $F_2 \cap G_2$ are still finite we are done.

$\square$

**Definition 3.19** ((Set-)Ultrafilters)**.** Let $X$ be a CAT(0) cube complex. We say that a subset $\alpha \subset \mathcal{H} \coloneqq \mathcal{H}(X)$ satisfies:

1. the *partial choice condition* if $\alpha \cap \alpha^* = \varnothing$,

2. the *choice condition* if $\alpha \cap \alpha^* = \varnothing$ and $\alpha \sqcup \alpha^* = \mathcal{H}$ and

3. the *consistency condition* if whenever $\mathfrak{h} \in \alpha$ and $\mathfrak{k} \in \mathcal{H}$ such that $\mathfrak{h} \subset \mathfrak{k}$, then $\mathfrak{k} \in \alpha$.

A *(set-)ultrafilter* is a set $\alpha \subset \mathcal{H}$ that satisfies the choice condition and the consistency condition. We denote by $\mathcal{U}_s(X) \subset \operatorname{Pot}(\mathcal{H}(X))$ the set of all (set-)ultrafilters and equip it with the subspace topology.

**Lemma 3.20.** *The space* $\mathcal{U}_s(X)$ *is a Hausdorff space.*

*Proof.* Let $\alpha, \beta \in \mathcal{U}_s(X)$ and $\alpha \neq \beta$. Then there exists $\mathfrak{h} \in \alpha$ such that $\mathfrak{h}^* \in \beta$ (choice condition). Next, let us define

$$\begin{aligned}U &\coloneqq \mathcal{C}(\mathfrak{h}\ ) \cap \mathcal{U}_s(X) \quad \text{and} \\ V &\coloneqq \mathcal{C}(\mathfrak{h}^*) \cap \mathcal{U}_s(X).\end{aligned}$$

By construction, both sets are open and we have $\alpha \in U$ and $\beta \in V$. Furthermore, no (set-)ultrafilter can contain both $\mathfrak{h}$ and $\mathfrak{h}^*$. Thus, $U \cap V = \varnothing$. $\square$

**Theorem 3.21.** *The spaces $\mathcal{U}(X)$ and $\mathcal{U}_s(X)$ are homeomorphic.*

*Proof.* Consider the map

$$\begin{aligned} f\colon \mathcal{U}(X) &\to \mathcal{U}_s(X), \\ (\mathfrak{h}_i)_{i\in\hat{\mathcal{H}}(X)} &\mapsto \{\mathfrak{h}_i \mid i \in \hat{\mathcal{H}}(X)\}. \end{aligned}$$

This map is well-defined and bijective. Next, let us show that it is continuous. For $1 \leq i \leq n$ and $1 \leq j \leq m$ let $\mathfrak{h}_i, \mathfrak{h}'_j \in \mathcal{H}(X)$ and set

$$U \coloneqq \mathcal{C}(\{\mathfrak{h}_1, \dots, \mathfrak{h}_n\}, \{\mathfrak{h}'_1, \dots, \mathfrak{h}'_k\}) \cap \mathcal{U}_s(X)$$

which is a basic open set. Thus, we have

$$\begin{aligned} f^{-1}(U) &= \{\alpha \in \mathcal{U}(X) \mid \mathfrak{h}_i \in \alpha \text{ and } \mathfrak{h}'_j \notin \alpha\} \\ &= \{\alpha \in \mathcal{U}(X) \mid \mathfrak{h}_i, \mathfrak{h}'^*_j \in \alpha\} \\ &= \mathcal{U}(\mathfrak{h}_1, \dots, \mathfrak{h}_n, \mathfrak{h}'^*_1, \dots, \mathfrak{h}'^*_k), \end{aligned}$$

which is a basic open set in $\mathcal{U}(X)$. However, this already suffices to show that $f$ is an homeomorphism. Indeed, every closed set $A \subset \mathcal{U}(X)$ is compact (since $\mathcal{U}(X)$ is) and as $f$ is continuous $f(A)$ is also compact. Lastly, as $\mathcal{U}_s(X)$ is Hausdorff $f(A)$ is also closed. This finishes the proof. □

**Remark 3.22.** With the above theorem in place, we can switch viewpoints whenever necessary. Actually, whenever convenient we will confuse the two and stop to distinguish between ultrafilters and set-ultrafilters.

## 3.4 Intervals & terminal elements

In this section we will introduce *intervals* of ultrafilters. These are special subsets of CAT(0) cube complexes. If they are interpreted as a complex in their own right, they can be embedded into some $\mathbb{R}^d$ with its standard cubulation (see Theorem 3.26).

Additionally, we will introduce *terminal elements*. The existence or non-existence of these is one of the main technical tools in the proof of our main result. We will see that intervals can have at most finitely many terminal elements.

**Definition 3.23.** Let $\alpha, \beta \in \bar{X}$. The *interval* $[\alpha, \beta]$ is defined as

$$[\alpha, \beta] \coloneqq \bigcap_{\mathfrak{h}\in\alpha\cap\beta} \mathfrak{h} \subset X.$$

**Lemma 3.24.** *Let $X$ be a finite-dimensional CAT(0) cube complex and $\alpha \in \bar{X}$ an ultrafilter. If $\alpha^*$ is an ultrafilter, then $X = [\alpha, \alpha^*]$.*

*Proof.* Since no halfspace can be contained in both $\alpha$ and $\alpha^*$, we have an empty intersection and thus $X = [\alpha, \alpha^*]$. $\square$

**Example 3.25** (Euclidean space)**.** The above lemma immediately shows that every $\mathbb{R}^d$ with its standard cubulation is an interval. Indeed, every hyperplane $\hat{\mathfrak{h}}$ is orthogonal to a coordinate axis given by a unit vector $e_i$. We choose the halfspace $\mathfrak{h}$ such that there exists a $K > 0$ such that $\lambda e_i \in \mathfrak{h}$ for all $\lambda > K$. This is an ultrafilter $\alpha$ and the opposite $\alpha^*$ is given by all the halfspaces such that there exists a $K > 0$ such that $-\lambda e_i \in \mathfrak{h}$ for all $\lambda > K$. This is again an ultrafilter.

**Theorem 3.26** ([Bro+09, Theorem 1.14])**.** *Let $X$ be a CAT(0) cube complex of finite dimension $d$ and $[\alpha, \beta] \subset X$ an interval. Then $[\alpha, \beta]$ is isometrically embeddable in $\mathbb{R}^d$ considered as a CAT(0) cube complex equipped with the edge metric.*

**Definition 3.27.** Let $\alpha$ and $\beta$ be two ultrafilters. The *set of separating halfspaces* of $\alpha$ and $\beta$ is defined as

$$\mathcal{H}(\alpha, \beta) := \{\mathfrak{h} \in \mathcal{H} \mid \mathfrak{h} \in \alpha \text{ and } \mathfrak{h}^* \in \beta \text{ or vice versa}\}.$$

**Remark 3.28.** Indeed, we have that $\mathcal{H}(\alpha, \beta) = \mathcal{H}([\alpha, \beta])$, i. e. the halfspaces separating $\alpha$ and $\beta$ are exactly the halfspaces of the interval $[\alpha, \beta]$.

**Lemma 3.29** ([Bro+09, Lemma 1.16])**.** *Let $X$ be a CAT(0) cube complex of dimension $d < \infty$ and $\alpha, \beta \in \bar{X}$. Then we have*

$$\mathcal{H}(\alpha, \beta) = P_1 \sqcup \cdots \sqcup P_d,$$

*where each $P_i$ contains all the halfspaces whose associated hyperplanes are parallel. Some of the $P_i$ might be empty.*

**Remark 3.30.** By construction, each of the non-empty $P_i$ contains exactly two chains. The first by choosing an arbitrary element and considering all halfspaces that are comparable to this element. The second by involution on this chain.

**Corollary 3.31** ([Fer16, Corollary 2.8])**.** *Let $X$ be a CAT(0) cube complex of (finite) dimension $d$ and $\alpha, \beta \in \bar{X}$. Then the set*

$$\{(\gamma, \delta) \in \bar{X} \times \bar{X} \mid [\gamma, \delta] = [\alpha, \beta]\}$$

*contains at most $2^d$ elements.*

*Proof.* As we noted in Remark 3.28, $\mathcal{H}(\alpha, \beta) = \mathcal{H}([\alpha, \beta])$. Because of the Roller duality, it is enough to work on the pocset of halfspaces. By Lemma 3.29 we have the decomposition

$$\mathcal{H}(\alpha, \beta) = P_1 \sqcup \cdots \sqcup P_d.$$

Any other elements $\gamma, \delta \in \bar{X}$ with the same set of halfspaces must admit the same decomposition. Let $\mathfrak{h}, \mathfrak{k} \in P_i$ and $\mathfrak{h} \in \alpha$. Hence, $\mathfrak{h}^* \in \beta$. If $\mathfrak{h} \subset \mathfrak{k}$, then $\mathfrak{k} \in \alpha$ because of the consistency condition. If $\mathfrak{k} \subset \mathfrak{h}$, then $\mathfrak{k}^* \in \beta$ because of the consistency condition and again we have $\mathfrak{k} \in \alpha$. We see that $\alpha$ contains a chain in $P_i$ and $\beta$ contains the opposite chain (under involution, see Remark 3.30).

However, the same is true for any other pair $(\gamma, \delta)$ having the same decomposition $P_i$. Since each $P_i$ contains 0 or 2 chains, we see that there are at most $2^d$ possible choices of how these two chains might be divided between $\gamma$ and $\delta$. □

**Definition 3.32.** Let $\mathcal{H}$ be the pocset of halfspaces of a CAT(0) cube complex and $\mathcal{H}' \subset \mathcal{H}$ a subset. An element $\mathfrak{h} \in \mathcal{H}'$ is called

- *minimal in* $\mathcal{H}'$ if for every $\mathfrak{k} \in \mathcal{H}'$ we have either $\mathfrak{k} \pitchfork \mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{k}$ or $\mathfrak{h} \subset \mathfrak{k}^*$,
- *maximal in* $\mathcal{H}'$ if $\mathfrak{h}^*$ is minimal in $\mathcal{H}'$,
- *terminal in* $\mathcal{H}'$ if it is either minimal or maximal in $\mathcal{H}'$.

Let $\tau\colon \operatorname{Pot}(\mathcal{H}) \to \operatorname{Pot}(\mathcal{H})$ be the map that assigns to each subset of $\mathcal{H}$ its set of terminal elements.

**Example 3.33** (DCC ultrafilters)**.** Every ultrafilter satisfying the descending chain condition contains minimal (and hence terminal) elements. If the CAT(0) cube complex is finite-dimensional, there can be at most finitely many transverse halfspaces and hence only finitely many minimal elements. In general, we cannot say anything about maximal elements in ultrafilters.

**Example 3.34** (Euclidean space)**.** In $\mathbb{R}^d$ with its standard cubulation, we know that all halfspaces are parallel to coordinate axes and along each axis we can have at most one minimal and one maximal element. This implies that any subset of halfspaces of $\mathbb{R}^d$ can have at most $2d$ terminal elements. With the help of Theorem 3.26 this reasoning can be extended to intervals.

**Lemma 3.35** ([CFI16, Lemma 4.12])**.** *Let $\alpha$ and $\beta$ be two ultrafilters and $\mathfrak{h} \in \tau(\alpha)$. Then $\mathfrak{h} \notin \beta$ if and only if $\mathfrak{h} \in \tau(\mathcal{H}(\alpha, \beta))$.*

*Proof.* If $\mathfrak{h} \in \beta$, then $\mathfrak{h}$ does not separate $\alpha$ and $\beta$. Hence, $\mathfrak{h} \notin \mathcal{H}(\alpha, \beta)$ and also $\mathfrak{h} \notin \tau(\mathcal{H}(\alpha, \beta))$. Conversely, assume $\mathfrak{h} \notin \tau(\mathcal{H}(\alpha, \beta))$. If $\mathfrak{h} \notin \mathcal{H}(\alpha, \beta)$, then $\mathfrak{h} \in \beta$. Otherwise, $\mathfrak{h}$ is not a terminal element in $\mathcal{H}(\alpha, \beta)$. However, this is impossible since $\mathfrak{h}$ is terminal in $\alpha$. □

## 3.5 Embeddings of the Roller compactification

In Remark 3.6 we outlined the connection between discrete pocsets and CAT(0) cube complexes. We saw that each discrete pocset has a unique CAT(0) cube complex associated to it. Now, let $X$ be any CAT(0) cube complex and $\mathcal{H}$ its pocset of halfspaces. Then any involution invariant subset $\mathcal{H}' \subset \mathcal{H}$ is a pocset in its own right with associated CAT(0) cube complex $X'$. It is natural to ask under which circumstances it is possible to embed $X'$ into $X$. This section will establish a sufficient (but not necessarily necessary) condition by introducing the notion of a *lifting decomposition*. We will need this construction later in Section 6.4 in order to embed the Roller compactification of certain subcomplexes into the Roller compactification of the parent-complex. See Lemma 5.15.

Unless noted otherwise $X$ is a connected, locally countable, finite-dimensional CAT(0) cube complex.

**Definition 3.36.** Let $\mathcal{H}' \subset \mathcal{H} := \mathcal{H}(X)$ be an involution invariant subset of halfspaces. A *lifting decomposition* of $\mathcal{H}'$ is a choice of a subset $W \subset \mathcal{H} \setminus \mathcal{H}'$ satisfying the partial choice and consistency condition (see Definition 3.19) and such that $\mathcal{H} = \mathcal{H}' \sqcup W \sqcup W^*$.

**Lemma 3.37** ([CFI16, Lemma 2.6])**.** *Let $\mathcal{H}' \subset \mathcal{H} := \mathcal{H}(X)$ be an involution invariant subset of halfspaces. Assume that $\mathcal{H}'$ admits a lifting decomposition $\mathcal{H} = \mathcal{H}' \sqcup W \sqcup W^*$. Then there is a continuous injective map*

$$\begin{aligned} i\colon \bar{X}' := \bar{X}(\mathcal{H}') &\to \bar{X}, \\ \alpha &\mapsto \alpha \sqcup W, \end{aligned}$$

*whose image is given by $i(\bar{X}') = \cap_{\mathfrak{h} \in W} \mathcal{C}(\mathfrak{h})$ (c.f. Definition 3.17).*

*Furthermore, if $\mathcal{H}' = \varnothing$, then $i(\bar{X}')$ is a point. If $W$ contains an infinite descending chain, then $i(\bar{X}') \subset \partial X$.*

*Proof.* We will work with the power set definition of the Roller compactification.

Since $\mathcal{H}'$ is involution invariant, it is a pocset in its own right and therefore we can construct a unique CAT(0) cube complex $X'$ with $\mathcal{H}'$ as its set of halfspaces (c. f. Remark 3.6). First, we need to show that the above construction of the map is well-defined. Let $\alpha' \in \bar{X}'$. We claim that $\alpha := \alpha' \sqcup W$ is an ultrafilter in $\bar{X}$. First, we see that $\alpha$ satisfies the choice condition. Indeed,

$$\begin{aligned} \alpha \cap \alpha^* &= (\alpha' \sqcup W) \cap (\alpha^* \sqcup W^*) \\ &= (\alpha' \cap \alpha'^*) \sqcup (\alpha' \cap W^*) \sqcup (W \cap \alpha'^*) \sqcup (W \cap W^*) \\ &= \varnothing \end{aligned}$$

and

$$\begin{aligned} \alpha \sqcup \alpha^* &= (\alpha' \sqcup \alpha'^*) \sqcup W \sqcup W^* \\ &= \mathcal{H}' \sqcup W \sqcup W^* \\ &= \mathcal{H}. \end{aligned}$$

For the consistency condition, we proceed as follows. Let $\mathfrak{h} \in \alpha$ and $\mathfrak{k} \in \mathcal{H}$ such that $\mathfrak{h} \subset \mathfrak{k}$. We have three cases:

1. If $\mathfrak{k} \in \mathcal{H}'$, then $\mathfrak{k} \in \alpha' \subset \alpha$ and we are done.

2. If $\mathfrak{k} \in W \subset \alpha$, there is nothing to prove.

3. If $\mathfrak{k} \in W^*$, we will find a contradiction. Indeed, we have $\mathfrak{k}^* \subset \mathfrak{h}^*$ and $W$ satisfies the consistency condition. Hence, $\mathfrak{h}^* \in W$ and, equivalently, $\mathfrak{h} \in W^*$. However, $\alpha \cap W^* = \varnothing$ which is absurd.

If $\alpha' \neq \beta'$ in $\mathcal{H}'$, then also $\alpha \neq \beta$ in $\mathcal{H}$ and $i$ is injective. Next, let us consider the continuity of $i$. Consider two finite subsets $F_1, F_2 \subset \mathcal{H}$. These can be decomposed into $F_i = G_i \sqcup E_i$, where $G_i \subset \mathcal{H}'$ and $E_i \subset W \sqcup W^*$. Then we have

$$i^{-1}(C(F_1, F_2)) = C(G_1, G_2)$$

and $i$ is continuous.

In order to compute the image, let us first show that $i(\bar{X}') \subset \cap_{\mathfrak{h} \in W} \mathcal{C}(\mathfrak{h})$. Indeed, we have

$$\alpha' \cap W = \bigcap_{\mathfrak{h} \in \alpha \sqcup W} \mathcal{C}(\mathfrak{h}) \subset \bigcap_{\mathfrak{h} \in W} \mathcal{C}(\mathfrak{h}).$$

Conversely, if $\alpha \in \cap_{\mathfrak{h} \in W} \mathcal{C}(\mathfrak{h})$, then $W \subset \alpha$. Additionally, $(\alpha \setminus W) \cap W^* = \varnothing$ because of the choice condition. Hence, $\alpha' := \alpha \setminus W \subset \mathcal{H}'$. We claim that $\alpha'$ is an ultrafilter. The choice condition is satisfied, since it is satisfied by $\alpha$. So let $\mathfrak{h} \in \alpha'$ and $\mathfrak{k} \in \mathcal{H}'$ such that $\mathfrak{h} \subset \mathfrak{k}$. Hence, $\mathfrak{k} \in \alpha$ and $\mathfrak{k} \notin W$. This shows that $\mathfrak{k} \in \alpha'$ and $\alpha'$ also satisfies the consistency condition.

Since we are only interested in connected cube complexes, $\mathcal{H}' = \varnothing$ implies that $X'$ is only a single point. The same is true for $\bar{X}'$ and its image under $i$.

If $W$ contains an infinite descending chain, so does $\alpha' \sqcup W$ and thus $\alpha' \sqcup W \in \partial X$. $\square$

# 4 Measure theoretic preliminaries

Measure theory will play an important role in the main proof. Hence, we have this dedicated chapter introducing the necessary notation.

This chapter is divided into three sections. Section 4.1 collects general facts about measures, measure theory and functional analysis. Section 4.2 introduces the notion of *weighted halfspaces* which are central to our problem. We will highlight some of their elementary properties. Section 4.2 is the most important of the three. Section 4.3 contains a collection of unrelated lemmas proving measurability for some maps which will be used later on. All of these maps are technical in nature and the same is true for the proofs.

## 4.1 Properties of probability measures and functional analysis

This section is a conglomeration of facts about measures and their connection to certain spaces of continuous functions. We will need deep measure theoretic and functional analytic results, which we will mostly state without proof. All of the results will culminate in Section 5.4 in Theorem 5.51 which is the first big step towards our boundary map.

### Probability measures

**Definition 4.1** (Signed and probability measures)**.** Let $(B, \Sigma)$ be a measure space. A map $\mu\colon \Sigma \to \mathbb{R}$ is called a *signed measure* if it is *$\sigma$-additive*, i. e.

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

for arbitrary $A_i \in \Sigma$ such that $A_i \cap A_j = \varnothing$ whenever $i \neq j$. Here we mean that the right hand side needs to converge.

We call a signed measure $\mu$ a *measure*, if $\mu(A) \geq 0$ for every $A \in \Sigma$ and a *probability measure*, if it is a measure and $\mu(X) = 1$.

$A \in \Sigma$ is called *null*, if $\mu(A) = 0$ and *conull* or *full measure* if its complement is null.

**Remark 4.2.** In this thesis we are mostly interested in probability measures and hence our measures are always finite. Thus, we defined signed measures only with image in $\mathbb{R}$, ignoring infinities. Since we have this restrictions in place, the above definition of (signed) measures in fact satisfies all the standard conditions, in particular $\mu(\varnothing) = 0$.

**Definition 4.3** (Positive, negative and total variation)**.** For any $A \in \Sigma$ and any signed measure $\mu$ the *total variation* $|\mu|(A)$ is defined as

$$|\mu|(A) \coloneqq \sup\left\{ \sum_{i\in\mathbb{N}} \mu(A_i) \,\middle|\, A_i \in \Sigma \text{ and } A = \bigsqcup_{i\in\mathbb{N}} A_i \right\}.$$

The *positive* and *negative variation* are defined as

$$\mu^\pm := \frac{1}{2}(|\mu| \pm \mu).$$

**Proposition 4.4** ([Rud91, Ch. 6.1]). *The maps $|\mu|$ and $\mu^\pm$ are measures on $(X, \Sigma)$ and $|\mu|(X) < \infty$ holds.*

Often the $\sigma$-algebras we are considering stem from a topology on our space, i. e. are the Borel $\sigma$-algebra of its topology. In this case we would also like our measures to be related to the topology of our space. Hence, we need the following definitions:

**Definition 4.5** (Borel measures and support)**.** If $\Sigma$ is the Borel $\sigma$-algebra of a topology on $X$, then a measure $\mu$ is called *Borel*, if every $x \in X$ has an open neighborhood $U \subset X$ such that $\mu(U) > 0$.

