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Fixed points of Hamiltonian diffeomorphisms on closed symplectically aspherical manifolds

Irene Elisabeth Seifert

Matr. 383505

Supervisors: Prof. Dr. Albers
Dr. Jungsoo Kang

Mathematical Institute, University of Münster, Germany

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CHAPTER 1

Introduction

In this work the following theorem will be proved:

Theorem 1.1: *If a Hamiltonian diffeomorphism Φ on a closed symplectically aspherical manifold (M, ω) has only finitely many fixed points, then it has at least two of different symplectic action.*

The existence of at least one fixed point follows from the following version of the Arnold conjecture:

Theorem 1.2 (Arnold conjecture, proved by Floer in [3])**:** *Let $H : S^1 \times M \to \mathbb{R}$ be a non-degenerate Hamiltonian function on a closed symplectically aspherical manifold (M, ω), and let $\Phi_H^1 : M \to M$ be the time-one-map of the corresponding flow. Then*

$$\#\{\textit{fixed points of } \Phi_H^1\} \geq \sum_{k=0}^{dimM} b_k(M),$$

where the $b_k := dim\ H_k(M, \mathbb{Q})$ are the Betti numbers of M.

A degenerate Hamiltonian can be approximated by non-gegenerate ones. However, in the approximation process fixed points can collide, so that in general we get only one fixed point.

Floer proved the Arnold conjecture for closed monotone symplectic manifolds (this includes symplectically aspherical ones) in [3] in a way that many others adapted for more general cases: He established Floer homology $HF_*(H; M)$ for non-degenerate Hamiltonians H and then showed that the resulting homology theory does not depend on H nor on ω and that it coincides with singular homology. The methods involved are the study of J-holomorphic curves and moduli spaces solutions of a PDE called the Floer equation, combining a geometrical and a variational approach.

The existence of two fixed points of different action appears as a side product in an article by Matthias Schwarz (see [13]). He proves a much more elaborate statement, using Floer homology and the action spectrum.

The goal of this work is to give a somewhat shorter proof for the existence of at least two fixed points of a Hamiltonian diffeomorphism which does not need Floer homology: We start with a J-holomorphic sphere and deform it to a Floer cylinder of positive energy. This Floer cylinder is a gradient flow line connecting different critical points of the symplectic action functional $\mathcal{A}_H$ on the space of contractible loops, and so there have to be at least two such critical points, which correspond to the fixed points of the Hamiltonian diffeomorphism.

Chapter 2 explains the basic notions which are needed to understand the statement. It also covers certain facts about Floer cylinders that will appear in the other chapters. In chapter 3, the main theorem is stated and its proof is outlined. In chapters 4, 5, 6 and 7 we go into the technical details of the proof.

I would like to thank my advisors for all their ideas, instructions and explanations, my fellow students for proofreading and discussions and my husband, family and friends for existing.

CHAPTER 2

Preliminaries

In this first chapter we will define the basic notion that are needed to understand the main theorem. After that we discuss some important properties of Floer cylinders.

2.1 Hamiltonian diffeomorphisms

Hamiltonian diffeomorphisms are diffeomorphisms on symplectic manifolds which arise as flows of certain vector fields.

Definition 2.1: A **time-independent Hamiltonian function** on a manifold M is a smooth map $H : M \to \mathbb{R}$. A 1-periodic **time-dependent Hamiltonian function** on M is a smooth map $H : S^1 \times M \to \mathbb{R}$.

Now consider a closed symplectic manifold (M, ω), that is a compact smooth manifold M without boundary equipped with a closed non-degenerate 2-form ω.

Definition 2.2: Let $H : M \to \mathbb{R}$ be a Hamiltonian function. The equation

$$\omega(X_H, \cdot) = -dH$$

defines a vector field X_H on M which is called the **Hamiltonian vector field**. If the function $H : \mathbb{R} \times M \to \mathbb{R}$ is time-dependent, we get a time-dependent Hamiltonian vector field X_{H_t} by the equation

$$\omega(X_{H_t}, \cdot) = -dH_t.$$

Here $H_t = H(t, \cdot)$ and we sometimes also write $X_{H_t} = X_H(t, \cdot)$.

Recall that the flow of a (possibly time-dependent) vector field X on a manifold M is a $(-\varepsilon, \varepsilon)$-family of diffeomorphisms

$$\Phi^t_X : M \to M$$

given by $\frac{d}{dt}\Phi^t_X = X(t, \Phi^t_X)$ and $\Phi^0_X = \mathrm{id}_M$ for some ε. If one can find an $\mathbb{R}$-family with these properties, then the flow is said to be defined for all times.

Definition 2.3: Let H be a Hamiltonian function on M (time-dependent or not) and X_H the corresponding Hamiltonian vector field. Its flow is called the **Hamiltonian flow**. A diffeomorphism $\Phi : M \to M$ is called **Hamiltonian** if it arises as $\Phi^t_H := \Phi^t_{X_H}$ for some $t \in \mathbb{R}$ and some (possibly time-dependent) Hamiltonian function H on M.

Since M is closed, the Hamiltonian flow is defined for all time.

Remark 2.4: Every Hamiltonian diffeomorphism is automatically a symplectomorphism, which means that it preserves the symplectic structure:

$$\begin{aligned}\frac{d}{dt}(\Phi_H^t)^*\omega &= (\Phi_H^t)^*\mathcal{L}_{X_{H_t}}\omega \\ &= (\Phi_H^t)^*(d\iota_{X_{H_t}}\omega + \iota_{X_{H_t}}d\omega) \\ &= (\Phi_H^t)^*(-dH_t + \iota_{X_{H_t}}0) = 0\end{aligned}$$

for all t. With $\Phi_H^0 = id_M$ it follows that $(\Phi_H^t)^*\omega = \omega$.

Lemma 2.5: *For every Hamiltonian diffeomorphism $\Phi : M \to M$ there is a periodic time-dependent Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ such that Φ is the time-one-map of the corresponding flow, that is $\Phi = \Phi_H^1$.*

Proof. Assume that $\Phi = \Phi_K^{t_0}$ for some $t_0 \in \mathbb{R}_{\geq 0}$ and some function $K : S^1 \times M \to \mathbb{R}$. Choose a monotone smooth function $\rho : [0,1] \to [0, t_0]$ with $\rho \equiv 0$ in a neighbourhood of 0 and $\rho \equiv t_0$ in a neighbourhood of 1 and define

$$\begin{aligned}&H : S^1 \times M \to \mathbb{R} \\ &H(t,x) := \rho'(t) \cdot K(\rho(t), x).\end{aligned}$$

This is well-defined since $\rho' \equiv 0$ near 0 and near 1. Now one can compute the following:

$$\frac{d}{dt}\Phi_K^{\rho(t)} = \rho'(t)X_{K_{\rho(t)}}\left(\Phi_K^{\rho(t)}\right) = X_{H_t}\left(\Phi_K^{\rho(t)}\right)$$

Together with $\Phi_K^{\rho(0)} = \Phi_K^0 = id_M$ this implies that $\Phi_K^{\rho(t)}$ is the flow of X_H and thus $\Phi_K^{\rho(t)} = \Phi_H^t$ for all t, especially for $t = 1$. $\square$

We write $\mathcal{P}(H) := \left\{\gamma : S^1 \to M \,\middle|\, \forall t \in S^1 : \dot\gamma(t) = X_{H_t}(\gamma(t))\right\}$ for the space of 1-perodic Hamiltonian flow lines of H and observe that there is a one-to-one-correspondence

$$\begin{aligned}\operatorname{Fix}\Phi_H^1 &\longleftrightarrow \mathcal{P}(H) \\ x &\longmapsto \left(t \mapsto \Phi_H^t(x)\right) \\ \gamma(0) &\longleftarrow\!\shortmid \gamma\end{aligned}$$

between fixed points of the Hamiltonian diffeomorphism Φ_H^1 and 1-periodic Hamiltonian flow lines.

2.2 J-holomorphic curves

Definition 2.6: Let M be a (smooth) manifold. An **almost complex structure** J on M is an endomorphism

$$J : TM \to TM$$

of the tangent bundle such that $J^2 = -id_{TM}$. In other words, J is a smooth family of vector space isomorphisms

$$J_p : T_pM \to T_pM$$

with $J_p^2 = -id_{T_pM}$ for all points $p \in M$.

Definition 2.7: Let M_1, M_2 be manifolds with almost complex structures J_1, J_2. A map $u : M_1 \to M_2$ is called (J_1, J_2)-**holomorphic** if

$$du \circ J_1 = J_2 \circ du, \tag{2.1}$$

where $du : TM_1 \to TM_2$ is the differential of u.

Remark 2.8: The differential of any function u can be written as

$$du = \partial_{(J_1,J_2)}u + \bar{\partial}_{(J_1,J_2)}u,$$

where

$$\partial_{(J_1,J_2)}u := \tfrac{1}{2}\left(du - J_2 \circ du \circ J_1\right)$$

is holomorphic and

$$\bar{\partial}_{(J_1,J_2)}u := \tfrac{1}{2}\left(du + J_2 \circ du \circ J_1\right)$$

is anti-holomorphic. Note that u is (J_1, J_2)-holomorphic if and only if the anti-holomorphic part of its differential vanishes.

If M_1 is a manifold of real dimension 2 and we are given local coordinates $z = s + it \in \mathbb{C}$ (such that the complex structure J_1 is given by multiplication with $i \in \mathbb{C}$ in this chart), we can derive a local equation out of (2.1): The vector fields $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ form a basis of the tangent bundle of $\mathbb{C}$. We use the notation $\partial_s u := du\left(\frac{\partial}{\partial s}\right)$, $\partial_t u := du\left(\frac{\partial}{\partial t}\right)$. Equation (2.1) then reads as

$$\partial_t u = J(u)\partial_s u. \tag{2.2}$$

Often the almost complex structure on the domain manifold M_1 is clear from the context, while the one on the target manifold M_2 is the one we are interested in. Then we mention only the almost complex structure J on M_2 and talk about J-holomorphic maps. If both almost complex structures are clear from the context, J-holomorphic maps are also called **pseudo-holomorphic**.

A J-holomorphic map on a (real) 2-dimensional domain or its image is sometimes also referred to as a J-holomorphic curve. These curves have rigidity properties similar to the ones of classically holomorphic maps.

Definition 2.9: Let (M, ω) be a symplectic manifold. An almost complex structure J on M is **compatible** with ω if

$$g_J(v, w) := \omega(Jv, w)$$

defines a Riemannian metric on M. In this case we write $|\cdot|_J$ for the corresponding norm on TM.

The space of compatible almost complex structures on a symplectic manifold (M, ω) is known to be non-empty and contractibe.

Remark 2.10: The differential du of a map $u : \Sigma \to M$ between manifolds consists of linear maps

$$du(z) : T_z\Sigma \to T_{u(z)}M$$

for all $z \in M$; one can call it a 1-form with values in the pullback bundle u^*TM. If j_Σ is an almost complex structure on Σ, the manifold (M, ω) is symplectic and J is an ω-compatible almost complex structure on M we can define a norm on the space of linear maps $L : T_z\Sigma \to T_{u(z)}M$ by

$$\|L\|_J := |\zeta|^{-1}\sqrt{|L(\zeta)|_J^2 + |L(j_\Sigma\zeta)|_J^2}$$

for some $0 \neq \zeta \in T_z\Sigma$ – one can show that this expression is independent of ζ.

Definition 2.11: The **energy** of a J-holomorphic curve $u : \Sigma \to M$ in a symplectic manifold (M, ω) with compatible almost complex structure J is defined as

$$E(u) = \frac{1}{2}\int_\Sigma \|du\|_J^2 \, d\,vol_\Sigma.$$

If there are global holomorphic coordinates (s, t) on Σ, this can be written as

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Sigma |\partial_s u|_J^2 + |\partial_t u|_J^2 \; dt \wedge ds \\ &= \int_\Sigma |\partial_s u|_J^2 \; dt \wedge ds. \end{aligned}$$

Sometimes we are interested in the energy of a curve over a subdomain $G \in \Sigma$, then we write

$$E(u; G) := \frac{1}{2}\int_G \|du\|_J^2 \, d\,vol_\Sigma.$$

Lemma 2.12: *For a J-holomorphic curve $u : \Sigma \to M$, one has*

$$E(u) = -\int_\Sigma u^*\omega.$$

Proof. In local holomorphic coordinates (s, t):

$$\begin{aligned} |\partial_s u|_J^2 + |\partial_t u|_J^2 \; dt \wedge ds &= g_J(\partial_s u, \partial_s u) + g_J(\partial_t u, \partial_t u) \; dt \wedge ds \\ &= \omega(J(u)\partial_s u, \partial_s u) + \omega(J(u)\partial_t u, \partial_t u) \; dt \wedge ds \\ &= \omega(\partial_t u, \partial_s u) + \omega(-\partial_s u, \partial_t u) \; dt \wedge ds \\ &= -\omega(\partial_s u, \partial_t u) - \omega(\partial_s u, \partial_t u) \; dt \wedge ds \\ &= -2u^*\omega \end{aligned}$$

Thus the integrals are the same. □

2.3 Removal of singularities

One of the nice properties of J-holomorphic curves is the following famous theorem about removability of singularities:

Theorem 2.13: *Let (M, ω) be a closed symplectic manifold and let J be an ω-compatible almost complex structure on M. If $u : D^2 \backslash \{0\} \to M$ is a J-holomorphic curve of finite energy $E(u) < \infty$, then it extends to a J-holomorphic map $D^2 \to M$.*

A proof can be found in [7], pages 76ff.

Corollary 2.14: *Let (M, ω) be a closed symplectic manifold and J an ω-compatible almost complex structure on M. If $u : \mathbb{C} \to M$ is a J-holomorphic curve of finite energy $E(u) < \infty$, then it extends to a smooth map $S^2 = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} \to M$.*

Proof. We use the map

$$\begin{aligned} \mathbb{C}\backslash B_1(0) &\longrightarrow D^2\backslash\{0\} \\ z &\longmapsto \frac{1}{z} \end{aligned}$$

which continues to a biholomorphic identification

$$f : (\mathbb{C}\backslash B_1(0)) \cup \{\infty\} \longrightarrow D^2.$$

If $u : \mathbb{C} \to M$ is J-holormophic, then so is

$$v := u \circ f^{-1} : D^2\backslash\{0\} \longrightarrow M$$

with the same energy and we can use theorem 2.13 to get a J-holomorphic continuation $\hat{v} : D^2 \to M$. Define a continuation of u by

$$\begin{aligned} &\hat{u} : \mathbb{C} \cup \{\infty\} \longrightarrow M \\ &\hat{u}(z) = \begin{cases} u(z) & \text{if } z \in B_1(0) \\ \hat{v}(f(z)) & \text{if } z \in (\mathbb{C}\backslash B_1(0)) \cup \{\infty\} \end{cases} \end{aligned}$$

– this is well-defined and holomorphic because both components are holomorphic and they agree on the overlap. □

2.4 The symplectic action functional

It is possible to see fixed points of Hamiltonian diffeomorphisms on symplectic manifolds as critical points of an action functional on the loop space. For this to be well-defined as a real-valued map one has to assume something more about the manifold.

Definition 2.15: Let (M, ω) be a symplectic manifold. The pair (M, ω) is called **symplectically aspherical** if for all smooth maps $u : S^2 \to M$ one has

$$\int_{S^2} u^*\omega = 0.$$

Remark 2.16: Let $\Pi \subseteq H_2(M; \mathbb{Z})$ be the image of the Hurewicz homomorphism

$$h_2 : \pi_2(M) \to H_2(M; \mathbb{Z})$$

and denote by $[\omega]$ the (de Rham) cohomology class of the symplectic form. Then $[\omega]$ can be evaluated on elements of Π via the Kronecker pairing. In this notation, (M, ω) is symplectically aspherical if and only if

$$[\omega]\,(x) = 0$$

for every $x \in \Pi$.

Example: Let (M, ω) be a symplectic manifold with vanishing second homotopy group, that is $\pi_2(M) = \{0\}$. Then the image of the second Hurewicz homomorphism is trivial and so M is symplectically aspherical.

For more examples of symplectically aspherical manifolds, see for instance [5] and [4].

Remark 2.17: The symplectic asphericity condition implies that every J-holomorphic sphere in M has to be constant: Let $u : S^2 \to M$ be J-holomorphic. Then

$$\tfrac{1}{2} \int_{S^2} \|du\|_J^2 \, d\,vol_{S^2} = E(u) = -\int_{S^2} u^*\omega = 0,$$

so the derivatives of u vanish and it has to be constant.

Now fix a Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ on a symplectically aspherical manifold (M, ω). By

$$\mathcal{L}M = \mathcal{C}^\infty_{\text{contr}}(S^1, M)$$

we denote the space of smooth contractible loops in M. The **symplectic action functional** $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ is defined by

$$\mathcal{A}_H(x) := \int_{D^2} \bar{x}^*\omega - \int_0^1 H(t, x(t))\, dt.$$

Here,

$$\bar{x} : D^2 \to M$$

is a capping of the loop x, that is a smooth map from the two-dimensional unit disk D^2 into M that agrees with x on the boundary. When we think of the 1-sphere as $S^1 = \mathbb{R}/\mathbb{Z}$ and of the disk as $D^2 \subseteq \mathbb{C}$, this means that $\bar{x}(e^{it}) = x(t)$ for all $t \in S^1$.

Lemma 2.18: *The symplectic action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ is well-defined.*

Proof. Suppose we have a loop $x \in \mathcal{L}M$ with two cappings $\bar{x}, \widehat{x}$. Without loss of generality we can assume that

$$\bar{x}(re^{it}) = x(t) = \widehat{x}(re^{it}) \text{ for } r \in [1-\varepsilon, 1]$$

for some $\varepsilon > 0$. Gluing D^2 and $-D^2$ (the disk with the reversed orientation) along their boundaries yields a manifold S which is diffeomorphic to the sphere. Then

$$\begin{aligned} u : S &\longrightarrow M \\ z &\longmapsto \begin{cases} \bar{x}(z) & \text{if } z \in D^2 \\ \widehat{x}(z) & \text{if } z \in -D^2 \end{cases} \end{aligned}$$

defines a smooth map and

$$\int_{D^2} \bar{x}^*\omega + \int_{-D^2} \widehat{x}^*\omega = \int_S u^*\omega = 0.$$

Thus $\int_{D^2} \bar{x}^*\omega = -\int_{-D^2} \widehat{x}^*\omega = \int_{D^2} \widehat{x}^*\omega$ and hence the definition of $\mathcal{A}_H$ does not depend on the choice of capping. □

Proposition 2.19: *A loop $x \in \mathcal{L}M$ is a critical point of the symplectic action functional if and only if it is a 1-periodic flow line of the Hamiltonian vector field given by H on M. This means that one has*

$$\mathrm{Crit}\mathcal{A}_H = \mathcal{P}_{\mathrm{contr}}(H)$$

where $\mathcal{P}_{\mathrm{contr}}(H) \subseteq \mathcal{P}(H)$ is the subset of contractible flow lines.

Proof. In order to find the critical points of $\mathcal{A}_H$, we have to compute its linearization

$$d\mathcal{A}_H(x) \ : \ T_x\mathcal{L}M \longrightarrow T_{\mathcal{A}_H(x)}\mathbb{R} \cong \mathbb{R}$$

at a point $x \in \mathcal{L}M$. The tangent space

$$T_x\mathcal{L}M = \Big\{\xi : S^1 \to TM \text{ smooth } \ \big| \ \xi(t) \in T_{x(t)}M \text{ for all } t \in S^1\Big\}$$

is the space of all smooth vector fields along x. We claim that

$$d\mathcal{A}_H(x)(\xi) = \int_0^1 \omega\Big(X_{H_t}(x(t)) - \dot{x}(t), \xi(t)\Big)\, dt$$

for $\xi \in T_x\mathcal{L}M$. To see this, consider a path

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{L}M \\ s &\longmapsto x_s \end{aligned}$$

with $x_0 = x$ and $\frac{d}{ds}\big|_{s=0} x_s = \xi$. As we have to compute the values of $\mathcal{A}_H$ along the path we also need cappings

$$\bar{x}_s : D^2 \longrightarrow M$$

for all x_s. Write $\bar{\xi} := \frac{d}{ds}\big|_{s=0}\bar{x}_s$ and note that this is a vector field along the disk which agrees with ξ on the boundary of the disk. Compute:

$$\begin{aligned}
d\mathcal{A}_H(x)(\xi) &= \frac{d}{ds}\bigg|_{s=0} \mathcal{A}_H(x_s) \\
&= \frac{d}{ds}\bigg|_{s=0} \int_{D^2} \bar{x}_s^*\omega - \frac{d}{ds}\bigg|_{s=0} \int_0^1 H_t(x_s(t))\,dt \\
&= \int_{D^2} \frac{d}{ds}\bigg|_{s=0} \bar{x}_s^*\omega - \int_0^1 \frac{d}{ds}\bigg|_{s=0} H_t(x_s(t))\,dt \\
&= \int_{D^2} \bar{x}_0^*(\mathcal{L}_{\bar{\xi}}\omega) - \int_0^1 dH_t(x(t))(\xi(t))\,dt \\
&= \int_{D^2} \bar{x}_0^*(d(\iota_{\bar{\xi}}\omega) + \iota_{\bar{\xi}}(d\omega)) - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_{D^2} \bar{x}_0^*(d(\iota_{\bar{\xi}}\omega)) - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_{D^2} d(\bar{x}_0^*(\iota_{\bar{\xi}}\omega)) - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_{\partial D^2} \bar{x}_0^*(\iota_{\bar{\xi}}\omega) - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_0^1 x^*(\iota_{\xi}\omega) - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_0^1 \omega\Big(\xi(t), \dot{x}(t)\Big)\,dt - \int_0^1 \omega\Big(\xi(t), X_{H_t}(x(t))\Big)\,dt \\
&= \int_0^1 \omega\Big(\xi(t), \dot{x}(t) - X_{H_t}(x(t))\Big)\,dt
\end{aligned}$$

With this formula it is obvious that $x \in \mathcal{L}M$ is a critical point of the symplectic action functional if and only if $\dot{x}(t) = X_{H_t}(x(t))$ for all $t \in S^1$, which means that x is a Hamiltonian flow line. □

Remark 2.20: The Riemannian metrics g_J on M define a metric $\widehat{g}_J$ on the loop space $\mathcal{L}M$ by

$$\widehat{g}_J(\xi, \eta) := \int_0^1 g_J(\xi(t), \eta(t))\,dt$$

for $\xi, \eta \in T_x\mathcal{L}M$. Since we computed

$$\begin{aligned}
d\mathcal{A}_H(x)(\xi) &= \int_0^1 \omega\Big(\xi(t), \dot{x}(t) - X_{H_t}(x(t))\Big)\,dt \\
&= \int_0^1 \omega\left(J(x(t))\xi(t), J(x(t))\Big(\dot{x}(t) - X_{H_t}(x(t))\Big)\right)\,dt \\
&= \int_0^1 g_J\left(\xi(t), J(x(t))\Big(\dot{x}(t) - X_{H_t}(x(t))\Big)\right)\,dt \\
&= \widehat{g}_J\left(\xi, J(x)\Big(\dot{x} - X_H(x)\Big)\right) \\
&= \widehat{g}_J\left(J(x)\Big(\dot{x} - X_H(x)\Big), \xi\right),
\end{aligned}$$

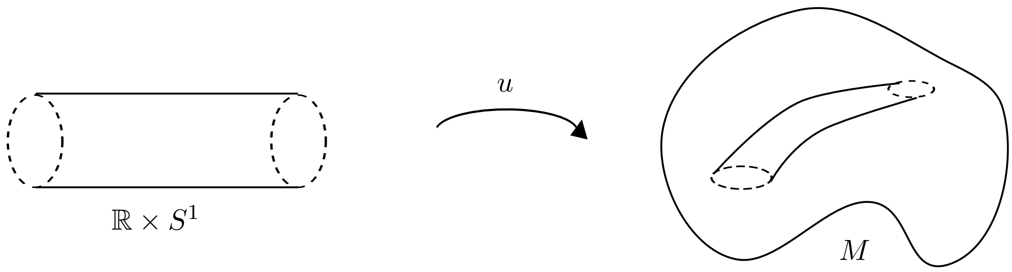

Figure 2.1: Sketch of $u : \mathbb{R} \times S^1 \to M$.

the gradient of $\mathcal{A}_H$ with respect to this metric is given by

$$\nabla_{\hat{g}_J} \mathcal{A}_H(x) = J(x)\Big(\dot{x} - X_H(x)\Big).$$

2.5 Floer cylinders

Floer cylinders are negative gradient flow lines of the symplectic action functional with respect to the metric $\hat{g}_J$. A negative gradient flow line of $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ is a smooth path

$$u : \mathbb{R} \longrightarrow \mathcal{L}M$$

such that there are $x_-, x_+ \in \mathrm{Crit}(\mathcal{A}_H)$ with

$$\lim_{s \to \pm\infty} u(s) = x_\pm$$

and

$$\frac{d}{ds} u + \nabla_{\hat{g}_J} \mathcal{A}_H(u) = 0.$$

It can be interpreted as a map

$$u : \mathbb{R} \times S^1 \longrightarrow M$$

(sketched in figure 2.1) such that each $u(s, \cdot)$ is a contractible loop, satisfying

$$\lim_{s \to \pm\infty} u(s, \cdot) = x_\pm(\cdot)$$

uniformly and the equation

$$\partial_s u(s,t) + J(u(s,t))\Big(\partial_t u(s,t) - X_{H_t}(u(s,t))\Big) = 0 \tag{2.3}$$

for all $(s,t) \in \mathbb{R} \times S^1$. This partial differential equation will be called the Floer equation and solutions will be called Floer cylinders.

