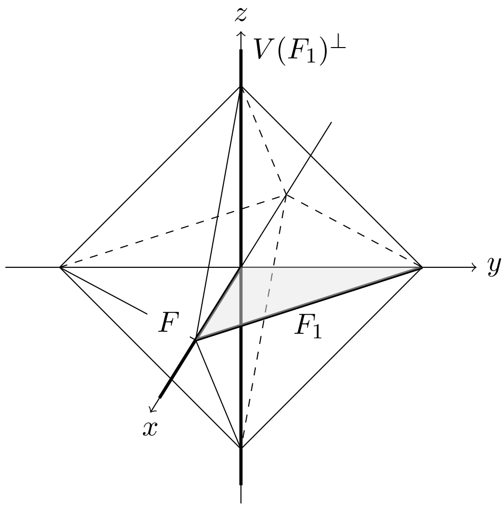
Fakultät für Mathematik und Informatik
Ruprecht-Karls-Universität Heidelberg

Diploma Thesis

Horofunction Compactification of Finite-Dimensional Normed Spaces and of Symmetric Spaces



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Abstract

In the first part of this thesis we consider the horofunction compactification of finite-dimensional normed real spaces and show that the boundary of the compactification has the shape of the dual unit ball if the norm of the space is polyhedral. We will characterise the sequences converging to some Busemann point and see that only the limiting direction and an eventual parallel shift of the sequence have influence on this Busemann point.
In the second part of the thesis we examine symmetric spaces with Finsler metrics and their horofunction compactification by using the results of the first part and we make a short comparison with the Furstenberg compactification.

Zusammenfassung

In dem ersten Teil dieser Arbeit befassen wir uns mit der Horofunktions-Kompaktifizierung von endlich dimensionalen normierten reellen Räumen und zeigen, dass der Rand der Kompaktifizierung die Form des dualen Einheitsballs hat, wenn die Norm auf dem Raum polyederförmig ist. Wir geben eine Charakterisierung der Folgen an, die gegen einen Busemannpunkt konvergieren und sehen, dass nur die beschränkende Richtung und eine eventuelle Parallelverschiebung der Folge Einfluss auf diesen Busemannpunkt haben.
Im zweiten Teil der Arbeit untersuchen wir symmetrische Räume mit Finsler-Metrik und deren Horofunktions-Kompaktifizierung, indem wir die Ergebnisse des ersten Teils verwenden, und geben einen kurzen Ausblick auf die Fürstenberg-Kompaktifizierung.

Contents

1 Introduction

This thesis deals with horofunction compactifications of certain manifolds.

Let X be a topological space. If X is not compact, there are several possibilities to compactify it. One of the most basic compactifications is the (Alexandroff) one-point compactification. But there are far more ways to compactify X, some of them homeomorphic to each other, some not.

In this thesis we will consider spaces with additional structure, namely finite-dimensional normed real spaces and symmetric spaces. We will examine their horofunction compactification. It is also called “Busemann compactification”, if every point of the horoboundary is a Busemann point.

This thesis can be split into three main parts. The first part (chapters 2 and 3) introduces the basic concepts needed such as the horofunction compactification and Busemann points of a metric space (X, d), following [Wal10]. Most of the background knowledge presented consists of convex analysis and the presentation is based on the books [Bee93] and [Roc70]. Other important concepts are the definition of extreme sets of a convex set and the construction of the dual unit ball B° of a polyhedral unit ball B defining the norm of our space.
The horofunction compactification is defined by the embedding of X into the space of continuous functions $C(X)$ via the map

$$\begin{aligned} \psi : X &\longrightarrow C(x) \\ z &\longmapsto \psi_z \end{aligned}$$

where $\psi_z(x) := d(x, z) - d(p_0, z)$ and p_0 is a basepoint of X. There are some special horofunctions, called Busemann points, which can be realised as limits of almost-geodesics.