The support of a Borel measure $\mu$ is defined to be the set

$$\operatorname{supp}(\mu) := \{x \in X \mid \forall\, U \subset X \text{ open with } x \in U \colon \mu(U) > 0\}.$$

**Definition 4.6** (Regular measures)**.** A Borel measure $\mu$ is called *inner regular* if

$$\mu(A) = \sup\{\mu(K) \mid K \subset A \text{ compact}\}$$

and *outer regular* if

$$\mu(A) = \inf\{\mu(U) \mid U \supset A \text{ open}\}.$$

If it is both it is called *regular*. A signed Borel measure $\mu$ is called *regular* if $|\mu|$ is regular.

Lastly, let us recall the definition of measurable maps and how to pushforward measures along these maps:

**Definition 4.7** (Measurable maps)**.** Let $(A, \Sigma_A)$ and $(B, \Sigma_B)$ be two measure spaces and $f\colon A \to B$ a map. $f$ is called *measurable* if $f^{-1}(S) \in \Sigma_A$ for every $S \in \Sigma_B$.

Let $\mu$ be a (signed) measure on $A$ and $f$ measurable. Then $f$ is called *essentially constant (with regard to $\mu$)* if there exists a conull set $S \in \Sigma_A$ such that $f|_S$ is constant. The *pushforward of $\mu$ along $f$* is defined via

$$(f_*\mu)(S) := \mu(f^{-1}(S))$$

for every $S \in \Sigma_B$.

**Remark 4.8.** Clearly, $f_*\mu$ is again a (signed) measure.

**Lemma 4.9.** *Let $X$ be a topological space and $\mu$ an inner regular measure on $X$. Then for every measurable set $A \subset X \setminus \operatorname{supp}(\mu)$ we have that $\mu(A) = 0$. In particular, if $\mu$ is non-zero the set $\operatorname{supp}(\mu)$ is not empty.*

*Proof.* We fix a measurable $A \subset X \setminus \operatorname{supp}(\mu)$. Since $\mu$ is inner regular, we find a sequence of compact sets $K_n \subset A$ such that $\mu(K_n) \xrightarrow{n\to\infty} \mu(A)$. Let $x \in K_n$, then $x \notin \operatorname{supp}(\mu)$ and there exists $U_x \subset X$ open such that $x \in U_x$ and $\mu(U_x) = 0$. Since $K_n$ is compact, we find an $l = l(n) \in \mathbb{N}$ and finitely many $x_1, \ldots, x_l \in K_n$ such that $K_n \subset \bigcup_{i=1}^{l} U_{x_i}$. Hence, we have

$$\mu(K_n) \leq \sum_{i=1}^{l} \mu(U_{x_i}) = 0$$

for every $n \in \mathbb{N}$. By the convergence, we obtain $\mu(A) = 0$.

If $\operatorname{supp}(\mu)$ were empty, then the set $X \setminus \operatorname{supp}(\mu)$ would be measurable of full (non-zero) measure, which is a contradiction. □

**Functional analytic preliminaries**

**Definition 4.10.** Let $X$ be a topological space. The vector space $C_0(X)$ of *continuous functions vanishing at infinity* is defined via

$$C_0(X) := \{f \in C(X) \mid \forall \varepsilon > 0\ \exists K \subset X \text{ compact}\colon f|_{X\setminus K} < \varepsilon\}.$$

**Theorem 4.11** (Riesz-Markow representation, [Rud91, Theorem 6.19]). *Let $X$ be a locally compact Hausdorff space. Every bounded linear functional $\Phi$ on $C_0(X)$ is represented by a unique regular signed Borel measure $\mu$ in the sense that*

$$\Phi f = \int_X f \, \mathrm{d}\mu$$

*for every element $f \in C_0(X)$. Moreover, we have $\|\Phi\| = |\mu|(X)$. In other words, there exists an isometry of normed vector spaces between $X^*$ the (topological) dual of $X$ equipped with the operator norm $\|\cdot\|$ and $M_s(X)$ the space of signed measures equipped with total variation $|\cdot|(X)$ as norm.*

**Theorem 4.12** (Banach-Alaoglu, [Rud87]). *Let $X$ be a topological vector space and $V \subset X$ a neighborhood of 0. Then*

$$K := \{\Phi \in X^* \mid |\Phi x| \leq 1 \quad \forall x \in V\},$$

*is weak∗-compact.*

**Corollary 4.13.** *If $X$ is a compact metric space, then the space of all regular probability measures $\mathcal{P}(X)$ is weak∗-compact and contained in the unit ball of all regular signed measures $M_s(X)$.*

*Proof.* Note that $C_0(X)$ together with the supremum norm is a Banach space. Let $V$ be the unit ball in $C_0(X)$. Theorem 4.12 yields that the unit ball $B \subset M_s(X) \cong C_0(X)^*$ is weak$*$-compact. For each probability measure $\mu$ we have $|\mu|(X) = \mu(X) = 1$. Thus, it follows that $P(X) \subset B$. Now, we only need to show that $P(X)$ is weak$*$-closed in $B$. However,

$$\begin{aligned} P &:= \left\{ \mu \in M_s(X) \,\middle|\, \int_X f \,\mathrm{d}\mu \geq 0 \quad \forall f \geq 0 \right\}, \\ N &:= \left\{ \mu \in M_s(X) \,\middle|\, \int_X \mathrm{d}\mu = \int_X \chi_X \,\mathrm{d}\mu = 1 \right\}, \\ \mathcal{P}(X) &= B \cap P \cap N. \end{aligned}$$

The second set ensures that the measure is positive and the last enforces the normalization. These are all the necessary restrictions for a probability measure. Additionally, these sets are clearly weak$*$-closed. □

We will close the section with a result concerning the separability of the vector space of continuous functions:

**Lemma 4.14** ([Con90, Theorem V.6.6])**.** *If $X$ is a compact metric space, then the vector space of continuous functions $C(X)$ equipped with the supremum norm is separable.*

## 4.2 Weighted halfspaces

This section defines the main technical tool for our main proof, namely *weighted halfspaces*. These are special subsets of the pocset of halfspaces $\mathcal{H}$ of a CAT(0) cube complex, which are defined using probability measures on the Roller compactification. In order to understand the following section, it is important to keep the notation introduced in Sections 2.4 and 3.2 in mind.

Unless noted otherwise, $X$ is a connected, locally countable, finite-dimensional CAT(0) cube complex with pocset of halfspaces $\mathcal{H}$ and Roller compactification $\bar{X}$. Recall the topology on $\bar{X}$ introduced in Definition 3.17 with open sets $\mathcal{C}(\mathfrak{h})$ for every $\mathfrak{h} \in \bar{X}$.

**Definition 4.15.** Let $\mu$ be a regular probability measure on $\bar{X}$. We define

$$\begin{aligned} H_\mu &:= \{\mathfrak{h} \in \mathcal{H}(X) \mid \mu(\mathcal{C}(\mathfrak{h})) = \mu(\mathcal{C}(\mathfrak{h}^*))\}, \\ H_\mu^+ &:= \{\mathfrak{h} \in \mathcal{H}(X) \mid \mu(\mathcal{C}(\mathfrak{h})) > 1/2\}, \\ H_\mu^- &:= \{\mathfrak{h} \in \mathcal{H}(X) \mid \mu(\mathcal{C}(\mathfrak{h})) < 1/2\} \text{ and} \\ H_\mu^\pm &:= \{\mathfrak{h} \in \mathcal{H}(X) \mid \mu(\mathcal{C}(\mathfrak{h})) \neq 1/2\}. \end{aligned}$$

The above four sets are called *balanced, heavy, light* and *unbalanced halfspaces* respectively.

**Lemma 4.16** ([CFI16, Lemma 4.6])**.** *Let $\mu$ and $\nu$ be regular probability measures on $\bar{X}$.*

1. *$H_\mu$ is closed under involution. Also, involution is a bijection between $H_\mu^+$ and $H_\mu^-$.*

2. *There is the following partition* $\mathcal{H}(X) = H_\mu \sqcup H_\mu^\pm = H_\mu \sqcup H_\mu^+ \sqcup H_\mu^-$.

3. *If* $\mathfrak{h}, \mathfrak{k} \in H_\mu$ *(resp.* $H_\mu^+$ *or* $H_\mu^-$*), then either* $\mathfrak{h} \pitchfork \mathfrak{k}$ *or (after possibly switching* $\mathfrak{h}$ *and* $\mathfrak{k}$*) the interval* $[\mathfrak{h}, \mathfrak{k}]$ *lies in* $H_\mu$ *(resp.* $H_\mu^+$ *or* $H_\mu^-$*).*

4. *If* $H_\mu$ *and* $H_\nu$ *are both non-empty and* $H_\mu \cap H_\nu = \varnothing$*, then* $H_\mu \cap H_\nu^+ \neq \varnothing$ *and* $H_\mu \cap H_\nu^- \neq \varnothing$.

5. *If* $\mathfrak{h}, \mathfrak{k} \in H_\mu$ *are two parallel halfspaces with* $\mathfrak{h} \subset \mathfrak{k}$*, then* $\mu(\mathcal{C}(\mathfrak{h}^*) \cap \mathcal{C}(\mathfrak{k})) = 0$.

6. *The assignments* $\mu \mapsto H_\mu$, $\mu \mapsto H_\mu^+$ *and* $\mu \mapsto H_\mu^-$ *are* $\mathrm{Aut}(X)$*-equivariant for the natural actions on* $\mathcal{P}(\bar{X})$ *and* $\mathrm{Pot}(\mathcal{H}(X))$.

*Proof.* 1. – 3. are clear from the definitions and the additivity of the measure.

For 4. we see that $H_\mu \subset H_\nu^+ \cap H_\nu^-$. Since $H_\mu$ is invariant under involution, but $H_\nu^+$ and $H_\nu^-$ get interchanged, we see that both intersections must be non-empty.

For 5. we have

$$\begin{aligned}\frac{1}{2} = \mu(\mathcal{C}(\mathfrak{k})) &= \mu(\mathcal{C}(\mathfrak{k}) \cap \mathcal{C}(\mathfrak{h}^*)) + \mu(\mathcal{C}(\mathfrak{k}) \cap \mathcal{C}(\mathfrak{h}))\\ &= \mu(\mathcal{C}(\mathfrak{k}) \cap \mathcal{C}(\mathfrak{h}^*)) + \mu(\mathcal{C}(\mathfrak{h}))\\ &= \mu(\mathcal{C}(\mathfrak{k}) \cap \mathcal{C}(\mathfrak{h}^*)) + \frac{1}{2},\end{aligned}$$

where we have $\mathcal{C}(\mathfrak{h}) \subset \mathcal{C}(\mathfrak{k})$ because we have $\mathfrak{h} \subset \mathfrak{k}$ and ultrafilters satisfy the consistency condition. Hence, $\mu(\mathcal{C}(\mathfrak{h}^*) \cap \mathcal{C}(\mathfrak{k})) = 0$. Assertion 6 follows again easily from the definitions. □

**Lemma 4.17** ([CFI16, Lemma 4.7])**.** *The complex* $\bar{X}(H_\mu)$ *is an interval (in the sense of Definition 3.23).*

*Proof.* Let

$$\begin{aligned} p\colon \bar{X} &\to \bar{X}(H_\mu),\\ \alpha &\mapsto \alpha \cap H_\mu\end{aligned}$$

be the projection. Since $p$ is continuous, we can have $p_*\mu$. Next, choose $\alpha \in \mathrm{supp}(p_*\mu)$, i. e. every open neighborhood of $\alpha$ must have non-zero measure. The set $\mathrm{supp}(p_*\mu)$ is not empty by Lemma 4.9. We claim that $\alpha^*$ is also an ultrafilter. It automatically satisfies the choice condition, so we only need to check consistency. Let $\mathfrak{h} \in \alpha^*$ and $\mathfrak{k} \in H_\mu$ such that $\mathfrak{h} \subset \mathfrak{k}$. Assume that $\mathfrak{k}$ is not in $\alpha^*$. Then $\mathcal{C}(\mathfrak{h}^*) \cap \mathcal{C}(\mathfrak{k})$ is an open neighborhood of $\alpha$ in $\bar{X}(H_\mu)$ and its measure positive. However, by Lemma 4.16(5), we know that the measure is zero. Hence, we have $\mathfrak{k} \in \alpha^*$ and we can conclude the proof thanks to Lemma 3.24. □

**Lemma 4.18.** *For any* $H \subset H_\mu$*, we have that* $|\tau(H)| < \infty$ *(see Definition 3.32).*

*Proof.* By Lemma 4.17 and Theorem 3.26, we know that $\bar{X}(H_\mu)$ is an interval and is embeddable in some $\mathbb{R}^d$. The observation in Example 3.34 shows that any subset $H \subset H_\mu \subset \mathcal{H}(\mathbb{R}^d)$ has at most $2d$ terminal elements. □

**Lemma 4.19.** *Let $X$ be a finite-dimensional CAT(0) cube complex with pocset of halfspaces $\mathcal{H}$ and $Y$ one of its irreducible factors with pocset of halfspaces $\mathcal{H}'$. Then the projection*

$$\begin{aligned} p\colon \bar{X} &\to \bar{Y}, \\ \alpha &\mapsto \alpha \cap \mathcal{H}' \end{aligned}$$

*is continuous. If $\mu$ is a regular probability measure on $\bar{X}$, we have*

$$H_{p_*\mu} = H_\mu \cap \mathcal{H}'$$

*Proof.* Let $F, G \subset \mathcal{H}'$ be finite. Then

$$p^{-1}(\mathcal{C}_{\mathcal{H}'}(F, G)) = \mathcal{C}_{\mathcal{H}}(F, G) \subset \mathcal{H}$$

and $p$ is continuous. Hence, the pushforward of the measure is well-defined and for every $\mathfrak{h} \in \mathcal{H}'$ we have

$$(p_*\mu))(\mathcal{C}_{\mathcal{H}'}(\mathfrak{h})) = \mu(p^{-1}(\mathcal{C}_{\mathcal{H}'}(\mathfrak{h}))) = \mu(\mathcal{C}_{\mathcal{H}}(\mathfrak{h}))$$

proving the last equality. □

## 4.3 Measurability of certain maps

In the main proof of our theorem ergodicity will play a central role (see Section 5.3). We will mostly use it in the form that measurable $\Gamma$-invariant maps have to be essentially constant. Hence, we need all our important maps to be measurable. These proofs are mainly technical and have been collected in this section. The results are mostly stand-alone and there is no deeper connection between them.

Unless noted otherwise, $X$ is a locally countable, finite-dimensional CAT(0) cube complex with $\mathcal{H} := \mathcal{H}(X)$ its pocset of halfspaces and $\bar{X}$ its Roller compactification.

**Lemma 4.20.** *Let $N$ be a countable set. Then*

$$\begin{aligned} f\colon \operatorname{Pot}(N) &\to \mathbb{N} \cup \{\infty\}, \\ A &\mapsto |A| \end{aligned}$$

*is measurable, where $\mathbb{N} \cup \{\infty\}$ is equipped with the discrete topology.*

*Proof.* We will see that a basis of the topology is mapped to measurable sets via the preimage. First, let us consider $n \in \mathbb{N}$. Then we have

$$\begin{aligned} f^{-1}(\{n\}) &= \{A \subset N \mid |A| = n\} \\ &= \bigcup_{\substack{A \subset N \\ |A|=n}} \left( \bigcap_{\substack{F \subset N \\ |F|<\infty}} \mathcal{C}(A, F) \right), \end{aligned}$$

where $\mathcal{C}(A, F)$ is a cylinder set as defined in Definition 3.17. Since the set of all finite subsets of $N$ is countable, the above preimage is measurable as it is a countable union and intersection of measurable sets.

Lastly, we consider $f^{-1}(\{\infty\})$. However, here we have

$$f^{-1}(\{\infty\}) = N \setminus \left( \bigcup_{n=0}^{\infty} f^{-1}(\{n\}) \right)$$

which is measurable as the complement of a measurable set. □

**Lemma 4.21.** *Let $\tau\colon \operatorname{Pot}(\mathcal{H}) \to \operatorname{Pot}(\mathcal{H})$ be the map assigning to each subset of $\mathcal{H}$ its set of terminal elements (c. f. Definition 3.32). Then $\tau$ is measurable.*

*Proof.* We take an arbitrary cylinder set $\mathcal{C}(F_1, F_2)$ and are interested in the preimage

$$\begin{aligned} \tau^{-1}(\mathcal{C}(F_1, F_2)) &= \{H \subset \mathcal{H} \mid F_1 \subset \tau(H) \text{ and } \forall \mathfrak{h} \in F_2\colon \mathfrak{h} \in H \ \Rightarrow\ \mathfrak{h} \notin \tau H\} \\ &= \{H \subset \mathcal{H} \mid F_1 \subset \tau(H)\} \cap \{H \subset \mathcal{H} \mid \forall \mathfrak{h} \in F_2\colon \mathfrak{h} \in H \ \Rightarrow\ \mathfrak{h} \notin \tau(H)\} \\ &=: T \cap N. \end{aligned}$$

We now decompose $T$ as follows:

$$\begin{aligned} T &= \bigcap_{\mathfrak{h} \in F_1} \{H \subset \mathcal{H} \mid \mathfrak{h} \in \tau(H)\} \\ &= \bigcap_{\mathfrak{h} \in F_1} (\{H \subset \mathcal{H} \mid \mathfrak{h} \in H \text{ minimal}\} \cup \{H \subset \mathcal{H} \mid \mathfrak{h} \in H \text{ maximal}\}) \\ &= \bigcap_{\mathfrak{h} \in F_1} \left( \bigcap_{\substack{\mathfrak{k} \in \mathcal{H} \\ \mathfrak{k} \subset \mathfrak{h}}} \mathcal{C}(\{\mathfrak{h}\}, \{\mathfrak{k}\}) \cup \bigcap_{\substack{\mathfrak{k} \in \mathcal{H} \\ \mathfrak{h} \subset \mathfrak{k}}} \mathcal{C}(\{\mathfrak{h}\}, \{\mathfrak{k}\}) \right). \end{aligned}$$

This set is measurable as it is a countable intersection and union of measurable sets. Next let us decompose $N$:

$$\begin{aligned} N &= \bigcup_{F \subset F_2} \{H \subset \mathcal{H} \mid F \subset H \setminus \tau(H) \text{ and } (F_2 \setminus F) \cap H = \varnothing\} \\ &= \bigcup_{F \subset F_2} \left(\{H \subset \mathcal{H} \mid F \subset H \setminus \tau(H)\} \cap \mathcal{C}(\varnothing, F_2 \setminus F)\right). \end{aligned}$$

We will now show that every set where a finite subset of elements, that are not terminal, is measurable. This will conclude the proof. This can be achieved via an induction over $n = |F|$. The case $n = 0$ is clear, since any $\sigma$-algebra needs to contain the whole set. Assume the assertion is true for every finite subset $F \subset \mathcal{H}$ of $n$ elements. If $F$ contains $n+1$ elements, fixing $\mathfrak{h} \in F$ and $\tilde{F} := F \setminus \{\mathfrak{h}\}$, we do the following decomposition:

$$\begin{aligned} \{H \subset \mathcal{H} \mid F \subset \mathcal{H} \setminus \tau(H)\} &= \{H \subset \mathcal{H} \mid \{\mathfrak{h}\} \cup \tilde{F} \subset H \setminus \tau(H)\} \\ &= \{H \subset \mathcal{H} \mid \tilde{F} \subset H \setminus \tau(H)\} \setminus \{H \subset \mathcal{H} \mid \mathfrak{h} \in \tau(H)\}. \end{aligned}$$

The first set is measurable by induction hypothesis, the second one is measurable by our computations above. All in all we obtain that $\tau^{-1}(\mathcal{C}(F_1, F_2))$ is measurable. □

**Lemma 4.22** ([CFI16, Lemma A.1]). *Let $I \subset [0,1]$ be a subinterval that is either open, closed or half open. Let $H_\mu^I := \{h \in \mathcal{H}(X) \mid \mu(\mathcal{C}(h)) \in I\}$. Then the map*

$$\begin{aligned} \mathcal{P}(\bar{X}) &\to \mathrm{Pot}(\mathcal{H}(X)), \\ \mu &\mapsto H_\mu^I \end{aligned}$$

*is measurable with respect to the weak$*$-topology on $\mathcal{P}(X)$.*

*Proof.* In Section 3 we defined a topology on the power set (c. f. Definition 3.17 and Proposition 3.18). Hence, we need to show that the preimages of the basic open sets are mapped to measurable sets in $\mathcal{P}(\bar{X})$. So let $F_1, F_2 \subset \mathcal{H}(X)$ be finite and consider the cylinder set $\mathcal{C}(F_1, F_2)$. Then the preimage is given by the set

$$K(F_1, F_2) = \{\mu \in \mathcal{P}(\bar{X}) \mid H_\mu^I \in \mathcal{C}(F_1, F_2)\}.$$

For now, let us consider the sets $E_I(\mathfrak{h}) := \{\mu \in \mathcal{P}(\bar{X}) \mid \mu(\mathcal{U}(\mathfrak{h})) \in I\}$. We will now show that these sets are measurable. We know that $\bar{X} = \mathcal{U}(\mathfrak{h}) \sqcup \mathcal{U}(\mathfrak{h}^*)$. Hence, $\tilde{\mathfrak{h}} := \mathcal{U}(\mathfrak{h})$ is both open and closed in $\bar{X}$. Thus, the indicator function $\chi_{\tilde{\mathfrak{h}}}$ is continuous. However, the weak$*$-topology is defined such that each map

$$\begin{aligned} T_f \colon \mathcal{P}(\bar{X}) &\to \mathbb{R}, \\ \mu &\mapsto \int_X f \,\mathrm{d}\mu \end{aligned}$$

is continuous (for each $f \in C(\bar{X})$). Hence, $T := T_{\chi_{\tilde{\mathfrak{h}}}}$ is continuous and thus also measurable. Furthermore, we have that $T^{-1}(I) = E_I(\mathfrak{h})$. The interval $I$ is measurable, so the same is true for $E_I(\mathfrak{h})$.