Definition 2.21: The **energy** of a smooth function $u : \mathbb{R} \times S^1 \to M$ is defined as

$$\begin{aligned} E(u) :&= \tfrac{1}{2} \int_{-\infty}^{\infty} \int_0^1 |\partial_s u(s,t)|_J^2 + |\partial_t u(s,t) - X_{H_t}(u(s,t))|_J^2 \, dtds \\ &= \int_{-\infty}^{\infty} \int_0^1 |\partial_s u(s,t)|_J^2 \, dtds \\ &= \int_{-\infty}^{\infty} \|\partial_t u_s - X_H(u_s)\|_{L^2(S^1,M)}^2 \, ds \end{aligned}$$

Remark 2.22: If $u : \mathbb{R} \times S^1 \to M$ is a Floer cylinder with energy $E(u) = 0$, then this means that $|\partial_s u|_J^2$ vanishes everywhere and so does $|\partial_t u - X_H(u)|_J^2$. Therefore $u(s,t)$ does not depend on s and $\partial_t u = X_H(u)$, so each $u_s := u(s,\cdot)$ equals $x_- = x_+$, where $x_\pm$ are the limit loops as above.

Let

$$\mathcal{M}_F := \left\{ u : \mathbb{R} \times S^1 \to M \;\middle|\; \begin{array}{c} u \text{ satisfies (2.3)}, E(u) < \infty, \\ \text{each } u_s := u(s,\cdot) \text{ is contractible} \end{array} \right\}$$

be the moduli space of all Floer cylinders.

Theorem 2.23 (Gromov compactness)**:** *The space $\mathcal{M}_F$ is compact in the $\mathcal{C}^\infty_{loc}$-topology, that is the topology of convergence with all derivatives on all compact subsets.*

A proof of theorem 2.23 can be found in [2], it is theorem 6.2.1 there.

Remark 2.24: $\mathbb{R}$ acts from the right on $\mathcal{M}_F$:

$$\begin{aligned} \mathcal{M}_F \times \mathbb{R} &\longrightarrow \mathcal{M}_F \\ (u,\sigma) &\longmapsto u \cdot \sigma, \end{aligned}$$

where $u \cdot \sigma(s,t) := u(s+\sigma, t)$. It is obvious that $u \cdot \sigma$ again solves the Floer equation and that it has the same energy as u.

2.6 Convergence at the ends of the cylinder

We are interested in flow lines that come from one critical point $x_- \in \mathrm{Crit}\mathcal{A}_H$ and converge to another one, $x_+ \in \mathrm{Crit}\mathcal{A}_H$.

Proposition 2.25: *Let $u : \mathbb{R} \times S^1 \to M$ be a solution of* (2.3) *and suppose there are $x_\pm \in \mathrm{Crit}\mathcal{A}_H$ such that*

$$\lim_{s \to \pm\infty} u(s,\cdot) = x_\pm(\cdot)$$

in $\mathcal{L}M$. Then

$$E(u) = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+)$$

– in particular, the energy of u is finite.

Proof. Write $u_s := u(s,\cdot)$ and understand it as a loop. Such a loop is freely homotopic to x_- and x_+ via u and so it is contractible. Then:

$$\begin{aligned} E(u) &= \tfrac{1}{2} \int_{-\infty}^{\infty} \int_0^1 |\partial_s u(s,t)|_J^2 + |\partial_t u(s,t) - X_{H_t}(u(s,t))|_J^2 \, dtds \\ &= \int_{-\infty}^{\infty} \int_0^1 |\partial_s u(s,t)|_J^2 \, dtds \\ &= \int_{-\infty}^{\infty} \int_0^1 g_J\Big(\partial_s u(s,t), \partial_s u(s,t)\Big) \, dtds \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_0^1 g_J\left(J(u)\Big(X_{H_t}(u(s,t)) - \partial_t u(s,t)\Big), \partial_s u(s,t)\right)\, dtds \\
&= \int_{-\infty}^{\infty} \hat{g}_J\left(J(u_s)\Big(X_H(u_s) - \dot{u}_s\Big), \partial_s u_s\right)\, ds \\
&= -\int_{-\infty}^{\infty} \hat{g}_J\Big(\nabla_{\hat{g}_J}\mathcal{A}_H(u_s), \partial_s u_s\Big)\, ds \\
&= -\int_{-\infty}^{\infty} d\mathcal{A}_H(u_s)(\partial_s u_s)\, ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\left(\mathcal{A}_H(u_s)\right)\, ds \\
&= -\left(\lim_{s\to\infty}\mathcal{A}_H(u_s) - \lim_{s\to-\infty}\mathcal{A}_H(u_s)\right) \\
&= \mathcal{A}_H\left(\lim_{s\to-\infty} u_s\right) - \mathcal{A}_H\left(\lim_{s\to\infty} u_s\right) \\
&= \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+)
\end{aligned}
$$

□

It would be nice to have the reversed statement as well: that a solution of (2.3) with finite energy converges to critical points of $\mathcal{A}_H$ at the ends. Indeed, in the case of a non-degenerate Hamiltonian[1], such maps converge to periodic Hamiltonian flow lines exponentially fast (see [11], proposition 1.21). In general, exponential convergence cannot be hoped for, but luckily if $\#\mathcal{P}(H) < \infty$ there is still some kind of convergence. In proving this, we follow Audin and Damian ([2], chapter 6) for the rest of this section.

We start with a result about the value of the action functional on the loops $u_s := u(s,\cdot)$ as s tends to $\pm\infty$.

Proposition 2.26: *Let $u : \mathbb{R}\times S^1 \to M$ satisfy equation* (2.3) *and $E(u) < \infty$. Assume also that every loop*

$$u_s := u(s,\cdot)$$

is contractible. Then there are $x_\pm \in \mathrm{Crit}\mathcal{A}_H$ such that

$$\lim_{s\to\pm\infty}\mathcal{A}_H(u_s) = \mathcal{A}_H(x_\pm).$$

Proof. We prove convergence for $s\to\infty$, the other case is analogous. Consider the function

$$
\begin{aligned}
\mathbb{R} &\longrightarrow \mathbb{R} \\
s &\longmapsto \mathcal{A}_H(u_s).
\end{aligned}
$$

[1] A Hamiltonian function H is called non-degenerate if for every $P \in M$ the number 1 is not an eigenvalue of the linearized time-one-map $d\Phi_H^1(P)$ or, equivalently, if the graph of Φ_H^1 intersects the diagonal transversely in $M \times M$.

Its derivative

$$\begin{aligned}
\frac{d}{ds}\mathcal{A}_H(u_s) &= d\mathcal{A}_H(u_s)\left(\tfrac{d}{ds}u_s\right)\\
&= \hat{g}_J\Big(J(u_s)(X_H(u_s)-\dot{u}_s), \partial_s u_s\Big)\\
&= -\hat{g}_J(\partial_s u_s, \partial_s u_s)\\
&= -\|\partial_s u_s\|_J^2
\end{aligned}$$

is negative everywhere. This means that the function is decreasing and thus it is enough to show that there is $x_+ \in \mathrm{Crit}\mathcal{A}_H$ and a sequence $(s_k)_{k\in\mathbb{N}}$ of real numbers such that $\lim_{k\to\infty} s_k = \infty$ and

$$\lim_{k\to\infty} \mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(x_+).$$

It is

$$\begin{aligned}
\infty > E(u) &= \int_{-\infty}^{\infty}\left(\int_0^1 |\partial_t u(s,t) - X_{H_t}(u(s,t))|_J^2\, dt\right) ds\\
&= \int_{-\infty}^{\infty} \|\partial_t u_s - X_H(u_s)\|_{L^2(S^1,M)}^2\, ds
\end{aligned}$$

and thus there is a sequence $(s_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} s_k = \infty$ and

$$\lim_{k\to\infty} \|\partial_t u_{s_k} - X_H(u_{s_k})\|_{L^2(S^1,M)} = 0.$$

Here, the L^2-norm of a loop is derived from the norm given by J on the tangent space of M. To simplify the situation, remember that M can be embedded in some Euclidean space $\mathbb{R}^N$ and that the J-norm is therefore equivalent to the restriction of the Euclidean norm onto M. This means that we have

$$\lim_{k\to\infty} \|\partial_t u_{s_k} - X_H(u_{s_k})\|_{L^2(S^1,\mathbb{R}^N)} = 0.$$

Since X_H is bounded on the compact manifold M, there must be a constant B such that

$$\|\partial_t u_{s_k}\|_{L^2(S^1,\mathbb{R}^N)} < B$$

for all $k \in \mathbb{N}$. For any two $t_0, t_1 \in S^1 = \mathbb{R}/\mathbb{Z}$ we compute the following:

$$\begin{aligned}
\|u_{s_k}(t_1) - u_{s_k}(t_0)\| &= \int_{t_0}^{t_1} \dot{u}_{s_k}(t)\, dt\\
&\le \sqrt{t_1 - t_0}\cdot \|\dot{u}_{s_k}\|_{L^2(S^1,\mathbb{R}^N)}\\
&\le \sqrt{t_1 - t_0}\cdot B
\end{aligned}$$

This means that $(u_{s_k})_k$ is equicontinuous. Besides that, $(u_{s_k}(t))_k$ is relatively compact in M for all $t \in S^1$. By the Arzelà-Ascoli theorem we conclude that the $(u_{s_k})_k$ converge in the $\mathcal{C}^0$-topology to a limit $x_+ : S^1 \to M$.

The next step is to show that this continuous limit is smooth. Again recollect that we can work in $\mathbb{R}^N$ and thus for $t \in S^1$ the following term is defined.

$$\begin{aligned}
x_+(t) - x_+(0) - \int_0^t X_{H_t}(x_+(\tau))d\tau &= \lim_{k\to\infty}\left(u_{s_k}(t) - u_{s_k}(0) - \int_0^t X_{H_t}(x_+(\tau))d\tau\right)\\
&= \lim_{k\to\infty}\left(\int_0^t \partial_t u_{s_k}(\tau)d\tau - \int_0^t X_{H_t}(x_+(\tau))d\tau\right)\\
&= \lim_{k\to\infty}\left(\int_0^t \partial_t u_{s_k}(\tau) - X_{H_t}(u_{s_k}(\tau))d\tau\right)\\
&\quad + \lim_{k\to\infty}\left(\int_0^t X_{H_t}(u_{s_k}(\tau)) - X_{H_t}(x_+(\tau))d\tau\right)\\
&= \lim_{k\to\infty}\left(\int_0^t \partial_t u_{s_k}(\tau) - X_{H_t}(u_{s_k}(\tau))d\tau\right) + 0\\
&= 0 + 0
\end{aligned}$$

The penultimate equality holds because x_+ is the pointwise limit of the u_{s_k}; for the last one we have to use the Cauchy-Schwarz inequality for the inner product on the L^2-space:

$$\begin{aligned}
\left|\int_0^t \partial_t u_{s_k}(\tau) - X_{H_t}(u_{s_k}(\tau))d\tau\right| &= \left|\int_0^1 1_{[0,t]}(\tau)\left(\partial_t u_{s_k}(\tau) - X_{H_t}(u_{s_k}(\tau))\right)d\tau\right|\\
&\leq \|1_{[0,t]}\|_{L^2(S^1)} \cdot \|\partial_t u_{s_k} - X_{H_t}(u_{s_k})\|_{L^2(S^1)}\\
&= \sqrt{t} \cdot \|\partial_t u_{s_k} - X_{H_t}(u_{s_k})\|_{L^2(S^1)}
\end{aligned}$$

This does indeed tend to zero for $k \to \infty$. But now, having established the formula

$$x_+(t) - x_+(0) = \int_0^t X_{H_t}(x_+(\tau))d\tau$$

for all values of $t \in S^1$, we can deduce from continuity that x_+ is $\mathcal{C}^1$, from $\mathcal{C}^1$ that it is $\mathcal{C}^2$ and so on, getting on the whole that x_+ is indeed of class $\mathcal{C}^\infty$. This gives also convergence in the $\mathcal{C}^\infty$-topology.

As a last step it remains to show that indeed

$$\lim_{k\to\infty} \mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(x_+).$$

It is clear that

$$\lim_{k\to\infty}\int_0^1 H_t(u_{s_k}(t))\,dt = \int_0^1 H_t(x_+(t))\,dt,$$

but we need an argument why

$$\lim_{k\to\infty}\int_{D^2} \bar{u}_{s_k}^*\omega = \int_{D^2} \bar{x}_+^*\omega$$

for cappings $\bar{u}_{s_k}$ and $\bar{x}_+$ of u_{s_k} and x_+.

First let us see what would happen if $\omega = d\lambda$ was exact. Then we would compute the following:

$$\begin{aligned}
\int_{D^2} \bar{u}_{s_k}^* \omega - \int_{D^2} \bar{x}_+^* \omega &= \int_{D^2} \bar{u}_{s_k}^* d\lambda - \int_{D^2} \bar{x}_+^* d\lambda \\
&= \int_{\partial D^2} \bar{u}_{s_k}^* \lambda - \int_{\partial D^2} \bar{x}_+^* \lambda \\
&= \int_{S^1} u_{s_k}^* \lambda - \int_{S^1} x_+^* \lambda \\
&= \int_{S^1} \lambda(\dot{u}_{s_k}(t)) dt - \int_{S^1} \lambda(\dot{x}_+(t))\, dt \\
&= \int_{S^1} \lambda(\dot{u}_{s_k}(t) - \dot{x}_+(t))\, dt \\
&= \int_{S^1} \lambda(\dot{u}_{s_k}(t) - X_{H_t}(u_{s_k}(t)))\, dt + \int_{S^1} \lambda(X_{H_t}(u_{s_k}(t)) - \dot{x}_+(t))\, dt \\
&= \int_{S^1} \lambda(\dot{u}_{s_k}(t) - X_{H_t}(u_{s_k}(t)))\, dt + \int_{S^1} \lambda(X_{H_t}(u_{s_k}(t)) - X_{H_t}(x_+(t)))\, dt
\end{aligned}$$

The second of these integrals tends to zero as $k \to \infty$ since x_+ is the pointwise limit of u_{s_k}. For the first we get

$$\begin{aligned}
\left| \int_{S^1} \lambda(\dot{u}_{s_k}(t) - X_{H_t}(u_{s_k}(t)))\, dt \right| &\leq \sup \|\lambda\| \cdot \|\dot{u}_{s_k} - X_{H_t}(u_{s_k})\|_{L^1(S^1)} \\
&\leq \text{const} \cdot \sup \|\lambda\| \cdot \|\dot{u}_{s_k} - X_{H_t}(u_{s_k})\|_{L^2(S^1)}
\end{aligned}$$

because S^1 is compact, so this integral also tends to 0.

Now we use this result for exact ω and the symplectic asphericity to prove the general result.

Choose a neighbourhood $\mathcal{U}$ of $x_+(S^1)$ in M which is a deformation retract of $x_+(S^1)$ so that $\omega|_{\mathcal{U}}$ is exact. For k big enough, u_{s_k} is contained in $\mathcal{U}$. Choose a homotopy from u_{s_k} to x_+ which is contained in $\mathcal{U}$ – it can be interpreted as a map

$$h_k : [0,1] \times S^1 \to M.$$

Glue this map with $\bar{u}_{s_k}$ at the left boundary component and with $\bar{x}_+$ at the right one to a map

$$D^2 \cup \Big([0,1] \times S^1\Big) \cup (-D^2) \to M$$

in a way that it gives a smooth function

$$v : S^2 \to M.$$

For this function we compute

$$0 = \int_{S^2} v^* \omega = \int_{D^2} \bar{u}_{s_k}^* \omega + \int_{[0,1]\times S^1} h_k^* \omega - \int_{D^2} \bar{x}_+^* \omega$$

and thus get

$$\int_{D^2} \bar{x}_+^* \omega - \int_{D^2} \bar{u}_{s_k}^* \omega = \int_{[0,1]\times S^1} h_k^* \omega.$$

All this lies in $\mathcal{U}$ where ω is exact, so we can use the previous computation. □

As a next step, we show that the loops u_s accumulate as $s \to \pm\infty$.

Proposition 2.27: *Again let $u : \mathbb{R} \times S^1 \to M$ be a finite energy solution of the Floer equation* (2.3) *such that each $u(s, \cdot)$ is a contractible loop and let $(s_k)_{k\in\mathbb{N}}$ be a sequence of real numbers tending to ∞ (or $-\infty$). Then there is a subsequence still denoted by $(s_k)_{k\in\mathbb{N}}$ and a critical point $x_+ \in \mathrm{Crit}\mathcal{A}_H$ (or $x_- \in \mathrm{Crit}\mathcal{A}_H$) such that*

$$\lim_{k\to\infty} u_{s_k} = x_+$$

(or $\lim_{k\to-\infty} u_{s_k} = x_-$) in the $\mathcal{C}^\infty$-topology.

Proof. We will show the case $s_k \to \infty$, the other case is analogous. Remember the $\mathbb{R}$-action on $\mathcal{M}_F$ and define $u_k \in \mathcal{M}_F$ for all $k \in \mathbb{N}$ by

$$u_k := u \cdot s_k.$$

Since by theorem 2.23 $\mathcal{M}_F$ is compact, after the choice of a subsequence $(u_k)_{k\in\mathbb{N}}$ converges to some $v \in \mathcal{M}_F$.

Fix any $s_0 \in \mathbb{R}$ and write $v(s_0, \cdot) =: v_{s_0}$ as a loop. Then

$$\mathcal{A}_H(v_{s_0}) = \lim_{k\to\infty} \mathcal{A}_H(u_{s_0+s_k}) = \lim_{s\to\infty} \mathcal{A}_H(u_s),$$

since this last limit exists by proposition 2.26. This holds for all possible $s_0 \in \mathbb{R}$, and thus

$$\begin{aligned} E(v) &= \mathcal{A}_H\left(\lim_{s_0\to\infty} v_{s_0}\right) - \mathcal{A}_H\left(\lim_{s_0\to-\infty} v_{s_0}\right) \\ &= \lim_{s_0\to\infty} \mathcal{A}_H(v_{s_0}) - \lim_{s_0\to-\infty} \mathcal{A}_H(v_{s_0}) \\ &= 0. \end{aligned}$$

By remark 2.22, we can think of v as of a periodic orbit $x_+ \in \mathcal{P}(\mathcal{H})$ and from

$$\lim_{k\to\infty} u_k(s,t) = v(s,t) = x_+(t)$$

we get the desired result. ☐

Proposition 2.28: *Assume that $\#\mathcal{P}(H) < \infty$. Then for every $u \in \mathcal{M}_F$ there are loops $x_\pm \in \mathrm{Crit}\mathcal{A}_H$ such that*

$$\lim_{s\to\pm\infty} u(s,\cdot) = x_\pm(\cdot)$$

in the $\mathcal{C}^\infty$-topology.

Proof. Again we only show the case $s \to \infty$. On $\mathcal{L}M$ one has a distance function d_∞ which induces the $\mathcal{C}^\infty$-topology. For $x \in \mathcal{L}M$, let $B_\varepsilon(x)$ denote the open ball of radius ε around x. Since $\#\mathrm{Crit}\mathcal{A}_H < \infty$ we can choose $\varepsilon > 0$ so small that all ε-balls around critical values of $\mathcal{A}_H$ are disjoint. Let

$$U_\varepsilon = \bigcup_{x\in\mathrm{Crit}\mathcal{A}_H} B_\varepsilon(s)$$

be their union. Consider now the cylinder $u \in \mathcal{M}_F$. For ε small enough, there is $s_\varepsilon \in \mathbb{R}$ such that

$$u\left([s_\varepsilon, \infty) \times S^1\right) \subset U_\varepsilon,$$

for otherwise there would be $\varepsilon_0 > 0$ and a sequence $(s_k)_{k\in\mathbb{N}}$ tending to ∞ such that

$$u_{s_k} \notin U_{\varepsilon_0}$$

for all $k \in \mathbb{N}$, which contradicts proposition 2.27. But $u\left([s_\varepsilon, \infty) \times S^1\right)$ is connected and U_ε is a disjoint union of balls, so $u\left([s_\varepsilon, \infty) \times S^1\right)$ must be contained in one of them, say $B_\varepsilon(x_+)$. This holds for all ε and we always get a ball around the same $x_+ \in \mathrm{Crit}\mathcal{A}_H$, so

$$\lim_{s\to\infty} u(s, \cdot) = x_+(\cdot).$$

□

In fact, the derivative of a Floer cylinder in $\mathbb{R}$-direction tends to 0 for $s \to \pm\infty$.