The next two chapters (4 and 5) deal with the horofunction compactification of finite-dimensional normed spaces. We will explain a paper of Walsh ([Wal07]) where he proves the following two results:

Theorem 4.0.32 *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. Then the set of Busemann points of V is the set $\{f^*_{E,p} \mid E$ is a proper extreme set of $B^\circ, p \in V\}$, where f^* denotes the Legendre-Fenchel transform of f.*

Theorem 4.0.33 *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. Then every horofunction of V is a Busemann point if and only if the set of extreme sets of the dual unit ball B° is closed in the Painlevé-Kuratowski topology.*

The functions $f_{E,p}$ in the first theorem are special affine functions on V^*. In this sense the first theorem gives us a way of finding the Busemann points of a compactification explicitly by computing the functions $f^*_{E,p}$. As mentioned above, every Busemann point is a horofunction but not necessarily vice versa. The nice point of the second theorem is that it gives a criterion when they coincide and when not.
After these rather theoretical results, we will apply them to many examples in which we determine the horofunction compactification. Most of the time we will consider $\mathbb{R}^2$ equipped

with a polyhedral norm or a norm induced by fixing the convex unit ball. I went this way and gained enough intuition to see the general behaviour of sequences converging in the horofunction compactification and a way to construct the boundary geometrically. This work resulted in the formulation of Theorem 5.6.7, a classification of these sequences and their corresponding Busemann points. The theorem is presented after plenty of examples which shall help the reader to get a feeling for the procedure.

In the last two chapters (chapters 6 and 7) we will temporarily leave the finite-dimensional normed spaces and turn to symmetric spaces with Finsler metrics. That means we will consider classical symmetric spaces, that is, Riemannian manifolds with a certain symmetry, which are provided with an additional Finsler structure. As we are mainly dealing with the Lie theoretical description of symmetric spaces, in most of the cases it does not make any difference whether we have a Riemannian structure or not.

In saying this we already mentioned one of the great benefits of symmetric spaces: besides being a Riemannian manifold with symmetry, each symmetric space M is diffeomorphic to some G/K, the set of left cosets of the group G of isometries on M and a certain closed subgroup $K \subseteq G$. As the group G carries the structure of a Lie group, we can consider its Lie algebra $\mathfrak{g}$. Moreover the treatment of symmetric spaces can be turned into a treatment of Lie algebras and we get the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{p} \cong T_{p_0}M$ and $\mathfrak{k}$ the Lie algebra of K. By the root space decomposition with respect to a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, we obtain the Weyl group W and the Weyl chambers of $\mathfrak{a}$. As a subspace of $\mathfrak{p}$, $\mathfrak{a}$ is a finite-dimensional normed space where the norm, a W-invariant convex ball, stands in one-to-one correspondence with the Finsler metric on M. At this point we come back to our results of the first part. We will use them to compactify $SL(3,\mathbb{R})/SO(3)$ and $Sp(4,\mathbb{R})/U(4)$ and see (related to [GJT98]) which choice of Finsler structure leads to a horofunction compactification isomorphic to the Furstenberg compactification.

I would like to thank my advisor, Professor Anna Wienhard, for all her help and her intensive and motivating mentoring during the last year and additionally for this really interesting subject of my thesis I immersed myself deeper and deeper with steadily growing enthusiasm. Her lecture about symmetric spaces and the talks with her have also contributed to it. This thesis wouldn't have been like this and it wouldn't have been such a nice time working on it without the personal, technical and corrective help of my friends, especially of Eike Fokken, who had to endure long discussions especially about converging sequences. Thank you all.

2 Preliminaries

2.1 Painlevé-Kuratowski Topology

We will follow [Bee93] and [Roc70] for an introduction on convex analysis.