Together with the following observation this finishes the proof:

$$K(F_1, F_2) = \left( \bigcap_{\mathfrak{h} \in F_1} E_I(\mathfrak{h}) \right) \cap \left( \bigcap_{\mathfrak{h} \in F_2} E_I(\mathfrak{h})^c \right).$$

□

**Lemma 4.23.** *Let $\alpha, \beta \in \bar{X}$. Let $\mathcal{H}(\alpha, \beta)$ be the set of halfspaces separating $\alpha$ from $\beta$ (c. f. Definition 3.27). Then the map*

$$\begin{aligned} f \colon \bar{X} \times \bar{X} &\to \mathrm{Pot}(\mathcal{H}), \\ (\alpha, \beta) &\mapsto \mathcal{H}(\alpha, \beta) \end{aligned}$$

*is measurable.*

*Proof.* We will see that the map is even continuous. Take finite subsets $F, G \subset \mathcal{H}$ and consider the following preimage

$$\begin{aligned} f^{-1}(\mathcal{C}(F, G)) =& \{(\alpha, \beta) \in \bar{X} \times \bar{X} \mid F \subset \mathcal{H}(\alpha, \beta)\} \\ & \cap \{(\alpha, \beta) \in \bar{X} \times \bar{X} \mid G \cap \mathcal{H}(\alpha, \beta) = \varnothing\} \\ =:& S \cap N. \end{aligned}$$

We will consider $S$ and $N$ separately:

$$S = \bigcup_{F' \subset F} \big( \mathcal{C}(F' \cup (F \setminus F')^*, \varnothing) \times \mathcal{C}(F'^* \cup F \setminus F', \varnothing) \big)$$

and

$$N = \bigcap_{\mathfrak{h} \in G} \left[ (\mathcal{C}(\mathfrak{h}) \times \mathcal{C}(\mathfrak{h})) \cup (\mathcal{C}(\mathfrak{h}^*) \times \mathcal{C}(\mathfrak{h}^*)) \right].$$

Then $f^{-1}(\mathcal{C}(F, G))$ is open as a finite intersection of open sets. □

**Lemma 4.24.** *The map*

$$\begin{aligned} \mathcal{P}(\bar{X}) &\to \mathbb{N} \cup \{\infty\}, \\ \mu &\mapsto |(H_\mu \times H_\mu) \cap \mathcal{S}|, \end{aligned}$$

*where*

$$\mathcal{S} := \{(\mathfrak{h}, \mathfrak{k}) \in \mathcal{H} \times \mathcal{H} \mid \mathfrak{h} \textit{ and } \mathfrak{k} \textit{ are strongly separated}\},$$

*is measurable.*

*Proof.* We decompose the map as follows

$$\begin{aligned}\mathcal{P}(\bar{X}) \xrightarrow{f} \mathcal{P}(\bar{X})^2 \xrightarrow{g} \mathrm{Pot}(\mathcal{H}^2) \quad &\xrightarrow{h} \mathrm{Pot}(\mathcal{H}^2) \qquad \xrightarrow{j} \mathbb{N} \cup \{\infty\},\\ \mu \mapsto (\mu, \mu) \quad \mapsto (H_\mu, H_\mu) &\mapsto H_\mu \times H_\mu \cap \mathcal{S} \mapsto |H_\mu \times H_\mu \cap \mathcal{S}|.\end{aligned}$$

The map $f$ is continuous and hence measurable. The map $g$ is measurable, because the map on each factor is measurable by Lemma 4.22. The map $j$ is measurable by Lemma 4.20. We are left with $h$. Consider the two finite subsets $F, G \subset \mathcal{H}^2$. Then the preimage of the associated cylinder set is given by

$$\begin{aligned} h^{-1}(\mathcal{C}(F,G)) :=& \{(H,K) \in \mathrm{Pot}(\mathcal{H})^2 \mid F \subset H \times K \wedge G \subset (H \times K)^c \\ & \wedge (\mathfrak{h}, \mathfrak{k}) \in H \times K \text{ is strongly separated}\} \\ =& \left( \bigcap_{(\mathfrak{h},\mathfrak{k}) \in F} \mathcal{C}(\mathfrak{h}) \times \mathcal{C}(\mathfrak{k}) \right) \\ & \cap \left( \bigcap_{(\mathfrak{h},\mathfrak{k}) \in G} \mathcal{C}(\mathfrak{h})^c \times \mathrm{Pot}(\mathcal{H}) \cup \mathrm{Pot}(\mathcal{H}) \times \mathcal{C}(\mathfrak{k})^c \right) \\ & \cap \left( \bigcap_{\substack{(\mathfrak{h},\mathfrak{k}) \in \mathcal{H}^2 \\ \text{not str. sep.}}} \mathcal{C}(\mathfrak{h}) \times \mathcal{C}(\mathfrak{k}) \right)^c, \end{aligned}$$

which is a countable union of measurable sets. Hence, our map is measurable as a composition of measurable maps. ☐

**Lemma 4.25.** *Let $A$ be a set. The maps*

$$\begin{aligned} f \colon \mathrm{Pot}(A) \times \mathrm{Pot}(A) &\to \mathrm{Pot}(A) \\ (H,K) &\mapsto H \cap K \end{aligned}$$

*and*

$$\begin{aligned} g \colon \mathrm{Pot}(A) \times \mathrm{Pot}(A) &\to \mathrm{Pot}(A) \\ (H,K) &\mapsto H \cup K \end{aligned}$$

*are measurable with regard to the Borel-$\sigma$-algebra of the cylinder topology.*

*Proof.* Let $F, G \subset A$ be two finite subsets. Then we have

$$\begin{aligned} f^{-1}(\mathcal{C}(F,G)) =& \{(H,K) \in \mathrm{Pot}(A)^2 \mid F \subset H \cap K \wedge G \subset H^c \cup K^c\} \\ =& \bigcap_{f \in F} \mathcal{C}(f) \times \mathcal{C}(f) \\ & \cap \bigcap_{g \in G} (\mathcal{C}(g)^c \times \mathrm{Pot}(A) \cup \mathrm{Pot}(A) \times \mathcal{C}(g)^c) \end{aligned}$$

and

$$\begin{aligned} g^{-1}(\mathcal{C}(F,G)) =& \{(H,K) \in \mathrm{Pot}(A)^2 \mid F \subset H \cup K \wedge G \subset H^c \cap K^c\} \\ =& \bigcap_{f \in F} (\mathcal{C}(f) \times \mathrm{Pot}(A) \cup \mathrm{Pot}(A) \times \mathcal{C}(f)) \\ & \cap \bigcap_{g \in G} \mathcal{C}(g)^c \times \mathcal{C}(g)^c. \end{aligned}$$

Hence, both maps are measurable. □

**Lemma 4.26.** *The map*

$$\begin{aligned} \mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) &\to \mathrm{Pot}_f(\mathcal{H}), \\ (\mu, \nu) &\mapsto \tau([H_\mu^+ \cap H_\nu] \cup [H_\nu^+ \cap H_\mu]) \end{aligned}$$

*is measurable.*

*Proof.* We know that the diagonal embedding and the product of measurable functions is again measurable. Hence, the only interesting thing to see is that the intersection and union preserve measurability. However, this has been proven in Lemma 4.3. Together with Lemma 4.21 this proves the assertion. □

# 5 Group actions on CAT(0) cube complexes and strong $\Gamma$-boundaries

This chapter is divided into four sections which cover two general topics. The first two sections deal with the actions of a group $\Gamma$ on a CAT(0) cube complex $X$. First, we will convince ourselves that a group action via automorphisms always extends to an action on $\bar{X}$ by homeomorphisms. This will be accomplished in Section 5.1. In Section 5.2, we will introduce two properties for group actions on $X$, namely *non-elementarity* (introduced in [CFI16]) and *essentiality* (introduced in [CS11]). Interestingly, these two properties are rather of a different kind. The first is more concerned with the CAT(0) structure, namely, it excludes the existence of any finite orbits in $X$ or in the visual boundary $\partial_{\sphericalangle} X$. Whereas the second is more concerned with the combinatorial structure of the halfspaces of $X$. Here we want orbits of the group to get arbitrarily far away from each hyperplane (on both of its sides).

The last two sections (Sections 5.3 and 5.4) are concerned with the definition of *strong* $\Gamma$*-boundaries.* They can be thought of as generalized Furstenberg-Poisson boundaries for certain random walks on nice groups $\Gamma$ (see Example 5.48 or [Kai03]). They are defined via two properties. The first is a strengthening of the notion of *ergodicity* and the second is *amenability.* Hence, in Section 5.3 we will first recall the notion of (standard) ergodicity. Our two most important consequences are encoded in Lemmas 5.33 and 5.36. Afterwards, in Section 5, we will introduce *ergodicity with coefficients* and relate it to the notion of essentiality defined before (Corollary 5.42). Lastly, we will introduce *amenability.* This property guarantees the existence of certain measurable and almost everywhere $\Gamma$-equivariant maps from the strong $\Gamma$-boundary to the dual of a separable Banach space. In our case we will use it to construct a map to $\mathcal{P}(\bar{X})$, the set of regular probability measures on $\bar{X}$. The result is recorded in Corollary 5.52. The notion of a strong $\Gamma$-boundary was first introduced by Monod and Shalom [MS04]. Many of the concepts of ergodicity with coefficients and amenability were also present in Monod's thesis [Mon01]. Amenability was first introduced by Zimmer [Zim78]. For an overview we would recommend the treatise by Kaimanovich [Kai03].

## 5.1 Extending a group action to the Roller boundary

In this section we would like to remark on how one can extend a group action on a CAT(0) cube complex $X$ to its Roller compactification $\bar{X}$. For that matter, let $\Gamma$ be a group with an action $\Gamma \to \mathrm{Aut}(X)$, where $X$ is a finite-dimensional CAT(0) cube complex. The group $\mathrm{Aut}(X)$ consists of the combinatorial automorphisms of $X$ (c. f. Definition 2.40).

The following proposition collects some facts about how combinatorial isomorphisms act on $X$. The notation can be found in Sections 2.4 and 3.

**Proposition 5.1.** *Let $g \in \mathrm{Aut}(X)$. Then the following holds:*

1. *if* $\hat{\mathfrak{h}} \in \hat{\mathcal{H}}(X)$*, then* $g\hat{\mathfrak{h}} \in \hat{\mathcal{H}}(X)$*,*
2. *if* $\mathfrak{h} \in \mathcal{H}(X)$*, then* $g\mathfrak{h} \in \mathcal{H}(X)$*,*
3. *for every* $\mathfrak{h} \in \mathcal{H}(X)\colon\ g(\mathfrak{h}^*) = (g\mathfrak{h})^*$*,*
4. *if* $\mathfrak{h}, \mathfrak{h}' \in \mathcal{H}(X)\colon\ \mathfrak{h} \subset \mathfrak{h}'$*, then* $g\mathfrak{h} \subset g\mathfrak{h}'$*,*
5. *if* $\alpha \in \bar{X}$*, then* $g\alpha \in \bar{X}$ *and*
6. *if* $\alpha$ *satisfies the descending chain condition, then so does* $g\alpha$*.*

*Proof.* The first statement is an immediate consequence of the fact that $g$ is an isometry. This leads directly to statement 2 and 3. For statement 4 we only need that $g$ is a bijection. Statements 5 and 6 are then simple applications of 4. □

With the above proposition in place, we see that each group action $\Gamma \to \mathrm{Aut}(X)$ immediately leads to an action $\Gamma \to \mathrm{Perm}(\bar{X})$. However, this is not yet what we want. We would prefer the image to lie in the homeomorphisms of $\bar{X}$. This will be accomplished by the following observation:

**Proposition 5.2.** *Let* $g \in \mathrm{Aut}(X)$ *and*

$$\mathcal{U} := \mathcal{U}(\mathfrak{h}_1, \ldots, \mathfrak{h}_n) \subset \bar{X}$$

*a basic open set. Then we have*

$$g^{-1}\mathcal{U} = \mathcal{U}(g^{-1}\mathfrak{h}_1, \ldots, g^{-1}\mathfrak{h}_n).$$

*Hence,* $g \in \mathrm{Homeo}(\bar{X})$*.*

*Sketch.* $\mathfrak{h}_i$ lies in $\alpha$ if and only if $g\mathfrak{h}_i$ lies in $g\alpha$. □

We arrive at the following result:

**Theorem 5.3.** *Let* $\Gamma$ *be a group and* $\Gamma \to \mathrm{Aut}(X)$ *a group action on a CAT(0) cube complex* $X$*. Then this action extends to an action* $\Gamma \to \mathrm{Homeo}(\bar{X})$ *on the Roller compactification.*

## 5.2 Non-elementary and essential group actions

Not every group action on a CAT(0) cube complex leads to our desired boundary map. There are two additionally properties we need to demand in order for our construction to work. These two are introduced in this section. The first is *non-elementarity* which is concerned with fixed points of the group in $X$ and in the visual boundary of $X$ (see Definition 2.12). The second is *essentiality* which is concerned with the interplay of the group with the halfspaces $\mathcal{H}$ of $X$. The section will close with some results by Caprace and Sageev [CS11].

### 5.2.1 Non-elementary group actions

**Definition 5.4** ((Non-)elementary action)**.** A group action $\Gamma \to \mathrm{Aut}(X)$ is called *elementary* if there exists a finite orbit of the action on $X \sqcup \partial_{\triangleleft} X$. Otherwise the action is called *non-elementary.*

The above definition has one interesting immediate consequence:

**Proposition 5.5.** *If $\Gamma \to \mathrm{Aut}(X)$ acts non-elementary on $X$, then the vertex set $V(X)$ of $X$ together with the edge metric is unbounded, i. e. for every $v \in V(X)$ and every $K > 0$ there exists a $w \in V(X)$ such that $d(v, w) > K$. In particular, $V(X)$ is infinite.*

The previous observation already provides many examples for elementary actions by considering any group action on a finite cube complex. Let us consider an example of a cube complex with infinitely many vertices:

**Example 5.6** (Euclidean space)**.** An example of an elementary action is $X := \mathbb{R}^d$ with its standard cubulation and any cyclic subgroup of $\mathbb{Z}^d$ acting by translations. This action has no finite orbits in $\mathbb{R}^d$, but every point at infinity is fixed. Indeed, two rays define the same point at infinity if and only if they are parallel. However, a translated ray is still parallel to the untranslated ray.

**Example 5.7** (A tree)**.** As an example for a non-elementary action we consider the universal cover $\tilde{X}$ of $X := S^1 \vee S^1$ as depicted in Figure 5.1 (we note that this is also the Cayley graph of the free group with two generators). The space $X$ is not a cube complex. However, its universal cover $\tilde{X}$ is a tree and hence a CAT(0) cube complex. Let $\Gamma := \mathrm{Deck}(\tilde{X}/X)$ be the group of *deck transformations* (for an introduction to covering space theory and deck transformations see for example [Hat01, Section 1.3]). This group acts by combinatorial maps. Indeed, each vertex is mapped to a vertex, since fibers are preserved by deck transformations and each edge is again mapped to an edge (even more: the signing as in Figure 5.1 is preserved) because of the path lifting property.

The universal cover is always a normal covering. This implies that $\Gamma$ acts transitively on the vertices. Hence, $\Gamma$ has no finite orbit in $\tilde{X}$. Now, consider any geodesic ray $c$ represented by its vertices $(v_i)_{i\in\mathbb{N}}$. Then each edge $e_i := \overline{v_i v_{i+1}}$ has an assigned signing by an arrow and after translating by a deck transformation, this signing will be the same. Without loss of generality assume that $e_i$ has signing $\rightarrow$. We choose one of the neighbors $w_i$ of $v_i$ such that $\overline{v_i w_i}$ has signing $\twoheadrightarrow$. Then we choose $g_i \in \Gamma$ such that $g_i v_i = w_i$. We claim that $g_i c$ and $c$ are not equivalent. Since we can drop any finite number of starting vertices, it is enough to show the assertion in the case $i = 1$. If $g_1 c$ and $c$ are equivalent, then there exists an $m \in \mathbb{N}$ such that $g_1 v_m$ is a vertex in $c$, call it $v_n$. There is a path connecting the two, namely

$$g_1 v_m, g_1 v_{m-1}, \ldots, g_1 v_1, v_1, \ldots, v_n.$$

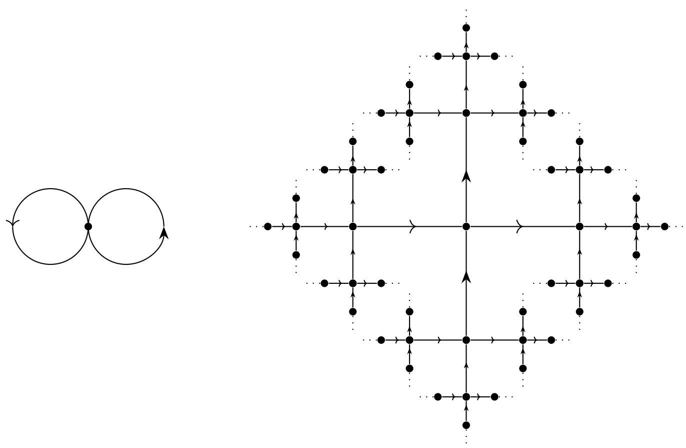

Figure 5.1: The pointed sum of two spheres $X := S^1 \vee S^1$ and its universal cover$\tilde{X}$. The arrows indicate which edges in the universal cover correspond to which loop in $X$.

Since we are in a tree, this is the unique path. Hence, $g_1c$ must transverse $\overline{v_1w_1} = \overline{v_1(g_1v_1)}$. However, every vertex is at most visited once by $g_1c$ and its first vertex is $w_1$. So we need $m = 1$ and $g_1e_1 = \overline{v_1w_1}$. This is impossible, because deck transformations conserve signing of edges. So we established that $g_ic$ and $c$ are always inequivalent, showing that any orbit in the visual boundary is infinite.

### 5.2.2 Essential group actions

**Definition 5.8** (Essential halfspaces)**.** A halfspace $\mathfrak{h} \in \mathcal{H}$ is called $\Gamma$-*essential* if for some $x \in X$ the orbit in $\mathfrak{h}$, $\Gamma x \cap \mathfrak{h}$, is *not* a bounded distance away from $\hat{\mathfrak{h}}$. A hyperplane $\hat{\mathfrak{h}} \in \hat{\mathcal{H}}$ is called $\Gamma$-*essential* if both its halfspaces are $\Gamma$-essential. It is called *half-essential* if only one of its halfspaces is $\Gamma$-essential.

Ess$(X, \Gamma)$ denotes the set of all essential hyperplanes. Accordingly, nEss$(X, \Gamma)$ denotes the set of all non-essential hyperplanes, leading to

$$\hat{\mathcal{H}}(X) = \mathrm{Ess}(X, \Gamma) \sqcup \mathrm{nEss}(X, \Gamma).$$

The above definition leads to the following consequence:

**Proposition 5.9.** *The sets* Ess$(X, \Gamma)$ *and* nEss$(X, \Gamma)$ *are* $\Gamma$*-invariant.*

**Definition 5.10** (The essential core)**.** The *essential core* is the CAT(0) cube complex corresponding to the the pocset of halfspaces associated to Ess$(X, \Gamma)$.

**Proposition 5.11** ([CS11, Proposition 3.5])**.** *Let $X$ be a finite-dimensional CAT(0) cube complex and let $\Gamma \leq \mathrm{Aut}(X)$. Assume that at least one of the following two conditions is satisfied:*

1. *$\Gamma$ has finitely many orbits of hyperplanes or*

2. *$\Gamma$ has not fixed point at infinity.*

*Then the essential core of $X$ is unbounded if and only if $\Gamma$ has no fixed point. In that case the essential core embeds as a $\Gamma$-invariant convex subcomplex $Y$ of $X$.*

**Definition 5.12** (Essential action)**.** A group action $\Gamma \to \mathrm{Aut}(X)$ is called *essential* if the essential core of $\Gamma$ is the whole space $X$.

**Example 5.13** (Euclidean space)**.** Consider $X := \mathbb{R}^d$ with the standard cubulation and the action of $\Gamma := \mathbb{Z}^d$ on it via translations. This action respects the cube complex structure. Additionally, every hyperplane in $X$ is a hyperplane $\hat{\mathfrak{h}} \in \hat{\mathcal{H}}(X)$ in the usual Euclidean sense. The translates of any vertex get arbitrarily far away from $\mathfrak{h}$ on either side. Hence, $\Gamma$ acts essentially on $X$.

### 5.2.3 Consequences of non-elementary and essential group actions

This paragraph contains some consequences and characterizations of non-elementary and essential group actions. Most of these have been found by Caprace and Sageev [CS11] and we refer the reader to this text for the proofs.

Lemmas 5.14 and 5.15 relate the Roller boundary of a subcomplex to the Roller boundary of the whole complex. This is important for Corollary 6.21.