Proposition 2.29: *Suppose that $\#\mathcal{P}(H) < \infty$ and $u \in \mathcal{M}_F$. Then*

$$\lim_{s\to\pm\infty} \partial_s u(s,t) = 0.$$

Proof. Again, we only prove the case $s \to +\infty$. Let $x_+ \in \mathrm{Crit}\mathcal{A}_H$ be critical points such that $\lim_{s\to\infty} u(s, \cdot) = x_+(\cdot)$. We first show that

$$\lim_{s\to\pm\infty} \partial_t u(s, \cdot) = \dot{x}_+$$

in the $\mathcal{C}^\infty$-topology. Assume that this is not the case. Then there is a sequence $(s_k)_k \subset \mathbb{R}$ tending to ∞ such that $\partial_t u(s_k, \cdot)$ does not tend to $\dot{x}_+$. Define Floer cylinders u_k by

$$u_k(s,t) := u(s_k + s, t),$$

then $u_k \in \mathcal{M}_F$ for all $k \in \mathbb{N}$ and

$$\lim_{s\to\infty} u_k(s, \cdot) = x_+(\cdot)$$

– first in the $\mathcal{C}^0$-topology only, but then by elliptic regularity (see corollary 5.21 in chapter 5) also in the $\mathcal{C}^\infty$-topology. But this is a contradiction to what we assumed about the sequence $(s_k)_{k\in\mathbb{N}}$.

Now, since u is a solution of the Floer equation (2.3) we compute

$$\begin{aligned}
\lim_{s\to\pm\infty} \partial_s u(s,t) &= \lim_{s\to\pm\infty} -J(u(s,t))\left(\partial_t u(s,t) - X_{H_t}(u(s,t))\right) \\
&= -J(x_+(t))\left(\dot{x}_+(t) - X_{H_t}(x_+(t))\right) \\
&= 0.
\end{aligned}$$

□

CHAPTER 3

The main theorem

In this chapter we state our main theorem and outline its proof.

Theorem 3.1: *Let (M, ω) be a closed symplectically aspherical manifold. Assume that $H : S^1 \times M \to \mathbb{R}$ is a Hamiltonian function (degenerate or not) such that $\#\mathcal{P}(H) < \infty$. Then there are $x, y \in \mathcal{P}_{\mathrm{contr}}(H) = \mathrm{Crit}\mathcal{A}_H \subseteq \mathrm{Fix}\,\Phi_H^1$ with*

$$\mathcal{A}_H(x) \neq \mathcal{A}_H(y).$$

The strategy is to prove this theorem by interpolating between holomorphic spheres – about which we know how they look like – and Floer cylinders which converge to critical points of the action functional at the ends.

3.1 The perturbed Floer equation

For $R \in \mathbb{R}_{\geq 0}$, there is a diffeomorphism

$$\psi_R : [-R-1, R+1] \times S^1 \longrightarrow Z_R \subset S^2$$

from the closed cylinder onto a subset Z_R of the 2-sphere. Choose such subsets and diffeomorphisms for all $R \in \mathbb{R}_{\geq 0}$ in a way that the following hold:

- For $r < R$, $Z_r \subset Z_R$.
- For $r < R$, $\psi_R|_{[-r-1,r+1]} = \psi_r$.
- $\bigcup_{R \in \mathbb{R}_{\geq 0}} Z_R = S^2 \backslash \{z_-, z_+\}$ for two points $z_-, z_+ \in S^2$, that means the finite cylinders Z_R exploit $S^2 \backslash \{z_-, z_+\}$.

On each $[-R-1, R+1] \times S^1$ we have coordinates (s,t) and the complex structure i inherited from $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. These give coordinates (s,t) and a complex structure i on Z_R.

This means that on Z_R we can use coordinates (s,t) and that on S^2 there is a complex structure i_R that on Z_R coincides with the pullback (by ψ_R^{-1}) of i. What is more, these complex structures can be made to agree with each other on $\bigcup_{R \in \mathbb{R}_{\geq 0}} Z_R$ and can be continued onto S^2: It is known that the infinite half-cylinder $(-\infty, 0] \times S^1$ is biholomorphic to the punctured disk $D^2 \backslash \{0\} \subset \mathbb{C}$ via

$$\begin{aligned}(-\infty, 0] \times S^1 &\longrightarrow D^2 \backslash \{0\}\\ (s,t) &\longmapsto e^{-2\pi(s+it)},\end{aligned}$$

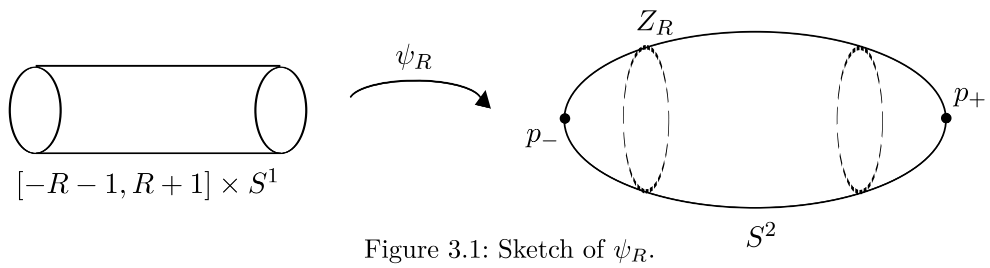

Figure 3.1: Sketch of ψ_R.

and removing the puncture corresponds to adding the point z_- to the punctured half-sphere $\bigcup_{R\in\mathbb{R}_{\geq 0}} Z_R \cap \{s \leq 0\}$. In the same way, use that the infinite half-cylinder $[0,\infty)\times S^1$ also is biholomorphic to the punctured disk to continue the complex structure to z_+. All in all, we have cylindrical coordinates (s,t) and the usual complex structure i on $S^2\backslash\{z_-,z_+\}$, and it continues to an almost-complex structure on the whole sphere S^2.

Remark 3.2: Using these (s,t)-coordinates on $S^2\backslash\{z_-,z_+\}$ for a function $u : S^2 \to M$, we can understand each $u_s := u(s,\cdot)$ as a loop. Such a loop is automatically contractible, and when s tends to $\pm\infty$ the loops converge to constant loops $t \mapsto u(z_\pm)$.

Furthermore we choose a family of smooth functions $\beta_R : \mathbb{R} \to [0,1]$ smoothly varying in $R \in [0,\infty)$ with

- $\beta_R(s) = 0$ for $s \leq -R+\delta$ and for $s \geq R-\delta$ for some $\delta > 0$,
- $\beta_R(s) = 1$ for $-R+1 \leq s \leq R-1$ (this can only happen for $R \geq 1$),
- $\beta_0 \equiv 0$ and
- $\frac{d}{ds}\beta_R(s)$ is bounded uniformly in R and s.

Note that $\lim_{R\to\infty} \beta_R(s) = 1$ for all $s \in \mathbb{R}$.

For maps $u : \mathbb{R}\times S^1 \to M$ we can now consider the following perturbed Floer equation:

$$\partial_s u(s,t) + J(u(s,t))\Big(\partial_t u(s,t) - \beta_R(s) X_{H_t}(u(s,t))\Big) = 0 \tag{3.1}$$

Solutions u solve the usual Floer equation inside Z_{R-1} and they are J-holomorphic outside Z_R. This is a good start – but since in the interpolation we want to start from J-holomorphic spheres, we need such an equation for curves $u : S^2 \to M$.

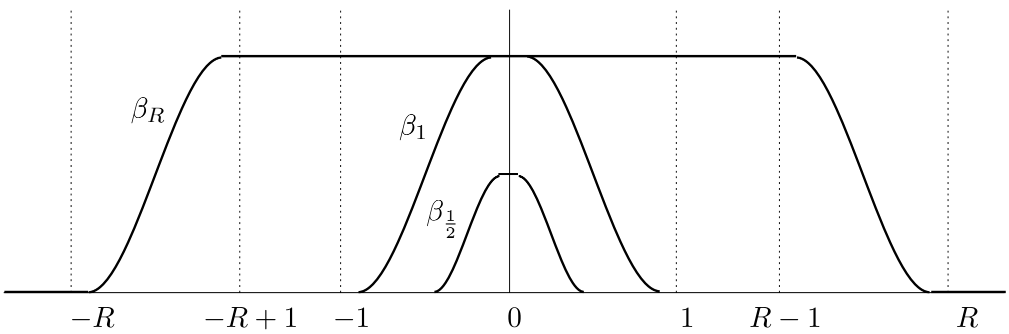

Figure 3.2: Sketch of $\beta_R : \mathbb{R} \to \mathbb{R}$ for several values of R.

For this, let τ be a 1-form on S^2 which agrees with dt on $S^2\backslash\{z_-, z_+\}$, and understand each β_R as a smooth function on the sphere by setting

$$\beta_R : S^2 \to \mathbb{R}$$
$$\beta_R(z) := \begin{cases} \beta_R(s) & \text{if } z = (s,t) \in S^2\backslash\{z_-, z_+\} \\ 0 & \text{if } z = z_\pm. \end{cases}$$

Equation (3.1) now extends to curves $u : S^2 \to M$ as follows:

$$\Big(du - \tau \otimes \beta_R X_H(u)\Big)^{(0,1)} = 0 \tag{3.2}$$

Here, $du - \tau \otimes \beta_R X_H(u)$ is a 1-form on S^2 with values in the pull-back bundle u^*TM and

$$\Big(du - \tau \otimes \beta_R X_H(u)\Big)^{(0,1)} := \Big(du - \tau \otimes \beta_R X_H(u)\Big) + J(u)\Big(du - \tau \otimes \beta_R X_H(u)\Big) \circ i$$

means its antiholomorphic part (with respect to i on S^2 and J on M). On $S^2\backslash\{z_-, z_+\}$, the equations (3.1) and (3.1) coincide, while in local holomorphic coordinates (s,t) around z_- or z_+ equation (3.2) reduces to equation (2.2).

Definition 3.3: The **$\mathbb{R}$-energy** of a solution $u : S^2 \to M$ of the perturbed Floer equation (3.2) for $R \in \mathbb{R}_{\geq 0}$ is defined as

$$E_R(u) := \tfrac{1}{2} \int_{S^2} \|du(z) - \tau(z)\beta_R(z) X_{H(z)}(u(z))\|_J^2 \; dvol_{S^2}.$$

(If we write $z = (s,t)$ in holomorphic coordinates, then $H(z)$ only depends on t.) The norm $\|\cdot\|_J$ used here is again the one from remark 2.10.

Remark 3.4: The set $\{z_-, z_+\}$ has measure 0, therefore

$$\begin{aligned} E_R(u) &= \tfrac{1}{2} \int_{-\infty}^{\infty} |\partial_s u(s,t)|^2 + |\partial_t u(s,t) - \beta_R(s) X_{H_t}(u(s,t))|^2 \; dtds \\ &= \int_{-\infty}^{\infty} |\partial_s u(s,t)|^2 \; dtds. \end{aligned}$$

The first equality is true for every $u : S^2 \to M$, the second only for solutions of (3.2).

Since S^2 is compact, the energy of a solution $u : S^2 \to M$ will always be finite. We will see in section 4.1 of chapter 4 that there also is a uniform bound on the energy of all such solutions.

The definition of $E_R(u)$ depends on $R \in \mathbb{R}_{\geq 0}$. But X_H is bounded and β_R is bounded uniformly in R (since M and S^1 are compact). This means that a uniform bound on the R-energy E_R implies a uniform bound on the 0-energy E_0.

3.2 Definition of the moduli spaces

Theorem 3.1 will be proved by analysing the space of solutions of the perturbed Floer equation (3.2). Recall the situation of this theorem: (M, ω) is a closed symplectically aspherical manifold and $H : S^1 \times M \to \mathbb{R}$ is a Hamiltonian function. Choose an ω-compatible almost complex structure J on M.

For every point $P \in M$ define the following moduli spaces:

$$\mathcal{M}_R(P) := \left\{ u : S^2 \to M \text{ smooth } \middle| \begin{array}{r} \left(du - \tau \otimes \beta_R X_H(u)\right)^{(0,1)} = 0, \\ [u] = 0 \in \pi_2(M), u(0,0) = P \end{array} \right\}$$

$$\widehat{\mathcal{M}}(P) := \left\{ (R, u) \;\middle|\; R \in \mathbb{R}_{\geq 0} \text{ and } u \in \mathcal{M}_R(P) \right\}$$

The condition about the homotopy class will be important later. For one of these moduli spaces we can directly see that what it looks like:

Lemma 3.5: *For every $P \in M$, $\mathcal{M}_0(P)$ consists of exactly one point, namely the constant sphere u_P through P.*

Proof. Since $\beta_0 \equiv 0$, every element of $\mathcal{M}_0(P)$ is a J-holomorphic sphere. By remark 2.17 it has to be constant. On the other hand, for every $P \in M$ the constant J-holomorphic sphere

$$u_P(z) \equiv P \text{ for all z} \in S^2$$

is certainly J-holomorphic with $[u_P] = 0 \in \pi_2(M)$. □

In general, $\widehat{\mathcal{M}}(P)$ does not have to be a manifold. But we will see later that near the pair $(0, u_P)$ it really has the structure of a smooth 1-dimensional manifold with boundary $\{(0, u_P)\} \cong \mathcal{M}_(P)$.

3.3 Outline of the proof

In chapter 4 we will prove the following result:

Proposition 3.6: *Assume that $\#\mathcal{P}(H) < \infty$ and that the Hamiltonian diffeomorphism Φ_H^1 has no fixed points $x, y \in \mathcal{P}_{\text{contr}}(H) = \text{Crit}\mathcal{A}_H \subseteq \text{Fix } \Phi_H^1$ such that $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$. Since $\#\mathcal{P}(H) < \infty$ we can choose a point $P \in M$ which is not a fixed point of Φ_H^1. Then the moduli space $\widehat{\mathcal{M}}(P)$ is compact.*

Chapter 5 explains the setup needed for chapters 6 and 7, in which we can establish the following:

Proposition 3.7: *Assume that $\widehat{\mathcal{M}}(P)$ is compact for a point $P \in M$. Then there is a compact 1-dimensional manifold $\widehat{\mathcal{M}}_\lambda(P)$ with boundary diffeomorphic to $\mathcal{M}_0(P) = \{u_P\}$.*

With these results, it is now easy to prove our main theorem 3.1:

Proof of theorem 3.1. Assume that $\#\mathcal{P}(H) < \infty$ and that the Hamiltonian diffeomorphism Φ_H^1 has no fixed points $x, y \in \mathcal{P}_{\text{contr}}(H) = \text{Crit}\mathcal{A}_H \subseteq \text{Fix } \Phi_H^1$ such that $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.

Choose a point $P \in M$ which is not a fixed point of Φ_H^1. By proposition 3.6, $\widehat{\mathcal{M}}(P)$ is compact. Therefore we can apply proposition 3.7 to get a compact 1-dimensional manifold $\widehat{\mathcal{M}}_\lambda(P)$ with boundary $\mathcal{M}_0(P) = \{u_P\}$ consisting of exactly one point – but such manifolds do not exist. $\square$

CHAPTER 4

Compactness

The purpose of this chapter is to give a proof of proposition 3.6. We rephrase it in a way that suits more the way we will prove it:

Proposition 4.1 (new formulation of proposition 3.6)**:** *For $P \in M$ which is not a fixed point of Φ_H^1, the moduli space $\widehat{\mathcal{M}}(P)$ is compact or there are $x, y \in \mathcal{P}_{\text{contr}}(H) = \text{Crit}\mathcal{A}_H \subseteq$ Fix Φ_H^1 with*

$$\mathcal{A}_H(x) \neq \mathcal{A}_H(y).$$

The most important means for showing compactness is the theorem of Arzelà and Ascoli, a proof of which can be found for example in [6]:

Theorem 4.2 (Arzelà-Ascoli)**:** *Let X be a compact Hausdorff space, Y a metric space, and $(f_k)_{k\in\mathbb{N}}$ a sequence of continuous functions $f_k : X \to Y$. If $(f_k)_{k\in\mathbb{N}}$ is equicontinuous and pointwise relatively compact, then it has a converging subsequence.*

This theorem gives a limit only in the $\mathcal{C}^0$-topology and of course we want to have more regularity here. This is why we will work with weak derivatives (explained in section 5.3) and then use several elliptic regularity results from section 5.5.

4.1 An energy bound

Proposition 4.3: *There is a constant $C > 0$ such that for all $P \in M$ and all pairs $(R, u) \in \widehat{\mathcal{M}}(P)$ the following estimate holds:*

$$E_R(u) \leq C$$

Proof. For fixed $R \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}$, we define the following variation of the action functional on the space of contractible loops of M:

$$\mathcal{A}_{R,s} : \mathcal{L}M \longrightarrow \mathbb{R}$$
$$\mathcal{A}_{R,s}(x) := \int_{D^2} u^*\omega - \int_0^1 \beta_R(s) H_t(x(t))\, dt,$$

where $u : D^2 \to M$ is a capping of x. Exactly as in lemma 2.18 and remark 2.20 from chapter 2, $\mathcal{A}_{R,s}$ is well defined because of the symplectic asphericity of M, and its gradient (with respect to the metric $\hat{g}_J$ on the loop space) is given by

$$\nabla_{\hat{g}_J} \mathcal{A}_{R,s}(x) = J(x)\Big(\dot{x} - \beta_R(s) X_H(x)\Big).$$

Now for $(R,u) \in \widehat{\mathcal{M}}(P)$, understand each $u(s,\cdot) =: u_s$ as a loop and compute the energy:

$$\begin{aligned}
E_R(u) &= \tfrac{1}{2}\int_{-\infty}^{\infty}\int_0^1 |\partial_s u(s,t)|_J^2 + |\partial_t u(s,t) - \beta_R(s) X_{H_t}(u(s,t))|_J^2 \; dtds \\
&= \int_{-\infty}^{\infty}\int_0^1 |\partial_s u(s,t)|_J^2 \; dtds \\
&= \int_{-\infty}^{\infty}\int_0^1 g_J\Big(\partial_s u(s,t), \partial_s u(s,t)\Big) \; dtds \\
&= \int_{-\infty}^{\infty}\int_0^1 g_J\left(-J(u(s,t))\Big(\partial_t u(s,t) - \beta_R(s) X_{H_t}(u(s,t))\Big), \partial_s u(s,t)\right) \; dtds \\
&= -\int_{-\infty}^{\infty} \hat{g}_J\left(J(u_s)\Big(\dot{u}_s - \beta_R X_H(u_s)\Big), \partial_s u_s\right) \; ds \\
&= -\int_{-\infty}^{\infty} \hat{g}_J\Big(\nabla_{\hat{g}_J}\mathcal{A}_{R,s}(u_s), \partial_s u_s\Big) \; ds \\
&= -\int_{-\infty}^{\infty} d\mathcal{A}_{R,s}(u_s)(\partial_s u_s) \; ds \\
&= -\int_{-\infty}^{\infty} \frac{d}{ds}\Big(\mathcal{A}_{R,s}(u_s)\Big) + \left(\frac{d}{ds}\mathcal{A}_{R,s}\right)(u_s) \; ds \\
&= -\left(\lim_{s\to\infty}\mathcal{A}_{R,s}(u_s) - \lim_{s\to-\infty}\mathcal{A}_{R,s}(u_s)\right) + \int_{-\infty}^{\infty}\left(\frac{d}{ds}\mathcal{A}_{R,s}\right)(u_s) \; ds && (4.1) \\
&= -(0-0) - \int_{-\infty}^{\infty}\int_0^1 \left(\frac{d}{ds}\beta_R(s)\right) H_t(u_s(t)) \; dtds && (4.2) \\
&= \int_{-\infty}^{0}\int_0^1 \underbrace{\left(-\frac{d}{ds}\beta_R(s)\right)}_{\le 0} H_t(u_s(t)) \; dtds + \int_0^{\infty}\int_0^1 \underbrace{\left(-\frac{d}{ds}\beta_R(s)\right)}_{\ge 0} H_t(u_s(t)) \; dtds \\
&\le \int_{-\infty}^{0}\int_0^1 \left(-\frac{d}{ds}\beta_R(s)\right) \min H_t \; dtds + \int_0^{\infty}\int_0^1 \left(-\frac{d}{ds}\beta_R(s)\right) \max H_t \; dtds \\
&= -\Big(\beta_R(0) - \lim_{s\to-\infty}\beta_R(s)\Big)\int_0^1 \min H_t \; dt - \Big(\lim_{s\to\infty}\beta_R(s) - \beta_R(0)\Big)\int_0^1 \max H_t \; dt \\
&= -\beta_R(0)\int_0^1 \min H_t \; dt + \beta_R(0)\int_0^1 \max H_t \; dt \\
&\le \|H\|_{\text{Hofer}}
\end{aligned}$$

In the step from (4.1) to (4.2) we used that u_s converges to constant loops u_+, u_- for $s \to \pm\infty$ (see remark 3.2) and that the first term of the above action functional vanishes for constant loops while the second vanishes for $s \to \pm\infty$. □

Remark 4.4: The Hofer norm of a Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ is defined by

$$\|H\|_{\text{Hofer}} := \int_0^1 \left(\max_M H_t - \min_M H_t\right) dt.$$

4.2 The bubbling phenomenon

The Arzelà-Ascoli theorem is only applicable for sequences of functions which are equicontinuous. Thus we need the first derivatives of all $u \in \widehat{\mathcal{M}}(P)$ in all directions to be uniformly bounded. Note that the J-norm of the periodic vector field X_{H_t} is bounded since M and S^1 are compact and that also $|\beta_R|$ is bounded. This means that for a function which solves a perturbed Floer equation

$$\left(du - \tau \otimes \beta_R X_H(u)\right)^{(0,1)} = 0$$

the existence of a uniform bound on directional derivatives in one direction implies the existence of a uniform bound on directional derivatives in all directions.

The following phenomenon is known as 'bubbling off' of holomorphic spheres:

Proposition 4.5: *Suppose there is a sequence $(u_k, R_k)_{k\in\mathbb{N}} \subseteq \widehat{\mathcal{M}}(P)$ for some $P \in M$ with*

$$\max_{z\in S^2} \|du_k(z)\|_J \to \infty$$

for $k \to \infty$. Then there is a subsequence converging after reparametrization to a non-constant J-holomorphic sphere in M.

Proof. For all k, choose $z_k \in S^2$ such that $\|du_k(z_k)\| = \max_{z\in S^2} \|du_k(z)\|_J$. Since S^2 is compact, there is a subsequence (which we again denote by $(z_k)_{k\in\mathbb{N}}$) converging to a point $z_* \in S^2$. Passing again to another subsequence, we can assume that $(R_k)_{k\in\mathbb{N}}$ converges to some $R_* \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Choose local holomorphic coordinates (s,t) in a neighbourhood Ω of z_* in a way that $z_* = (0,0)$. Without loss of generality we can assume that all z_k are in Ω. Thus we can write $z_k = (s_k, t_k)$ for all k. Write

$$c_k := |\partial_s u_k(s_k, t_k)|_J \in \mathbb{R},$$

then $c_k \to \infty$ for $k \to \infty$.