Definition 2.1.1 A set Λ is called *directed* by a relation $\geq$ if the relation is reflexive, transitive and for each pair $\{\lambda_1, \lambda_2\} \subseteq \Lambda$ there is a $\lambda_3 \in \Lambda$ such that $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. A *net* in a set X based on a directed set Λ is a function $a : \Lambda \longrightarrow X$. In particular a sequence is a net where $\Lambda = \mathbb{N}$ [1] ordered as usual.
We will write a_λ for $a(\lambda)$ and $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ for the net.

Definition 2.1.2 Let Λ be a directed set. A subset $\Sigma \subseteq \Lambda$ is called

(i) *residual* $\Leftrightarrow \exists \lambda \in \Lambda \; \forall \sigma \in \Lambda : (\sigma \in \Sigma \Leftrightarrow \sigma \geq \lambda)$.

(ii) *cofinal* $\Leftrightarrow \forall \lambda \in \Lambda \; \exists \sigma \in \Sigma : \lambda \leq \sigma$.

Example 2.1.3 A residual set in $\mathbb{Z}$ is a set which consists of all numbers bigger than some $\lambda \in \mathbb{Z}$. A subset of $\mathbb{Z}$ is cofinal if and only if it is unbounded above.

Definition 2.1.4 Let X be a Hausdorff space and $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ a net of subsets of X. Let $x_0 \in X$ be a point. Then

(i) x_0 is called a *limit point* of $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ if each neighbourhood of x_0 intersects A_λ for all λ in some residual subset of Λ.

(ii) x_0 is called a *cluster point* of $\langle A_\lambda \rangle_{\lambda \in \Lambda}$ if each neighbourhood of x_0 intersects A_λ for all λ in some cofinal subset of Λ.

(iii) $\operatorname{Li} A_\lambda := \{\text{limit points of } \langle A_\lambda \rangle\}$ is called the *lower closed limit* of $\langle A_\lambda \rangle$.

(iv) $\operatorname{Ls} A_\lambda := \{\text{cluster points of } \langle A_\lambda \rangle\}$ is called the *upper closed limit* of $\langle A_\lambda \rangle$.

Obviously we have $\operatorname{Li} A_\lambda \subseteq \operatorname{Ls} A_\lambda$.

Example 2.1.5 Let X be the real line $\mathbb{R}$. Define

$$A_n := \begin{cases} [-n, -\frac{1}{n}] & \text{for } n \in \mathbb{N} \text{ even;} \\ [\frac{1}{n}, n] & \text{for } n \in \mathbb{N} \text{ odd.} \end{cases}$$

We already saw that a residual subset of $\mathbb{N}$ consists of all numbers greater or equal to some $\lambda \in \mathbb{N}$. Therefore $\operatorname{Li} A_n = \{0\}$, because for each neighbourhood of 0, there is an $N \in \mathbb{N}$ big enough such that the neighbourhood intersects all A_n with $n \geq N$. There cannot be another limit point x in $\operatorname{Li} A_n$, because x would have a neighbourhood not intersecting either the positive or the negative intervals.
Since a cofinal subset of $\mathbb{N}$ is an arbitrary subset unbounded above, $\operatorname{Ls} A_n = \mathbb{R}$, because for each $y \in \mathbb{R}$ every neighbourhood of y intersects all intervals A_n with n either even or odd and big enough.

[1] In this thesis, $0 \notin \mathbb{N}$.

Proposition 2.1.6 ([Bee93, Prop. 5.2.2]) *Let X be a Hausdorff space and $\langle A_\lambda \rangle_{\lambda\in\Lambda}$ be a net of sets in X. Then*

$$\operatorname{Li} A_\lambda = \bigcap_{\Sigma\subseteq\Lambda \text{ cofinal}} \operatorname{cl}(\bigcup_{\lambda\in\Sigma} A_\lambda)$$

and

$$\operatorname{Ls} A_\lambda = \bigcap_{\Sigma\subseteq\Lambda \text{ residual}} \operatorname{cl}(\bigcup_{\lambda\in\Sigma} A_\lambda).$$

Hence $\operatorname{Li} A_\lambda$ *and* $\operatorname{Ls} A_\lambda$ *are both closed sets.*

Definition 2.1.7 Let X be a Hausdorff space and $\langle A_\lambda \rangle$ a net of sets in X. Let furthermore A be a closed set in X.
We say $\langle A_\lambda \rangle$ is *Painlevé-Kuratowski convergent* to A, if $\operatorname{Li} A_\lambda = \operatorname{Ls} A_\lambda = A$.
In this case, we write $A = \text{K-lim}\, A_\lambda$ or $A_\lambda \xrightarrow{K} A$.