**Lemma 5.14** ([CS11, Lemma 3.1])**.** *Let $X$ be a CAT(0) cube complex and let $\Gamma \leq \mathrm{Aut}(X)$ and $Y \subset X$ be a $\Gamma$-invariant convex subcomplex. Then each hyperplane of $Y$ extends to a unique hyperplane of $X$ such that there is a natural inclusion $\hat{\mathcal{H}}(Y) \subset \hat{\mathcal{H}}(Y)$.*

**Lemma 5.15.** *Let $X$ be a CAT(0) cube complex and let $\Gamma \leq \mathrm{Aut}(X)$ and $Y \subset X$ be a $\Gamma$-invariant convex subcomplex. Let $\mathcal{H}$ and $\mathcal{H}'$ be the pocset of halfspaces associated to $X$ and $Y$. Then there exists a natural lifting decomposition*

$$W := \{\mathfrak{h} \in \mathcal{H} \mid Y \subset \mathfrak{h}\}$$

*and the associated continuous inclusion*

$$\iota \colon \bar{Y} \to \bar{X}$$

*is $\Gamma$-equivariant. Furthermore, we have $\iota(\partial Y) \subset \partial X$.*

*Proof.* Clearly, $W$ satisfies the partial choice and consistency. Then, by Lemma 5.14, we have

$$\mathcal{H} = \mathcal{H}' \sqcup W \sqcup W^*.$$

Applying Lemma 3.37, we obtain the lifting decomposition

$$\begin{aligned} i\colon \bar{Y} &\to \bar{X}, \\ \alpha &\mapsto \alpha \sqcup W. \end{aligned}$$

Since $Y$ is $\Gamma$-invariant, the same is true for $W$. Hence, $i$ is $\Gamma$-equivariant.

Lastly, if $\alpha$ contains an infinite descending chain, so does $\alpha \sqcup W$. □

The following four results (Lemmas 5.16, 5.18 and 5.17 and Theorem 5.19) give characterizations of essential and/or non-elementary group actions. These give us a handle on the special interplay between our group $\Gamma$ and the pocset of halfspaces $\mathcal{H}$. All of these results will be used throughout the remainder of this thesis.

**Lemma 5.16** (Skewering Lemma, [CS11, Proposition 3.2]). *Let $X$ be a finite-dimensional CAT(0) cube complex, $\hat{\mathfrak{h}} \in \hat{\mathcal{H}}(X)$ and $\Gamma \leq \mathrm{Aut}(X)$. Then the following are equivalent:*

1. $\hat{\mathfrak{h}} \in \mathrm{Ess}(X, \Gamma)$,
2. $X(\Gamma \cdot \hat{\mathfrak{h}})$ *is unbounded and*
3. $\Gamma$ skewers $\hat{\mathfrak{h}}$, *i. e. there exists $g \in \Gamma$ and $n \in \mathbb{N}$ such that for one $\mathfrak{h}$ of the two halfspaces of $\hat{\mathfrak{h}}$ we have $g^n\mathfrak{h} \subsetneq \mathfrak{h}$.*

**Lemma 5.17** ([CFI16, Lemma 2.28]). *Let $X$ be a finite-dimensional CAT(0) cube complex and let $\Gamma \to \mathrm{Aut}(X)$ be a non-elementary action. Then the $\Gamma_0$-action on the irreducible factors of the essential core is also non-elementary and essential, where $\Gamma_0$ is the finite index subgroup preserving the decomposition in irreducible factors.*

*Proof.* Let $Y \subset X$ be the essential factor and $Y = Y_1 \times \cdots \times Y_m$ its decomposition into irreducible factors. Let $\Gamma_0$ be the finite index subgroup of $\Gamma$ preserving this decomposition.

We will first show that the $\Gamma_0$-action on each $Y_i$ is essential. By construction, $\Gamma$ acts essentially on $Y$ and since $\Gamma_0$ has finite index in $\Gamma$ the same is true for $\Gamma_0$. We would like to apply Lemma 5.16. For this we note that any halfspace $\mathfrak{h}_i \in \mathcal{H}(Y_i)$ defines a unique halfspace in $Y$ via

$$\mathfrak{h} := Y_1 \times \cdots \times Y_{i-1} \times \mathfrak{h}_i \times Y_{i+1} \times \cdots \times Y_k.$$

Using this each hyperplane $\hat{\mathfrak{h}}_i = \{\mathfrak{h}_i, \mathfrak{h}_i^*\} \in \hat{\mathcal{H}}(Y_i)$ defines an associated hyperplane $\hat{\mathfrak{h}} = \{\mathfrak{h}, \mathfrak{h}^*\}$. Since $\Gamma_0$ acts essentially on $Y$, Lemma 5.16 assures the existence of $g \in \Gamma_0$ and $n \in \mathbb{N}$ such that $g^n\mathfrak{h} \subsetneq \mathfrak{h}$ (after possibly switching $\mathfrak{h}$ and $\mathfrak{h}^*$). We have $g^n\mathfrak{h}_i \subsetneq \mathfrak{h}_i$ and using the same lemma in the other direction, we obtain that $\Gamma_0$ acts essentially on each $Y_i$.

Secondly, we will show that $Y_i$ does not have a fixed point in the visual boundary. We will achieve this by contraposition. Assume that $\Gamma_0 \leq \Gamma$ has a finite orbit. Passing to a further finite index subgroup, which we will still call $\Gamma_0$, we can assume that $\Gamma_0$ has a fixed point at infinity. However, then we can apply Lemma 2.16 and see that we have a $\Gamma_0$-fixed point in $\partial_\triangleleft Y \subset \partial_\triangleleft X$. By the finite index, $\Gamma$ must have a finite orbit at infinity. □

**Lemma 5.18** ([CS11, Double Skewering Lemma]). *Let $X$ be a finite-dimensional CAT(0) cube complex and $\Gamma \leq \mathrm{Aut}(X)$ be a group acting essentially and without fixed point at infinity. Then for any two halfspaces $\mathfrak{k} \subset \mathfrak{h}$, there exists $g \in \Gamma$ such that $g\mathfrak{h} \subsetneq \mathfrak{k} \subset \mathfrak{h}$.*

**Theorem 5.19** (Flipping Lemma, [CS11, Theorem 4.1]). *Assume that $X$ is a finite-dimensional CAT(0) cube complex and let $\Gamma \leq \mathrm{Aut}(X)$ be any subgroup. Let $\mathfrak{h} \in \mathcal{H}(X)$ such that $\mathfrak{h}^* \not\subset g\mathfrak{h}$ for each $g \in \Gamma$. Then $\Gamma$ has a fixed point in the visual boundary or $\mathfrak{h}$ is not essential with regard to $\Gamma$.*

Caprace and Sageev [CS11] showed that the existence of strongly separated hyperplanes is closely related to the irreducibility of the complex. This is the content of the next proposition:

**Proposition 5.20** ([CS11, Proposition 5.1]). *Let $X$ be a finite-dimensional and unbounded CAT(0) cube complex such that $\mathrm{Aut}(X)$ acts essentially and without fixed point at infinity. Then the following conditions are equivalent:*

1. *$X$ is irreducible,*
2. *there is a pair of strongly separated hyperplanes, and*
3. *for every halfspace $\mathfrak{h}$ there is a pair of strongly separated halfspaces $\mathfrak{h}_i$ such that $\mathfrak{h}_1 \subset \mathfrak{h} \subset \mathfrak{h}_2$.*

The remainder of this paragraph builds towards Corollary 5.24 and for the remainder of this thesis, knowledge of this result is quite sufficient. We will see that a non-elementary group action prevents the existence of any factors in the irreducible decomposition that are intervals (in the sense of Definition 3.23). This result is directly used in the proof of our main result (Theorem 6.20).

**Proposition 5.21** ([CS11, Proposition 3.6]). *Let $X$ be a finite-dimensional CAT(0) cube complex and let $\Gamma \leq \mathrm{Aut}(X)$. Let $(Y_a)_{a\in A}$ be a $\Gamma$-invariant family of closed convex subsets of $X$. If for any finite subset $B \subset A$ the intersection $\bigcap_{a\in B} Y_a$ is non-empty, then either $\bigcap_{a\in A} Y_a$ is a non-empty $\Gamma$-invariant subspace or $\bigcap_{a\in A} \partial_{\triangleleft} Y \subset \partial_{\triangleleft} X$ contains a finite $\Gamma$-orbit. In particular, in this case $\Gamma$ acts elementary on $X$.*

**Lemma 5.22** ([Fer16, Lemma 3.19]). *If $X$ is an interval, then $\mathrm{Aut}(X)$ is elementary.*

*Proof.* By Corollary 3.31, we know that there are only finitely many $\alpha, \beta \in \bar{X}$ such that $X = [\alpha, \beta]$. Since $X$ is clearly $\mathrm{Aut}(X)$-invariant, we see that $\mathrm{Aut}(X)$ acts by permutations on the finite set

$$I = \{(\alpha, \beta) \in \bar{X}^2 \mid X = [\alpha, \beta]\}.$$

Hence, we find a finite index subgroup $\Gamma_0 \leq \mathrm{Aut}(X)$ fixing each element in $I$. Let $(\alpha, \beta) \in I$. By construction, we have $g\alpha = \alpha$ for all $g \in \Gamma_0$. Hence, the collection of halfspaces given by $\alpha$ is $\Gamma_0$-invariant. Take $\mathfrak{h}_1, \dots, \mathfrak{h}_n \in \alpha$. If any of the $\hat{\mathfrak{h}}_i$ are parallel, we have seen in the proof of Corollary 3.31 that the corresponding halfspaces in $\alpha$ form a chain. Hence, their intersection is never empty. If the two halfspaces are transverse, their intersection is also non-empty. Hence, we can invoke Theorem 2.53 to see that $\bigcap_{i=1}^n \mathfrak{h}_i$ is non-empty as well. We can now apply Proposition 5.21 and see that $\Gamma_0$ acts elementary in $X$. Since $\Gamma_0$ has finite index in $\mathrm{Aut}(X)$ every finite orbit of $\Gamma_0$ leads to a finite orbit in $\mathrm{Aut}(X)$. Hence, $\mathrm{Aut}(X)$ acts elementary as well. □

**Remark 5.23.** With the previous lemma, Example 3.25 shows that we could have chosen every group action by combinatorial automorphisms on $\mathbb{R}^d$ in Example 5.6, not only cyclic subgroups of $\mathbb{Z}^d$.

**Corollary 5.24** ([Fer16, Corollary 3.21])**.** *If $\Gamma \to \mathrm{Aut}(X)$ is non-elementary, then no factor in the irreducible decomposition of $X$ is an interval.*

*Proof.* Let $Y$ be one of the factors and $\Gamma_0 \leq \Gamma$ the finite index subgroup preserving the decomposition. By Lemma 5.17, we know that $\Gamma_0$ acts non-elementary on each factor, in particular on $Y$. However, $\Gamma_0 \leq \mathrm{Aut}(Y)$ and hence $\mathrm{Aut}(Y)$ has to act non-elementary, too. Otherwise, we would find finite orbits of $\mathrm{Aut}(Y)$, which would directly lead to finite orbits of $\Gamma_0$. However, then Lemma 5.22 finishes the proof. □

## 5.3 (Doubly) ergodic group actions

We need properties of our group action not only on our complex, but also on its so called strong boundary (which will be introduced in the next section). One of these properties is *ergodicity*. We will mostly use it to make certain maps essentially constant. One other guise we need it in is that ergodic actions of finite groups lead to purely atomic spaces (c. f. Definition 5.32 and Lemma 5.33). Lastly, we will introduce *standard Borel* and *Lebesgue spaces* and prove one last technical lemma making use of this definition and Mackey's point realization (Theorem 5.35).

Unless noted otherwise, $\Gamma$ will denote a countable, discrete group.

**Definition 5.25** (Measure class preserving action)**.** Let $(B, \Sigma)$ be a measurable space. We can define an equivalence relation on all measures on $\Sigma$ via $\mu \sim \nu$ if and only if the null-sets of $\mu$ and $\nu$ coincide. An equivalence class $[\mu]$ is called a *measure class.* If a group $\Gamma$ acts by measurable transformations on $B$, then $\Gamma$ *preserves measure classes* if for every measure $\mu$ of $\Sigma$ we have that $\mu(A) = 0$ implies that $\mu(g^{-1}A) = 0$ for every $A \in \Sigma$ and $g \in \Gamma$.

**Lemma 5.26.** *Let $(B, \Sigma, \mu)$ be a measure space on which a countable group $\Gamma$ acts by measurable and measure class preserving transformations. Let $B_0 \in \Sigma$ be a conull subspace. If $\mu \neq 0$, then there exists $x \in M$ such that $\Gamma \cdot x \subset B_0$.*

*Proof.* We define

$$A := \{x \in B_0 \mid \exists g \in \Gamma\colon gx \in B_0^c\} = B_0 \cap \left( \bigcup_{g \in \Gamma} g^{-1} B_0^c \right) \subset \bigcup_{g \in \Gamma} g^{-1} B_0^c.$$

Since $\Gamma$ acts measure class preserving, we have that $\mu(gB_0^c) = 0$ for every $g \in \Gamma$. As $\Gamma$ is countable, we have that $\mu(A) = 0$. Hence, $B_0 \setminus A$ has full measure. In particular it is not empty and every $x \in B_0 \setminus A$ satisfies $\Gamma \cdot x \subset B_0$. □

**Definition 5.27** ((Doubly) ergodic action)**.** Let $(B, \Sigma, \mu)$ be a probability space with a group $\Gamma$ acting by measurable and measure class preserving transformations. Then the action is called *ergodic* if one of the two equivalent conditions is satisfied (c. f. Lemma 5.28):

1. for every $E \in \Sigma$ such that $g^{-1}E = E$ for each $g \in \Gamma$ we have $\mu(E) = 0$ or $\mu(E) = 1$, or

2. every measurable $\Gamma$-invariant map $f\colon B \to \mathbb{R}$ is essentially constant.

The action is called *doubly ergodic* if the diagonal action on $B \times B$ equipped with the product measure is ergodic.

**Lemma 5.28.** *The two statements in Definition 5.27 are equivalent.*

*Proof.* We assume 1. Let $f\colon B \to \mathbb{R}$ be measurable and $\Gamma$-invariant. Then for every $c \in \mathbb{R}$ the set $f^{-1}(c)$ is measurable and $\Gamma$-invariant. Hence, it has either measure 0 or 1. Since $B$ is a disjoint union of all these preimages, we see that there exists exactly one $c \in \mathbb{R}$ such that $f^{-1}(c)$ has full measure, which means that $f$ is essentially constant.

Now, we assume 2. Consider a $\Gamma$-invariant measurable set $E$. Then its indicator function $\chi_E$ is a measurable, $\Gamma$-invariant map and hence essentially constant. This implies that $\mu(E) \in \{0, 1\}$. □

The first definition of ergodicity shows that *transitive* group actions (i. e. for every $x, y \in B$ there exists a $g \in \Gamma$ such that $gx = y$) are automatically ergodic (as long as they act measurably and measure class preserving). The following is a less pathological example:

**Example 5.29** (Bernoulli space)**.** Consider the Bernoulli space $B := \{0, 1\}^{\mathbb{Z}}$ with the product $\sigma$-algebra stemming from the discrete topology on each of the sets $\{0, 1\}$. We equip this space with the measure $\mu$ that comes from the uniform distribution on each factor. Let $\mathbb{Z}$ act on this space via a shift operation. Then Klenke [Kle14, Example 20.26] shows that this system is ergodic.

**Proposition 5.30.** *Every doubly ergodic action is ergodic.*

*Sketch.* Consider the second criterion together with the (equivariant and measurable) projection from the product to the first factor. □

**Lemma 5.31.** *Let $A$ and $B$ be measurable spaces with a measurable group action $\Gamma$. Furthermore, let $f\colon A \to B$ be a measurable $\Gamma$-equivariant map and $\mu$ a measure on $A$. If $\Gamma$ acts ergodically on $(A, \mu)$, then $\Gamma$ acts ergodically on $(B, f_*\mu)$, where $f_*\mu$ is the pushforward measure (see Definition 4.7).*

*Proof.* We will apply the first criterion for ergodicity. Let $E \subset B$ be measurable such that $g^{-1}E = E$ for every $g \in \Gamma$. Then we have:

$$f^{-1}(E) = f^{-1}(g^{-1}E) = g^{-1}f^{-1}(E)$$

because of the equivariance. Thus, by ergodicity on $A$ we have $\mu(f^{-1}(E)) \in \{0, 1\}$. However, $\mu(f^{-1}(E))$ is exactly the definition of $f_*\mu(E)$. ☐

**Definition 5.32.** Let $(M, \Sigma, \mu)$ be a measure space. A set $B \in \Sigma$ is called *atomic* if $\mu(B) > 0$ and for all measurable $A \subset B$ either $\mu(A) = 0$ or $\mu(A) = \mu(B)$. The space $M$ is called *purely atomic* if there exists a partition of $M$ consisting of atomic sets.

**Lemma 5.33.** *Let $(M, \Sigma, \mu)$ be a measure space and $\Gamma$ a finite group acting ergodically on it. Then $M$ is purely atomic.*

*Proof.* We will find the above mentioned partition. Start by considering the following set:

$$\Lambda \coloneqq \{\tilde{A} \in \Sigma \mid \mu(\tilde{A}) > 0\}.$$

This set is clearly partially ordered under inclusion and not empty. Consider a descending chain $A_1 \supset A_2 \supset A_3 \supset \dots$ in $\Lambda$. Then $A_0 \coloneqq \cap_i A_i$ is also measurable, and since

$$\mu\left(\bigcup_{g\in\Gamma} gA_0\right) = \lim_{k\to\infty} \mu\left(\bigcap_{i=1}^{k} \bigcup_{g\in\Gamma} gA_i\right) = \lim_{k\to\infty} \mu\left(\bigcup_{g\in\Gamma} gA_k\right) = 1$$

at least one of the sets $gA_0$ has non-zero measure for some $g \in \Gamma$. Next, we note that $\Gamma$ acts measure class preserving. Hence, if any of the sets $gA_0$ has zero measure, then all of them have zero measure. All in all we obtain $A_0 \in \Lambda$. Thus, we have found a lower bound for our chain. Applying Zorn's Lemma we find a minimal element $A \in \Lambda$, i. e. for every $B \in \Lambda$ such that $B \subset A$ we have $B = A$. We note that the above reasoning also shows that if $B \in \Lambda$, then $gB \in \Lambda$ for every $g \in \Gamma$ by means of the measure class preserving action. Observe that for each $g \in \Gamma$, the set $gA$ is also a minimal element. Indeed, if $B \subset gA$ then $g^{-1}B \subset A$. Hence, $g^{-1}B = A$ and multiplying again we find $B = gA$.

Let us consider the case that $A$ is $\Gamma$-invariant. Then by ergodicity, we have $\mu(A) = 1$. We claim that in this case $M$ is atomic. Take any $B \in \Sigma$. Then

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c) = \mu(B \cap A),$$

since $A^c$ is a null-set. We see that $B \cap A \subset A$ and thus either $\mu(B \cap A) = 0$ or $B \cap A \in \Lambda$ and hence $B \cap A = A$ and $\mu(B) = 1 = \mu(M)$.

If $A$ is not $\Gamma$-invariant, we can consider the sets $A \cap gA$ for each $g \in \Gamma$. Whenever $\mu(A \cap gA) > 0$, we have $A = A \cap gA = gA$, since both $A$ and $gA$ are minimal in $\Lambda$. Thus, there exists at least one $g \in \Gamma$ such that $\mu(A \cap gA) = 0$, otherwise $A$ would be $\Gamma$-invariant. Let $g_1, \ldots, g_l \in \Gamma$ all these group elements. We define:

$$\begin{aligned} B_1 &\coloneqq A \setminus g_1 A \\ B_i &\coloneqq B_{i-1} \setminus g_i A \quad \forall i = 2, \ldots, l. \end{aligned}$$

We claim that $\mu(B_i) = \mu(A) > 0$ for each $i$. Indeed, by induction we have:

$$\begin{aligned} \mu(B_1) = \mu(A) - \mu(A \cap g_1 A) = \mu(A) \quad \text{and} \\ \mu(A) \geq \mu(B_i) = \mu(B_{i-1}) - \mu(B_{i-1} \cap g_i A) \geq \mu(A) - \mu(A \cap g_i A) = \mu(A). \end{aligned}$$

Hence, $B_i \in \Lambda$ and $B_i = A$ for each $i$ and we have:

$$A \cap g_i A = B_i \cap g_i A = (B_{i-1} \setminus g_i A) \cap g_i A = \varnothing.$$

All in all we have that

$$\bigcup_{g \in \Gamma} gA = \bigsqcup_{i=0}^{l} g_i A,$$

where we set $g_0 = e$. Thus, this set has full measure. If we now define

$$\begin{aligned} B_0 &\coloneqq M \setminus \left( \bigsqcup_{i=1}^{l} g_i A \right) \supset A \\ B_i &\coloneqq g_i A \qquad \forall i = 1, \ldots, l, \end{aligned}$$

then $M = \sqcup_i B_i$ and as before we can show that each of these sets is atomic. □

**Definition 5.34.** A measurable space $(B, \Sigma)$ is called a *standard Borel space* if it is isomorphic (as a measurable space) to a measurable set $E \subset X$, where $X$ is a complete separable metric space equipped with its Borel $\sigma$-algebra.

A measure space $(B, \Sigma, \mu)$ is called a *Lebesgue space* if it is a standard Borel space and $\mu$ is a regular probability measure.