Choose $\varepsilon > 0$ small enough that for all $k \in \mathbb{N}$, the ball $B_\varepsilon(s_k, t_k)$ of radius ε is contained in Ω. Define a reparametrized map

$$\begin{aligned} v_k : B_{\varepsilon\cdot c_k}(0,0) &\to M \\ v_k(s,t) &= u_k\left(s_k + \frac{s}{c_k}, t_k + \frac{t}{c_k}\right), \end{aligned}$$

then one has

$$\begin{aligned} \partial_s v_k(s,t) &= \partial_s u_k\left(s_k + \frac{s}{c_k}, t_k + \frac{t}{c_k}\right) \cdot \frac{1}{c_k} \\ \partial_t v_k(s,t) &= \partial_t u_k\left(s_k + \frac{s}{c_k}, t_k + \frac{t}{c_k}\right) \cdot \frac{1}{c_k}. \end{aligned}$$

This gives

$$\begin{aligned} |\partial_s v_k(0,0)|_J &= |\partial_s u_k(s_k,t_k)|_J \cdot \frac{1}{c_k} = 1, \\ \max |\partial_s v_k|_J &= \max |\partial_s u_k(s,t)|_J \cdot \frac{1}{c_k} = 1 \end{aligned}$$

and (this is a case of conformal reparametrization)

$$
\begin{aligned}
E_{R_k}(v_k; B_{\varepsilon\cdot c_k}(0,0)) &= \int_{(s,t)\in B_{\varepsilon\cdot c_k}(0,0)} |\partial_s v_k(s,t)|_J^2 \, dsdt \\
&= \int_{(s,t)\in B_{\varepsilon\cdot c_k}(0,0)} \left| \partial_s u_k \left(s_k + \tfrac{s}{c_k}, t_k + t\frac{t}{c_k} \right) \cdot \tfrac{1}{c_k} \right|_J^2 dsdt \\
&= \int_{(s,t)\in B_{\varepsilon}(s_k,t_k)} |\partial_s u_k(s,t)|_J^2 \, dsdt \\
&= E_{R_k}(u_k; B_\varepsilon(s_k,t_k)) \\
&\le E_{R_k}(u_k) \\
&\le \text{const},
\end{aligned}
$$

where the constant does not depend on k. The last estimate follows from proposition 4.3.

We will now see what kind of differential equation the v_k satisfy. For the sake of shorter notation, fix an integer k and write $\tilde{s} := s_k + \frac{s}{c_k}$ and $\tilde{t} := t_k + \frac{t}{c_k}$ for $(s,t) \in \mathbb{R} \times S^1$. Since u_k satisfies (3.1) for R_k, we have

$$
\begin{aligned}
\partial_s v_k(s,t) &= \partial_s u_k(\tilde{s},\tilde{t}) \cdot \frac{1}{c_k} \\
&= -J(u_k(\tilde{s},\tilde{t})) \left(\partial_t u_k(\tilde{s},\tilde{t}) - \beta_{R_k}(\tilde{s}) X_{H_{\tilde{t}}}(u_k(\tilde{s},\tilde{t})) \right) \cdot \frac{1}{c_k} \\
&= -J(u_k(\tilde{s},\tilde{t}))\partial_t u_k(\tilde{s},\tilde{t}) \cdot \frac{1}{c_k} + J(u_k(\tilde{s},\tilde{t}))\beta_{R_k}(\tilde{s}) X_{H_{\tilde{t}}}(u_k(\tilde{s},\tilde{t})) \cdot \frac{1}{c_k}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\partial_s v_k(s,t) + J(v_k(s,t)) \cdot \partial_t v_k(s,t) - J(v_k(s,t)) \cdot \frac{1}{c_k} \beta_{R_k}(\tilde{s}) X_{H_{\tilde{t}}}(v_k(s,t)) \\
&\quad = -J(u_k(\tilde{s},\tilde{t}))\partial_t u_k(\tilde{s},\tilde{t}) \cdot \frac{1}{c_k} + J u_k(\tilde{s},\tilde{t}))\beta_{R_k}(\tilde{s}) X_{H_{\tilde{t}}}(u_k(\tilde{s},\tilde{t})) \cdot \frac{1}{c_k} \\
&\qquad + J(u_k(\tilde{s},\tilde{t}))\partial_t u_k(\tilde{s},\tilde{t}) \cdot \frac{1}{c_k} - J(u_k(\tilde{s},\tilde{t})) \cdot \frac{1}{c_k} \beta_{R_k}(\tilde{s}) X_{H_{\tilde{t}}}(u_k(\tilde{s},\tilde{t})) \\
&\quad = 0. \qquad (4.3)
\end{aligned}
$$

(We might not be using the cylinder coordinates here which we used to define the function β_R. But as the reader will see, the only thing we need to know about $\beta_{R_k}(\tilde{s})$ in this computation is that its absolute value is less than or equal to 1.)

Let $K \subset \mathbb{C}$ be a compact subset. Then there is $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ the set K is contained in the ball $B_{\varepsilon\cdot c_k}(0,0)$. The image of each $B_{\varepsilon\cdot c_k}(0,0)$ lies in the compact manifold M and since we have $\max |\partial_s v_k|_J = 1$ for all k, the sequence v_k is equicontinuous. Thus the Arzela-Ascoli theorem 4.2 says that a subsequence of $\left(v_k\big|_K \right)_{k\ge k_0}$ converges in the $\mathcal{C}^0$-topology to a continuous limit

$$v\big|_K : K \to M.$$

As a continuous function on a compact domain, $v\big|_K$ is of class $L^p(K,M)$ for every number p and thus we can talk about weak partial derivatives $\partial_s v\big|_K$ and $\partial_t v\big|_K$ of $v\big|_K$. These are almost everywhere pointwise limits of the derivatives of the $v_k\big|_K$.

Now fix $(s,t) \in K$ instead and let k tend to infinity in equation (4.3). Note that $|\beta_{R_k}(\tilde{s})| \leq 1$ independently of k and that X_H is bounded. Hence for almost all[1] $(s,t) \in K$ in the limit we get the following:

$$\partial_s v\big|_K(s,t) + J\left(v\big|_K(s,t)\right) \cdot \partial_t v\big|_K(s,t) = 0 \tag{4.4}$$

Now, by elliptic regularity (see corollary 5.21 in chapter 5), $v\big|_K$ is smooth on the interior of K and hence

$$(v_k\big|_K)_k \longrightarrow v\big|_K$$

with all derivatives.

We now define a smooth function

$$\begin{aligned} &v : \mathbb{C} \longrightarrow M \\ &v(s,t) := v\big|_K(s,t) \quad \text{for some compact set } K \subset \mathbb{C} \text{ with } (s,t) \in K \end{aligned}$$

– it is clear that $(v_k)_k$ converges to v with all derivatives on all compact subsets of $\mathbb{C}$. Moreover, by equation (4.4), v is J-holomorphic on $\mathbb{C}$.

Besides, we know

$$\begin{aligned} |\partial_s v(0,0)|_J &= \lim_{k\to\infty} |\partial_s v_k(0,0)|_J = 1 \\ \max |\partial_s v|_J &= \lim_{k\to\infty} \max |\partial_s v_k|_J = 1 \end{aligned}$$

and for the energy of v one computes

$$E_{R_*}(v) = \lim_{k\to\infty} E_{R_k}(v_k; B_{\varepsilon\cdot c_k}(0,0)) = \lim_{k\to\infty} E_{R_k}(u_k; B_\varepsilon(s_k,t_k)) \leq \lim_{k\to\infty} E_{R_k}(u_k) < \infty.$$

By the removable singularity theorem (cf. corollary 2.14), v extends to a J-holomorphic sphere

$$v : S^2 = \mathbb{C} \cup \{\infty\} \to M.$$

Since $|\partial_s v(0,0)|_J = 1$, it is not constant.

□

Because of the symplectic asphericity of M, there can only be constant J-holomorphic spheres ('bubbles') and thus we get the uniform bounds on the derivatives as a corollary.

Corollary 4.6: *For every $P \in M$ there is a constant $C > 0$ such that for all $(u,R) \in \widehat{\mathcal{M}}(P)$ one has*

$$\max_{z\in S^2} \|du_k(z)\|_J < C.$$

Proof. If not, proposition 4.5 gives a non-constant J-holomorphic sphere in M. But since we assumed M to be symplectically aspherical, there can be no such (see remark 2.17). □

Remark 4.7: In the proof of proposition 4.5 we did in no way use the point constraint $u(0,0) = P$. This means that the constant can be chosen uniformly for all $P \in M$.

[1] Note that we do not know so far whether $\partial_s v\big|_K$ and $\partial_t v\big|_K$ are continuous, so they need not be pointwise limits in all points.

4.3 Proof of proposition 4.1

Proof of proposition 4.1. Take a sequence $(R_k, u_k)_{k\in\mathbb{N}} \subseteq \widehat{\mathcal{M}}(P)$. We have to find either a converging subsequence or two fixed points x_-, x_+ of $\Phi^1_{H_t}$ with different symplectic action.

Case 1: $(R_k)_{k\in\mathbb{N}}$ has a converging subsequence.

Without loss of generality assume that $R_k \to R_* \in \mathbb{R}_{\geq 0}$ for $k \to \infty$.

S^2 is compact, and since the first derivatives of the u_k are uniformly bounded we have equicontinuity and thus we can use the Arzelà-Ascoli theorem 4.2. This tells us that there is a subsequence (still denoted by $(u_k)_{k\in\mathbb{N}}$) converging in the $\mathcal{C}^0$-topology to a continuous limit

$$u : S^2 \to M.$$

For the derivatives of the u_k we cannot use the Arzelà-Ascoli theorem again since they do not have to be equicontinuous. But since S^2 is compact and the partial derivatives $\partial_s u_k$ are continuous, they are p-integrable for any chosen $p \in [1, \infty)$. L^p-spaces are complete. For that reason there is a subsequence of the $\partial_s u_k$ converging to an L^p-function $\partial_s u$. This L^p-function is a weak partial derivative of $u : S^2 \to M$. In exactly the same way (possibly passing to another subsequence) we also get a weak partial derivative $\partial_t u$.

Every u_k satisfies the perturbed Floer equation for R_k:

$$\Big(du_k - \tau \otimes \beta_{R_k} X_H(u_k)\Big)^{(0,1)} = 0$$

In local cylindrical coordinates (s,t) on the cylinder Z_{R_k} this means

$$\partial_s u_k(s,t) + J(u_k(s,t))\Big(\partial_t u_k(s,t) - \beta_{R_k}(s) X_{H_t}(u_k(s,t))\Big) = 0$$

for all (s,t). Now, letting k tend to infinity, we can use the weak derivatives to get an equation which may not hold everywhere, but at least almost everywhere:

$$\partial_s u(s,t) + J(u(s,t))\Big(\partial_t u(s,t) - \beta_{R_*}(s) X_{H_t}(u(s,t))\Big) = 0$$

Hence u is a weak solution of the perturbed Floer equation for R_*. By elliptic regularity (see proposition 5.22), u has to be smooth. Then $u \in \mathcal{M}_{R_*}(P)$ and $(R_*, u) \in \widehat{\mathcal{M}}(P)$. So the original sequence $(R_k, u_k)_{k\in\mathbb{N}}$ has a subsequence which converges in $\widehat{\mathcal{M}}(P)$.

Case 2: $(R_k)_{k\in\mathbb{N}}$ does not have a converging subsequence.

Then without loss of generality we can assume that $R_k \to \infty$ for $k \to \infty$.

For every fixed positive integer $T \in \mathbb{N}$ consider the compact cylinder

$$Z_T = \Psi_T\Big([-T-1, T+1] \times S^1\Big) \subset S^2$$

and the restricted sequence $\Big(u_k|_{Z_T}\Big)_k$. By the Arzelà-Ascoli theorem 4.2 there is a subsequence (still denoted by $(u_k)_k$) such that $\Big(u_k|_{Z_T}\Big)_k$ converges in the $\mathcal{C}^0$-topology to a continuous limit

$$u_T : Z_T \to M.$$

Since u_T is a continuous function on a compact topological space, it has weak partial derivatives. These are the L^p-limits of the partial derivatives of the $u_k\big|_{Z_T}$. Each $u_k\big|_{Z_T}$ satisfies equation (3.1), and so for u_T we get

$$\partial_s u_T + J(u_T)\Big(\partial_t u_T - X_H(u_T)\Big) = 0$$

on Z_T. By elliptic regularity (see corollary 5.21), u_T is smooth. We can pull it back to $[-T-1, T+1] \times S^1$ with Ψ_T and thus get a smooth function

$$\begin{aligned} &v_T : [-T-1, T+1] \times S^1 \longrightarrow M \\ &v_T = u_T \circ \Psi_T \end{aligned}$$

which is the limit of the sequence $\Big(u_k\big|_{Z_T} \circ \Psi_T\Big)_{k\in\mathbb{N}}$. By choosing the subsequences of $(u_k)_k$ for all $T \in \mathbb{N}$ successively, we can assume that each is a subsequence of the previous and thus for $T_1 < T_2$,

$$v_{T_2}\big|_{Z_{T_1}} = v_{T_1}.$$

This is why we can define a function $v : \mathbb{R} \times S^1 \to M$ by

$$v(s,t) := v_T(s,t) \quad \text{for some } T > s$$

– surely, this function is again smooth, and since our coordinates on each Z_T were chosen to agree with those on the cylinder, v satisfies

$$\partial_s v + J(v)\Big(\partial_t v - X_H(v)\Big) = 0$$

everywhere on $\mathbb{R} \times S^1$. Moreover, the diagonal sequence

$$\left(u_k\Big|_{Z_k} \circ \Psi_k : [-k-1, k+1] \times S^1 \longrightarrow M\right)_{k\in\mathbb{N}}$$

converges to v in the $\mathcal{C}^\infty_{loc}$-topology, that is with all derivatives on all compact subsets of the infinite cylinder $\mathbb{R} \times S^1$.

Fix $s \in \mathbb{R}$ and consider the loop $v_s = v(s,\cdot)$. It is the limit of loops which come from maps $S^2 \to M$, and as such it is contractible.

From proposition 4.3 we know that there is a constant $C > 0$ such that

$$\tfrac{1}{2}\int_{-\infty}^{\infty} |\partial_s u_k(s,t)|^2 + |\partial_t u_k(s,t) - \beta_{R_k}(s) X_{H_t}(u_k(s,t))|^2 \, dtds = E_{R_k}(u_k) \leq C$$

for all $k \in \mathbb{N}$. Again because of the choice of our coordinates, in the limit we get

$$\tfrac{1}{2}\int_{-\infty}^{\infty} |\partial_s v(s,t)|^2 + |\partial_t v(s,t) - X_{H_t}(v(s,t))|^2 \, dtds = E_\infty(v) = E(v) \leq C,$$

where $E(v)$ denotes the energy of Floer cylinders as in definition 2.21, and so $v \in \mathcal{M}_F$. By proposition 2.28 there are $x_\pm \in \mathrm{Crit}\mathcal{A}_H$ such that

$$\lim_{s\to\pm\infty} v(s,\cdot) = x_\pm(\cdot).$$

By proposition 2.25 it is

$$E(v) = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+).$$

The goal was to find $x_\pm \in \mathrm{Crit}\mathcal{A}_H$ of different symplectic action, and if $E(v) \neq 0$ we are done. But if $E(v) = 0$, then by remark 2.22 v is constant. Because of

$$u(0,0) = \lim_{k\to\infty} u_k(0,0) = \lim_{k\to\infty} P = P$$

this would mean that $u \equiv P$ and thus that $x_\pm \equiv P$. But this cannot be the case since we assumed that P was not a fixed point of Φ_H^1. So $E(u) \neq 0$ and thus

$$\mathcal{A}_H(x_+) \neq \mathcal{A}_H(x_-).$$

If for all possible choices of sequences $(R_k, u_k)_{k\in\mathbb{N}} \subseteq \widehat{\mathcal{M}}(P)$ we happen to stay in case 1, $\widehat{\mathcal{M}}(P)$ is compact. If case 2 happens for at least one sequence, we find two fixed points of Φ_H^1, which, understood as loops in M, have different symplectic action.

□

CHAPTER 5

Analytic Setup

We obtained the moduli space $\mathcal{M}_R(P)$ as the space of all smooth maps $u : S^2 \to M$ which satisfy $u(0,0) = P$, $[u] = 0$ and

$$\left(du - \tau \otimes \beta_R X_H(u)\right)^{0,1} = 0.$$

For a fixed function u, the expressions du, $\tau \otimes \beta_R X_H(u)$ and $(du - \tau \otimes \beta_R X_H(u))^{0,1}$ denote a collection of linear maps

$$T_z S^2 \longrightarrow T_{u(z)} M$$

for all $z \in S^2$ – this means they are 1-forms with values in the bundle u^*TM over S^2. The moduli space $\mathcal{M}_R(P)$ is the intersection of the section

$$u \mapsto \left(du - \tau \otimes \beta_R X_H(u)\right)^{0,1}$$

in this bundle with the zero section. In a finite-dimensional setting, the implicit function theorem can be used to gather information about such intersection. The bundles concerned here are infinite-dimensional. There is an infinite-dimensional version of the implicit function theorem, though, which we can use – but since it holds in a Banach setting only, there is some more work to do.

5.1 Fredholm operators

We need the definition of a Fredholm operator and some of its properties. This and much more can be found in [6].

Definition 5.1: Let X and Y be Banach spaces. A bounded linear operator $L : X \to Y$ is said to have the **Fredholm** property (or is simply called **Fredholm operator**) if

- the image of L is closed in Y,
- $\dim \ker L < \infty$, and
- $\dim \operatorname{coker} L = \operatorname{codim} \operatorname{im} L < \infty$.

In this case, its Fredholm index is

$$\operatorname{ind} L := \dim \ker L - \dim \operatorname{coker} L.$$

Note if the dimensions of kernel and cokernel are finite, then the image is automatically closed. Moreover, when X and Y are of finite-dimension every linear operator $L : X \to Y$ is automatically Fredholm, and its index is

$$\mathrm{ind}(L) = \mathrm{dim}X - \mathrm{dim}Y.$$

Proposition 5.2: *Let X and Y be Banach spaces and let $K : X \to Y$ be a compact operator. Then a bounded linear operator $L : X \to Y$ has the Fredholm property if and only if $L + K$ has, and in this case it is*

$$\mathrm{ind}(L) = \mathrm{ind}(L + K).$$

5.2 Fredholm maps

The notion of a smooth Banach manifold is a generalization of the notion of a finite dimensional smooth manifold.

Definition 5.3: A smooth **Banach manifold** is a topological space X with charts into a Banach space around each point in a way that the transition functions are smooth.

Between Banach manifolds we can consider Fredholm maps.

Definition 5.4: Let X and Y be Banach manifolds. A smooth map $f : X \to Y$ is called a **Fredholm map** if for all $x \in X$

$$df(x) : T_xX \longrightarrow T_{f(x)}Y$$

is a Fredholm operator.

Remark 5.5: The set of Fredholm operators between two spaces is open in the set of all bounded linear operators, and the Fredholm index is a continuous function from this set to the integers – in particular, it is locally constant. Since T_xX and $T_{f(x)}Y$ depend on $x \in X$ continuously, also the function

$$x \longmapsto \mathrm{ind}\ df(x)$$

is locally constant. So if X is connected, the Fredholm index of $df(x)$ does not depend on the choice of $x \in X$ and we can define the Fredholm index of f to be

$$\mathrm{ind}\ f := \mathrm{ind}\ df(x)$$

for any $x \in X$.

Definition 5.6: Let $f : X \to Y$ be a smooth map between Banach manifolds (not necessarily Fredholm). A point $y \in Y$ is called a **regular value** of f if for all $x \in f^{-1}(\{y\})$

$$df(x) : T_xX \longrightarrow T_{f(x)}Y$$

is surjective and has a bounded right inverse.

Remark 5.7: If $df(x)$ is surjective, then surely there is a linear operator $Q : T_{f(x)}Y \to T_xX$ with $df(x) \circ Q = id_{T_{f(x)}Y}$ – the only question is if Q is bounded or not. If f is Fredholm and $df(x) \circ Q = id_{T_{f(x)}Y}$, then

$$T_xX = \ker df(x) \oplus \operatorname{im} Q,$$

where $\ker df(x)$ is finite-dimensional and hence $\operatorname{im} Q$ is closed. So

$$df(x)\big|_{\operatorname{im} Q} : \operatorname{im} Q \longrightarrow T_{f(x)}Y$$

is a bijective bounded linear operator between Banach spaces and hence by the Bounded Inverse theorem its inverse

$$Q : T_{f(x)}Y \longrightarrow \operatorname{im} Q$$

is bounded. So if $f : X \to Y$ is a Fredholm map and $y \in Y$ is a point such that for all $x \in f^{-1}(\{y\})$ the differential $df(x) : T_xX \to T_{f(x)}Y$ is surjective, then y is a regular value of f.

Theorem 5.8 (Implicit function theorem)**:** *Let X and Y be Banach manifolds. If $y \in Y$ is a regular value of $f : X \to Y$, then $N := f^{-1}(\{y\})$ is a smooth manifold and*

$$T_xN = \ker df(X)$$

for all $x \in N$. In particular, if f is Fredholm and X is connected, N is a smooth manifold of dimension $\dim N = \operatorname{ind} f$.

A proof can be found in [7], it is theorem A.3.3 there.

In the case of sections in a vector bundle $E \to B$, the corresponding notions are constructed on the vertical differential, that is the differential in fibre direction: A section $s : B \to E$ is Fredholm if kernel and cokernel of the vertical differential

$$d^v s(x) : T_xB \to E_x$$

(E_x denotes the fibre over x) are finite-dimensional, and 0 is a regular value if this vertical differential is surjective and has a bounded right inverse for all $x \in B$ with $s(x) = (x, 0)$.

5.3 Sobolev spaces

Spaces of smooth maps are not complete. That is why we are going to work with Sobolev spaces. The introduction we give here is more or less taken from the appendix of [7]; for more details see for example [1].

Let $U \subset \mathbb{R}^n$ be a bounded open subset and choose a number $p \in [1, \infty)$.

Definition 5.9:

- For measurable functions $u : U \to \mathbb{R}$,

$$\|u\|_{L^p(U)} := \left(\int_{x\in U} |u(x)|^p dx\right)^{\frac{1}{p}}$$

defines a seminorm which is called the L^p-norm.

- Two measurable functions $u, v : U \to \mathbb{R}$ are called equivalent if

$$\|u - v\|_{L^p(U)} = 0.$$

This defines an equivalence relation. The quotient space

$$L^p(U) := \{\text{measurable functions } u : U \to \mathbb{R} \text{ with finite } L^p\text{-norm}\} / \text{equivalence}$$

is a vector space; its elements are called **L^p-functions** and often thought of as functions which are defined up to changes on sets of measure zero.

- For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ of order $|\alpha| := \sum_{i=1}^n \alpha_i$, let

$$d^\alpha f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} f$$

denote the α-th derivative of f, if existent.