Lemma 2.1.8 ([Bee93, Lemma 5.2.4]) *Let A be a closed subset of a Hausdorff space X and let $\langle A_\lambda \rangle$ be a net of subsets of X.*
Then $A_\lambda \xrightarrow{K} A$ if and only if $A \subseteq \operatorname{Li} A_\lambda$ and $\operatorname{Ls} A_\lambda \subseteq A$.

Lemma 2.1.9 ([Bee93, Prop. 5.2.9]) *Let X be a Hausdorff space, $\langle A_n \rangle_{n\in\mathbb{N}}$ a net of subsets and A a closed set.*
Then $A_n \xrightarrow{K} A$ if and only if the following conditions both hold:

- $\forall x \in A : \exists x_n \in A_n : x_n \to x.$
- *if $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ is a subsequence of $\mathbb{N}$ with $n_k < n_{k+1}$ and $x_k \in A_{n_k}$ $\forall k \in \mathbb{N}$ then $(x_k) \to x$ implies that $x \in A$.*

2.2 Convex Analysis

In the following let X always be a finite-dimensional normed linear space over $\mathbb{R}$.

Definition 2.2.1 A subset $A \subseteq X$ is called *convex* if $\alpha a + (1-\alpha)b \in A$ for all $\alpha \in [0,1]$ and $a, b \in A$.

Theorem 2.2.2 ([Bee93, Thm. 1.4.1]) *Let X and Y be normed linear spaces.*

(i) The sum of two convex subsets of X is convex.

(ii) Let $A \subseteq X$ be convex and α be a scalar. Then αA, $\operatorname{cl} A$ and $\operatorname{int} A$ are also convex.

(iii) Let $f : X \to Y$ be an affine function and $A \subseteq X$ be convex. Then the image $f(A) \subseteq Y$ is also convex.

(iv) Let $\{A_i \mid i \in I\}$ be a family of convex subsets of X. Then the intersection $\bigcap_{i\in I} A_i$ is also convex.

(v) Let $\{A_i \mid i \in I\}$ be a family of convex subsets of X. If $\forall i, j \in I\ \exists k \in I : A_i \cup A_j \subseteq A_k$ (i.e. $\{A_i\}$ is directed by inclusion), then $\bigcup_{i\in I} A_i$ is convex.

Definition 2.2.3 Let $A \subseteq X$ be a subset. The *convex hull* of A is the smallest convex subset of X containing A. We will denote this set by $\operatorname{conv} A$.

Lemma 2.2.4 *Let X be a normed linear space and $A \subseteq X$ a non-empty subset. Then*

$$\operatorname{conv} A = \left\{ \sum_{i=1}^{k} \mu_i a_i \,\middle|\, k \in \mathbb{N}, a_i \in A, \mu_i \geq 0 \text{ and } \sum_{i=i}^{k} \mu_i = 1 \right\}$$

Proof. We set $M := \{\sum_{i=1}^{k} \mu_i a_i \mid k \in \mathbb{N}, a_i \in A, \mu_i \geq 0$ and $\sum_{i=i}^{k} \mu_i = 1\}$.