**Theorem 5.35** (Mackey's point realization, [Mac62, p. 330], [Zim84, Corollary B.6])**.** *Let $(B, \Sigma, \vartheta)$ be a Lebesgue space. Let a locally compact, second countable group $\Gamma$ act on $B$ by measurable transformations. Let $\Lambda$ be a sub-$\sigma$-algebra of $\Sigma$ which is $\Gamma$-invariant, and such that for any $A \in \Lambda$ and $g \in \Gamma$ we have $\vartheta(A) = 0$ if and only if $\vartheta(gA) = 0$. Then there exists a Lebesgue space $(B', \Sigma', \vartheta')$ on which $\Gamma$ acts by measurable transformations and a $\Lambda$-measurable $\Gamma$-equivariant map $p\colon B \to B'$ such that $p_*\vartheta = \vartheta'$. Additionally, $p$ induces a bijection of $\Sigma'$ and $\Lambda$.*

**Lemma 5.36** ([CFI16, Lemma 4.3]). *Let $\Gamma$ be a group acting on a Lebesgue space $(B, \vartheta)$. If $\Gamma$ acts doubly ergodically on $B$, then every finite index subgroup $\Gamma_0 \leq \Gamma$ acts ergodically on $B$.*

*Proof.* We proceed by contradiction. Assume that $\Gamma$ acts doubly ergodic on $B$, but that there exists a finite index subgroup $\Gamma_0 \leq \Gamma$ which does not act ergodically on $B$. We can find a finite index normal subgroup of $\Gamma$ within $\Gamma_0$, which still acts non-ergodically on $B$. Without loss of generality, we can assume that $\Gamma_0$ is normal in $\Gamma$.

We consider the set

$$\Lambda := \{A \subset B \text{ measurable} \mid gA = A \quad \forall g \in \Gamma_0\}.$$

This is a $\sigma$-subalgebra. Since $\Gamma_0$ is normal it inherits a $\Gamma$-action. Applying Mackey's point realization (Theorem 5.35), we find a Lebesgue space $(B_0, \Sigma_0, \vartheta_0)$ and a measurable $\Gamma$-equivariant map $p\colon B \to B_0$, which induces a bijection on the two $\sigma$-algebras $\Lambda$ and $\Sigma_0$ and $\vartheta_0 = p_*\vartheta$. Via this pushforward, $\Gamma$ acts (doubly) ergodically on $B_0$ and on the $\sigma$-algebra $\Sigma_0$. We find a well-defined group action $\bar{\Gamma} := {}^{\Gamma}/_{\Gamma_0}$, which is still ergodic, because all elements of the algebra are $\Gamma_0$-invariant.

However, applying Lemma 5.33 this implies that $B_0$ is purely atomic. If $B_0$ is atomic, then $\Gamma_0$ would act ergodically on $B$. Indeed, any $A \in \Lambda$ corresponds to exactly one $A_0 \in \Sigma_0$ such that $p^{-1}(A_0) = A$ and hence

$$\vartheta(A) = \vartheta_0(A_0) \in \{0, 1\}.$$

This contradicts the fact that we assumed that $\Gamma_0$ does not act ergodically on $B$.

Therefore, there exists an atomic subset $A_0 \subset B$ with $0 < \vartheta_0(A_0) < 1$. We consider $A := p^{-1}(A_0)$ and also in this case $0 < \vartheta(A) < 1$. We claim that the set

$$X := \bigcup_{\bar{g} \in \bar{\Gamma}} gA \times gA \subset B \times B$$

is neither null nor conull. However, we will see that this is a contradiction, since $X$ is $\Gamma$-invariant and $\Gamma$ is assumed to act doubly ergodic. In order to see this, we first note that $X$ is well-defined, as $A$ is $\Gamma_0$-invariant. Thus, the action of $\Gamma$ factors through $\bar{\Gamma}$. Additionally, $X$ is not null as it contains $A \times A$. Lastly, we will show that up to a null set $A \times A^c$ is contained in $X^c$. Indeed, we have:

$$(\vartheta \times \vartheta)((A \times A^c) \cap X) \leq \sum_{\bar{g} \in \bar{\Gamma}} \vartheta(A \cap gA) \cdot \vartheta(A^c \cap gA).$$

Now $A \cap gA$ still lies in $\Lambda$. Also note that on $\Lambda$, $A$ is atomic. Hence, we have $\vartheta(A \cap gA) = 0$ or $\vartheta(A \cap gA) = \vartheta(A)$. So either $\vartheta(A \cap gA)$ or $\vartheta(A^c \cap gA)$ are 0 and the right-hand side of the above equation vanishes. $\square$

## 5.4 Strong $\Gamma$-boundaries

Finally, we are in a position to define *strong* $\Gamma$*-boundaries* for certain topological groups $\Gamma$. These are group theoretic objects and they will be the domain of our boundary map (Theorem 6.20). Indeed, at the end of this section, we will be able to construct the first half of this map. However, we first need to introduce two more group action properties. We need to generalize *ergodic group actions* as defined in the previous sections to *ergodic group actions with coefficients.* Afterwards, we will define *amenable group actions.* This property guarantees the existence of certain measurable maps from the space the group is acting on into the dual of certain Banach spaces (on which $\Gamma$ also has to act). Both definitions are rather technical in nature. However, their two main applications are simpler to grasp.

The first main application consists in Corollary 5.40, namely that ergodicity with coefficients implies ergodicity in the regular sense. We will provide a few applications of this.

The second main application consists in Theorem 5.51 and Corollary 5.52, in which we construct a measurable $\Gamma$-equivariant map $\psi\colon B \to \mathcal{P}(\bar{X})$, where $B$ is a strong $\Gamma$-boundary and $\mathcal{P}(\bar{X})$ is the set of all regular probability measures on $\bar{X}$. This is the only place in the entire proof, where we use the amenability of the group action.

### 5.4.1 Doubly ergodic group action with coefficients

Here we will strengthen the notion of ergodicity. The main application is Corollary 5.42. We will often encounter measurable, $\Gamma$-equivariant maps that take values in the finite subsets of some set. Corollary 5.42 ensures that these maps are always essentially constant with the empty set as essential value.

**Definition 5.37** (Doubly ergodic action with coefficients)**.** Let $\Gamma$ be a group and $(B, \Sigma, \vartheta)$ a Lebesgue space endowed with a measure class preserving $\Gamma$-action. The action of $\Gamma$ on $B$ is *doubly ergodic with coefficients* if any weak$*$-measurable $\Gamma$-equivariant map $B \times B \to E^*$ is essentially constant, where $E^*$ is the topological dual of any separable Banach space $E$ on which $\Gamma \to \mathrm{Isom}(E)$ acts by isometries.

**Remark 5.38.** Since we have an action of $\Gamma$ on $E$ by isometries, we also get an action of $\Gamma$ on $E^*$ via the adjoint.

**Lemma 5.39** ([BFS06, Section 2.a])**.** *Let* $\Gamma$ *act doubly ergodic with coefficients on* $B$*. Then for every measure preserving ergodic* $\Gamma$*-space* $(X, \mu)$*, the space* $B \times B \times X$ *is ergodic.*

**Corollary 5.40.** *If a group action is doubly ergodic with coefficients, then it is doubly ergodic in the usual sense.*

*Proof.* We choose a singleton for $X$ and apply Lemma 5.39. □

**Lemma 5.41** ([CFI16, Lemma 4.4]). *Let $C$ be a countable set with a $\Gamma$-action and $(B, \vartheta)$ a Lebesgue space with a measure class preserving $\Gamma$-action that is in addition doubly ergodic with coefficients. If $\psi\colon B \times B \to C$ or $\psi\colon B \to C$ is a $\Gamma$-equivariant measurable map, then $\psi$ is essentially constant.*

*Proof.* It satisfies to prove the assertion for $B \times B$. For $B$ we concatenate $\psi$ with the projection $p\colon B \times B \to B$ onto the first factor. By construction of the product measure, we have $p_*(\vartheta \times \vartheta) = \vartheta$ which finishes the proof in this case.

Since $\Gamma$ acts ergodically on $B \times B$, the same is true for the action on $C$ equipped with the pushforward measure $\mu := \psi_*(\beta \times \beta)$. Next, we choose representatives $(y_n)_{n\in\mathbb{N}}$ of the equivalence classes of ${}^{\operatorname{im}\psi}\!/\!_{\Gamma}$. Indeed, since $C$ is countable, we really only need countably many representatives. With this we have:

$$\operatorname{im}\psi = \bigsqcup_{n\in\mathbb{N}} \Gamma \cdot y_n$$

and thus

$$1 = \mu(\operatorname{im}\psi) = \sum_{n\in\mathbb{N}} \mu(\Gamma \cdot y_n).$$

However, each $\Gamma \cdot y_n$ is $\Gamma$-invariant and by ergodicity we obtain $\mu(\Gamma \cdot y_n) \in \{0, 1\}$. All in all we see that there exists exactly one $n \in \mathbb{N}$ such that $\mu(\Gamma \cdot y_n) = 1$. We define $D := \Gamma \cdot y_n$ for this $n$ and observe that $\Gamma$ acts transitively on this countable set.

First, we consider the case that $D$ is finite. In this case we find a finite index subgroup $\Gamma_0 \leq \Gamma$ which acts trivially on $D$. Furthermore, by the Lemma 5.36, we know that $\Gamma_0$ still acts ergodically on $D$. As previously, we can decompose $D$ via

$$1 = \mu(D) = \sum_{x\in D} \mu(\{x\}).$$

By the trivial action, each of these atomic spaces is $\Gamma_0$-invariant and hence for exactly one $x \in D$ we have $\mu(\{x\}) = 1$. Hence, $\psi$ is essentially constant with essential value $x$.

Lastly, we need to consider the case where $D$ is infinite. Indeed, we will show that this cannot happen. We consider the Bernoulli space $A := \{0, 1\}^D$ together with the standard Bernoulli measure $\lambda$ [c. f. Kle14, p. 29]. Since $\Gamma$ acts transitively on $D$, the action of $\Gamma$ on $A$ via $g\chi_S := \chi_{gS}$, where $S$ is any subset of $D$ and $g \in \Gamma$ and $\chi_S$ is an indicator function, is ergodic [c. f. Kle14, Example 20.26]. By Lemma 5.39, $B \times B \times A$ is ergodic. We can consider the following map

$$\begin{aligned} f\colon B \times B \times A &\to \mathbb{R}, \\ (x, y, \chi_S) &\mapsto \chi_S(\psi(x, y)). \end{aligned}$$

The map $f$ is $G$-invariant under the diagonal action and hence essentially constant. Denote this value by $y \in \{0, 1\}$. Then by Fubini's theorem we have that the map

$$g\colon B \times B \to \mathbb{R},$$
$$(x, y) \mapsto \int_A f(x, y, \chi_S)\, \mathrm{d}\mu(\chi_S)$$

exists for almost all $(x, y) \in B \times B$ and is also essentially constant with value $y$. Fixing a value $(x_0, y_0)$ for which this is true, we see that $\chi_S(x_0, y_0) = y$ for almost all $\chi_S \in A$. However, by construction of the standard Bernoulli measure on $A$, we have that

$$\mu(\{\chi_S \in A \mid \chi_S(d) = 1\}) = \mu(\{\chi_S \in A \mid \chi_S(d) = 0\}) = 1/2$$

for every $d \in D$. This is a contradiction to the previous statement for $d = \psi(x_0, y_0)$. Hence, $D$ cannot be infinite and we are done. □

**Corollary 5.42** ([CFI16, Cor. 4.5]). *Let $\mathcal{H}$ be the pocset of halfspaces of a connected, locally countable, finite-dimensional CAT(0) cube complex. Let $\mathrm{Pot}_f(\mathcal{H}) \subset \mathrm{Pot}(\mathcal{H})$ be the set containing only finite subsets of $\mathcal{H}$. Let $(B, \Sigma, \vartheta)$ be a Lebesgue space with a measure class preserving $\Gamma$-action that is in addition doubly ergodic with coefficients. If there exists a $\Gamma$-equivariant measurable map $B \times B \to \mathrm{Pot}_f(\mathcal{H})$ or if there exists a $\Gamma$-equivariant measurable map $B \to \mathrm{Pot}_f(\mathcal{H})$, whose image is not essentially $\varnothing$, then the $\Gamma$-action on $X$ is not essential.*

*Proof.* By Corollary 2.57, we know that $\mathcal{H}$ is countable. We choose an enumeration $\mathfrak{h}_n$ of $\mathcal{H}$ and for each $n \in \mathbb{N}$ we define

$$\mathcal{H}_n := \{\mathfrak{h}_1, \dots, \mathfrak{h}_n\}$$

We can then write

$$\mathrm{Pot}_f(\mathcal{H}) = \bigcup_{n\in\mathbb{N}} \{A \subset \mathcal{H}_n\}$$

showing that $\mathrm{Pot}_f(\mathcal{H})$ is countable. By Lemma 5.41, the map is essentially constant. Since $\Gamma$ is countable, we can apply Lemma 5.26 and find a finite orbit $\Gamma \cdot \hat{\mathfrak{h}}$. Then $X(\Gamma \cdot \hat{\mathfrak{h}})$ is finite. By Lemma 5.16, $\hat{\mathfrak{h}}$ is not essential and the group action is neither. □

### 5.4.2 Amenable group action

Now, we introduce amenable group actions. The details are rather technical. However, the only result we need is encoded in Theorem 5.51 and its corollary (Corollary 5.52). For the remainder of the thesis an understanding of these two results is sufficient and if need be, they can be thought of as an alternative definition of amenability.

In the rest of the section, we will use the following notation, unless stated otherwise:

- $\Gamma$ denotes a second countable, locally compact group,

- $E$ denotes a separable Banach space,
- $E_1^*$ denotes the unit ball in the (topological) dual of $E$, and
- $S$ denotes a standard Borel space.

We also assume that $\Gamma$ acts on $S$ preserving measure classes.

**Definition 5.43** (Borel field)**.** For each $s \in S$ consider a non-empty convex weak$*$-compact subspace $A_s \subset E_1^*$. Then $(A_s)_{s\in S}$ will be called a *Borel field of compact convex sets* if $\{(s, \lambda) \mid \lambda \in A_s\}$ is a Borel subset of $S \times E_1^*$.

**Definition 5.44** ((Left) cocycle)**.** Let $M$ be a topological group equipped with its Borel $\sigma$-algebra. Then a *(left) cocycle* is a measurable map

$$\alpha\colon \Gamma \times S \to M$$

such that $\alpha(gh, s) = \alpha(g, hs) \cdot \alpha(h, s)$ for all $g, h \in \Gamma$ and almost all $s \in S$.

**Remark 5.45.** Each element $T \in \text{Isom}(E)$ gives rise to a homeomorphism $T^*$ of $E_1^*$ via $(T^*\Phi)(x) := \Phi(Tx)$ for every $x \in E$. Thus, every cocycle $\alpha\colon S \times \Gamma \to \text{Isom}(E)$ gives rise to a cocycle $\alpha^*\colon S \times \Gamma \to \text{Homeo}(E^*)$ via $\alpha^*(g, s) = (\alpha(g, s)^{-1})^*$.

With this remark in place, we can turn to the final definition of this paragraph:

**Definition 5.46** (Amenable group action)**.**

- Let $\alpha\colon \Gamma \times S \to \text{Isom}(E)$ be a cocycle. A Borel field $(A_s)_{s\in S}$ is called $\alpha$*-invariant* if $\alpha^*(g, s)A_s = A_{gs}$ for each $g \in \Gamma$ and almost all $s \in S$.
- The $\Gamma$-action on $S$ is called *amenable* if for every separable Banach space $E$, every Borelian (left) cocycle $\alpha\colon \Gamma \times S \to \text{Isom}(E)$ and every $\alpha$-invariant Borel field $(A_s)_{s\in S}$, there exists a Borel map $\varphi\colon S \to E_1^*$ such that $\varphi(s) \in A_s$ for almost all $s$ and for each $g \in \Gamma$ we have $\alpha^*(g, s)\varphi(s) = \varphi(gs)$ almost everywhere.

### 5.4.3 Strong $\Gamma$-boundary

In this paragraph we will plug the previous two notions, namely *ergodicity with coefficients* and *amenability*, together to define *strong* $\Gamma$*-boundaries*. We will see that these boundaries exist for a broad enough spectrum of groups (see Remark 5.49) and that they are well-behaved under transitioning to finite index subgroups (Theorem 5.50).

**Definition 5.47** (Strong $\Gamma$-boundary)**.** Let $\Gamma$ be a second countable, locally compact group. A Lebesgue space $(B, \Sigma, \vartheta)$ is called a *strong* $\Gamma$*-boundary* if there is a group action of $\Gamma$ on $B$ by measurable transformations, and this action is:

1. amenable, and

2. doubly ergodic with coefficients.

**Example 5.48** (Furstenberg-Poisson boundary)**.** In his paper, Kaimanovich [Kai03, Theorem 3] showed that the Furstenberg-Poisson boundary of any spread out, non-degenerate, symmetric random walk on a locally compact, second countable group $\Gamma$ is a strong $\Gamma$-boundary. A measure $\mu$ on $\Gamma$ is called *spread out* (or *étalée*) if there exists a convolution power $\mu^{*n}$ which is not singular with respect to the Haar measure class on $\Gamma$, i. e. there is no partition $\Gamma = A \sqcup B$ such that $\mu^{*n}$ is zero on all measurable subsets of $A$ and the Haar measure class is zero on all measurable subsets of $B$. The measure is called *non-degenerate* if the minimal closed semigroup $S \subset \Gamma$ with $\mu(S) = 1$ is all of $\Gamma$. A measure is called *symmetric*, if $\mu = f_*\mu$, where $f$ is the continuous map given by inversion on $\Gamma$. A random walk is called spread out, non-degenerate or symmetric if the same is true for the measure $\mu$ of the associated transition probability. For details please refer to [Kai03, Section 3].

If $\Gamma$ is a free group, then the Furstenberg-Poisson boundary is isomorphic to the visual (and hence, to the Roller) boundary of the Cayley tree of $\Gamma$. More generally, Kaimanovich [Kai00] showed that if the first moment of the transition measure is finite and it is non-degenerate, then for every hyperbolic group the Gromov boundary is equivalent to the Furstenberg-Poisson boundary.

**Remark 5.49.** The previous example is the one most often encountered. Indeed, in his paper Kaimanovich [Kai03] used the Furstenberg-Poisson boundary to prove that every locally compact, second countable, $\sigma$-compact group (and in particular, every countable, discrete group) admits a strong $\Gamma$-boundary.

In the following, we will need that strong $\Gamma$-boundaries are well behaved under going over to finite index subgroups:

**Theorem 5.50** ([Mon01, Lemma 5.4.3], [Kai03, Proposition 3.2.4])**.** *Let $\Gamma$ be a countable, discrete group and $\Gamma_0 \leq \Gamma$ a finite index subgroup. If $(B, \Sigma, \vartheta)$ is a strong $\Gamma$-boundary, then it is also a strong $\Gamma_0$-boundary.*

**Theorem 5.51.** *Let $(B, \Sigma, \vartheta)$ be a strong $\Gamma$-boundary and $X$ a compact metric space with a continuous $\Gamma$-action. Then there exists a measurable map $\varphi \colon B \to \mathcal{P}(X)$ which is $\Gamma$-equivariant almost everywhere and where $\mathcal{P}(X)$ is the set of all regular probability measures on $X$.*

*Proof.* Let $C(X)$ be the space of continuous functions from $X$ to $\mathbb{R}$. This is a Banach space with respect to the supremum norm. By Lemma 4.14, it is also separable. Furthermore, there exists a group action of $\Gamma$ on $C(X)$ via $(gf)(x) := f(g^{-1}x)$ where $g \in \Gamma$, $x \in X$ and $f \in C(X)$. This action is clearly via isometries. Also for $\mu \in \mathcal{P}(X)$ we define $(g\mu)(A) := \mu(g^{-1}A)$ for every $g \in \Gamma$ and $A \in \Sigma$. Then the dual pairing established in the Riesz-Markow representation theorem (Theorem 4.11) yields

$$\langle gf, \mu \rangle = \langle f, g^{-1}\mu \rangle$$

or, in other words, $g^* = g^{-1}$. Next, consider

$$\begin{aligned} \alpha\colon \Gamma \times B &\to \mathrm{Isom}(C(X)), \\ (g, b) &\mapsto g. \end{aligned}$$

This is a left cocycle. Since $X$ is compact, we have $C(X) = C_0(X)$ (c. f. Definition 4.10). Thus, using the Riesz-Markow representation theorem (Theorem 4.11), we obtain that $C(X)^*$ is isomorphic to $M_s(X)$ the set of all regular signed measures on $X$. By Corollary 4.13, we know that $\mathcal{P}(X)$ is weak$*$-compact and contained in the unit ball of $M_s(X)$. Furthermore, $\mathcal{P}(X)$ is convex and non-empty (take any normalized Dirac measure) and we set $A_b \coloneqq \mathcal{P}(X)$ for all $b \in B$. This is in fact an $\alpha$-invariant Borel field. Since $B$ is a strong $\Gamma$-boundary, the $\Gamma$-action is amenable and we obtain a measurable map $\varphi\colon B \to C(X)^*_1$ such that $\varphi(b) \in A_b = \mathcal{P}(X)$, i. e. $\varphi\colon B \to \mathcal{P}(X)$ (which is still measurable). Lastly, we have

$$\begin{aligned} \varphi(gb) &= \alpha^*(g, b)\varphi(b) \\ &= \left(\alpha(g, b)^{-1}\right)^* \varphi(b) \\ &= \left(g^{-1}\right)^* \varphi(b) \\ &= g\varphi(b) \end{aligned}$$

for almost all $b \in B$ and every $g \in \Gamma$. □

**Corollary 5.52.** *Let $X$ be a finite-dimensional CAT(0) cube complex and $\bar{X}$ its Roller compactification. Let $\Gamma \to \mathrm{Aut}(X)$ be a group acting on $X$ and $B$ a strong $\Gamma$-boundary. Then there exists a measurable map $\varphi\colon B \to \mathcal{P}(\bar{X})$ which is $\Gamma$-equivariant almost everywhere and where $\mathcal{P}(\bar{X})$ is the set of regular probability measures on $\bar{X}$.*

*Additionally, $\mathcal{P}(\bar{X})$ inherits a probability measure via the pushforward from $B$ and the group action of $\Gamma$ on $\mathcal{P}(\bar{X})$ is doubly ergodic with coefficients with respect to this measure.*

*Proof.* Corollary 3.8 establishes that $\bar{X}$ is a compact metrizable space. Furthermore, the $\Gamma$-action on $X$ extends to a $\Gamma$-action on $\bar{X}$(c. f. Theorem 5.3). Thus, all conditions for Theorem 5.51 are satisfied and we get the desired map $\varphi\colon B \to \mathcal{P}(\bar{X})$. □

# 6 The boundary map

Finally, we can construct the boundary map as in Theorem 6.20. This process will take the whole chapter. The actual construction will be achieved in Section 6.1. However, we will need the additional assumption that $H_\mu$ is empty for almost all $\mu \in \mathcal{P}(\bar{X})$ (see Lemma 6.1). Afterwards, we will prove that the boundary map (if it exists) will only take values in the Roller boundary $\partial X$ instead of in the whole Roller compactification $\bar{X}$. Then we will have to prove that $H_\mu$ is indeed empty for almost all $\mu$. Section 6.2 will deal with the rather simple case $0 < |H_\mu| < \infty$. Section 6.3 will construct the tools to deal with the case $|H_\mu| = \infty$. This case is the most involved and we will have to split it into further subcases in order to deal with it. The detailed strategies for all the cases can be found at the start of each paragraph. Section 6.4 contains the statement (Theorem 6.20) and the proof of our main theorem. Additionally, it contains a slight generalization in the form of Corollary 6.21.