- A function $v \in L^p(U)$ is called the α-th **weak derivative** of a function $u \in L^p(U)$ if

$$\int_{x \in U} v(x) \cdot \phi(x)\, dx = \int_{x \in U} u(x) \cdot d^\alpha \phi(x)\, dx$$

for all compactly supported test functions $\phi \in \mathcal{C}_c^\infty(U)$. (If u has a representative which has a continuous α-th derivative $d^\alpha u$, then this derivative satisfies the above equation due to integration by parts.) If such a v exists, we denote it by $d^\alpha u$.

- For $k \in \mathbb{N}$ define the **Sobolev space**

$$W^{k,p}(U) := \left\{ u \in L^p(U) \,\middle|\, \forall \alpha \text{ with } |\alpha| \leq k \text{ there is } d^\alpha u \in L^p(U) \right\}$$

to be the space of all L^p-functions that have weak derivatives up to order k. For such a function, the **Sobolev-(k,p)-norm** is defined as

$$\|u\|_{k,p} := \|u\|_{W^{k,p}(U)} := \sum_{|\alpha| \leq k} \left(\int_{x \in U} |d^\alpha u(x)|^p dx \right)^{\frac{1}{p}}.$$

Remark 5.10: If $K \subset U$ is a compact subset, then by the Hölder inequality one has

$$\|u\|_{L^1(K)} \leq \|u\|_{L^p(K)} \cdot \|1\|_{L^q(K)} \text{ for } q \in [1, \infty) \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

so every L^p function has finite L^1-norm over K and the integrals in the definition of weak derivatives are finite.

There is another characterization of Sobolev spaces which does not use the notion of weak derivatives:

Proposition 5.11: *$W^{k,p}(U)$ is the completion of $\mathcal{C}^\infty(\bar{U})$ with respect to the $W^{k,p}$-norm.*

This characterization also implies the next theorem, which is the reason why we work with Sobolev spaces:

Theorem 5.12: *$W^{k,p}(U)$ is a Banach space.*

Still, in the end we are interested in functions which are smooth or at least continuous. The famous Sobolev embedding theorems give some information about the existence of differentiable representatives:

Theorem 5.13 (Sobolev embedding theorem)**:** *Let $U \subseteq \mathbb{R}^n$ be a bounded open subset and $k \in \mathbb{N}$, $p \in [1, \infty)$ numbers such that $kp > n$. Then there is a constant $C = C(k, p, U) > 0$ such that*

$$\|u\|_{\mathcal{C}^{k-1-\lfloor \frac{n}{p} \rfloor}} \leq C \cdot \|u\|_{k,p}$$

for all $u \in \mathcal{C}^\infty(\bar{U})$. In particular, there is a continuous embedding

$$W^{k,p}(U) \hookrightarrow \mathcal{C}^{k-1-\lfloor \frac{n}{p} \rfloor}(U)$$

and moreover, this embedding is compact.

In particular this means that in the case $kp > n$ all $W^{k,p}$-functions can be represented by continuous functions and that the following holds:

Corollary 5.14: *Choose any $p \in [1, \infty)$. Then*

$$\mathcal{C}^\infty(\overline{U}) = \bigcap_{k=1}^{\infty} W^{k,p}(U).$$

So far we defined $W^{k,p}$ spaces only for functions from an open subset U of $\mathbb{R}^n$ into the real numbers. It is easy to generalize it for functions $u : U \to \mathbb{R}^m$, $m \in \mathbb{N}$, and from that we can generalize to functions $u : N \to \mathbb{R}^m$ for a smooth compact manifold N.

Definition 5.15:

- Let $u : U \to \mathbb{R}^m$ be a measurable function. Write $u = (u_1, \ldots, u_m)$ with $u_1, \ldots, u_m : U \to \mathbb{R}$. Then the **$W^{k,p}$-norm** of u is

  $$\|u\|_{k,p} := \|u\|_{W^{k,p}(U,\mathbb{R}^m)} := \sum_{j=1}^{m} \|u_j\|_{W^{k,p}(U)}.$$

  (Strictly speaking, it only is a seminorm until we pass to equivalence classes.)

- Fix a finite cover of the compact manifold N by charts

  $$\phi_i : U_i \to B_i$$

  for $i = 1, \ldots, r$, where $U_i \subseteq N$ are open subsets and $B_i \subseteq \mathbb{R}^{\dim N}$ are open balls. Then for $u : N \to \mathbb{R}^m$ we can define

  $$\|u\|_{k,p} := \|u\|_{W^{k,p}(N,\mathbb{R}^m)} := \sum_{i=1}^{r} \|u \circ \phi_i^{-1}\|_{W^{k,p}(B_i,\mathbb{R}^m)}.$$

- The space of **Sobolev-(k,p)-functions** from N to $\mathbb{R}^m$ is

  $$W^{k,p}(N, \mathbb{R}^m) := \text{completion of } \mathcal{C}^\infty(N, \mathbb{R}^m) \text{ with respect to the } W^{k,p}\text{-norm.}$$

Remark 5.16: The definition of $\|u\|_{W^{k,p}(N,\mathbb{R}^m)}$ depends on the choice of cover and charts. But the norms we get for different choices are all equivalent and so the definition of $W^{k,p}(N,\mathbb{R}^m)$ does not depend on any choices.

For defining $W^{k,p}$-spaces of functions from one manifold to another, one fixes an embedding of the target manifold into some big Euclidean space. This only is well defined, though, in the case that $k \cdot p > \dim N$. For our purpose it is enough to generalize for sections in a bundle over a manifold, and for that one does not need an embedding into Euclidean space.

Definition 5.17: Let $E \to N$ be a vector bundle of rank m over a compact manifold N. Fix a finite cover by charts

$$\phi_i : U_i \to B_i, \qquad i = 1, \dots, r$$

as before and trivializations

$$B_i \times \mathbb{R}^m \cong U_i \times \mathbb{R}^m \cong E\big|_{U_i}$$

for $i = 1, \dots, r$. Let $u : N \to E$ be a section. We can understand $u \circ \phi_i^{-1} : B_i \to E$ as $u \circ \phi_i^{-1} : B_i \to \mathbb{R}^m$ via the trivialization and then define

$$\|u\|_{k,p} := \|u\|_{W^{k,p}(N,E)} := \sum_{i=1}^{r} \|u \circ \phi_i^{-1}\|_{W^{k,p}(B_i,\mathbb{R}^m)}.$$

Again, this does depend on the choice of charts and trivializations, but the norms one gets for all these choices are equivalent and the space

$$W^{k,p}(N,E) := \text{completion of } \mathcal{C}^\infty(N,E) \text{ with respect to the } W^{k,p}\text{-norm}$$

of $W^{k,p}$-sections of the bundle $E \to N$ does not depend on any choices.

Remark 5.18: As in corollary 5.14, one can use generalized versions of the Sobolev embedding theorem 5.13 to show that for $k \cdot p > n$ every $W^{k,p}$-function is continuous and that

$$\mathcal{C}^\infty(N,E) = \bigcap_{k=1}^{\infty} W^{k,p}(N,E).$$

Remark 5.19: There are local versions of the Sobolev spaces: A function $u : N \to \mathbb{R}^n$ is an element of $W^{k,p}_{loc}(N,\mathbb{R}^m)$ if $u\big|_K \in W^{k,p}(K,\mathbb{R}^m)$ for every compact subset $K \subseteq N$. This is often the space one works with, because in the case of a non-compact domain N constant functions do not belong to $W^{k,p}(N,\mathbb{R}^m)$.

5.4 Two Banach bundles

In order to be able to use the implicit function theorem 5.8 we have to describe the moduli spaces $\mathcal{M}_R(P)$ and $\widehat{\mathcal{M}}(P)$ as regular level sets of Fredholm maps between Banach manifolds. Denote by

$$\begin{aligned} \mathcal{B} : &= \left\{ u \in W^{1,p}(S^2, M) \mid [u] = 0 \in \pi_2(M), u(0,0) = P \right\} \\ &= \overline{\left\{ u \in \mathcal{C}^\infty(S^2, M) \mid [u] = 0 \in \pi_2(M), u(0,0) = P \right\}}^{W^{1,p}} \end{aligned}$$

the Banach manifold of all $W^{1,p}$-maps from S^2 to M for some fixed $p > 2$ which belong to the right homotopy class and satisfy the point constraint. The value of the Sobolev function u at the point $(0,0) \in S^2$ is to be understood as the value of the continuous representative of u given by the Sobolev embedding theorem 5.13.

For every $u \in \mathcal{B}$ there is a Banach space

$$\mathcal{E}_u := \left\{ \eta \;\middle|\; \eta \text{ is a 1-form of class } L^p \text{ on } S^2 \text{ with values in } u^*TM \right\}$$

and together these form a bundle

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{B} \\ \eta &\longmapsto u \text{ for } \eta \in \mathcal{E}_u \end{aligned}$$

with fibre $\mathcal{E}_u$ over $u \in \mathcal{B}$. The zero section in this bundle will be denoted by $\mathcal{O}_{\mathcal{E}}$. For $R \in \mathbb{R}_{\geq 0}$ we define a section

$$\begin{aligned} \mathcal{F}_R : \mathcal{B} &\longrightarrow \mathcal{E} \\ u &\longmapsto \Big(du - \tau \otimes \beta_R X_H(u) \Big)^{0,1} \end{aligned}$$

– it is now clear that $\mathcal{M}_R(P) \subseteq \mathcal{F}_R^{-1}(\mathcal{O}_{\mathcal{E}})$. Indeed, by elliptic regularity of $\mathcal{F}_R$ (see proposition 5.22 in the next section), we will see that in fact there is equality:

$$\mathcal{M}_R(P) = \mathcal{F}_R^{-1}(\mathcal{O}_{\mathcal{E}})$$

Now write

$$\widehat{\mathcal{B}} := \mathbb{R}_{\geq 0} \times \mathcal{B}$$

and consider the bundle

$$\begin{aligned} \widehat{\mathcal{E}} &\longrightarrow \widehat{\mathcal{B}} \\ \eta &\longmapsto (R,u) \text{ for } \eta \in \mathcal{E}_{(R,u)} \end{aligned}$$

with fibre

$$\widehat{\mathcal{E}}_{(R,u)} = \left\{ (R,u,\eta) \;\middle|\; \eta \text{ is a 1-form on } S^2 \text{ with values in } u^*TM \right\} \qquad \cong \mathcal{E}_u$$

over $(R,u) \in \widehat{\mathcal{B}}$. In this bundle the zero section will be denoted by $\mathcal{O}_{\widehat{\mathcal{E}}}$, and there is a section

$$\begin{aligned} \widehat{\mathcal{F}} : \widehat{\mathcal{B}} &\longrightarrow \widehat{\mathcal{E}} \\ (R,u) &\longmapsto \Big(du - \tau \otimes \beta_R X_H(u) \Big)^{0,1} = \mathcal{F}_R(u). \end{aligned}$$

Again, it is obvious that $\widehat{\mathcal{M}}(P) \subseteq \widehat{\mathcal{F}}^{-1}(\mathcal{O}_{\widehat{\mathcal{E}}})$ and by elliptic regularity of $\widehat{\mathcal{F}}$ (see proposition 5.22) we will see that in fact

$$\widehat{\mathcal{M}}(P) = \widehat{\mathcal{F}}^{-1}(\mathcal{O}_{\widehat{\mathcal{E}}}).$$

5.5 Elliptic regularity

In this section we want to see that indeed all weak solutions of the differential equation

$$\Big(du - \tau \otimes \beta_R X_H(u)\Big)^{0,1} = 0$$

are smooth and that the $W^{k,p}$-topology coincides with the $\mathcal{C}^\infty$-topology on our moduli spaces. Roughly speaking, this results from the fact that a function has higher regularity (that is, 'more derivatives') than its derivatives. A famous example is the Laplace equation

$$\Delta u = 0$$

for a locally integrable function $u : U \to \mathbb{R}^2$, where $U \subseteq \mathbb{R}^2 \cong \mathbb{C}$ is an open subset and Δ denotes the Laplace operator $\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$. Weyl's lemma states that every weak solution u of this equation is smooth and thus a strong solution. Similar results hold for the Cauchy-Riemann operator.

In [7], McDuff and Salamon state and prove a version which also respects boundary conditions and time-dependent almost complex structures. Without these, it reads as follows:

Proposition 5.20 (Proposition B.4.9 in [7])**:** *Let $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ be an open subset, $l \in \mathbb{N}_{>0}$ a positive integer and $p > 2$. Assume $J \in W^{l,p}(\Omega, \mathbb{R}^{2n\times 2n})$ is such that $J^2 = -id_{\mathbb{R}^{2n}}$. Then for every $k \in \{0, \dots, l\}$ the following holds: If $u \in L^p_{loc}(\Omega, \mathbb{R}^{2n})$ and $\eta \in W^{k,p}_{loc}(\Omega, \mathbb{R}^{2n})$ satisfy*

$$\partial_s u + J(u)\partial_t u = \eta \tag{5.1}$$

in the weak sense, then u is an element of $W^{k+1,p}_{loc}(\Omega, \mathbb{R}^{2n})$.

Corollary 5.21: *Weak solutions of the Cauchy-Riemann equation*

$$\partial_s u + J(u)\partial_t u = 0$$

and of the Floer equation

$$\partial_s u + J(u)\Big(\partial_t u - X_H(u)\Big) = 0$$

are smooth.

Proof. The function $\eta \equiv 0$ lies in $W^{k,p}$ for every k and p, so the above proposition can be applied for every $k \in \mathbb{N}$ and smoothness follows from corollary 5.14. For the function $\eta := J(u)X_H(u)$ we use a bootstrapping argument: It is the composition of smooth functions (which are bounded and have bounded derivatives on each compact subdomain) with the $W^{1,p}$-function u and so it is of class $W^{1,p}$. Therefore by proposition 5.20 u is of class $W^{2,p}$. But now η also is of class $W^{2,p}$ and we can use the same argument to show that u is of class $W^{3,p}$. This bootstrapping argument shows that u is of class $W^{k,p}$ for every $k \in \mathbb{N}$ and thus, by corollary 5.14, it is smooth. $\square$

Now we can use this for our special setting.

Proposition 5.22: *It is* $\mathcal{M}_R = \mathcal{F}_R^{-1}(\mathcal{O}_{\mathcal{E}})$ *and* $\widehat{\mathcal{M}} = \widehat{\mathcal{F}}^{-1}(\mathcal{O}_{\widehat{\mathcal{E}}})$.

Proof. We have to show that every $u \in W^{1,p}(S^2, M)$ which satisfies

$$\Big(du - \tau \otimes \beta_R X_H(u)\Big)^{0,1} = 0$$

for some $R \in \mathbb{R}_{\geq 0}$ is smooth. Since differentiability is a local property it is enough to show this in local coordinates. So we can assume that $u \in W^{1,p}(\Omega, M)$ for some $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ and the equation reduces to

$$\partial_s u + J(u)\Big(\partial_t u - \beta_R X_H(u)\Big) = 0,$$

which can be written as

$$\partial_s u + J(u)\partial_t u = \beta_R J(u) X_H(u).$$

The vector field $\eta := \beta_R J(u) X_H(u) : \Omega \to \mathbb{R}^2$ is not smooth, but as a composition of smooth maps (which are bounded and have bounded derivatives on the compact set $\bar{\Omega}$) with u it is of class $W^{1,p}$. As above, we now start a bootstrapping argument: By proposition 5.20 we first get that u is of class $W^{2,p}$. But then so is η and thus we use the proposition again to get that u is of class $W^{3,p}$ – and so on, which inductively means that u is of class $W^{k,p}$ for every p and thus by remark 5.18 it is smooth.

$\square$

CHAPTER 6

Fredholm Analysis

It is the goal of this chapter to show that both $\mathcal{F}_0$ and $\widehat{\mathcal{F}}$ are Fredholm sections in the bundles $\mathcal{E} \to \mathcal{B}$ and $\widehat{\mathcal{E}} \to \widehat{\mathcal{B}}$ respectively. For this, we need their linearizations.

6.1 The linearized operators

The tangent space of $\mathcal{B}$ at $u \in \mathcal{B}$ is given by all vector fields of class $W^{k,p}$ along u which vanish in the point $(0,0)$, that is

$$T_u\mathcal{B} = \left\{ \xi : S^2 \to TM \;\middle|\; \xi \text{ is of class } W^{1,p}, \forall z \in S^2 : \xi(z) \in T_{u(z)}M, \xi(0,0) = 0 \in T_P M \right\}.$$

Again we have to say what is meant by the equation $\xi(0,0) = 0$ in the case of a Sobolev function ξ. But as in the definition of $\mathcal{B}$, we know from the Sobolev emdedding theorem 5.13 that there is a continuous representative of ξ and we can demand that the equation hold for this representative.

Since we are working with a section in a bundle, the interesting part of the linearization is the vertical differential. Along the zero section $\mathcal{O}_{\mathcal{E}}$, the tangent space of the total space $\mathcal{E}$ at $(u,0)$ is a direct sum

$$T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u$$

and using the projection

$$\pi_u : T_u\mathcal{B} \oplus \mathcal{E}_u \longrightarrow \mathcal{E}_u$$

we can write the vertical differential of $\mathcal{F}_0$ at $u \in \mathcal{M}_0(P)$ as

$$D_u := \pi_u \circ d\mathcal{F}_0(u) : T_u\mathcal{B} \longrightarrow \mathcal{E}_u.$$

Outside the zero section $\mathcal{O}_{\mathcal{E}}$, one can also split

$$T_{(u,\eta)}\mathcal{E} = (T_{(u,\eta)}\mathcal{E})^h \oplus (T_{(u,\eta)}\mathcal{E})^v,$$

into horizontal and vertical subspaces, and the vertical subspace can be canonically defined by

$$(T_{(u,\eta)}\mathcal{E})^v := T_\eta\mathcal{E}_u \cong \mathcal{E}_u.$$

The horizontal subspace, though, depends on the complement we choose. This corresponds to the choice of a connection in the bundle $\mathcal{E} \to \mathcal{B}$.

A connection on $\mathcal{E} \to \mathcal{B}$ comes from a connection on the tangent bundle $TM \to M$. It will be useful to work with one which preserves the almost-complex structure J, so we choose the connection

$$\widetilde{\nabla}_v X := \nabla_v X - \tfrac{1}{2} J(\nabla_v J) X$$

where ∇ is the Levi-Civita connection of the Riemannian metric g_J.

Lemma 6.1: *For each $u \in \mathcal{B}$, the vertical differential of $\mathcal{F}_0$ at u is given by*

$$D_u \xi = \tfrac{1}{2}\Big(\nabla \xi + J(u) \circ \nabla \xi \circ i\Big) - \tfrac{1}{2} J(u)(\nabla_\xi J)(u) \partial_J u$$

for $\xi \in T_u \mathcal{B}$.

Proof. For understanding this expression, first note that $D_u \xi$ is meant to be a 1-form on S^2 with values in $u^* TM$. Inserting a vector field ζ on S^2 into $\nabla \xi$ means

$$\nabla_\zeta \xi := \nabla_{du(\zeta)} \xi.$$

Let $(u_\alpha)_\alpha \subseteq \mathcal{B}$ be a family of maps such that $u_0 = u$ and

$$\frac{d}{d\alpha}\bigg|_{\alpha=0} u_\alpha = \xi.$$

In the following we shortly write $\widetilde{\nabla}_\alpha := \widetilde{\nabla}_{\frac{d}{d\alpha} u_\alpha}$ and $\nabla_\alpha := \nabla_{\frac{d}{d\alpha} u_\alpha}$. Also note that $J(u_\alpha)$ anticommutes with $J(u_\alpha)(\nabla_\alpha J)(u_\alpha)$: It is

$$\begin{aligned} -(\nabla_\alpha J)(u_\alpha) &= (\nabla_\alpha - J)(u_\alpha) \\ &= (\nabla_\alpha J \cdot J \cdot J)(u_\alpha) \\ &= -(\nabla_\alpha J)(u_\alpha) + J(u_\alpha)(\nabla_\alpha J)(u_\alpha) J(u_\alpha) - (\nabla_\alpha J)(u_\alpha), \end{aligned}$$

so

$$(\nabla_\alpha J)(u_\alpha) = J(u_\alpha)(\nabla_\alpha J)(u_\alpha) J(u_\alpha)$$

and thus

$$J(u_\alpha) \cdot J(u_\alpha)(\nabla_\alpha J)(u_\alpha) = -J(u_\alpha)(\nabla_\alpha J)(u_\alpha) \cdot J(u_\alpha).$$

Now we can compute D_u as follows.

$$\begin{aligned} D_u \xi &= \frac{d}{d\alpha}\bigg|_{\alpha=0} (du_\alpha)^{0,1} \\ &= \frac{d}{d\alpha}\bigg|_{\alpha=0} \tfrac{1}{2}\left(du_\alpha + J(u_\alpha) \circ du_\alpha \circ i\right) \\ &= \tfrac{1}{2} \widetilde{\nabla}_\alpha \left(du_\alpha + J(u_\alpha) \circ du_\alpha \circ i\right) \bigg|_{\alpha=0} \\ &= \tfrac{1}{2}\left(\widetilde{\nabla}_\alpha du_\alpha + J(u_\alpha) \circ \widetilde{\nabla}_\alpha du_\alpha \circ i\right) \bigg|_{\alpha=0} \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\left(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha \circ i\right)\Big|_{\alpha=0} \\
&\quad - \tfrac{1}{2}\cdot\tfrac{1}{2}\left(J(u_\alpha)(\nabla_\alpha J)(u_\alpha)du_\alpha + J(u_\alpha)J(u_\alpha)(\nabla_\alpha J)(u_\alpha)du_\alpha\circ i\right)\Big|_{\alpha=0} \\
&= \tfrac{1}{2}\left(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha \circ i\right)\Big|_{\alpha=0} \\
&\quad - \tfrac{1}{4}\left(J(u_\alpha)(\nabla_\alpha J)(u_\alpha)du_\alpha - J(u_\alpha)(\nabla_\alpha J)(u_\alpha)J(u_\alpha)du_\alpha\circ i\right)\Big|_{\alpha=0} \\
&= \tfrac{1}{2}\left(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha \circ i\right)\Big|_{\alpha=0} \\
&\quad - \tfrac{1}{4}\left(J(u_\alpha)(\nabla_\alpha J)(u_\alpha)\underbrace{(du_\alpha - J(u_\alpha)du_\alpha\circ i)}_{=2\partial_J u_\alpha}\right)\Big|_{\alpha=0} \\
&= \tfrac{1}{2}\left(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha \circ i\right)\Big|_{\alpha=0} - \tfrac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u \\
&= \tfrac{1}{2}\left(\nabla\xi + J(u)\circ\nabla\xi\circ i\right) - \tfrac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u
\end{aligned}
$$

In the last step we used that the Levi-Civita connection ∇ is torsion free. More precisely, note that for any fixed $z \in S^2$ and $\zeta(z) \in T_z S^2$ one can choose a path $t \mapsto z(t)$ such that $z(0) = z$ and $\frac{d}{dt}\Big|_{t=0} z(t) = \zeta(z)$, then one has

$$
du_\alpha(\zeta(z)) = \frac{d}{dt}\Big|_{t=0} u_\alpha(z(t))
$$

and thus

$$
\begin{aligned}
&(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha\circ i)\Big|_{\alpha=0}(\zeta(z)) \\
&\quad = \left(\nabla_{\frac{d}{d\alpha}u_\alpha} du_\alpha + J(u_\alpha)\circ\nabla_{\frac{d}{d\alpha}u_\alpha} du_\alpha\circ i\right)\Big|_{\alpha=0}(\zeta(z)) \\
&\quad = \left(\nabla_{\frac{d}{d\alpha}u_\alpha(z(t))}\frac{d}{dt}u_\alpha(z(t)) + J(u_\alpha)\circ\nabla_{\frac{d}{d\alpha}u_\alpha(z(t))}\frac{d}{dt}u_\alpha(z(t))\circ i\right)\Big|_{t=0}\Big|_{\alpha=0} \\
&\quad = \left(\nabla_{\frac{d}{dt}u_\alpha(z(t))}\frac{d}{d\alpha}u_\alpha(z(t)) + J(u_\alpha)\circ\nabla_{\frac{d}{dt}u_\alpha(z(t))}\frac{d}{d\alpha}u_\alpha(z(t))\circ i\right)\Big|_{t=0}\Big|_{\alpha=0} \\
&\quad = \nabla_{du(\zeta(z))}\xi(z) + J(u)\circ\nabla_{du(\zeta(z))}\xi(z)\circ i.
\end{aligned}
$$

This holds for all $z \in S^2$ and $\zeta(z) \in T_z S^2$, hence

$$
(\nabla_\alpha du_\alpha + J(u_\alpha)\circ\nabla_\alpha du_\alpha\circ i)\Big|_{\alpha=0} = \nabla\xi + J(u)\circ\nabla\xi\circ i
$$

as 1-forms. □

Since $\widehat{\mathcal{B}} = \mathbb{R}_{\geq 0}\times\mathcal{B}$ is a product, its tangent space at $(R,u)\in\widehat{\mathcal{B}}$ equals $\mathbb{R}\times T_u\mathcal{B}$. For the tangent space of $\widehat{\mathcal{E}}$ at the zero section again we have a splitting

$$
T_{(R,u,0)}\widehat{\mathcal{E}} = T_{(R,u)}\widehat{\mathcal{B}}\oplus\mathcal{E}_u
$$

which we can use to define the vertical differential

$$\widehat{D}_{(R,u)} = \pi_{(R,u)} \circ d\widehat{\mathcal{F}}(R,u) : \mathbb{R} \times T_u\mathcal{B} \longrightarrow \mathcal{E}_u$$

for $(R,u) \in \widehat{\mathcal{M}}(P)$. At points $(R,u) \notin \widehat{\mathcal{M}}(P)$, the splitting into horizontal and vertical subspaces again depends on the choice of our connection $\widetilde{\nabla}$.