"$\subseteq$" We show that M is convex and contains A. As conv$A$ is the smallest subset of X containing A, it follows that conv$A \subseteq M$ by definition of the convex hull.
For convexity of M let $m_1, m_2 \in M$ be arbitrary. Then $\exists a_i, b_j \in A$; $\mu_i, \nu_j \geq 0$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ with $\sum_{i=1}^{k} \mu_i = 1; \sum_{j=1}^{l} \nu_j = 1$ such that

$$m_1 = \sum_{i=1}^{k} \mu_i a_i; \qquad m_2 = \sum_{j=1}^{l} \nu_j b_j.$$

We have to show, that $\alpha m_1 + (1-\alpha) m_2 \in M \; \forall \alpha \in [0,1]$. Let $\alpha \in [0,1]$.

$$\begin{aligned} \alpha m_1 + (1-\alpha) m_2 &= \alpha \sum_{i=1}^{k} \mu_i a_i + (1-\alpha) \sum_{j=1}^{l} \nu_j b_j \\ &= \sum_{i=1}^{k+l} \gamma_i c_i \end{aligned}$$

with

$$\gamma_i := \begin{cases} \alpha \mu_i & \text{for } i = 1, \ldots, k; \\ (1-\alpha)\nu_{i-k} & \text{for } i = k+1, \ldots, k+l. \end{cases}$$

and

$$c_i := \begin{cases} a_i & \text{for } i = 1 \ldots k; \\ b_{i-k} & \text{for } i = k+1 \ldots k+l. \end{cases}$$

As $\gamma_i \in [0,1]$, $c_i \in A \; \forall i = 1 \ldots k+l$ and

$$\sum_{i=1}^{k+l} \gamma_i = \alpha \sum_{i=1}^{k} \mu_i + (1-\alpha) \sum_{j=1}^{l} \nu_j = \alpha + 1 - \alpha = 1,$$

our last sum lies in M.
It is obvious that $A \subseteq M$. Altogether M is convex and contains A, so conv$A \subseteq M$ as stated above.

"$\supseteq$" Proof by induction over the length of the sum k, the case $k = 1$ is clear.
For $k = 2$ let $m = \mu_1 a_1 + \mu_2 a_2 \in M$.
As $\mu_2 = 1 - \mu_1$, it is $m = \mu_1 a_1 + (1-\mu_1) a_2 \in$ conv$A$ since conv$A$ is convex.
Assume now that $\sum_{i=1}^{k} \mu_i a_i \in$ conv$A$ for all $\mu_i \in [0,1]$ and $a_i \in A$ where $i = 1, \ldots, k$.
Let $m = \sum_{i=1}^{k+1} \mu_i a_i \in M$, that is $a_i \in A$, $\mu_i \geq 0$ and $\sum_{i=1}^{k+1} \mu_i = 1$. Without loss of generality let all $\mu_i > 0$. Define

$$s := \sum_{i=1}^{k} \mu_i.$$

Then

$$\sum_{i=1}^{k} \frac{\mu_i}{s} = 1 \text{ and } \frac{\mu_i}{s} \in [0,1] \; \forall i = 1, \ldots, k$$

and

$$\begin{aligned} m &= \sum_{i=1}^{k+1} \mu_i a_i \\ &= \sum_{i=1}^{k} \mu_i a_i + \mu_{k+1} a_{k+1} \\ &= s\left(\sum_{i=1}^{k} \frac{\mu_i}{s} a_i\right) + \mu_{k+1} a_{k+1} \\ &= sc + (1-s) a_{k+1} \in \operatorname{conv} A \end{aligned}$$

with $c := \sum_{i=1}^{k} \frac{\mu_i}{s} a_i \in \operatorname{conv} A$ by induction and $\mu_{k+1} = 1 - \sum_{i=1}^{k} \mu_i = 1 - s$. The last sum lies in conv$A$ by definition of the convex hull. □

Definition 2.2.5 Let K be an non-empty convex set. Then K is called a *cone*, if whenever $x \in K$ and $\alpha \geq 0$, then $\alpha x \in C$.
Let $C \subseteq \mathbb{R}^m$ be a convex set then we define

$$K_C := \{x \in \mathbb{R}^m \mid x = \alpha c \ \alpha \geq 0, c \in C\}$$

to be the smallest cone containing C.