## 6.1 The construction of the boundary map

In this section we will construct the actual boundary map first assuming that the set of balanced halfspaces $H_\mu$ (see Section 4.2) is empty for almost every regular probability measure $\mu$ (with respect to the measure mentioned in Corollary 5.52). Then we will prove that its image lies in the Roller boundary $\partial X$.

**Lemma 6.1.** *Let $X$ be a finite-dimensional, locally countable CAT(0) cube complex and $\Gamma$ a group with an action $\Gamma \to \operatorname{Aut}(X)$ that is essential and non-elementary. Furthermore, let $(B, \Sigma, \vartheta)$ be a strong $\Gamma$-boundary. If $H_\mu = \varnothing$ for almost all $\mu \in \mathcal{P}(\bar{X})$ with respect to the pushforward measure from $B$, then there exists a measurable map $\varphi\colon B \to \bar{X}$ which is $\Gamma$-equivariant almost everywhere.*

*Proof.* By Corollary 5.52, we have a measurable map $\psi\colon B \to \mathcal{P}(\bar{X})$ which is $\Gamma$-equivariant almost everywhere. Hence, we only need to find a map from $\mathcal{P}(\bar{X})$ to $\bar{X}$. We will first prove that if $H_\mu = \varnothing$, then $H_\mu^+$ is an ultrafilter. Indeed, since $\mathcal{H} = H_\mu^+ \sqcup H_\mu \sqcup H_\mu^-$ and $(H_\mu^+)^* = H_\mu^-$ (c. f. Lemma 4.16), we have the choice condition. For the consistency condition we only need to know that $\mathfrak{h} \subset \mathfrak{k}$ implies $\mathcal{C}(\mathfrak{h}) \subset \mathcal{C}(\mathfrak{k})$ and hence $\mu(\mathcal{C}(\mathfrak{h})) \leq \mu(\mathcal{C}(\mathfrak{k}))$.

By assumption, the set $\mathcal{E} := \{\mu \in \mathcal{P}(\bar{X}) \mid H_\mu = \varnothing\}$ has full measure. Since $\psi$ is only well-defined up to a null set, we can concatenate it with the map

$$\begin{aligned} \xi\colon \mathcal{E} &\to \bar{X}, \\ \mu &\mapsto H_\mu^+. \end{aligned}$$

By Lemma 4.22 applied to the interval$(1/2, 1]$, this map is measurable and $\Gamma$-equivariant almost everywhere.

All in all we have that $\xi \circ \psi$ is our desired map $\varphi$. $\square$

We will now prove that any $\varphi$ as above only takes values in $\partial X$. This will be accomplished with the next lemma:

**Lemma 6.2** ([CFI16, Lemma 4.11]). *Let $X$ be a finite-dimensional, locally countable CAT(0) cube complex. Let $\Gamma$ be a discrete, countable group with an essential and non-elementary action $\Gamma \to \mathrm{Aut}(X)$, $(B, \nu)$ a Lebesgue space on which $\Gamma$ acts doubly ergodic with coefficients. If $\varphi\colon B \to \bar{X}$ is a measurable map which is $\Gamma$-equivariant almost everywhere, then $\varphi$ takes values in the non-terminating ultrafilters of $X$.*

*Proof.* Consider the map

$$\begin{aligned} B &\to \mathbb{N} \cup \{\infty\}, \\ x &\mapsto |\tau(\varphi(x))|, \end{aligned}$$

which is measurable (Lemma 4.21) and $\Gamma$-invariant ($\tau$ was defined in Definition 3.32). By ergodicity, it is essentially constant with essential value $M$. If we show that $M = 0$, then the image of $\varphi$ contains only non-terminating ultrafilters (up to measure 0) and we are done.

For this purpose, let us consider the following map

$$\begin{aligned} B \times B &\to \mathbb{N} \cup \{\infty\}, \\ (x, y) &\mapsto |\tau(\mathcal{H}(\varphi(x), \varphi(y)))|. \end{aligned}$$

It is measurable as it is a composition of measurable maps (consider Lemma 4.23) and $\Gamma$-invariant. Again we obtain an essential value $N$. By Remark 3.28, we have that $N < \infty$ and hence $\tau(\mathcal{H}(\varphi(x), \varphi(y)))$ takes values in $\mathrm{Pot}_f(\mathcal{H})$. By Corollary 5.42, this would mean that the action of $\Gamma$ is inessential, unless $N = 0$.

Lastly, we will show that this is incompatible with the case $M > 0$. Contrarily, assume $M > 0$, then we could find $x_0 \in B$ such that $|\tau(\varphi(x_0))| > 0$ and a set $B_0 \subset B$ of full measure such that $\tau(\mathcal{H}(\varphi(x_0), \varphi(y))) = \varnothing$ for all $y \in B_0$. By Lemma 3.35, for all $\mathfrak{h} \in \tau(\varphi(x_0))$, we have $\mathfrak{h} \in \varphi(y)$.

However, by Lemma 5.26, $B_0$ contains a $\Gamma$-orbit, i. e. there exists a $y \in B_0$ such that $g\mathfrak{h} \in \varphi(y)$ for every $g \in \Gamma$. Now, $\Gamma$ acts non-elementary and essential. By Theorem 5.19, we find $g \in \Gamma$ such that $g\mathfrak{h} \subset \mathfrak{h}^*$, but by consistency, we would then have $\mathfrak{h}^* \in \varphi(y)$ which is a contradiction to the choice condition of ultrafilters. □

## 6.2 The case $0 < |H_\mu| < \infty$

So far we have seen that if $H_\mu$ is empty for almost all $\mu$, we find our desired map with all the necessary properties. We will now prove that if $|H_\mu|$ is finite, then it is already 0. So after this section we will be left with the case that $|H_\mu|$ is infinite. The following two lemmas capture the precise ideas. The main argument in the proof of Lemma 6.4 is Corollary 5.42, which gives us a contradiction to the essentiality of the $\Gamma$-action.

**Lemma 6.3.** *The map*

$$\mathcal{P}(\bar{X}) \to \mathbb{N} \cup \{\infty\},$$
$$\mu \mapsto |H_\mu|$$

*is essentially constant.*

*Proof.* The map is $\Gamma$-invariant and measurable as a concatenation of measurable maps (c.f. Section 4.3). Since the group action is ergodic with regard to the pushforward measure on $\mathcal{P}(\bar{X})$ from $B$, we see that the map is essentially constant. $\square$

**Lemma 6.4.** *If $|H_\mu|$ is essentially constant and not infinite, then $H_\mu$ is empty for almost all $\mu \in \mathcal{P}(\bar{X})$.*

*Proof.* We consider the map

$$\mathcal{P}(\bar{X}) \to \mathrm{Pot}_f(\mathcal{H}),$$
$$\mu \mapsto H_\mu.$$

This map is measurable and $\Gamma$-equivariant. Hence, by Corollary 5.42, we know that its image has to be essentially $\varnothing$ in order for our $\Gamma$-action to be essential. $\square$

All in all we see that if we can show that $|H_\mu|$ is finite for almost all $\mu \in \mathcal{P}(\bar{X})$, we are done.

## 6.3 The case $|H_\mu| = \infty$

This is the most involved case. The following two paragraphs will contain all the technical details in order to exclude it. We will divide this case into two subcases, namely $|H_\mu \cap H_\nu| = 0$ and $|H_\mu \cap H_\nu| = \infty$ for almost all $\mu$ and $\nu$. The strategy is always to find a contradiction to the essentiality of the group action, the non-elementarity of the group action or to the fact that the complex is finite-dimensional. The central result of Paragraph 6.3.1 is Proposition 6.6. The central results of Paragraph 6.3.2 are Proposition 6.13 and Proposition 6.14. Lemma 6.15 and Lemma 6.16 are also used in the main proof, but their complete content is given there. For the two subcases to make sense, we need the following lemma:

**Lemma 6.5.** *The map*

$$\mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) \to \mathbb{N} \cup \{\infty\},$$
$$(\mu, \nu) \mapsto |H_\mu \cap H_\nu|$$

*is essentially constant.*

*Proof.* This map is again measurable and $\Gamma$-invariant (c. f. Section 4.3) and hence essentially constant by the doubly ergodic action of $\Gamma$ on $\mathcal{P}(\bar{X})$. $\square$

### 6.3.1 The case $|H_\mu \cap H_\nu| = 0$

Here we will prove that $|H_\mu| = \infty$ and $|H_\mu \cap H_\nu| = 0$ for almost all $\mu, \nu \in \mathcal{P}(\bar{X})$ cannot happen in our setting. More precisely, we will see that $X$ cannot be finite-dimensional. The precise statement is captured in the next proposition. It is the only result of this paragraph that will be used in the main proof of the theorem. The remainder of this paragraph is only necessary to understand the proof of this proposition.

Unless noted otherwise, $X$ is a connected, locally countable, finite-dimensional CAT(0) cube complex and $\Gamma$ a discrete, countable group.

**Proposition 6.6** ([CFI16, Proposition 4.10])**.** *If for almost all $\mu, \nu \in \mathcal{P}(\bar{X})$ we have all of the following:*

- $|H_\mu| = |H_\nu| = \infty$,
- $H_\mu \cap H_\nu = \varnothing$, *and*
- $\tau(H_\mu \cap H_\nu^+) = \varnothing$,

*then $X$ contains cubes of arbitrarily large dimension.*

We will prove this proposition at the end of the paragraph. Our strategy is to construct a directed graph having measures as vertices. The following lemma will then give a condition under which two measures are joined by a (directed) edge. Afterwards, we can use a graph theoretic result (Lemma 6.12) showing that we find (finite) sets of pairwise transverse halfspaces with arbitrarily many elements. This leads to the desired cubes in Proposition 6.6.

**Lemma 6.7.** *Let $\mu, \nu \in \mathcal{P}(\bar{X})$ be two regular probability measures such that $H_\mu \cap H_\nu = \varnothing$ and such that there exists an infinite descending chain $\mathfrak{h}_n \in H_\mu^+$ and an infinite descending chain $\mathfrak{k}_m \in H_\nu^+$. Then there exists $C \in \mathbb{N}$ such that we have a decomposition*

$$\mathbb{N}_C \subset N_1 \sqcup N_j,$$

*where $j \in \{2, 3, 4\}$ and we have:*

$$\begin{aligned} \mathbb{N}_C &:= \{(n,m) \in \mathbb{N}^2 \mid n, m \geq C\}, \\ N_1 &:= \{(n,m) \in \mathbb{N}^2 \mid \mathfrak{h}_n \pitchfork \mathfrak{k}_m\}, \\ N_2 &:= \{(n,m) \in \mathbb{N}^2 \mid \mathfrak{h}_n^* \subset \mathfrak{k}_m\}, \\ N_3 &:= \{(n,m) \in \mathbb{N}^2 \mid \mathfrak{h}_n \subset \mathfrak{k}_m\}, \quad \textit{and} \\ N_4 &:= \{(n,m) \in \mathbb{N}^2 \mid \mathfrak{h}_n \supset \mathfrak{k}_m\}. \end{aligned}$$

*Proof.* There is the following decomposition:

$$\mathbb{N} \times \mathbb{N} = N_1 \sqcup N_2 \sqcup N_3 \sqcup N_4.$$

We will consider different cases, depending on which $N_j$ are not empty. If at most one of the $N_j$ $(j = 2, 3, 4)$ is not empty, we are done. So we deal with the cases where at least two of the sets are non-empty.

**Case** $N_2, N_3 \neq \varnothing$**:** We can take $(n_3, m_3) \in N_3$ and $(n, m) \in N_2$ and define

$$m' := \min\{m, m_3\}.$$

Then we have the following two inclusions

$$\mathfrak{h}_{n_3} \subset \mathfrak{k}_{m_3} \subset \mathfrak{k}_{m'} \quad \text{and} \quad \mathfrak{h}_n^* \subset \mathfrak{k}_m \subset \mathfrak{k}_{m'}.$$

If we have $n \geq n_3$, we have $\mathfrak{h}_n \subset \mathfrak{h}_{n_3} \subset \mathfrak{k}_{m'}$, which is impossible by the second inclusion above. Hence, fixing $m_3$ we define

$$A := \min\{n_3 \in \mathbb{N} \mid (n_3, m_3) \in N_3\}$$

and see that for all $n \geq A$ and any $m \in \mathbb{N}$ we have $(n, m) \notin \mathbb{N}_2$ or, in other words, we have

$$\{(n, m) \in N_2 \mid n \geq A_3\} = \varnothing.$$

If $N_4$ is empty, this is already sufficient to show that $\mathbb{N}_A = N_1 \sqcup N_3$. The case, when all three are not empty, will be handled below.

**Case** $N_3, N_4 \neq \varnothing$**:** Again we take $(n_3, m_3) \in N_3$ and $(n, m) \in N_4$. If $n \geq n_3$, we have

$$\mathfrak{k}_n \subset \mathfrak{h}_n \subset \mathfrak{h}_{n_3} \subset \mathfrak{k}_{m_3}.$$

This would imply that $\mathfrak{h}_n \in H_\mu \cap H_\nu$, since it is enclosed in two halfspaces which lie in $H_\nu$. However, we have $H_\mu \cap H_\nu = \varnothing$. Hence, we define as before

$$B := \min\{n_3 \in \mathbb{N} \mid (n_3, m_3) \in N_3\}$$

and see that if $n \geq B$, then for any $m \in \mathbb{N}$ we have $(n, m) \notin N_4$. If $N_2$ is empty, this implies $\mathbb{N}_B = N_1 \sqcup N_3$.

**Case** $N_2, N_4 \neq \varnothing$**:** This case is analogous to the first except that we flip the roles of $n$ and $m$. Hence, we define a constant

$$D := \min\{m_4 \mid (n_4, m_4) \in N_4\},$$

where $n_4 \in \mathbb{N}$ was chosen such that the above set is not empty. If $m \geq B$ and $n \in \mathbb{N}$ is arbitrary, we have $(n, m) \notin N_2$. As above, if $N_3$ is empty, this is sufficient to show that $\mathbb{N}_D = N_1 \sqcup N_4$.

**Case** $N_j \neq \varnothing \ \forall j \in \{2,3,4\}$**:** In this case we can use the constants defined above and set $C := \max\{A, B, D\}$. Then if $(n,m) \in \mathbb{N}_C \setminus N_1$, we have $m \geq B$ and hence $(n,m) \notin N_2$ and $n \geq D$ meaning $(n,m) \notin N_4$. All in all this leads to $\mathbb{N}_C = N_1 \sqcup N_3$.

□

**Lemma 6.8** ([CFI16, Lemma 4.13])**.** *Let $(\mu_i)_{i\in\mathbb{N}_0}$ be a sequence of probability measures in $\mathcal{P}(\bar{X})$ such that $H_{\mu_i} \cap H_{\mu_j} = \varnothing$ whenever $i \neq j$ and such that for each $i > 0$ there exists an infinite descending chain $\mathfrak{h}_n^i \in H_{\mu_0}^+ \cap H_{\mu_i}$. Then, (up to switching $i$ and $j$) any pair of measures $\mu_i$ and $\mu_j$ satisfies the following condition:*

*There exists $C(i,j) \in \mathbb{N}$ such that for every $n \geq C(i,j)$ there is an $M_n \geq C(i,j)$ such that if $m \geq M_n$, then $\hat{\mathfrak{h}}_m^j \pitchfork \hat{\mathfrak{h}}_n^i$.*

*Proof.* We fix two measures and call them $\mu$ and $\nu$. Let $\mathfrak{h}_n \in H_{\mu_0}^+ \cap H_\mu$ and $\mathfrak{k}_m \in H_{\mu_0}^+ \cap H_\nu$ be the corresponding infinite descending sequences.

By Lemma 6.7, we have $C \in \mathbb{N}$, $j \in \{2,3,4\}$ and a decomposition

$$\mathbb{N}_C = N_1 \sqcup N_j.$$

We will consider three cases:

**Case** $\mathbb{N}_C = N_1 \sqcup N_3$**:** If $N_3 \neq \varnothing$, we take $(n_0, m_0) \in N_3$ and define

$$\begin{aligned} M = M(n_0) &:= \max\{m \in \mathbb{N} \mid (n_0, m) \in N_3\} \\ &= \max\{m \in \mathbb{N} \mid \mathfrak{h}_{n_0} \subset \mathfrak{k}_m \subset \mathfrak{k}_{m_0}\}. \end{aligned}$$

$M$ is well-defined since the maximum is taken over a non-empty set (by choice of $(n_0, m_0)$) and the set is finite since two nested halfspaces contain only finitely many halfspaces in between (c. f. Lemma 2.58).

We see that if $m > M$, we have $(n_0, m) \in N_1$, which is what we wanted.

**Case** $\mathbb{N}_C = N_1 \sqcup N_2$**:** This case works completely analogous, with $\mathfrak{h}_{n_0}$ replaced by $\mathfrak{h}_{n_0}^*$.

**Case** $\mathbb{N}_C = N_1 \sqcup N_4$**:** If we switch the roles of $\mathfrak{h}_n$ and $\mathfrak{k}_m$ the proof goes as above and we are done.

□

### Preliminaries on directed graphs

We did not dedicate a whole section to directed graphs as they are only necessary to understand the above mentioned proposition. For the main proof, in depth knowledge of this paragraph is not necessary. We will only need the technical result that every complete directed graph has a subgraph with the same vertex set that is strictly upper triangular (Lemma 6.12).

**Definition 6.9.** A *directed graph* $\mathcal{G}(V, E)$ consists of two sets $V$ and $E$, its *vertex set* and *edge set* respectively and of two maps $s, t \colon E \to V$ associating to each edge its source and target vertex respectively. In our case there are no parallel edges allowed (antiparallel edges may occur) and we will not allow loops. This allows us to think of $E \subset V \times V$ and we will prefer writing $\overline{vw} \in E$ for two vertices $v, w \in V$. This has the further advantage of making the maps $s$ and $t$ obsolete. We will call $\mathcal{G}$ *complete* if it is complete as an undirected graph, i. e. each pair of vertices is joined by a single (undirected) edge. For each $v \in V$ we will denote by $o(v)$ the number of *outgoing edges* and by $i(v)$ the number of *incoming edges*. A complete directed finite graph is *strictly upper triangular* if there exists an enumeration $V = \{v_1, \dots, v_D\}$ such that for all $j = 1, \dots, D$ we have

$$\begin{aligned} o(v_j) &= D - j \quad \text{and} \\ i(v_j) &= j - 1. \end{aligned}$$

**Remark 6.10.** The name strictly upper triangular stems from the fact that the transition matrix for the graph with the given enumeration of the vertices is strictly upper triangular.

**Lemma 6.11** ([CFI16, Lemma A.6]). *If $\mathcal{G} := \mathcal{G}(V, E)$ is a complete directed finite graph with $|V| = D$, then there exists $v \in V$ such that $o(v) \geq \frac{D-1}{2}$.*

*Proof.* Since $\mathcal{G}$ is complete we have $o(v) + i(v) \geq D - 1$ for every $v \in V$ and summing over all vertices we obtain

$$\sum_{v \in V} o(v) + i(v) \geq D(D-1).$$

Since all edges that start somewhere have to end somewhere, we have:

$$\sum_{v \in V} o(v) = \sum_{v \in V} i(v),$$

leading to

$$\sum_{v \in V} o(v) \geq \frac{D(D-1)}{2}.$$

If $o(v)$ were smaller than $\frac{D-1}{2}$ for each $v \in V$, we would have that

$$\sum_{v \in V} o(v) < \frac{D(D-1)}{2},$$

which is a contradiction. Hence, there exists at least one $v \in V$ such that $o(v) \geq \frac{D-1}{2}$. $\square$

**Lemma 6.12** ([CFI16, Lemma A.8]). *Let $\mathcal{G} = \mathcal{G}(V, E)$ be a (not necessarily finite) complete, directed graph and $D \in \mathbb{N}$. If $|V| \geq 5^D$, there exist $D$ vertices $v_1, \dots, v_D$ and a subset $E_D \subset E$ such that $\mathcal{G}(\{v_1, \dots, v_D\}, E_D)$ is complete, directed and strictly upper triangular.*

*Proof.* We will prove this by induction, but need a slightly stronger statement. We will prove:

Let $N \in \mathbb{N}$ and $|V| \geq 5^N$. Then for each $D \leq N$ there exist $v_1, \ldots, v_D \in V$ and a subset $E_D \subset E$ such that $\mathcal{G}(\{v_1, \ldots, v_D\}, E_D)$ is complete, directed and strictly upper triangular. Furthermore, for the set

$$V_D := \{v \in V \setminus \{v_1, \ldots, v_D\} \mid \overline{v_i v} \in E \; \forall i = 1, \ldots, D\}$$

we have $|V_D| \geq 5^{N-D}$.