Lemma 6.2: *For $(R,u) \in \widehat{\mathcal{B}} = \mathbb{R}_{\geq 0} \times \mathcal{B}$, the vertical differential of $\widehat{\mathcal{F}}$ at (R,u) is given by*

$$\begin{aligned}\widehat{D}_{(R,u)}(r,\xi) = D_u\xi - r \cdot \Big(\tau \otimes \gamma_R \cdot X_H(u)\Big)^{0,1} + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\ + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}\end{aligned}$$

for $(r,\xi) \in T_{(R,u)}\widehat{\mathcal{B}}$, where $\gamma_R := \nabla_{\tilde{R}}\beta_{\tilde{R}}\big|_{\tilde{R}=R}$ is a function $S^2 \to \mathbb{R}$ with compact support inside the cylinder Z_R.

Proof. Again let $(u_\alpha)_\alpha \subseteq \mathcal{B}$ be a family of maps such that $u_0 = u$ and $\frac{d}{d\alpha}\big|_{\alpha=0} u_\alpha = \xi$, and let $(R_\alpha) \subseteq \mathbb{R}_{\geq 0}$ be numbers such that $R_0 = R$ and $\frac{d}{d\alpha}\big|_{\alpha=0} R_\alpha = r$. Then:

$$\begin{aligned}
\widehat{D}_{(R,u)}(r,\xi) &= \frac{d}{d\alpha}\Big|_{\alpha=0} \Big(du_\alpha - \tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big)^{0,1} \\
&= \frac{d}{d\alpha}\Big|_{\alpha=0} \tfrac{1}{2}\Big(du_\alpha + J(u_\alpha) du_\alpha \circ i\Big) \\
&\quad - \frac{d}{d\alpha}\Big|_{\alpha=0} \tfrac{1}{2}\Big(\tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big) + J(u_\alpha)\Big(\tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big) \circ i \\
&= D_u\xi - \tfrac{1}{2}\frac{d}{d\alpha}\Big|_{\alpha=0} \tau \otimes \beta_{R_\alpha} X_H(u_\alpha) + J(u_\alpha)\Big(\tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big) \circ i \\
&= D_u\xi - \tfrac{1}{2}\widetilde{\nabla}_\alpha \tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0} - \tfrac{1}{2}\widetilde{\nabla}_\alpha J(u_\alpha)\Big(\tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big) \circ i\Big|_{\alpha=0} \\
&= D_u\xi - \tfrac{1}{2}\widetilde{\nabla}_\alpha \tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0} - \tfrac{1}{2}J(u)\widetilde{\nabla}_\alpha\Big(\tau \otimes \beta_{R_\alpha} X_H(u_\alpha)\Big)\Big|_{\alpha=0} \circ i \\
&= D_u\xi - \tfrac{1}{2}\tau \otimes \widetilde{\nabla}_\alpha \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0} - \tfrac{1}{2}J(u)\left(\tau \otimes \widetilde{\nabla}_\alpha \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0}\right) \circ i \\
&= D_u\xi - \tfrac{1}{2}\tau \otimes \nabla_\alpha \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0} - \tfrac{1}{2}J(u)\left(\tau \otimes \nabla_\alpha \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0}\right) \circ i \\
&\quad + \tfrac{1}{4}\tau \otimes J(u)\left(\nabla_\alpha J\big|_{\alpha=0}\right)(u)\beta_R X_H(u) \\
&\quad + \tfrac{1}{4}J(u)\left(\tau \otimes J(u)\left(\nabla_\alpha J\big|_{\alpha=0}\right)(u)\beta_R X_H(u)\right) \circ i \\
&= D_u\xi - \left(\tau \otimes \nabla_\alpha \beta_{R_\alpha} X_H(u_\alpha)\big|_{\alpha=0}\right)^{0,1} \\
&\quad + \tfrac{1}{2}\left(\tau \otimes J(u)\left(\nabla_\alpha J\big|_{\alpha=0}\right)(u)\beta_R X_H(u)\right)^{0,1}
\end{aligned}$$

Here, $(\ldots)^{0,1}$ means taking the anti-holomorphic part with respect to i on S^2 and J on M. It is

$$\left(\nabla_\alpha J\big|_{\alpha=0}\right)(u) = \left(\nabla_{\frac{d}{d\alpha}u_\alpha} J\big|_{\alpha=0}\right)(u) = \nabla_\xi J(u)$$

and

$$\begin{aligned}\nabla_\alpha \beta_{R_\alpha} X_H(u_\alpha)\Big|_{\alpha=0} &= \left(\nabla_\alpha \beta_{R_\alpha}\Big|_{\alpha=0}\right) X_H(u) + \beta_R \left(\nabla_\alpha X_H(u_\alpha)\Big|_{\alpha=0}\right)\\ &= r \cdot \gamma_R \cdot X_H(u) + \beta_R \nabla_\xi X_H(u),\end{aligned}$$

where $\gamma_R := \nabla_{\tilde{R}} \beta_{\tilde{R}}\Big|_{\tilde{R}=R}$ is a function $S^2 \to \mathbb{R}$. If $z \in S^2$ does not belong to the cylinder Z_R, then $\beta_{\tilde{R}}(s) = 0$ for all $\tilde{R}$ near R, so $\gamma_R(z) = 0$ in that case. All in all we get

$$\begin{aligned}\widehat{D}_{(R,u)}(r,\xi) = D_u\xi - r \cdot \Big(\tau \otimes \gamma_R \cdot X_H(u)\Big)^{0,1} &+ \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1}\\ &+ \tfrac{1}{2}\Big(\tau \otimes J(u) \nabla_\xi J(u) \beta_R X_H(u)\Big)^{0,1}.\end{aligned}$$

□

6.2 The Riemann-Roch theorem

In the appendix of [7], McDuff and Salamon introduce the notion of complex linear and real linear Cauchy-Riemann operators. Such operators always have the Fredholm property.

Let $E \to \Sigma$ be a smooth complex vector bundle over a compact Riemannian surface Σ. Denote by $\mathcal{C}^\infty(\Sigma, E)$ the space of smooth sections in this bundle and by $\mathcal{C}^\infty(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$ the space of smooth anti-holomorphic 1-forms on Σ with values in E. The corresponding spaces of regularity $W^{k,p}$ will be denoted by $W^{k,p}(\Sigma, E)$ and $W^{k,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$. Moreover, let

$$\begin{aligned}&\bar\partial : \mathcal{C}^\infty(\Sigma, \mathbb{C}) \longrightarrow \mathcal{C}^\infty(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes (\Sigma \times \mathbb{C}))\\ &\bar\partial f = (df)^{0,1} = df + i \circ df \circ i\end{aligned}$$

be the usual Cauchy-Riemann operator.

Definition 6.3:

- A **smooth complex linear Cauchy-Riemann operator** on the bundle $E \to \Sigma$ is a $\mathbb{C}$-linear operator

  $$D : \mathcal{C}^\infty(\Sigma, E) \longrightarrow \mathcal{C}^\infty(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$

  which satisfies the Leibniz rule

  $$D(f\xi) = f(D\xi) + (\bar\partial f)\xi$$

  for all smooth sections $\xi \in \mathcal{C}^\infty(\Sigma, E)$ and all smooth functions $f : \Sigma \to \mathbb{C}$.

- Let $l \geq 1$ be a positive integer and $p > 1$ a number such that $lp > 2$. A **real linear Cauchy-Riemann operator of class** $W^{l-1,p}$ on $E \to \Sigma$ is an operator of the form

  $$D = D_0 + \alpha : W^{l,p}(\Sigma, E) \longrightarrow W^{l-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$

  where D_0 is a smooth complex linear Cauchy-Riemann operator on $E \to \Sigma$ and α is an element of $W^{l-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes \mathrm{End}_{\mathbb{R}}(E))$, that is an anti-holomorphic 1-form on Σ of class $W^{l-1,p}$ with values in the endomorphism bundle.

Real linear Cauchy-Riemann operators also satisfy the Leibniz formula

$$D(f\xi) = f(D\xi) + (\bar{\partial} f)\xi,$$

but only for real valued functions $f : \Sigma \to \mathbb{R}$.

Note that if D is a smooth complex linear Cauchy-Riemann operator, then it is real linear of every class $W^{l-1,p}$.

In [7], McDuff and Salamon formulate a version of the Riemann-Roch theorem for real linear Cauchy-Riemann operators with totally real boundary conditions. Without the boundary conditions, it can be stated as follows:

Theorem 6.4 (Riemann-Roch theorem, theorem C.1.10 in [7])**:** *Let $E \to \Sigma$ be a complex vector bundle of rank $m \in \mathbb{N}$ over a closed Riemannian surface $\Sigma$and D a real linear Cauchy-Riemann operator of class $W^{l-1,p}$ on $E \to \Sigma$, where $l \geq 1$ is a positive integer and $p > 1$ is a number such that $lp > 2$. Then for every integer $k \in \{1, \dots, l\}$ and every real number $q > 1$ such that $k - \frac{2}{q} \leq l - \frac{2}{p}$, the following holds:*

- *The operator*
$$D : W^{k,q}(\Sigma, E) \to W^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$$
has the Fredholm property. Its kernel is independent of the choice of k and q.
- *The Fredholm index of D (seen as a real linear operator) is given by*
$$\text{ind}\ (D) = m \cdot \chi(\Sigma) + 2\langle c_1(E), [\Sigma]\rangle,$$
where $\chi(\Sigma)$ is the Euler characteristic of Σ and $c_1(E)$ is the first Chern class of the bundle $E \to \Sigma$, so $\langle c_1(E), [\Sigma]\rangle$ is its first Chern number.

Remark 6.5:
- The case $k = l$ and $q = p$ is the one we are most interested in.
- Note that the index neither depends on the precise definition of the operator D nor on its regularity class $W^{l-1,p}$ as long as $lp > 2$; the index is an invariant of the bundle.

6.3 Implications for D_u

Lemma 6.6:
1. *For every $u \in \mathcal{B}$ the operator*
$$\bar{\partial}_u : W^{1,p}(S^2, u^*TM) \longrightarrow W^{0,p}(S^2, \Lambda^{0,1}T^*S^2 \otimes u^*TM) = \mathcal{E}_u$$
$$\bar{\partial}_u \xi = \tfrac{1}{2}\Big(\nabla\xi + J(u) \circ \nabla\xi \circ i\Big)$$
is a real linear Cauchy-Riemann operator of class $W^{0,p} = L^p$.
2. *For every $u \in \mathcal{B}$, if we understand D_u as an operator*
$$\begin{aligned} D_u : W^{1,p}(S^2, u^*TM) &\longrightarrow W^{0,p,(0,1)}(S^2, u^*TM) = \mathcal{E}_u \\ D_u\xi &= \tfrac{1}{2}\Big(\nabla\xi + J(u) \circ \nabla\xi \circ i\Big) - \tfrac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u \\ &= \bar{\partial}_u\xi - \tfrac{1}{2}J(u)(\nabla_\xi J)(u)\partial_J u \end{aligned}$$
on the bigger space $W^{1,p}(S^2, u^*TM)$ containing $T_u\mathcal{B}$ as a subspace, then it is a real linear Cauchy-Riemann operator of class $W^{0,p} = L^p$.

Proof. For the first claim we have to show that $\bar{\partial}_u$ splits into a smooth complex linear Cauchy-Riemann operator $\bar{\partial}_u^0$ and an anti-holomorphic 1-form α of class $W^{0,p} = L^p$ with values in the endomorphism bundle $\mathrm{End}_{\mathbb{R}}(u^*TM) \to S^2$.

Remember that the connection ∇ on $u^*TM \to S^2$ was induced by the Levi-Civita connection on the tangent bundle $TM \to M$, also denoted by ∇. It is a well-known fact that the space of connections on a vector bundle is affine – a proof of this can be found for example in [8]. Moreover, there exists a Hermitian connection on $TM \to M$ (with respect to the Hermitian metric induced by J and ω) which induces a Hermitian connection on $u^*TM \to S^2$. So we can write our connection ∇ on $u^*TM \to S^2$ as a sum $\nabla = \nabla_0 + A$ of a Hermitian connection ∇_0 and a section $A \in L^p(S^2, T^*S^2 \otimes \mathrm{End}_{\mathbb{R}}(u^*TM))$. We write $A(\cdot)\xi$ for the 1-form with values in u^*TM which results from applying this bundle endomorphism to a vector field ξ along u.

Then we have

$$\begin{aligned}
\bar{\partial}_u \xi &= \tfrac{1}{2}\Big((\nabla_0 + A)\xi + J(u) \circ (\nabla_0 + A)\xi \circ i\Big) \\
&= \tfrac{1}{2}\Big(\nabla_0 \xi + J(u) \circ \nabla_0 \xi \circ i\Big) + \tfrac{1}{2}\Big(A(\cdot)\xi + J(u) \circ A(\cdot)\xi \circ i\Big) \\
&= \bar{\partial}_u^0 \xi + \alpha\xi,
\end{aligned}$$

where $\bar{\partial}_u^0 \xi := \frac{1}{2}\Big(\nabla_0 \xi + J(u) \circ \nabla_0 \xi \circ i\Big)$ and $\alpha\xi := \frac{1}{2}\Big(A(\cdot)\xi + J(u) \circ A(\cdot)\xi \circ i\Big)$. It is clear that this defines an element $\alpha \in L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes \mathrm{End}_{\mathbb{R}}(u^*TM))$. We still have to verify that $\bar{\partial}_u^0$ is a complex linear Cauchy-Riemann operator, so let $f = f_1 + i \cdot f_2 : S^2 \to \mathbb{C}$ be a function and compute the following.

$$\begin{aligned}
\bar{\partial}_u^0(f\xi) &= \bar{\partial}_u^0\Big((f_1 + i \cdot f_2)\xi\Big) \\
&= \bar{\partial}_u^0\Big(f_1\xi + f_2 J(u)\xi\Big) \\
&= \tfrac{1}{2}\Big(\nabla_0 f_1\xi + J(u) \circ \nabla_0 f_1 \xi \circ i\Big) + \tfrac{1}{2}\Big(\nabla_0(f_2 J(u)\xi) + J(u) \circ \nabla_0(f_2 J(u)\xi) \circ i\Big) \\
&= \tfrac{1}{2}\,\Big(\Big(df_1(\cdot)\xi + f_1 \nabla_0 \xi\Big) + J(u) \circ \Big(df_1(\cdot)\xi + f_1\nabla_0\xi\Big) \circ i\Big) \\
&\quad + \tfrac{1}{2}\,\Big(\Big(df_2(\cdot)J(u)\xi + f_2\nabla_0(J(u)\xi)\Big) + J(u) \circ \Big(df_2(\cdot)J(u)\xi + f_2\nabla_0(J(u)\xi)\Big) \circ i\Big) \\
&= \tfrac{1}{2}\Big(df_1(\cdot)\xi + J(u) \circ df_1(\cdot)\xi \circ i\Big) + \tfrac{1}{2}\Big(f_1\nabla_0\xi + J(u) \circ f_1 \nabla_0 \xi \circ i\Big) \\
&\quad + \tfrac{1}{2}\Big(df_2(\cdot)J(u)\xi + J(u) \circ df_2(\cdot)J(u)\xi \circ i\Big) \\
&\quad + \tfrac{1}{2}\Big(f_2\nabla_0(J(u)\xi) + f_2 J(u) \circ \nabla_0(J(u)\xi) \circ i\Big) \\
&= \bar{\partial} f_1 \cdot \xi + f_1 \cdot \bar{\partial}_u^0 \xi + \bar{\partial} f_2 \cdot J(u)\xi \\
&\quad + \tfrac{1}{2}\Big(f_2\nabla_0(J(u)\xi) + f_2 J(u) \circ \nabla_0(J(u)\xi) \circ i\Big) \\
&= \bar{\partial} f_1 \cdot \xi + f_1 \cdot \bar{\partial}_u^0 \xi + \bar{\partial} f_2 \cdot J(u)\xi \\
&\quad + \tfrac{1}{2}\left(f_2\Big(\underbrace{(\nabla_0 J(u))}_{=0 \text{ since } \nabla_0 \text{ is Hermitian}}\xi + J(u)\nabla_0\xi\Big) + f_2 J(u) \circ \Big(\underbrace{(\nabla_0 J(u))}_{=0 \text{ since } \nabla_0 \text{ is Hermitian}}\xi + J(u)\nabla_0\xi\Big) \circ i\right) \\
&= \bar{\partial} f_1 \cdot \xi + f_1 \cdot \bar{\partial}_u^0 \xi + \bar{\partial} f_2 \cdot J(u)\xi + f_2 \cdot J(u)\tfrac{1}{2}\Big(\nabla_0\xi + J(u)\nabla_0\xi \circ i\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \bar{\partial} f_1 \cdot \xi + f_1 \cdot \bar{\partial}^0_u \xi + \bar{\partial} f_2 \cdot J(u)\xi + f_2 J(u) \bar{\partial}^0_u \xi \\
&= \Big(\bar{\partial} f_1 + \bar{\partial} f_2 \cdot J(u) \Big) \xi + \Big(f_1 + f_2 J(u) \Big) \cdot \bar{\partial}^0_u \xi \\
&= \bar{\partial} f \cdot \xi + f \cdot \bar{\partial}^0_u \xi
\end{aligned}
$$

So $\bar{\partial}^0_u$ really is a complex linear Cauchy-Riemann operator.

The second claim is now obvious since

$$\xi \longmapsto -\tfrac{1}{2} J(u) \left(\nabla_\xi J \right) (u) \partial_J u$$

is an $\mathbb{R}$-linear endomorphism of u^*TM,

$$-\tfrac{1}{2} J(u)(\nabla_\xi J)(u) \partial_J u$$

is an anti-holomorphic 1-form (since $J(u)$ anticommutes with $J(u)(\nabla J)(u)$ as we have seen earlier in this chapter) and the assignment

$$S^2 \ni z \longmapsto -\tfrac{1}{2} J(u(z))(\nabla J)(u(z)) \partial_J u(z)$$

is of class $W^{0,p} = L^p$. □

Corollary 6.7 (of the Riemann-Roch theorem 6.4)**:** *For every $u \in \mathcal{B}$, the operator*

$$D_u : W^{1,p}(S^2, u^*TM) \longrightarrow L^p(S^2, \Lambda^{0,1} T^* S^2 \otimes u^*TM) = \mathcal{E}_u$$

is a Fredholm operator of index $2n$.

Proof. The bundle $u^*TM \to S^2$ is of complex rank n because $2n$ is the real dimension of M. The Euler characteristic of S^2 is 2. For the Chern number we compute

$$\langle c_1(u^*TM), [S^2] \rangle = \int_{S^2} c_1(u^*TM) = \int_{S^2} u^* c_1(TM).$$

and remember that it does only depend on the homotopy class of u. This is where the condition $[u] = 0 \in \pi_2(M)$ comes in: For constant u the computation above gives 0, and thus for all other choices of u it also has to be 0. Now use theorem 6.4. □

But we are not really interested in D_u as an operator

$$W^{1,p}(S^2, u^*TM) \to L^p(S^2, \Lambda^{0,1} T^* S^2 \otimes u^*TM),$$

but as an operator $T_u\mathcal{B} \to \mathcal{E}_u$. The fibre $\mathcal{E}_u$ equals the space $L^p(S^2, \Lambda^{0,1} T^* S^2 \otimes u^*TM)$, but $T_u\mathcal{B}$ and $W^{1,p}(S^2, u^*TM)$ differ from each other by the point constraint:

$$T_u\mathcal{B} = \{\xi \in W^{1,p}(S^2, u^*TM) | \xi(0,0) = 0\}$$

Proposition 6.8: *For every $P \in M$ and every $u \in \mathcal{B}$ there is a vector space isomorphism*

$$W^{1,p}(S^2, u^*TM) \cong T_u\mathcal{B} \oplus T_P M.$$

Proof. Let $P \in M$ and $u \in \mathcal{B}$ be given. Fix a neighbourhood $U \subseteq M$ of $u(0,0) = P$ such that the tangent bundle of M is trivial over U, and fix another neighbourhood $V \subset U$ of P such that the closure of V is contained in the interior of U. Moreover we fix a smooth cut-off function

$$\begin{aligned} \rho : M &\longrightarrow [0,1] \\ \rho\Big|_V &\equiv 1 \\ \rho\Big|_{M\setminus U} &\equiv 0. \end{aligned}$$

Now let $\xi \in W^{1,p}(S^2, u^*TM)$ be given. Set $v_\xi(P) := \xi(0,0)$. Since the tanget bundle of M is trivial over U, we can uniquely extend $v_\xi(P)$ to a constant local vector field v_ξ on U. Setting $w_\xi := \rho \cdot v_\xi$ we can understand w as a smooth vector field on the whole manifold M.