We now want to state an important fact of convex analysis: the separation theorem. Let X be a finite-dimensional normed linear space[2], X^* its dual space, $y^* \in X^*$ nonzero and $\alpha \in \mathbb{R}$.

Definition 2.2.6 The sets $\{x \in X \mid y^*(x) \leq \alpha\}$ and $\{x \in X \mid y^*(x) \geq \alpha\}$ are called *closed half-spaces* determined by the hyperplane $(y^*)^{-1}(\alpha)$.

Definition 2.2.7 An oriented hyperplane $H \subseteq X$ is called a *supporting hyperplane* of a convex set $A \subseteq X$, if the following conditions are satisfied:

(i) $A \cap H \neq \emptyset$

(ii) A lies completely in one of the two closed half-spaces of X determined by H.

The supporting hyperplane H is called *non-trivial* if A is not contained in H.

Remark 2.2.8 A convex set A must have at least one point in its boundary to have a supporting hyperplane H. In particular, whenever A has a supporting hyperplane H, then $A \cap H \subseteq \partial A$.

Remark 2.2.9 We can describe a supporting hyperplane H to a convex set A in $\mathbb{R}^n$ as

$$H = \{x \in \mathbb{R}^n \mid \langle x|b\rangle = \alpha\}$$

with $b \in \mathbb{R}^n$, $b \neq 0$, and $\alpha \in \mathbb{R}$ such that $\langle x|b\rangle \leq \alpha$ for all $x \in A$ and $\langle x|b\rangle = \alpha$ for at least one $x \in A$.

Definition 2.2.10 Let $A, B \subseteq X$ be two convex sets.
We say that the hyperplane $(y^*)^{-1}(\alpha)$ *separates* A and B if $A \subseteq \{x \in X \mid y^*(x) \leq \alpha\}$ and $B \subseteq \{x \in X \mid y^*(x) \geq \alpha\}$ or vice versa.
We say that y^* *strongly separates* A and B if the set $\{\alpha \in \mathbb{R} \mid (y^*)^{-1}(\alpha) \text{ separates } A \text{ and } B\}$ is an interval $[\alpha_0, \alpha_1]$ with $\alpha_0 \neq \alpha_1$.

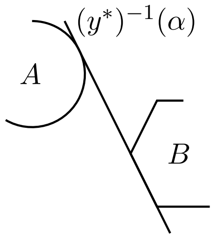


Figure 2.1: Separation

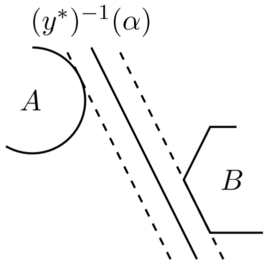


Figure 2.2: Strong separation

Remark 2.2.11 In the situation of the definition, the hyperplane $(y^*)^{-1}(\beta)$ strongly separates A and B for any β with $\alpha_0 < \beta < \alpha_1$.

We are now able to state the first separation theorem:

Theorem 2.2.12 ([Bee93, p. 22]) *Let $(X, \|\cdot\|)$ be a finite-dimensional normed linear space and $A, B \subseteq X$ non-empty convex subsets with $\operatorname{int} B \neq \emptyset$.*
If $A \cap \operatorname{int} B = \emptyset$, then there is a continuous linear functional on X separating A and B.
Dually, let $B \subseteq X^*$ be convex such that $\operatorname{int} B \neq \emptyset$ and $\emptyset \neq A \subseteq X^*$ another convex subset.
If $A \cap \operatorname{int} B = \emptyset$, then there is an $x \in X$ such that x separates A and B.

Definition 2.2.13 Let $M \subseteq \mathbb{R}^n$. M is said to be an *affine set*, if $(1-\lambda)x + \lambda y \in M$ for all $x, y \in M$ and $\lambda \in \mathbb{R}$.