Observe that it is sufficient to prove this statement for finite graphs. Indeed, for infinite graphs, we can always consider a finite subgraph with sufficiently many vertices. So we will reduce to the finite case.

**Base:** $D = 1$: By Lemma 6.11 we can find a $v_1 \in V$ such that

$$o(v_1) \geq \frac{|V| - 1}{2} \geq \frac{V}{5} \geq 5^{n-1}.$$

Then $\mathcal{G}(\{v_1\}, \varnothing)$ is clearly complete and upper triangular. Furthermore, we have that

$$V_1 = \{v \in V \mid \overline{v_1 v} \in E\},$$

since $\mathcal{G}$ does not contain loops. However, then we have $|V_1| = o(v_1) \geq 5^{N-1}$ and we are done.

**Inductive step:** $D \to D + 1$: By the induction hypothesis we find $\{v_1, \ldots, v_D\}$ and a subset $E_D \subset E$ such that $\mathcal{G}(\{v_1, \ldots, v_D\}, E_D)$ is complete and strictly upper triangular and $|V_D| \geq 5^{N-D}$. We consider the complete graph induced by $\mathcal{G}$ on the set $V_D \neq \varnothing$. Again by Lemma 6.11, we find a vertex $v_{D+1} \in V_D$ such that

$$o(v_{D+1}) \geq \frac{|V_D| - 1}{2} \geq \frac{|V_D|}{5} \geq 5^{N-(D+1)}.$$

By construction, this vertex is connected via an incoming edge to each of the $v_i$. If we set

$$E_{D+1} = E_D \cup \{\overline{v_i v_{D+1}} \mid \forall i = 1, \ldots, D\},$$

then $\mathcal{G}(\{v_1, \ldots, v_{D+1}\}, E_{D+1})$ is still complete and strictly upper triangular by construction. Additionally, we have

$$V_{D+1} = \{v \in V_D \mid \overline{v_{D+1} v} \in E\}$$

and thus $|V_{D+1}| = o(v_{D+1}) \geq 5^{N-(D+1)}$ which completes the induction.

$\square$

*Proof of Proposition 6.6.* Since $H_\mu \cap H_\nu^+$ has no minimal elements for almost all $\mu, \nu$, we can find a sequence $(\mu_i)_{i\in I}$ such that $H_{\mu_0}^+ \cap H_{\mu_i}$ contains an infinite descending chain, which we denote by $(\mathfrak{h}_n^i)_{n\in\mathbb{N}}$. Thus, we can apply Lemma 6.8 and find $C(i,j) \in \mathbb{N}$ such that for all $n \geq C(i,j)$ there is an $M \geq c(i,j)$ such that if $m > M_n$, we have $\hat{\mathfrak{h}}_n^i \pitchfork \hat{\mathfrak{h}}_m^j$ (after possibly switching $i$ and $j$).

Using this, we can construct a graph $\mathcal{G} := \mathcal{G}(V, E)$ with $V := \{\mu_i \mid i \in I\}$ and an edge from $\mu_i$ to $\mu_j$ if and only if the above mentioned $C(i,j)$ exists. With this, $\mathcal{G}$ becomes a directed graph and since $C(i,j)$ exists for $(i,j)$ or $(j,i)$ it is complete. Hence, we can apply Lemma 6.12 using any $D \in \mathbb{N}$ and find (after relabeling) $\mu_1, \ldots, \mu_D \in V$ such that we find a subset of edges $E_D$ such that $\mathcal{G}(V, E_D)$ becomes strictly upper triangular. This implies that for each $1 \leq i < j \leq D$ there exists $C(i,j)$ without needing to switch $i$ and $j$. We set

$$C := \max\{C(i,j) \mid 1 \leq i < j \leq D\} \quad \text{and}$$
$$M := \max\{M_C(i,j) \mid 1 \leq i < j \leq D\}.$$

Fixing $n = C$, for each $m \geq M$ we have that $\hat{\mathfrak{h}}_n^i \pitchfork \hat{\mathfrak{h}}_m^j$ for each $1 \leq i < j \leq D$. Fixing $m$, this leads to a set of $D$ transverse halfspaces. Since they intersect pairwise, by Theorem 2.53, the common intersection is not empty. An element in this intersection is in a cube, which has all these hyperplanes as midcubes. Hence, this cube has at least dimension $D$. Since $D$ was chosen arbitrarily, $X$ contains cubes of arbitrary dimension. $\square$

### 6.3.2 The case $|H_\mu \cap H_\nu| = \infty$

Here we will prove that $|H_\mu| = \infty$ and $|H_\mu \cap H_\nu| = \infty$ cannot happen in our case. The two main results of the paragraph are stated in the following two propositions.

Unless noted otherwise, $\Gamma$ is assumed to be a discrete, countable group and $X$ a connected, locally countable, finite-dimensional CAT(0) cube complex.

**Proposition 6.13** ([CFI16, Corollary 4.21])**.** *Let $X$ be irreducible. Assume that for almost every $\mu \in \mathcal{P}(\bar{X})$, there are no pairs of strongly separated halfspaces in $H_\mu$. If $H_\mu \cap H_\nu \neq \varnothing$ for almost every pair $(\mu, \nu)$, then the $\Gamma$-action is non-essential or $\Gamma$ has a fixed point in the visual boundary.*

We will find a contradiction to the Flipping Lemma (Theorem 5.19). We will see that if there are no strongly separated halfspaces in every$H_\mu$, then we can find two halfspaces $\mathfrak{h}, \mathfrak{k} \in \mathcal{H}$ such that for almost every $\mu$ and every $\mathfrak{l} \in H_\mu$ we have

$$\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{k}. \tag{6.1}$$

For the precise statement please see Lemma 6.18. Of course, this property prevents the flipping of $\mathfrak{l}$. The main work lies in establishing Equation (6.1).

**Proposition 6.14** ([Fer16, Corollary 3.32])**.** *Assume we have an essential and non-elementary action of $\Gamma$ on $X$, and $\Gamma_0 \leq \Gamma$ of finite index. If $\mathcal{H}' \subset \mathcal{H}$ is a non-empty sub-pocset such that*

- $\mathcal{H}'$ *is* $\Gamma_0$*-invariant, and*
- *if* $\mathfrak{h}, \mathfrak{h}' \in \mathcal{H}'$ *with* $\mathfrak{h} \subset \mathfrak{h}'$ *and* $\mathfrak{k} \in \mathcal{H}$ *such that* $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{h}'$*, then* $\mathfrak{k} \in \mathcal{H}'$*,*

*then either* $\mathcal{H}' = \mathcal{H}$ *or* $X = X' \times X''$ *and* $\mathcal{H}'$ *is the halfspace structure for* $X'$*.*

We will prove this with the help of strongly separated halfspaces. We will first consider the irreducible case and show that then $\mathcal{H} = \mathcal{H}'$ using an infinite sequence of strongly separated halfspaces in $\mathcal{H}'$. In the reducible case, we will apply the same reasoning on each irreducible factor separately. This leads to the product decomposition

For both propositions, the Double Skewering Lemma (Lemma 5.18) will be vital.

### Towards the proof of Proposition 6.13

As stated above, we will try to find a contradiction to the Flipping Lemma (Theorem 5.19). This is done by flanking any element $\mathfrak{l}$ in $H_\mu$ by two halfspaces thus preventing it from flipping.

**Lemma 6.15** ([CFI16, Lemma 4.18]). *Let* $X = X_1 \times \cdots \times X_n$ *be the decomposition of* $X$ *into irreducible factors and* $\mathcal{H} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_n$ *the associated decomposition of halfspaces. If* $H_\mu \cap \mathcal{H}_i$ *contains strongly separated halfspaces considered as a subset of* $\mathcal{H}_i$ *(c. f. Definition 2.54) for every* $i$*, then* $H_\mu^+$ *satisfies the descending chain condition.*

*Proof.* Let $(\mathfrak{h}_i)_{i \in \mathbb{N}}$ be a descending chain in $H_\mu^+$. Since all $\mathfrak{h}_i$ are parallel, they all lie in a common $\mathcal{H}_k$. Without loss of generality we assume $k = 1$. Now, take $\mathfrak{h}, \mathfrak{k} \in H_\mu \cap \mathcal{H}_1$ with $\mathfrak{h} \subset \mathfrak{k}$ strongly separated in $\mathcal{H}_1$ and define

$$P(\mathfrak{h}) := \{\mathfrak{l} \in H_\mu^+ \cap \mathcal{H}_1 \mid \mathfrak{h} \parallel \mathfrak{l}\}.$$

Because of the strong separation, each $\mathfrak{l} \in \mathcal{H}_1$ is parallel to $\mathfrak{h}$ or $\mathfrak{k}$. Hence, we have

$$H_\mu^+ \cap \mathcal{H}_1 = P(\mathfrak{h}) \cup P(\mathfrak{k}).$$

We return to our descending chain. By going over to a subsequence, we can assume that all halfspaces lie in either $P(\mathfrak{h})$ or $P(\mathfrak{k})$. Without loss of generality, we choose $P(\mathfrak{h})$. The case $\mathfrak{h}_n \subset \mathfrak{h}$ and $\mathfrak{h}_n \subset \mathfrak{h}^*$ cannot happen, as $\mu(\mathfrak{h}) = \mu(\mathfrak{h}^*) < \mu(\mathfrak{h}_n)$. In the case that $\mathfrak{h} \subset \mathfrak{h}_n$, we know by Lemma 2.58 that there are only finitely many halfspaces between the two. Hence, the chain must terminate. The same argument holds in the case $\mathfrak{h}^* \subset \mathfrak{h}_n$. □

**Lemma 6.16.** *Let* $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{l}$ *be three halfspaces in* $\mathcal{H}$ *and* $g \in \Gamma$*. Then*

1. *if* $(\mathfrak{h}, \mathfrak{k})$ *or* $(\mathfrak{k}, \mathfrak{l})$ *are strongly separated, then the same is true for* $(\mathfrak{h}, \mathfrak{l})$*, and*
2. $(\mathfrak{h}, \mathfrak{k})$ *is strongly separated if and only if the same is true for* $(g\mathfrak{h}, g\mathfrak{k})$*.*

*Proof.*

1. Assume that $\mathfrak{m}$ is transverse to both $\mathfrak{h}$ and $\mathfrak{l}$. Then we would have

$$\begin{aligned}
\mathfrak{m} \cap \mathfrak{k} &\supset \mathfrak{m} \cap \mathfrak{h} \neq \varnothing, \\
\mathfrak{m}^* \cap \mathfrak{k} &\supset \mathfrak{m}^* \cap \mathfrak{h} \neq \varnothing, \\
\mathfrak{m} \cap \mathfrak{k}^* &\supset \mathfrak{m} \cap \mathfrak{l}^* \neq \varnothing \quad \text{and} \\
\mathfrak{m}^* \cap \mathfrak{k}^* &\supset \mathfrak{m}^* \cap \mathfrak{l}^* \neq \varnothing.
\end{aligned}$$

   Hence, $\mathfrak{m}$ is transverse to $\mathfrak{k}$. This contradicts the assumption that $(\mathfrak{h}, \mathfrak{k})$ or $(\mathfrak{k}, \mathfrak{l})$ are strongly separated.

2. If $(\mathfrak{h}, \mathfrak{k})$ are strongly separated and we have any $\mathfrak{m} \in \mathcal{H}$, then $g^{-1}\mathfrak{m} \in \mathcal{H}$ and it is parallel to either $\mathfrak{h}$ or $\mathfrak{k}$. Hence, $\mathfrak{m}$ is parallel to either $g\mathfrak{h}$ or $g\mathfrak{k}$. The opposite direction is analogous.

□

**Lemma 6.17** ([CFI16, Lemma 4.19]). *Let $X$ be irreducible and $\Gamma \to \mathrm{Aut}(X)$ an essential and non-elementary group action. For every measure $\mu$ either $\hat{H}_\mu$ contains a pair of strongly separated hyperplanes or there exists a pair $\mathfrak{h} \in H_\mu^-$ and $\mathfrak{k} \in H_\mu^+$ of halfspaces such that the hyperplanes $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{k}}$ are strongly separated and for every $\mathfrak{l} \in H_\mu$ we have $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{h} \subset \mathfrak{l}^* \subset \mathfrak{k}$*

*Proof.* Suppose that $H_\mu$ does not contain strongly separated halfspaces. By Proposition 5.20, we find two strongly separated halfspaces $\mathfrak{k}_i$ such that $\mathfrak{k}_1 \subset \mathfrak{l} \subset \mathfrak{k}_2$. By Lemma 5.18, we find $g \in \Gamma$ such that

$$g\mathfrak{k}_1 \subset g\mathfrak{k}_2 \subset \mathfrak{k}_1 \subset \mathfrak{l} \subset \mathfrak{k}_2 \subset g^{-1}\mathfrak{k}_1 \subset g^{-1}\mathfrak{k}_2.$$

The arcs in the diagram connect strongly separated halfspaces which were identified using Lemma 6.16.

We would like to show that $g\mathfrak{k}_1 \in H_\mu^-$ and $g^{-1}\mathfrak{k}_2 \in H_\mu^+$. If neither $\mathfrak{k}_i$ is in $H_\mu$, we are done (since $\mathfrak{l}$ is in $H_\mu$). Suppose that $\mathfrak{k}_2$ is in $H_\mu$, then $\mathfrak{k}_1 \in H_\mu^-$, because $H_\mu$ contains no strongly separated halfspaces. Thus, by the additivity of the measure, we also have $g\mathfrak{k}_1 \in H_\mu^-$. Additionally, $g^{-1}\mathfrak{k}_2 \in H_\mu^+$ again since it is strongly separated from $\mathfrak{k}_2$. The case if $\mathfrak{k}_1 \in H_\mu$ can be proven similarly.

There is one additional step necessary. We define $\mathfrak{h} := g^2\mathfrak{k}_1 \in H_\mu^-$ and $\mathfrak{k} := g^{-2}\mathfrak{k}_2 \in H_\mu^+$, which are strongly separated by Lemma 6.16. Furthermore, we set $\mathfrak{k}_o := g\mathfrak{k}_1 \in H_\mu^-$ and $\mathfrak{k}_3 := g^{-1}\mathfrak{k}_2$. Then we have the following sequence:

$$\mathfrak{h} \subset \mathfrak{k}_0 \subset \mathfrak{k}_3 \subset \mathfrak{k},$$

where the pairs $(\mathfrak{h}, \mathfrak{k}_0)$, $(\mathfrak{k}_o, \mathfrak{k}_3)$ and $(\mathfrak{h}, \mathfrak{k})$ are strongly separated.

If we take any other $\mathfrak{l}' \in H_\mu$, then $\mathfrak{k} \not\subset \mathfrak{l}'$ and $\mathfrak{k} \not\subset \mathfrak{l}'^*$, because of the measure. Additionally, $\mathfrak{l}' \not\pitchfork \mathfrak{k}$, since it would be parallel to $\mathfrak{k}_3$ and as before we would have $\mathfrak{k}_3 \not\subset \mathfrak{l}'$ and $\mathfrak{k}_3 \not\subset \mathfrak{l}'^*$. Hence, $\mathfrak{l}'$ or $\mathfrak{l}'^*$ contains $\mathfrak{k}_3$ and thus $\mathfrak{k}$. All in all this shows that either $\mathfrak{l}'$ or $\mathfrak{l}'^*$ contains $\mathfrak{k}$. Up to renaming, we can assume that $\mathfrak{l}' \subset \mathfrak{k}$. A similar argument applied to $\mathfrak{h}$ and $\mathfrak{l}'$, shows that $\mathfrak{h} \subset \mathfrak{l}'$ and hence $\mathfrak{h} \subset \mathfrak{l}' \subset \mathfrak{k}$. □

**Lemma 6.18** ([CFI16, Lemma 4.20]). *Let $X$ be irreducible. Let $\mu_i \in \mathcal{P}(\bar{X})$ be a family of measures such that $\hat{H}_{\mu_i}$ does not contain strongly separated hyperplanes for all $i$. Assume that $H_{\mu_{i_0}} \cap H_{\mu_i} \neq \varnothing$ for all $i$ and a fixed $i_0$. Then there exists a pair of halfspaces $\mathfrak{h} \subset \mathfrak{k}$ such that $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{k}}$ are strongly separated and for every $\mathfrak{l} \in H_{\mu_j}$, we have $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{k}$ or $\mathfrak{h} \subset \mathfrak{l}^* \subset \mathfrak{k}$.*

*Proof.* We fix $\mu_0 := \mu_{i_0}$ and apply Lemma 6.17 to find two strongly separated halfspaces $\mathfrak{h}_2 \subset \mathfrak{h}_3$. By Lemma 5.18, we find a $g \in \Gamma$ such that $g\mathfrak{h}_3 \subset \mathfrak{h}_2 \subset \mathfrak{h}_3$. We set $\mathfrak{h}_0 := g^2\mathfrak{h}_2$, $\mathfrak{h}_1 := g\mathfrak{k}_2$, $\mathfrak{h}_4 := g^{-1}\mathfrak{h}_3$ and $\mathfrak{h}_5 := g^{-2}\mathfrak{h}_3$. Then we have the sequence

$$\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \mathfrak{h}_5.$$

Since $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are strongly separated, by Lemma 6.16, the above halfspaces are pairwise strongly separated.

We would like to show that $\mathfrak{h}_0 \subset \mathfrak{l} \subset \mathfrak{h}_5$ for each $\mathfrak{l} \in H_{\mu_i}$ and every $i$. This is in fact already true for every $\mathfrak{l} \in H_{\mu_i} \cap H_{\mu_0}$. By assumption, we know that the intersection is not empty and we fix a $\mathfrak{l}_i \in H_{\mu_i} \cap H_{\mu_0}$ for every $i$.

Now, for every $\mathfrak{l} \in H_{\mu_i}$, we see that $\mathfrak{l}$ can be parallel to at most one $\mathfrak{h}_i$, since they are pairwise strongly separated. We will consider the following cases:

**Case 1:** If $\mathfrak{l}$ is transverse to any $\mathfrak{h}_i$, where $1 \leq i \leq 4$, we are done, because in this case $\mathfrak{l}$ is parallel to $\mathfrak{h}_0$ and $\mathfrak{h}_5$.

**Case 2:** Assume that $\mathfrak{l}$ is transverse to $\mathfrak{h}_0$. We will see that this is impossible. In this case it is parallel to $\mathfrak{h}_1$ and we consider the following two subcases:

- We could have the chain $\mathfrak{l}' \subset \mathfrak{h}_1 \subset \mathfrak{h}_2 \subset \mathfrak{l}_i$, where $\mathfrak{l}'$ is either $\mathfrak{l}$ or its complement. In either case, we have $\mathfrak{l}', \mathfrak{l}_i \in H_{\mu_i}$ and thus the same is also true for the two enclosed halfspaces. However, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are strongly separated and $H_{\mu_i}$ does not contain pairs of strongly separated halfspace, so this cannot happen.
- The only other possibility is where $\mathfrak{h}_0 \subset \mathfrak{h}_1 \subset \mathfrak{l}'$, but then $\mathfrak{h}_0$ cannot be transverse to $\mathfrak{l}$.

Thus, $\mathfrak{l}$ can never be transverse to $\mathfrak{h}_0$.

**Case 3:** If $\mathfrak{l}$ is transverse to $\mathfrak{h}_5$ we find a contradiction as in Case 2.

**Case 4:** If $\mathfrak{l}$ is parallel to each $\mathfrak{h}_i$, then $\mathfrak{l}' \not\subset \mathfrak{h}_o$, since otherwise $H_{\mu_i}$ would again contain pairs of strongly separated halfspaces. By the same argument, $\mathfrak{h}_5 \not\subset \mathfrak{l}'$, where $\mathfrak{l}'$ is defined as above. As before we obtain $\mathfrak{h}_0 \subset \mathfrak{l} \subset \mathfrak{h}_5$ or $\mathfrak{h}_0 \subset \mathfrak{l}^* \subset \mathfrak{h}_5$.

□

*Proof of Proposition 6.13.* By the construction of the product measure, we find a measure $\mu_0$ and a subset $B_0 \subset \mathcal{P}(\bar{X})$ of full measure such that for all $\nu \in B_0$, we have

$$H_{\mu_0} \cap H_\nu \neq \varnothing.$$

By Lemma 6.18, we find two halfspaces $\mathfrak{h} \subset \mathfrak{k}$ such that for each hyperplane $\hat{\mathfrak{l}} \in \hat{\mathcal{H}}_\nu$ with $\nu \in B_0$, we that (after a possible involution of $\mathfrak{l}$) $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{k}$.

Since $\Gamma$ is countable, we can apply Lemma 5.26 and find a $\nu_0 \in B_0$ such that $\Gamma\nu_0 \subset B_0$.