Now we want to understand $\xi - w_\xi$ as a vector field along u via

$$\Big(\xi - w_\xi\Big)(z) := \xi(z) - w_\xi(u(z)) \in T_{u(z)}M$$

for all $z \in S^2$. In particular, $\xi - w_\xi$ satisfies

$$\Big(\xi - w_\xi\Big)(0,0) = \xi(0,0) - w_\xi(P) = 0 \in T_{u(z)}M$$

and so it is an element of $T_u\mathcal{B}$.

Define a map by

$$\begin{aligned} \Xi : W^{1,p}(S^2, u^*TM) &\longrightarrow T_u\mathcal{B} \oplus T_PM \\ \xi &\longmapsto \Big(\xi - w_\xi, \xi(0,0)\Big) \end{aligned}$$

– this is well-defined since all choices were fixed before the construction of w_ξ. Moreover, every step in the construction respected the vector space structures of $W^{1,p}(S^2, u^*TM)$, $T_u\mathcal{B}$ and T_PM, so Ξ surely is a linear map, and it is bounded. So it remains to show that Ξ is injective and surjective.

Let $\xi \in W^{1,p}(S^2, u^*TM)$ be such that $\Xi(\xi) = (0,0)$. Since $\xi(0,0) = 0$, the construction of w_ξ gives $w_\xi = 0$. But then from $\xi - w_\xi = 0$ it follows that $\xi = 0$. So Ξ is injective.

Take $(\zeta, v) \in T_u\mathcal{B} \oplus T_PM$. We want to construct a preimage of (ζ, v) under Ξ. Extend the vector $v \in T_PM$ to a smooth local vector field v over U and consider the smooth vector field $w := \rho \cdot v$ on M. Define a vector field $\zeta + w$ along u by

$$\Big(\zeta + w\Big)(z) := \zeta(z) + w(u(z))$$

for all $z \in S^2$. Then

$$\Big(\zeta + w\Big)(0,0) = \zeta(0,0) + w(u(0,0)) = 0 + w(P) = v,$$

so the second component of $\Xi(\zeta + w)$ equals v. For the first component, note that

$$v_{\zeta+w}(P) = \Big(\zeta + w\Big)(0,0) = \zeta(0,0) + w(P) = 0 + v = v$$

and that the construction of $w_{\zeta+w}$ out of $v_{\zeta+w}(P)$ equals the one from w out of v. Now we can compute $(\zeta+w)-w_{\zeta+w}$ at a point $z \in S^2$:

$$\begin{aligned}\Big((\zeta+w)-w_{\zeta+w}\Big)(z) &= \Big(\zeta+w\Big)(z) - w_{\zeta+w}(u(z)) \\ &= \zeta(z) + w(u(z)) - w_{\zeta-w}(u(z)) \\ &= \zeta(z)\end{aligned}$$

Thus $(\zeta+w)-w_{\zeta+w} = \zeta$ and hence $\Xi(\zeta+w) = (\zeta, v)$.

This completes the proof. □

Now we are ready to see that D_u is indeed a Fredholm operator on $T_u\mathcal{B}$.

Lemma 6.9: *For every $u \in \mathcal{B}$ the operator*

$$D_u : T_u\mathcal{B} \longrightarrow \mathcal{E}_u$$

is Fredholm of index 0.

Proof. From corollary 6.7 we know that

$$D_u : W^{1,p}(S^2, u^*TM) \longrightarrow L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u^*TM) = \mathcal{E}_u$$

is Fredholm of index $2n$, and from proposition 6.8 we know that

$$W^{1,p}(S^2, u^*TM) \cong T_u\mathcal{B} \oplus T_PM.$$

Since T_PM is of finite dimension $2n$, this means that both kernel and cokernel of

$$D_u : T_u\mathcal{B} \longrightarrow L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u^*TM) = \mathcal{E}_u$$

are still finite-dimensional, and so D_u has the Fredholm property.

$\mathcal{B}$ is a connected Banach manifold, so the index of D_u will be the same for all $u \in \mathcal{B}$. We compute it for the constant sphere $u_P \equiv P$.

Take $\xi \in W^{1,p}(S^2, u_P^*TM)$. Since u_P is a constant curve, ξ is just a function

$$\xi : S^2 \to T_PM.$$

Shifting this function by a vector $v \in T_PM$, we obtain another vector field $\xi + v \in W^{1,p}(S^2, u_P^*TM)$ which satisfies $D_u(\xi+v) = D_u\xi$. By choosing $v := -\xi(0,0)$ we achieve $(\xi+v)(0,0) = 0$, so that $\xi+v$ is an element of $T_{u_P}\mathcal{B}$ with $D_{u_P}(\xi+v) = D_{u_P}\xi$. This means that

$$\operatorname{im}\ (D_{u_p} : T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}) = \operatorname{im}\ \Big(D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM)\Big).$$

Now if ξ lies in the kernel of

$$D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM),$$

then so does $\xi + v$ for any choice of v. These are $2n$ dimensions which get lost when passing from $W^{1,p}(S^2, u_P^*TM)$ to $T_{u_P}\mathcal{B}$, and so we have

$$\begin{aligned}
&\dim \ker\ (D_{u_P} : T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}) \\
&\qquad = \dim \ker\ \Big(D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM)\Big) - 2n.
\end{aligned}$$

For the index we get

$$\begin{aligned}
&\operatorname{ind}\ (D_{u_P} : T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}) \\
&\quad = \dim \ker\ (D_{u_P} : T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}) - \dim \operatorname{coker}\ (D_{u_P} : T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}) \\
&\quad = \dim \ker\ \Big(D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM)\Big) - 2n \\
&\qquad - \dim \operatorname{coker}\ \Big(D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM)\Big) \\
&\quad = \operatorname{ind}\ \Big(D_{u_P} : W^{1,p}(S^2, u_P^*TM) \to L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes u_P^*TM)\Big) - 2n \\
&\quad = 0.
\end{aligned}$$

□

6.4 Implications for $\widehat{D}_{(R,u)}$

The operator $\widehat{D}_{(R,u)}$ itself does not fit the notion of a real linear Cauchy-Riemann operator as in definition 6.3, but we will see that it is closely related to one.

Lemma 6.10: *The operator*

$$\begin{aligned}
\widehat{D}_{(R,u)} &: \mathbb{R} \times T_u\mathcal{B} \longrightarrow \mathcal{E}_u \\
&(r, \xi) \longmapsto D_u\xi - r \cdot \Big(\tau \otimes \gamma_R \cdot X_H(u)\Big)^{0,1} + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\
&\qquad\qquad + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}
\end{aligned}$$

has the Fredholm property if and only if

$$\begin{aligned}
L_{(R,u)} &: T_u\mathcal{B} \to \mathcal{E}_u \\
&\xi \mapsto D_u\xi + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\
&\qquad + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}
\end{aligned}$$

does, and in that case

$$\operatorname{ind}\ \Big(\widehat{D}_{(R,u)}\Big) = \operatorname{ind}\ \Big(L_{(R,u)}\Big) + 1.$$

Proof. The operator

$$\begin{aligned}
K_{(R,u)} &: \mathbb{R} \longrightarrow \mathcal{E}_u \\
&r \longmapsto -r \cdot \Big(\tau \otimes \gamma_R \cdot X_H(u)\Big)^{0,1}
\end{aligned}$$

is compact. It follows from proposition 5.2 that $\widehat{D}_{(R,u)}$ has the Fredholm property if and only if

$$\begin{aligned}\underline{D} : \mathbb{R} \times T_u\mathcal{B} &\longrightarrow \mathcal{E}_u \\ (r,\xi) &\longmapsto D_u\xi + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\ &\qquad + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}\end{aligned}$$

does, and in that case they have the same index. This operator $\underline{D}$ in turn does have the Fredholm property if and only if

$$\begin{aligned}L_{(R,u)} : T_u\mathcal{B} &\longrightarrow \mathcal{E}_u \\ \xi &\longmapsto D_u\xi + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\ &\qquad + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}\end{aligned}$$

does, and in that case

$$\operatorname{ind}\ \Big(\widehat{D}_{(R,u)}\Big) = \operatorname{ind}\ (\underline{D}) = \operatorname{ind}\ \Big(L_{(R,u)}\Big) + 1$$

since the images are identical and the kernel of $\underline{D}$ has one more dimension than the kernel of $L_{(R,u)}$. □

This is why it is now enough to analyse $L_{(R,u)}$.

Lemma 6.11: *Understood as an operator*

$$\begin{aligned}L_{(R,u)} : W^{1,p}(S^2, u^*TM) &\longrightarrow \mathcal{E}_u \\ \xi &\longmapsto D_u\xi + \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} \\ &\qquad + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}\end{aligned}$$

on the bigger space $W^{1,p}(S^2, u^*TM)$ containing $T_u\mathcal{B}$ as a subspace, $L_{(R,u)}$ is a real linear Cauchy-Riemann operator of class $W^{0,p} = L^p$.

Proof. D_u is a real linear Cauchy-Riemann operator by lemma 6.6,

$$\xi \longmapsto \Big(\tau \otimes \beta_R \nabla_\xi X_H(u)\Big)^{0,1} + \tfrac{1}{2}\Big(\tau \otimes J(u)\nabla_\xi J(u)\beta_R X_H(u)\Big)^{0,1}$$

is an endomorphism of u^*TM, all terms are anti-holomorphic 1-forms and the assignment

$$z \longmapsto \Big(\tau_z \otimes \beta_R(z)\nabla X_{H_z}(u(z))\Big)^{0,1} + \tfrac{1}{2}\Big(\tau_z \otimes J(u(z))\nabla J(u(z))\beta_R(z) X_{H_z}(u(z))\Big)^{0,1}$$

is of class $W^{0,p} = L^p$. □

Corollary 6.12 (of the Riemann-Roch theorem 6.4)**:** *Understood as an operator*

$$L_{(R,u)} : W^{1,p}(S^2, u^*TM) \longrightarrow \mathcal{E}_u,$$

$L_{(R,u)}$ is a Fredholm operator of index $2n$.

Proof. The bundle $u^*TM \to S^2$ concerned here is still the same as in corollary 6.7, so the index formula from the Riemann-Roch theorem gives the same number. □

Now as in lemma 6.9 we have to see what happens when we add the point constraint.

Lemma 6.13: *For every* $(R, u) \in \mathbb{R}_{\geq 0} \times \mathcal{B}$*, the operator*

$$L_{(R,u)} : T_u\mathcal{B} \longrightarrow \mathcal{E}_u$$

is Fredholm of index 0*.*

Proof. The argumentation is exactly the same as in lemma 6.9: It follows from proposition 6.8 that kernel and cokernel of $L_{(R,u)}$ remain finite-dimensional when we pass from $W^{1,p}(S^2, u^*TM)$ to $T_u\mathcal{B}$. Then we compute the index for $(R, u) = (0, u_P)$, in which case $L_{(R,u)}$ equals D_u and so the computation is the same and gives $\operatorname{ind} L_{(R,u)} = 0$. Since $\mathbb{R}_{\geq 0} \times \mathcal{B}$ is connected, the index is then 0 for every pair (R, u). □

All in all, in this chapter we have seen that the vertical differential D_u of $\mathcal{F}_0$ at a point $u \in \mathcal{B}$ is Fredholm of index 0 and that the vertical differential $\widehat{D}_{(R,u)}$ of $\widehat{\mathcal{F}}$ at a point $(R, u) \in \mathbb{R}_{\geq 0} \times \mathcal{B}$ is Fredholm of index 1. This means that $\mathcal{F}_0$ and $\widehat{\mathcal{F}}$ are Fredholm sections of indices 0 and 1 in the bundles $\mathcal{E} \to \mathcal{B}$ and $\widehat{\mathcal{E}} \to \widehat{\mathcal{B}}$ respectively.

CHAPTER 7

Transversality

If we wanted to show that $\widehat{\mathcal{M}}(P)$ was a manifold as in proposition 3.7, the last step now would be to show that the section

$$\widehat{\mathcal{F}} : \widehat{\mathcal{B}} \longrightarrow \widehat{\mathcal{E}}$$

was transverse to the zero section and then use the implicit function theorem. Unfortunately, this does not need to be the case. But for our sake it is enough to see that a slight perturbation of $\widehat{\mathcal{M}}(P)$ is a manifold with the contradictory properties stated in proposition 3.7. For this we will construct a perturbation $\widehat{\mathcal{F}}_\lambda$ of the section $\widehat{\mathcal{F}}$ which is transverse to the zero section. In order for the resulting manifold to have the required properties, the perturbed section $\widehat{\mathcal{F}}_\lambda$ must satisfy the following:

- $\widehat{\mathcal{F}}_\lambda$ is a Fredholm section.
- $\mathrm{ind}(\widehat{\mathcal{F}}_\lambda) = \mathrm{ind}(\widehat{\mathcal{F}}) = 1$
- The zero set of $\widehat{\mathcal{F}}_\lambda$ is compact.
- The boundary of the zero set of $\widehat{\mathcal{F}}_\lambda$ is diffeomorphic to $\mathcal{M}_0(P)$.

The most important instrument of this chapter is an infinite-dimensional version of the theorem of Sard which he first stated in [14]. We need the following definition.

Definition 7.1: A subset of a topological space Y is said to be **of second category** (or 'residual') in the sense of Baire if it contains a countable intersection of subsets which are open and dense in Y.

Theorem 7.2 (Sard-Smale, theorem A.5.1 in [7])**:** *Let X and Y be separable Banach spaces and $U \subset X$ be an open set. Suppose that $f : U \to Y$ is a Fredholm map of class $\mathcal{C}^l$, where*

$$l \geq \max\{1, \mathrm{ind}(f) + 1\}.$$

Then the set

$$Y_{\mathrm{reg}}(f) := \{y \in Y \mid x \in U, f(x) = y \Rightarrow \mathrm{im}\ df(x) = Y\}$$

of regular values of f is residual in the sense of Baire.

It follows from remark 5.7 that $Y_{\mathrm{reg}}(f)$ with this definition is indeed the set of regular values of f as in definition 5.6.

7.1 At the boundary

We start with showing that the section $\widehat{\mathcal{F}}$ is transverse to the zero section at the boundary $\partial\widehat{\mathcal{B}} = \{0\} \times \mathcal{B}$. Since for every $P \in M$ the moduli space $\mathcal{M}_0(P)$ consists only of the constant sphere u_P through P, this means that $\widehat{D}_{(0,u_P)}$ is surjective.

Proposition 7.3: *For every $P \in M$, the operator $\widehat{D}_{(0,u_P)} : \mathbb{R} \times T_{u_P}\mathcal{B} \to \mathcal{E}_{u_P}$ is surjective.*

Proof. **Step 1: Reduction to D_{u_P}**

It is

$$\widehat{D}_{(0,u_P)}(r,\xi) = D_{u_P}\xi - r \cdot \Big(\tau \otimes \gamma_0 \cdot X_H(u_P)\Big)^{0,1}$$

for $(r,\xi) \in \mathbb{R} \times T_{u_P}\mathcal{B}$, where $\gamma_0 := \nabla_{\tilde{R}}\beta_{\tilde{R}}\big|_{\tilde{R}=0}$. This means that it is certainly enough to show that

$$\begin{aligned} D_{u_P} : T_{u_P}\mathcal{B} &\longrightarrow \mathcal{E}_{u_P} \\ \xi &\longmapsto \tfrac{1}{2}\Big(\nabla\xi + J(u_P) \circ \nabla\xi \circ i\Big) - \tfrac{1}{2}J(u_P)(\nabla_\xi J)(u_P)\partial_J u_P \end{aligned}$$

is surjective.

Step 2: Using that u_P is constant

Since u_P is constant, the second term in the formula above vanishes and we have

$$D_{u_P}\xi = \tfrac{1}{2}\left(\nabla\xi + J(u_P) \circ \nabla\xi \circ i\right).$$

What is more, a vector field $\xi \in T_{u_P}\mathcal{B}$ along the constant sphere u_P is just a map

$$\xi \in W^{1,p}(S^2, T_P M),$$

and identifying $(T_P M, J_P) \cong (\mathbb{R}^{2n}, J_{st}) \cong (\mathbb{C}^n, i)$ we can assume $\xi \in W^{1,p}(S^2, \mathbb{R}^{2n})$. An element η of the fibre $\mathcal{E}_{u_P}$ is then a 1-form of class L^p on S^2 with values in $\mathbb{R}^{2n}$.

Step 3: Reduction to a local equation

Take $\eta \in L^p(S^2, \Lambda^{0,1}T^*S^2 \otimes \mathbb{R}^{2n})$. We have to show that there exists $\xi \in W^{1,p}(S^2, \mathbb{R}^{2n})$ such that

$$\tfrac{1}{2}\left(\nabla\xi + J_{st} \circ \nabla\xi \circ i\right) = \eta.$$

But in fact it is enough to show this equation locally: Cover S^2 by charts and write η as the sum of 1-forms supported in the chart domains. If every such local form has a preimage under the operator D_{u_P}, then the sum of these preimages is a preimage of η.

For any vector field $\xi \in W^{1,p}(S^2, \mathbb{R}^{2n})$, in local coordinates $(s,t) \in U \subset \mathbb{R}^{2n}$ we have

$$\begin{aligned} D_u\xi\left(\tfrac{d}{ds}\right) &= \nabla_{\frac{d}{ds}}\xi + J(u)\nabla_{\frac{d}{dt}}\xi \\ &= \partial_s\xi + J(u)\partial_t\xi \end{aligned}$$

and

$$\begin{aligned} D_u\xi\left(\tfrac{d}{dt}\right) &= \nabla_{\frac{d}{dt}}\xi + J(u)\nabla_{-\frac{d}{ds}}\xi \\ &= \partial_t\xi - J(u)\partial_s\xi \\ &= -J(u)\left(\partial_s\xi + J(u_0)\partial_t\xi\right) \\ &= -J(u)D_u\xi\left(\tfrac{d}{ds}\right). \end{aligned}$$

Since η is an anti-holomorphic 1-form,

$$\eta\left(\tfrac{d}{dt}\right) = -J(u)\eta\left(\tfrac{d}{ds}\right).$$

So the vector field $\xi \in W^{1,p}(\mathbb{R}^2, \mathbb{R}^{2n})$ only has to satisfy

$$\partial_s\xi + J_{st}\partial_t\xi = \eta\left(\tfrac{d}{ds}\right)$$

on U.

Step 4: Local construction

The local construction can be done with the fundamental solution of the Cauchy-Riemann operator. Write ξ as

$$\xi = (\xi_1, \ldots, \xi_n)$$

with functions $\xi_k : U \to \mathbb{R}^{2n} \cong \mathbb{C}$ which are of class $W^{1,p}$ and

$$\eta\left(\tfrac{d}{ds}\right) = (\zeta_1, \ldots, \zeta_n)$$

with functions $\zeta_k : U \to \mathbb{R}^{2n} \cong \mathbb{C}$ of class L^p. Then we have to solve the equation

$$\partial_s\xi_k + i \cdot \partial_t\xi_k = \zeta_k$$

on U for $k = 1, \ldots, n$. Using the fundamental solution of the Cauchy-Riemann operator we get solutions

$$\xi_k(z) = \frac{1}{2\pi i}\int_U \frac{\zeta_k(w)}{w - z}\, dw\, d\bar{w}$$

for all k.

This completes the proof. □

In order to assure that the boundary of the perturbed moduli space really consists of exactly one point, we need the following lemma.

By lemma 7.5, we know that transversality not only holds in $(0, u_P)$, but also in a neighbourhood of $(0, u_P)$. Hence near $(0, u_P)$, the moduli space $\widehat{\mathcal{M}}(P)$ has the structure of a smooth manifold and we can talk about its tangent space in $(0, p)$.

Lemma 7.4: *The moduli space $\widehat{\mathcal{M}}(P)$ is not tangent to the boundary $\partial\widehat{\mathcal{B}} = \{0\} \times \mathcal{B}$.*

Proof. The intersection of $\widehat{\mathcal{M}}(P)$ and $\{0\} \times \mathcal{B}$ consists of one element, namely the pair $(0, u_P)$. Being tangential would then mean that

$$T_{(0,u_P)}\widehat{\mathcal{M}}(P) \subseteq \{0\} \times T_{u_P}\mathcal{B}.$$

But it is

$$T_{(0,u_P)}\widehat{\mathcal{M}}(P) = \left\{(r,\xi) \in \mathbb{R} \times T_u\mathcal{B} \;\middle|\; \widehat{D}_{(0,u_P)}(r,\xi) = 0\right\}$$

and for $(r,\xi) \in \mathbb{R} \times T_{u_P}\mathcal{B}$ one has

$$\widehat{D}_{(0,u_P)}(r,\xi) = D_{u_P}\xi - r \cdot \Big(\tau \otimes \gamma_0 \cdot X_H(u_P)\Big)^{0,1}$$

where we know from the proof of proposition 6.8 thatD_{u_P} is surjective. So for any choice of $r \neq 0$ we can find $\xi \in T_{u_P}\xi$ such that

$$D_{u_P}\xi = r \cdot \Big(\tau \otimes \gamma_0 \cdot X_H(u_P)\Big)^{0,1}$$

and thus $\widehat{D}_{(0,u_P)}(r,\xi) = 0$. Hence $T_{(0,u_P)}\widehat{\mathcal{M}}(P)$ is not contained in $\{0\} \times T_{u_P}\mathcal{B}$. □

7.2 Some general considerations

In this section we will see how to deform the section $\widehat{\mathcal{F}} : \widehat{\mathcal{B}} \to \widehat{\mathcal{E}}$ into a new section $\widehat{\mathcal{F}}_\lambda$ that is transverse to the zero section. The proof does not need any specific information about $\widehat{\mathcal{F}}$ other that than that it is a Fredholm section and its zero set is compact, so we can formulate it in a more general setting.

The idea is that we can 'fill the cokernel' of a Fredholm section $S_0 : B \to E$ with compact zero set in a certain way using a function $S : B \times \mathbb{R}^{k \cdot m} \to E$ for some $k, m \in \mathbb{N}$ and that by a genericity argument there is some $\lambda \in \mathbb{R}^{k \cdot m}$ such that $S_\lambda := S(\cdot, \lambda)$ has the desired properties.

We start with two lemmas.

Lemma 7.5: *Let X and Y be Banach spaces and let $A : X \to Y$ be a surjective bounded linear operator. Then there is a ball around A in the operator metric such that all elements of this ball are surjective.*

Proof. The proof follows an argument from [6]. By the Banach-Schauder theorem, A is open and thus maps the open ball $B_1(0; X) \subset X$ to an open subset of Y. This subset contains 0 and, because it is open, there is $\varepsilon > 0$ with

$$B_\varepsilon(0; Y) \subseteq A(B_1(0; X)).$$

By rescaling the norms we can achieve

$$\overline{B_1(0; Y)} \subseteq A(\overline{B_1(0; X)})$$

and by linearity it follows that

$$\overline{B_r(0;Y)} \subseteq A(\overline{B_r(0;X)})$$

for all $r > 0$. Now let $B_1(A)$ be the open ball of radius 1 around A taken with respect to the operator norm. We claim that every element $C \in B_1(A)$ is surjective.