Definition 2.2.14 Let $A \subseteq \mathbb{R}^n$. The *affine hull* aff$A$ is defined as the smallest affine set in $\mathbb{R}^n$ containing A.

Definition 2.2.15 The *relative interior* $\operatorname{ri} A$ of a set $A \subseteq \mathbb{R}^n$ is the interior which results when A is regarded as a subset of its affine hull aff$A$:

$$\operatorname{ri} A := \{x \in \operatorname{aff} A \mid \exists\, \epsilon > 0 : B_\epsilon(x) \cap \operatorname{aff} A \subseteq A\}.$$

In the same way we define the *relative boundary* of A as

$$\partial_{\text{rel}} A := (\operatorname{cl} A) \setminus (\operatorname{ri} A).$$

The important part in the definition of $\operatorname{ri} A$ is that the interior is regarded as a subset of aff$A$ and not of $\mathbb{R}^n$. This makes the difference between the relative and the normal interior. Here is a simple example to see that $\operatorname{ri} A \neq \operatorname{int} A$.

Example 2.2.16 Let $A = \{(t, 0) \in \mathbb{R}^2 \mid -2 \leq t \leq 2\}$ be as in the figure below.

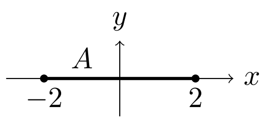


Figure 2.3: ri$A \neq$ int$A$

Then aff$A$ is the x-axis and $\operatorname{ri} A = \{(t, 0) \in \mathbb{R}^2 \mid -2 < t < 2\}$, whereas $\operatorname{int} A = \emptyset$.

[2]More generally, the following definitions also hold when X is a locally convex Hausdorff space, see [Bee93]

Remark 2.2.17

(i) If $A \subseteq \mathbb{R}^n$ is an n-dimensional convex subset[3], then aff$A = \mathbb{R}^n$.

(ii) The relative interior of a point is the point itself.

With this preparation we can now state another version of the separation theorem:

Theorem 2.2.18 ([Roc70, Thm. 11.6]) *Let C be a convex set and $D \subseteq C$ a convex subset. Then $D \cap \operatorname{ri} C = \emptyset$ if and only if there exists a non-trivial supporting hyperplane to C containing D.*

Definition 2.2.19 Let X be a metric space[4], $x_0 \in X$ and $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$.
f is called *lower semi-continuous at* x_0 if for all $\varepsilon > 0$ there is a neighbourhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon \ \forall x \in U$.
f is called *lower semi-continuous* if f is lower semi-continuous at every $x_0 \in X$.

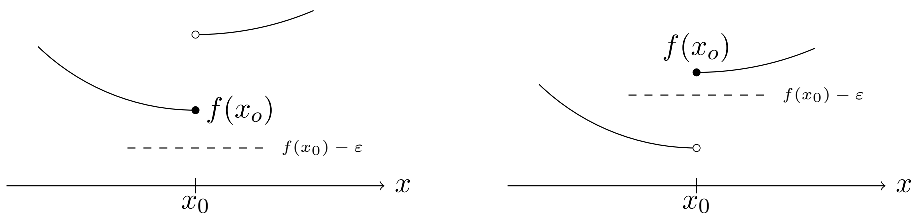


Figure 2.4: f lower semi-continuous at x_0 (left) and not semi-continuous (right)

Remark 2.2.20 In a metric space X with $E \subseteq X$, $x_0 \in E$ and $f : E \longrightarrow \mathbb{R} \cup \{\pm\infty\}$, this definition is equivalent to[5]:

$$\liminf_{x \to x_0} f(x) \geq f(x_0)$$

Lemma 2.2.21 ([Roc70, Thm. 7.1]) *Let $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$. Then:*

$$f \text{ is lower-semi continuous} \iff \text{all lower level sets } \{x \in X \mid f(x) \leq a\} \text{ for } a \in \mathbb{R} \text{ are closed.}$$
$$\iff \{x \in X \mid f(x) > a\} \subseteq X \text{ is open } \forall a \in \mathbb{R}.$$

Definition 2.2.22 Let $f : X \longrightarrow \mathbb{R}$ be a function. The *epigraph of* f is defined as

$$\operatorname{epi}(f) := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq f(x)\} \subseteq X \times \mathbb{R}.$$

Definition 2.2.23 A function $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is called *convex*, if $\operatorname{epi} f$ is a convex set of $X \times \mathbb{R}$.