We fix an $\mathfrak{l} \in H_{\nu_0}$. We know that $g\nu_0 \in B_0$. Hence, we have $\mathfrak{h} \subset g\mathfrak{l} \subset \mathfrak{k}$ for all $g \in \Gamma$. However, then $\mathfrak{l}^* \not\subset g\mathfrak{l}$ for all $g \in \Gamma$, because $\mathfrak{k}$ cannot contain both $\mathfrak{l}$ and $\mathfrak{l}^*$. The Flipping Lemma (Theorem 5.19) then finishes the proof. □

### Towards the proof of Proposition 6.14

**Lemma 6.19** ([Fer16, Lemma 3.31])**.** *Suppose that $X$ is irreducible with a non-elementary and essential action of the group $\Gamma$. Any non-empty sub-pocset $\mathcal{H}' \subset \mathcal{H}$ satisfying the following properties must be equal to $\mathcal{H}$:*

- *$\mathcal{H}'$ is $\Gamma$-invariant, and*
- *if $\mathfrak{h}, \mathfrak{h}' \in \mathcal{H}'$ with $\mathfrak{h} \subset \mathfrak{h}'$ and $\mathfrak{k} \in \mathcal{H}$ such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{h}'$, then $\mathfrak{k} \in \mathcal{H}'$.*

*Proof.* Let $\mathfrak{h} \in \mathcal{H}'$. By Proposition 5.20, we find $\mathfrak{k}, \mathfrak{l} \in \mathcal{H}$ such that $\mathfrak{k}$ and $\mathfrak{l}$ are strongly separated and we have

$$\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{l}.$$

By the Double Skewering Lemma (Lemma 5.18), we find $g \in \Gamma$ such that $g\mathfrak{l} \subsetneq \mathfrak{k}$. By Lemma 6.16, $g\mathfrak{h}$ and $\mathfrak{l}$ are. Then, $g^2\mathfrak{h}$ and $g\mathfrak{l}$ are strongly separated, too. And lastly, we obtain that $\mathfrak{h}' := g^2\mathfrak{h}$ and $\mathfrak{h}$ are strongly separated. Additionally, we note that $\mathfrak{h}' \in \mathcal{H}'$, since $\mathcal{H}'$ is $\Gamma$-invariant.

Now, let $\mathfrak{m} \in \mathcal{H}$ be any halfspace. Then $\mathfrak{m}$ is transverse to at most one of the two halfspaces $\mathfrak{h}$ and $\mathfrak{h}'$. Without loss of generality, we assume that $\mathfrak{m}$ is parallel to $\mathfrak{h}$. We have four possible cases: $\mathfrak{m} \subset \mathfrak{h}$, $\mathfrak{m}^* \subset \mathfrak{h}$, $\mathfrak{h} \subset \mathfrak{m}$ or $\mathfrak{h} \subset \mathfrak{m}^*$. However, in all of these cases, we can use the Double Skewering Lemma (Lemma 5.18) to enclose $\mathfrak{m}$ or $\mathfrak{m}^*$ between two elements of $\mathcal{H}'$. Hence, by assumption, we have $\mathfrak{m} \in \mathcal{H}'$ or $\mathfrak{m}^* \in \mathcal{H}'$. Since $\mathcal{H}'$ is closed under involution, this finishes the proof. □

*Proof of Proposition 6.14.* If $X$ is irreducible, we can apply Lemma 6.19 and find that $\mathcal{H} = \mathcal{H}'$. Otherwise, let

$$\mathcal{H} = \mathcal{H}_1 \sqcup \cdots \sqcup \mathcal{H}_n$$

be the decomposition into sets of pairwise transverse halfspaces corresponding to the irreducible decomposition. By Lemma 5.17, we can assume without loss of generality that $\Gamma_0$ preserves this decomposition. Then we can apply Lemma 6.19 on each of the sets $\mathcal{H}' \cap \mathcal{H}_i$ considered as subsets of the pocsets $\mathcal{H}_i$. Then the intersection is either empty, or we have $\mathcal{H}' \cap \mathcal{H}_i = \mathcal{H}_i$. With this we see that $X$ decomposes as the desired product. $\square$

## 6.4 The main theorem

Finally, we are in a position to prove our main theorem (Theorem 6.20). A slight generalization takes the form of Corollary 6.21. The proofs will use all the technical details of the previous sections. Additionally, we will make heavy use of the measurability results of Section 4.3, results for essential and non-elementary group actions on CAT(0) cube complexes in Section 5.2 and results concerning strong $\Gamma$-boundaries in the Sections 5.3 and 5.4.

**Theorem 6.20** ([CFI16, Theorem 4.1]). *Let $\Gamma$ be a discrete, countable group acting on a connected, locally countable, finite-dimensional CAT(0) cube complex $X$ via automorphisms. Assume the action is essential and non-elementary. If $(B, \Sigma, \vartheta)$ is a strong $\Gamma$-boundary, there exists a measurable map*

$$\varphi\colon B \to \partial X$$

*which is $\Gamma$-equivariant almost everywhere and which takes values in the non-terminating ultrafilters in $\partial X$.*

*Proof.* By Lemma 6.3, $|H_\mu|$ is essentially constant. If $H_\mu$ is finite for almost all $\mu$, by Lemma 6.4, $H_\mu$ is empty for almost all $\mu$. Hence, Lemma 6.1 and Lemma 6.2 lead to our desired map.

The only thing left to prove is that $H_\mu$ cannot be infinite. Contrarily, assume that it is. For every $\mu, \nu$ we consider the set $H_\mu \cap H_\nu$. By Lemma 6.5, their cardinality must be essentially constant and we consider the case that the sets are infinite for almost all $\mu, \nu \in \mathcal{P}(\bar{X})$. We define the set

$$\mathcal{E} \coloneqq \{(\mu, \nu) \in \mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) \mid H_\mu = H_\nu\}.$$

If $f\colon \mathcal{P}(\bar{X}) \to \mathrm{Pot}(\mathcal{H})$ is the map such that $\mu \mapsto H_\mu$, then $\mathcal{E} = (f \times f)^{-1}(\Delta)$, where $\Delta \subset \mathrm{Pot}(\mathcal{H})^2$ is the diagonal. The set $\mathrm{Pot}(\mathcal{H})$ is Hausdorff with regard to the cylinder topology. Hence, $\Delta$ is closed and measurable. The map $f \times f$ is also measurable and we obtain that the same is true for $\mathcal{E}$. Furthermore, $\mathcal{E}$ is $\Gamma$-invariant. By the doubly ergodic action of $\Gamma$ on $\mathcal{P}(\bar{X})$, it has either measure 0 or measure 1. Let us consider the two cases:

**Case** $\vartheta(\mathcal{E}) = 1$**:** Since $H_\mu = H_\nu$ for almost all $\mu$ and $\nu$, by the definition of the product measure, we can find $\mu_0 \in \mathcal{P}(\bar{X})$ and a full measure subset $M \subset \mathcal{P}(X)$ such that $H_\nu = H_{\mu_0}$ for all $\nu \in M$. By Lemma 5.26, we find $\nu \in M$ such that

$$H_{g\nu} = gH_\nu = H_{g\nu} = H_{\mu_0} = H_\nu$$

for every $g \in \Gamma$. Hence, $H_\nu$ is $\Gamma$-invariant. The set $H_\nu$ is additionally closed under involution. By Lemma 4.16, we can apply Proposition 6.14. Hence, either $\mathcal{H} = H_\mu$ or $X = X' \times X''$, where $X'$ has $H_\mu$ as its pocset of halfspaces. In both cases, the irreducible factors of $X$ would contain an interval, since $X' = X(H_\mu)$ is an interval by Lemma 4.17. However, by Corollary 5.24, this contradicts the fact that the $\Gamma$-action is non-elementary.

**Case** $\vartheta(\mathcal{E}) = 0$**:** In this case we have $H_\mu \neq H_\nu$ for almost all $\mu$ and $\nu$. We decompose $X$ into its irreducible factors

$$X = X_1 \times \cdots \times X_n$$

and $\mathcal{H}$ into the associated subsets of pairwise transverse halfspaces

$$\mathcal{H} = \mathcal{H}_1 \sqcup \cdots \sqcup H_n.$$

Furthermore, we denote by $\Gamma_0 \leq \Gamma$ the finite index subgroup respecting this decomposition. Then $\Gamma_0$ acts still non-elementary and essential on $X$ (Lemma 5.17). We define the set

$$\mathcal{S}_i := \{(\mathfrak{h}, \mathfrak{k}) \in \mathcal{H}_i \times \mathcal{H}_i \mid \mathfrak{h} \text{ and } \mathfrak{k} \text{ are strongly separated in } \mathcal{H}_i\}.$$

By Lemma 6.16, we see that this set is $\Gamma_0$-invariant and we can consider the following map

$$\begin{aligned} \mathcal{P}(\bar{X}) &\to \mathbb{N} \cup \{\infty\}, \\ \mu &\mapsto |(H_\mu \times H_\mu) \cap \mathcal{S}_i|, \end{aligned}$$

which is, by the above observation, $\Gamma_0$-invariant and also measurable (c. f. Lemma 4.24 together with Lemma 4.19). Hence, it is essentially constant (Lemma 5.36). We have two cases depending on the essential values $N_i$.

**Case 1:** The value $N_i > 0$ for all $i$, i. e. there are strongly separated hyperplanes in all $H_\mu \cap \mathcal{H}_i$. Hence, we can use Lemma 6.15 and see that $H_\mu^+$ satisfies the descending chain condition. This implies that the sets $H_\mu^+ \cap H_\nu$ contain terminal elements whenever the intersection is not empty. Furthermore, we know by Lemma 4.18 that they each contain at most finitely many terminal elements. However, then

$$\begin{aligned} \mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) &\to \mathrm{Pot}_f(\mathcal{H}), \\ (\mu, \nu) &\mapsto \tau([H_\mu^+ \cap H_\nu] \cup [H_\nu^+ \cap H_\mu]) \end{aligned}$$

is $\Gamma$-equivariant and measurable (c. f. Lemma 4.26) and Corollary 5.42 assures $H_\mu^+ \cap H_\nu = \varnothing$ for almost all $\mu$ and $\nu$. However, $H_\nu \neq H_\mu$ for almost all $\mu$ and $\nu$. Hence, there exists $\mathfrak{h} \in H_\nu \setminus H_\mu$ and thus $\mathfrak{h} \in H_\mu^\pm$. Without loss of generality, we can assume that it lies in $H_\mu^+$ and we see that $H_\mu^+ \cap H_\nu \neq \varnothing$ for almost all $\mu$ and $\nu$, which is a contradiction.

**Case 2:** The value $N_i = 0$ for at least one $i$. Without loss of generality, we can assume $N_1 = 0$. The space $X_1$ is irreducible, $H_{p_{1*}\mu} = H_\mu \cap \mathcal{H}_1$ does not contain pairs of strongly separated halfspaces. Furthermore, $H_{p_{1*}\mu} \cap H_{p_{1*}\nu} = \varnothing$ for almost all $\mu, \nu$. Hence, we can apply Proposition 6.13 and find a contradiction to the fact that $\Gamma_0$ is both essential and non-elementary.

We see that $H_\mu \cap H_\nu$ cannot be infinite. Thus, we have the map

$$\begin{aligned} \mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) &\to \mathrm{Pot}_f(\mathcal{H}), \\ (\mu, \nu) &\mapsto H_\mu \cap H_\nu, \end{aligned}$$

which is $\Gamma$-invariant and measurable (c. f. Section 4.3) and takes values in the finite subsets of $\mathcal{H}$. With the help of Corollary 5.42, we see that this implies that $H_\mu \cap H_\nu$ must be empty for almost all $\mu$ and $\nu$.

In this case we consider the map

$$\begin{aligned} \mathcal{P}(\bar{X}) \times \mathcal{P}(\bar{X}) &\to \mathbb{N} \cup \{\infty\}, \\ (\mu, \nu) &\mapsto |\tau([H_\mu \cap H_\nu^+] \cup [H_\nu \cap H_\mu^+])|, \end{aligned}$$

where $\tau$ is the map assigning the set of terminal elements to any subset of $\mathcal{H}$ (c. f. Definition 3.32). This map is again measurable (c. f. Lemmas 4.20 and 4.26) and $\Gamma$-invariant. Hence, it is essentially constant. Furthermore, by Lemma 4.18, the map takes only finite values. However, by Corollary 5.42, we have that $[H_\mu \cap H_\nu^+] \cup [H_\nu \cap H_\mu^+]$ contains no terminal elements for almost all $\mu$ and $\nu$. This allows us to apply Proposition 6.6, which leads to the impossibility of the case $H_\mu \cap H_\nu = \varnothing$ and hence of $|H_\mu| = \infty$. □

Before finishing this treatise, we would like to drop the essentiality condition on $\Gamma$. This is indeed possible. The only thing we lose is that we cannot be sure that the image lies in the non-terminating ultrafilters. However, it will still lie in $\partial X$. The precise statement is as follows:

**Corollary 6.21.** *Let $\Gamma \to \mathrm{Aut}(X)$ be a discrete, countable group acting on a connected, locally countable, finite-dimensional CAT(0) cube complex $X$. Assume that the action is non-elementary and denote by $Y$ the essential core of $X$. Then there exists a measurable map*

$$\varphi\colon B \to \partial Y \subseteq \partial X$$

*which is $\Gamma$-equivariant almost everywhere.*

*Proof.* Since there is no finite orbit of $\Gamma$ in the visual boundary, we also have no fixed point there. Hence, we can apply Proposition 5.11 which yields that the essential core $Y$ is not empty. As a convex subcomplex, we have $\partial_\sphericalangle Y \subset \partial_\sphericalangle X$ and we see that $\Gamma$ acts non-elementary on $Y$, too. By definition, the action $\Gamma$ on $Y$ is also essential. Hence, we can apply Theorem 6.20 and find the map

$$\varphi\colon B \to \partial Y.$$

With the help of Lemma 5.15, we can embed $\partial Y$ into $\partial X$ in a natural way that is compatible with the $\Gamma$-action. $\square$

## Acknowledgment

First and foremost I would like to thank Professor Anna Wienhard for making it possible for me to work on this topic and for advising this thesis. I would also like to thank her for her patience during my work on this thesis. Next, I would like to express my deep-felt thanks to Doctor Valentina Disarlo. She never hesitated to discuss any of my mathematical problems and the remarks she gave me regarding the text of my thesis were priceless. Thank you very much for the many hours you have invested and the great advice you gave me along the way. I would like to thank Professor Maria Beatrice Pozzetti and Professor Talia Fernós who were kind enough to discuss my open questions concerning the paper [CFI16]. Both of them provided me with very good pointers and references to continue my research. Naturally, I would also like to thank my second corrector for agreeing to reserve the time necessary to work through and evaluate this (not so short) treatise. Thanks is also due to Doctor Yves de Cornulier, Doctor Mikael de la Salle and Professor Uri Bader for a clarifying (online) discussion.[1] Of course, this thesis would contain far more typos than are currently present without my proofreaders. I would like to thank Sven Grützmacher, Jakob Schnell and Stefan Zentarra for the hard work and the great remarks. My family, especially my parents, Jochen and Susanne, and my brother Marc, supported me throughout all of my studies, both financially and emotionally. I find it extraordinary that they made it possible for me to spend my time at university as freely as I did. I was backed in whatever decision I took. Thank you so much for giving me this space and trusting me to pursue my own goals. Last, but by no means least, I would like to thank my girlfriend Sabrina Baier. She endured many discussions and explanations concerning this thesis and more often than not she provided me with very useful insight. Thank you for always lending me an open ear and for your very deep and even more precise feedback.

[1] `https://mathoverflow.net/questions/286522/conull-subspace-containing-orbit-of-an-ergodically-acting-group`

## References


[BFS06] Uri Bader, Alex Furman, and Ali Shaker. "Superrigidity, Weyl groups, and actions on the Circle". In: (May 2006). arXiv: 0605276 [math].

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999. ISBN: 3-540-64324-9. DOI: 10.1007/978-3-662-12494-9.

[Bro+09] Jacek Brodzki et al. "Property A and CAT(0) cube complexes". In: *J. Funct. Anal.* 256.5 (2009), pp. 1408–1431. ISSN: 0022-1236.

[CFI16] Indira Chatterji, Talia Fernós, and Alessandra Iozzi. "The median class and superrigidity of actions on CAT$(0)$ cube complexes". In: *J. Topol.* 9.2 (2016). With an appendix by Pierre-Emmanuel Caprace, pp. 349–400. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtu025.

[Con90] John B. Conway. *A course in functional analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399. ISBN: 0-387-97245-5.

[CS11] Pierre-Emmanuel Caprace and Michah Sageev. "Rank rigidity for CAT(0) cube complexes". In: *Geom. Funct. Anal.* 21.4 (2011), pp. 851–891. ISSN: 1016-443X.

[Eng89] Ryszard Engelking. *General topology*. Second. Vol. 6. Sigma Series in Pure Mathematics. Translated from the Polish by the author. Heldermann Verlag, Berlin, 1989, pp. viii+529. ISBN: 3-88538-006-4.

[Fer16] Talia Fernós. "The Furstenberg–Poisson boundary and CAT(0) cube complexes". In: *Ergodic Theory and Dynamical Systems* (2016), pp. 1–44. DOI: 10.1017/etds.2016.124.

[Gro87] Mikhail Gromov. "Hyperbolic groups". In: *Essays in group theory*. Vol. 8. Math. Sci. Res. Inst. Publ. Springer, New York, 1987, pp. 75–263. DOI: 10.1007/978-1-4613-9586-7_3.

[Hat01] Allen Hatcher. *Algebraic Topology*. 2001.

[HR31] Heinz Hopf and Willi Rinow. "Über den Begriff der vollständigen differentialgeometrischen Fläche". In: *Commentarii Mathematici Helvetici* 3 (1931), pp. 209–225.

[Jän94] Klaus Jänich. *Topologie*. Fourth. Springer-Lehrbuch. [Springer Textbook]. With a chapter by Theodor Bröcker. Springer-Verlag, Berlin, 1994, pp. x+239. ISBN: 3-540-57471-9.

[Kai00] Vadim A. Kaimanovich. "The Poisson formula for groups with hyperbolic properties". In: *Ann. of Math. (2)* 152.3 (2000), pp. 659–692. ISSN: 0003-486X.

[Kai03] Vadim A. Kaimanovich. “Double ergodicity of the Poisson boundary and applications to bounded cohomology”. In: *Geom. Funct. Anal.* 13.4 (2003), pp. 852–861. ISSN: 1016-443X.

[Kle14] Achim Klenke. *Probability theory.* Second. Universitext. Springer, London, 2014, pp. xii+638. ISBN: 978-1-4471-5360-3; 978-1-4471-5361-0.

[Kle93] Felix Klein. “Vergleichende Betrachtungen über neuere geometrische Forschungen”. In: *Math. Ann.* 43.1 (1893), pp. 63–100. ISSN: 0025-5831.

[Lea13] Ian J. Leary. “A metric Kan-Thurston theorem”. In: *J. Topol.* 6.1 (2013), pp. 251–284. ISSN: 1753-8416. DOI: 10.1112/jtopol/jts035.

[Mac62] George W. Mackey. “Point realizations of transformation groups”. In: *Illinois J. Math.* 6 (1962), pp. 327–335. ISSN: 0019-2082.

[Mar91] Grigori A. Margulis. *Discrete subgroups of semisimple Lie groups.* Vol. 17. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1991, pp. x+388. ISBN: 3-540-12179-X.

[Mon01] Nicolas Monod. *Continuous bounded cohomology of locally compact groups.* Thesis (Dr.sc.math.)–Eidgenoessische Technische Hochschule Zuerich (Switzerland). ProQuest LLC, Ann Arbor, MI, 2001, p. 236.

[MS04] Nicolas Monod and Yehuda Shalom. “Cocycle superrigidity and bounded cohomology for negatively curved spaces”. In: *J. Differential Geom.* 67.3 (2004), pp. 395–455. ISSN: 0022-040X.

[Rol12] Pascal Rolli. *Notes on CAT(0) Cube Complexes.* 2012.

[Rol99] Martin Roller. *Poc Sets, Median Algebras and Group Actions.* July 1999. arXiv: 1607.07747.

[Rud87] Walter Rudin. *Real and complex analysis.* Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416. ISBN: 0-07-054234-1.

[Rud91] Walter Rudin. *Functional analysis.* Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424. ISBN: 0-07-054236-8.

[Sag12] Michah Sageev. *CAT(0) cube complexes and groups (lecture notes).* 2012.

[Sag95] Michah Sageev. “Ends of group pairs and non-positively curved cube complexes”. In: *Proc. London Math. Soc. (3)* 71.3 (1995), pp. 585–617. ISSN: 0024-6115. DOI: 10.1112/plms/s3-71.3.585.

[ST76] Isadore M. Singer and John A. Thorpe. *Lecture notes on elementary topology and geometry.* Reprint of the 1967 edition, Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976, pp. viii+232.

[Thu82] William P. Thurston. “Three-dimensional manifolds, Kleinian groups and hyperbolic geometry”. In: *Bull. Amer. Math. Soc. (N.S.)* 6.3 (1982), pp. 357–381. ISSN: 0273-0979.

[Zim78] Robert J. Zimmer. “Amenable ergodic group actions and an application to Poisson boundaries of random walks”. In: *J. Functional Analysis* 27.3 (1978), pp. 350–372.

[Zim84] Robert J. Zimmer. *Ergodic theory and semisimple groups.* Vol. 81. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984, pp. x+209. ISBN: 3-7643-3184-4.

## Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe und die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht habe. Ferner versichere ich, dass die übermittelte elektronische Version in Inhalt und Wortlaut der gedruckten Fassung entspricht. Ich bin einverstanden, dass die elektronische Fassung anhand einer Plagiatssoftware auf Plagiate überprüft wird.

Heidelberg, den 13.12.2017 ______________________________