Take $y \in Y$. We have to construct an element $x \in X$ such that $Cx = y$. We define $\alpha := \|A - C\| < 1$ and $y_0 := y$ and without loss of generality assume that $\|y_0\| < 1$. By surjectivity of A choose $x_0 \in \overline{B_1(0;X)}$ such that $Ax_0 = y_0$. Define $y_1 := y_0 - Cx_0$. Then one has

$$\|y_1\| = \|(A-C)x_0\| = \|A-C\| \cdot \|x_0\| \leq \alpha$$

and thus we can find $x_1 \in \overline{B_\alpha(0;X)}$ with $Ax_1 = y_1$. Define $y_2 := y_1 - Cx_1$, then

$$\|y_2\| = \|(A-C)x_1\| = \|A-C\| \cdot \|x_1\| \leq \alpha^2.$$

Inductively, we get sequences $(x_n)_{n\in\mathbb{N}} \subseteq X, (y_n)_{n\in\mathbb{N}} \subseteq Y$ which satisfy the following for all $n \in \mathbb{N}$:

- $\|x_n\| \leq \alpha^n$, $\|y_n\| \leq \alpha^n$
- $Ax_n = y_n$
- $y_{n+1} = y_n - Cx_n = (A-C)x_n$

The sequence $\left(\sum_{n=0}^{N}\right)_{N\in\mathbb{N}}$ is a Cauchy sequence in X, so we can define $x := \sum_{n=0}^{\infty} x_n$. This element has finite norm

$$\|x\| = \sum_{n=0}^{\infty} \|x_n\| \leq \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

and it satisfies

$$Cx = \sum_{n=0}^{\infty} Cx_n = \sum_{n=0}^{\infty} y_n - y_{n+1} = y_0 = y.$$

□

Lemma 7.6: *Let $(L_t : X \to Y)_{t\in I}$ be a continuous family of Fredholm operators indexed by a topological space I. Then the function*

$$\begin{aligned} k : I &\longrightarrow \mathbb{N} \\ t &\longmapsto \text{ dim coker } L_t \end{aligned}$$

is upper semi-continuous (that is, it can jump down but not up).

Proof. Without loss of generality assume that the Banach spaces are real. Fix $t_0 \in I$, then

$$Y \cong \text{im } L_{t_0} \oplus \mathbb{R}^{k(t_0)}.$$

Let $p_{\text{im } L_{t_0}} : Y \longrightarrow \text{im } L_{t_0}$ be the projection. Surely

$$p_{\text{im } L_{t_0}} \circ L_{t_0} : X \longrightarrow \text{im } L_{t_0}$$

is surjective. By the previous lemma, for t in a small neighbourhood of t_0,

$$p_{\mathrm{im}\ L_{t_0}} \circ L_t : X \longrightarrow \mathrm{im}\ L_{t_0}$$

is still surjective. This means that

$$\dim \mathrm{im}\ L_t \geq \dim \mathrm{im}\ L_{t_0}$$

and so the cokernel of L_t has dimension less than or equal to $k(t_0)$. □

Now we can perturb the section in the desired way.

Proposition 7.7: *Let $E \to B$ be a Banach fibre bundle over a connected Banach manifold B, and assume that S_0 is a Fredholm section in this bundle such that $\mathcal{M} := S_0^{-1}(\mathcal{O}_E)$ is compact.*

Moreover, fix a closed subset $B_0 \subseteq B$ such that $d^v S_0(u)$ is surjective (and thus, by remark 5.7, has a bounded right inverse) for all $u \in \mathcal{M} \cap B_0$.

Then there is a perturbed section $S_\lambda : B \to E$ which satisfies

- *S_λ coincides with S_0 on B_0 and outside a compact neighbourhood of $\mathcal{M}$*
- *S_λ is Fredholm of index ind$(S_\lambda) =$ ind(S_0) and*
- *S_λ is transversal to the zero section $\mathcal{O}_E$.*

Proof. **Step 1: Definition of a universal section S**

Cover $\mathcal{M}$ by bounded open sets over which the bundle $E \to B$ is trivial. By compactness of $\mathcal{M}$ choose a finite subcovering $U_1, \dots, U_m$. Locally, we can assume that all linear operators $d^v S_0(u) : T_u B \to E_u$ are defined on the same Banach spaces. By lemma 7.6, around each $u \in B$ there is a neighbourhood where the dimension of the cokernel of $d^v S_0(u)$ does not jump up. Since $\mathcal{M}$ is compact, finitely many of these neighbourhoods are enough to cover it. Therefore the number

$$k := \max \left\{ \dim \mathrm{coker}\ \left(d^v S_0(u)\right) \middle| u \in \mathcal{M} \right\} \in \mathbb{N}$$

is well-defined. For $1 \leq j \leq m$, $1 \leq i \leq k$ we can choose sections s_{ij} in the bundle $E \to B$ such that

- s_{ij} vanishes outside of U_j,
- for each $u \in U_j \backslash B_0$, the classes of the vectors $s_{1j}(u), \dots, s_{kj}(u)$ generate the finite-dimensional vector space coker $(d^v S_0(u))$ and
- $s_{ij}(u) = 0$ for all $u \in B_0, 1 \leq i \leq k, 1 \leq j \leq m$

– this is possible because the bundle is trivial over each U_j and B_0 is closed. Now we define a smooth function

$$S : B \times \mathbb{R}^{k \cdot m} \longrightarrow E$$
$$S(u, \lambda) := S_0(u) + \sum_{j=1}^{m} \sum_{i=1}^{k} \lambda_{ij} \cdot s_{ij}(u),$$

where $\lambda = (\lambda_{ij})_{\substack{1\le i\le k\\ 1\le j\le m}} \in \mathbb{R}^{k\cdot m}$. This function respects the bundle structure – it is $S(u,\lambda) \in E_u$ for every pair (u,λ), and so $S_\lambda := S(\cdot,\lambda)$ defines a section for every $\lambda \in \mathbb{R}^{k\cdot m}$. It is clear from the definition of the s_{ij} that every S_λ coincides with S_0 on B_0 and outside $\bigcup_{j=1}^m U_j$.

Now consider the linearization of S. For $\xi \in T_uB, l \in T_\lambda\mathbb{R}^{k\cdot m}$ one has

$$dS(u,\lambda)(\xi,l) = dS_0(u)\xi + \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot s_{ij}(u) + \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot ds_{ij}(u)\xi$$

and since we are in a bundle we work again with the vertical differential

$$d^vS(u,\lambda)(\xi,l) = d^vS_0(u)\xi + \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot s_{ij}(u) + \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot d^vs_{ij}(u)\xi.$$

We have

$$d^vS_\lambda(u)(\xi) = d^vS(u,\lambda)(\xi,0) = d^vS_0(u)\xi + \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot d^vs_{ij}(u)(\xi)$$

and the operator

$$\xi \mapsto \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot d^vs_{ij}(u)\xi$$

is compact. In particular, $d^vS_\lambda(u)$ is a compact perturbation of $d^vS_0(u)$. By proposition 5.2 this means that $S_\lambda = S(\cdot,\lambda)$ is a Fredholm section and its index is

$$\operatorname{ind}(S_\lambda) = \operatorname{ind}(S_0).$$

Step 2: Transversality of the universal section S for small λ

It is

$$d^vS(u,0)(\xi,l) = d^vS_0(u)\xi + \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot s_{ij}(u)$$

and so $d^vS(u,0)$ is surjective for all $u \in \bigcup_{j=1}^m U_j$ – for $u \in B_0$ because of the assumption, for $u \notin B_0$ by definition of the s_{ij}.

Fix $u \in \bigcup_{j=1}^m U_j$. By lemma 7.5 there is $R_u > 0$ such that all linear operators

$$C : T_uB \times T_\lambda\mathbb{R}^{k\cdot m} \longrightarrow E_u$$

with

$$\|d^vS(u,0) - C\| < R_u$$

are surjective.

We could now use lemma 7.5 for every $u \in B$ to get an ε_u such that $d^vS(u,\lambda)$ is surjective for $\|\lambda\| < \varepsilon_u$. But unfortunaley the resulting map

$$u \longmapsto \varepsilon_u$$

would not be continuous, so there would not be any chance to find a minimum. This is why we have to be slightly more careful.

Locally around u pretend that B is a vector space and E is a trivial bundle, so that all vertical differentials $d^vS(u,\lambda) : T_uB \to E_u$ have the same domain and target. Then we have

$$\begin{aligned} d^vS(u,0)(\xi,l) - d^vS(u_2,\lambda)(\xi,l) = {} & d^vS_0(u)\xi + \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot s_{ij}(u) \\ & - d^vS_0(u_2)\xi - \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot s_{ij}(u_2) \\ & - \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot d^vs_{ij}(u_2)\xi \\ = {} & (d^vS_0(u) - d^vS_0(u_2))\,\xi + \sum_{j=1}^m\sum_{i=1}^k l_{ij}\cdot (s_{ij}(u) - s_{ij}(u_2)) \\ & - \sum_{j=1}^m\sum_{i=1}^k \lambda_{ij}\cdot d^vs_{ij}(u_2)\xi \end{aligned}$$

for u_2 near u and so

$$\begin{aligned} \|d^vS(u,0) - d^vS(u_2,\lambda)\| \quad & \leq \quad \|d^vS_0(u) - d^vS_0(u_2)\| + \sum_{j=1}^m\sum_{i=1}^k \|s_{ij}(u) - s_{ij}(u_2)\| \\ & \qquad + \sum_{j=1}^m\sum_{i=1}^k |\lambda_{ij}|\cdot\|d^vs_{ij}(u_2)\| \\ & \leq \quad \|d^vS_0(u) - d^vS_0(u_2)\| + \sum_{j=1}^m\sum_{i=1}^k \|s_{ij}(u) - s_{ij}(u_2)\| \\ & \qquad + \mathrm{const}\cdot\|\lambda\|\cdot\sum_{j=1}^m\sum_{i=1}^k \|d^vs_{ij}(u_2)\|. \end{aligned}$$

This means that there are $r_u > 0$ and $\varepsilon_u > 0$ such that if the three conditions

- $\|d^vS_0(u) - d^vS_0(u_2)\| < r_u$,
- $\sum_{j=1}^m\sum_{i=1}^k \|s_{ij}(u) - s_{ij}(u_2)\| < r_u$, and
- $\|\lambda\| < \varepsilon_u$

are satisfied, then $d^vS(u_2,\lambda)$ is surjective. The first two conditions are satisfied for all u_2 which lie in a ball $B_{\hat{r}_u}(u)$ of some radius $\hat{r}_u$ around u. These balls together form an open cover of $\bigcup_{j=1}^m U_j$ and thus of $\mathcal{M}$:

$$\mathcal{M} \subset \bigcup_{j=1}^m U_j \subset \bigcup_{u\in\bigcup_{j=1}^m U_j} B_{\hat{r}_u}(u)$$

By compactness of $\mathcal{M}$ there is a finite subcover, that is some number $d_{\max} \in \mathbb{N}$ and elements

$u_1, \ldots, u_{d_{\max}} \in \mathcal{M}$ such that

$$\mathcal{M} \subset \bigcup_{d=1}^{d_{\max}} B_{\hat{r}_{u_d}}(u_d).$$

Choose $\varepsilon := \min\{\varepsilon_{u_1}, \ldots, \varepsilon_{u_{d_{\max}}}\}$. Then for all $u \in \bigcup_{d=1}^{d_{\max}} B_{\hat{r}_{u_d}}(u_d)$ and all $\|\lambda\| < \varepsilon$,

$$d^v S(u, \lambda) : T_u B \times \mathbb{R}^{km} \to E_u$$

is surjective.

Define the subset

$$N_\varepsilon := \{(u, \lambda) \mid S(u, \lambda) = 0, \|\lambda\| < \varepsilon\} \subseteq B \times \mathbb{R}^{k \cdot m}$$

consisting of all zeros of all S_λ with $\|\lambda\| < \varepsilon$. Since every S_λ equals S_0 outside $\bigcup_{j=1}^m U_j$ and $\mathcal{M}$ is contained in $\bigcup_{j=1}^m U_j$, we have

$$N_\varepsilon \subset \bigcup_{j=1}^m U_j$$

and by possibly making ε even smaller we can achieve that

$$N_\varepsilon \subset \bigcup_{i=1}^{d_{\max}} B_{\hat{r}_{u_d}}(u_d),$$

so that $d^v S(u, \lambda)$ is surjective for all $(u, \lambda) \in N_\varepsilon$. Since we saw before that $d^v S(u, \lambda)$ is a Fredholm operator, the existence of a right inverse follows which remark 5.7. So the universal section S is transversal to the zero section.

Step 3: Selecting a regular value λ of $\pi\big|_{N_\varepsilon}$

Denote by $\pi : B \times \mathbb{R}^{k \cdot m} \to \mathbb{R}^{k \cdot m}$ the projection. We want to show that $\pi\big|_{N_\varepsilon} : N_\varepsilon \to \mathbb{R}^{km}$ is a Fredholm map between manifolds. Since $d^v S(u, \lambda)$ is surjective and has a bounded right inverse for $(u, \lambda) \in N_\varepsilon$, the implicit function therorem 5.8 asserts that N_ε is a manifold and its tangent space at (u, λ) is

$$T_{(u,\lambda)} N_\varepsilon = \ker \left(d^v S(u, \lambda) : T_u B \times T_\lambda \mathbb{R}^{k \cdot m} \longrightarrow E_u \right)$$

and thus, being the kernel of a Fredholm operator, it is finite-dimensional. But then the linearization of $\pi\big|_{N_\varepsilon}$ at $(u, \lambda) \in N_\varepsilon$, which is given by

$$\begin{aligned} d\pi\big|_{N_\varepsilon}(u, \lambda) : T_{(u,\lambda)} N_\varepsilon &\longrightarrow T_\lambda \mathbb{R}^{km} \cong \mathbb{R}^{km} \\ (\xi, l) &\longmapsto l, \end{aligned}$$

is simply a linear map between finite-dimensional manifolds.

By the theorem of Sard, the set of regular values of $\pi\big|_{\mathcal{N}_\varepsilon}$ is of second category; in particular, its intersection with the ball of radius ε around 0 is non-empty and hence we can choose a regular value $\lambda \in \mathbb{R}^{km}$ with $\|\lambda\| < \varepsilon$.

Step 4: Showing that $d^v S_\lambda$ is transverse to the zero section

Let $\lambda \in \mathbb{R}^{k\cdot m}$ be a regular value of $\pi|_{N_\varepsilon}$ with $\|\lambda\| < \varepsilon$. We claim that the corresponding section S_λ is transverse to the zero section.

Remember that

$$d^v S_\lambda(u)(\xi) = d^v S(u,\lambda)(\xi, 0)$$

for every $u \in B$, $\xi \in T_u B$.

Now fix $u \in B$ with $S_\lambda(u) = 0$. Because of our assumption on the norm of λ, the pair (u, λ) is a element of N_ε. Take $\eta \in E_u$. We have to find a preimage of η under $d^v S_\lambda(u)$.

Since $d^v S(u,\lambda)$ is surjective there is $(\xi_1, l) \in T_u B \times T_\lambda \mathbb{R}^{km}$ such that

$$d^v S(u,\lambda)(\xi_1, l) = \eta$$

and since $d\pi|_{N_\varepsilon}(u,\lambda)$ is surjective there is $\xi_2 \in T_u B$ with

$$(\xi_2, -l) \in T_{(u,\lambda)} N_\varepsilon,$$

which means that $d^v S(u,\lambda)(\xi, -l) = 0$.

Now:

$$\begin{aligned}
d^v S_\lambda(u)(\xi_1 + \xi_2) &= d^v S_\lambda(u)\xi_1 + d^v S_\lambda(u)\xi_2 \\
&= d^v S(u,\lambda)(\xi_1, 0) + d^v S(u,\lambda)(\xi_2, 0) \\
&= d^v S_0(u)\xi_1 + \sum_{j=1}^{m}\sum_{i=1}^{k} \lambda_{ij} \cdot d^v s_{ij}(u)\xi_1 \\
&\qquad + d^v S_0(u)\xi_2 + \sum_{j=1}^{m}\sum_{i=1}^{k} \lambda_{ij} \cdot d^v s_{ij}(u)\xi_2 \\
&= d^v S_0(u)\xi_1 + \sum_{j=1}^{m}\sum_{i=1}^{k} \lambda_{ij} \cdot d^v s_{ij}(u)\xi_1 \\
&\qquad + \sum_{j=1}^{m}\sum_{i=1}^{k} l_{ij} \cdot s_{ij}(u) - \sum_{j=1}^{m}\sum_{i=1}^{k} l_{ij} \cdot s_{ij}(u) \\
&\qquad + d^v S_0(u)\xi_2 + \sum_{j=1}^{m}\sum_{i=1}^{k} \lambda_{ij} \cdot d^v s_{ij}(u)\xi_2 \\
&= d^v S(u,\lambda)(\xi_1, l) + d^v S(u,\lambda)(\xi_2, -l) \\
&= \eta + 0 \\
&= \eta
\end{aligned}$$

So we have seen that $d^v S_\lambda(u)$ is surjective for every $u \in B$ with $S_\lambda(u) = 0$. We already know that S_λ is a Fredholm section, so by remark 5.7 this implies the existence of a bounded right inverse.

This completes the proof. $\square$

7.3 Back to the special setting

Now we can apply the results from section 7.2 to the setting of our proof.

Proposition 7.8 (Proposition 3.7 in another formulation)**:** *Assume that $\widehat{\mathcal{M}}(P)$ is compact for a point $P \in M$, and define the Banach manifold $\mathcal{B}$ with respect to this point P as in section 5.4. Then there is a section $\widehat{\mathcal{F}}_\lambda$ in the bundle $\widehat{\mathcal{E}} \to \mathbb{R}_{\geq 0} \times \mathcal{B}$ such that*

$$\widehat{\mathcal{M}}_\lambda(P) := \widehat{\mathcal{F}}_\lambda^{-1}(\mathcal{O}_{\widehat{\mathcal{E}}})$$

is a compact 1*-dimensional manifold with boundary*

$$\partial\widehat{\mathcal{M}}_\lambda(P) = \partial\widehat{\mathcal{M}}(P) \cong \mathcal{M}_0(P) = \{u_P\}.$$

Proof. From the previous chapters, we know that

$$\begin{aligned} \widehat{\mathcal{F}} : \mathbb{R}_{\geq 0} \times \mathcal{B} &\longrightarrow \widehat{\mathcal{E}} \\ (R, u) &\longmapsto \Big(du - \tau \otimes \beta_R X_H(u)\Big)^{0,1} \end{aligned}$$

is a Fredholm section, and we assumed that its intersection $\widehat{\mathcal{M}}(P)$ with the zero section is compact.

Remember that in proposition 7.3 we have seen that the vertical differential $\widehat{D}_{(0,u_P)}$ at the point $(0, u_P)$ is surjective. Then by lemma 7.5, the operator $\widehat{D}_{(R,u)}$ is surjective for every (R, u) in a small neighbourhood of $(0, u_P)$. Let $\mathcal{C}$ be a closed ball around $(0, u_P)$ which is contained in this neighbourhood.

As a closed subset of $\mathbb{R}_{\geq 0} \times \mathcal{B}$ we choose $(\{0\} \times \mathcal{B}) \cup \mathcal{C}$. Then

$$\widehat{\mathcal{M}}(P) \cap ((\{0\} \times \mathcal{B}) \cup \mathcal{C})$$

consists of pairs (R, u) such that the vertical differential $\widehat{D}_{(R,u)}$ is surjective.

Thus we can apply proposition 7.7 to obtain a perturbed Fredholm section $\widehat{\mathcal{F}}_\lambda : \widehat{\mathcal{B}} \to \widehat{\mathcal{E}}$ of index

$$\operatorname{ind}\left(\widehat{\mathcal{F}}_\lambda\right) = \operatorname{ind}\left(\widehat{\mathcal{F}}\right) = 1,$$

which is transversal to the zero section $\mathcal{O}_{\widehat{\mathcal{E}}}$ and coincides with $\widehat{\mathcal{F}}$ on $(\{0\} \times \mathcal{B}) \cup \mathcal{C}$ and outside a compact neighbourhood of $\widehat{\mathcal{M}}(P)$.

By the implicit function theorem 5.8, its zero set

$$\widehat{\mathcal{M}}_\lambda(P) := \widehat{\mathcal{F}}_\lambda^{-1}(\mathcal{O}_{\widehat{\mathcal{E}}})$$

is a smooth submanifold of $\mathbb{R}_{\geq 0} \times \mathcal{B}$ of dimension

$$\operatorname{ind}\widehat{\mathcal{F}}_\lambda = \operatorname{ind}\widehat{\mathcal{F}} = 1.$$

Because of $\widehat{\mathcal{F}}_\lambda(0, u) = \widehat{\mathcal{F}}(0, u)$ for all $u \in \mathcal{B}$,

$$\widehat{\mathcal{M}}_\lambda(P) \cap (\{0\} \times \mathcal{B}) \cong \mathcal{M}_0(P).$$

Since $\widehat{\mathcal{M}}_\lambda(P)$ is a smooth submanifold of $\mathbb{R}_{\geq 0} \times \mathcal{B}$, which has boundary $\partial(\mathbb{R}_{\geq 0} \times \mathcal{B}) = \{0\} \times \mathcal{B}$, its boundary is contained in $\{0\} \times \mathcal{B}$. The only possible boundary component is then

$$\widehat{\mathcal{M}}_\lambda(P) \cap (\{0\} \times \mathcal{B}) = \{(0, u_P)\}$$

and it follows from lemma 7.4 and the fact that we did not perturb the original section in a neighbourhood of $(0, u_P)$ that $(0, u_P)$ really is a boundary point:

$$\partial \widehat{\mathcal{M}}_\lambda(P) = \{(0, u_P)\}$$

It remains to show that $\widehat{\mathcal{M}}_\lambda(P)$ is compact. Let $(R_k, u_k)_{k \in \mathbb{N}} \subseteq \widehat{\mathcal{M}}_\lambda(P)$ be a sequence. Since $\widehat{\mathcal{F}}_\lambda \equiv \widehat{\mathcal{F}}$ outside a compact subset of $\mathbb{R}_{\geq 0} \times \mathcal{B}$ and $\widehat{\mathcal{M}}(P)$ is contained in this subset, also $\widehat{\mathcal{M}}_\lambda(P)$ is contained in it and thus the sequence $(R_k)_{k \in \mathbb{N}}$ is bounded from above. This means that there is a subsequence converging to $R_* \in \mathbb{R}_{\geq 0}$. Exactly as in the proof of proposition 4.1, from this it follows that there is $u_* \in \mathcal{B}$ such that $(R_*, u_*) \in \widehat{\mathcal{M}}_\lambda(P)$ and

$$\lim_{k \to \infty} (R_k, u_k) = (R_*, u_*).$$

□

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Münster, 09. März 2017

(Irene Elisabeth Seifert)