Lemma 2.2.24 ([Bee93, Thm. 1.3.3 and §5.3]) *Let $f : X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$. Then:*

(i) f is lower semi-continuous $\iff$ $\operatorname{epi} f \subseteq X \times \mathbb{R}$ is closed.

(ii) a sequence of lower semi-continuous functions converges to some lower semi-continuous function $\iff$ the associated sequence of epigraphs converges in the Painlevé-Kuratowski topology.

[3] A subset is called n-dimensional, if it cannot be embedded into an (affine) hyperplane.
[4] This definition also holds for a topological space X.
[5] $\liminf_{x \to x_0} f(x) := \sup \{ \inf\{f(x) \mid x \in E \cap U \setminus \{x_0\}\} | U \text{ open }, x_0 \in U, \ E \cap U \setminus \{x_0\} \neq \emptyset \}$.

2.3 Legendre-Fenchel Transformation

Let X be a Banach space and $f : X \longrightarrow \mathbb{R} \cup \{\infty\}$ a function. The Legendre-Fenchel transform f^* of f is defined as:

$$f^* : X^* \longrightarrow \mathbb{R} \cup \{\pm\infty\}; \qquad f^*(y) = \sup_{x \in X}(\langle y|x\rangle - f(x))$$

We will give a geometrical interpretation of the transformation of convex functions. A closed convex set can be uniquely described by its supporting hyperplanes. Analogously a convex function is uniquely determined by its lower supporting hyperplanes, which are exactly the hyperplanes defining the epigraph of our function (which is a convex set as noted above).

For a better understanding consider the one-dimensional case:
Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be convex. As $\mathbb{R}^* \cong \mathbb{R}$, we have $f^* : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ (in particular $f^*(y) < \infty$ for all $y \in \mathbb{R}$ by convexity) and for $m \in \mathbb{R}$:

$$f^*(m) = \sup_{x \in \mathbb{R}}[mx - f(x)]$$

As epi$f \subseteq \mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$, the mx-part defines a straight line with slope m, hence a hyperplane of dimension 1. We now have to find the point $x_0 \in \mathbb{R}$ where the (positive) distance between the straight line mx and the graph of the function (the boundary of the epigraph) is maximal. Then $f^*(m)$ is equal to this maximal distance. Because of

$$\sup_{x \in X}[\langle y|x\rangle - f(x)] = -\inf_{x \in X}[f(x) - \langle y|x\rangle]$$

this is equivalent to searching for the point where the distance is minimal and then take the negative of this value. In the example in figure 2.5 the latter is more convenient, as $f(x) \geq mx$.
One can also say that the value of the Legendre-Fenchel transform at m is the negative up or down shift of the supporting hyperplane to epi f at the point $f(x_0)$ such that it passes through the origin.

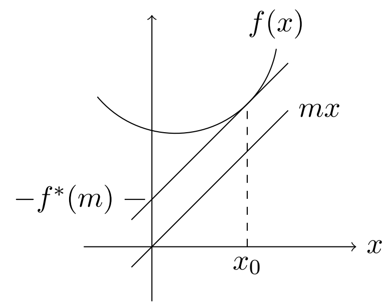


Figure 2.5: The Legendre-Fenchel transform $f^*(m)$ is the negative of the minimal y-axis intercept of a supporting hyperplane to epi f.

In higher dimensions, the main idea remains the same, with the sole difference that the straight line becomes a hyperplane of dimension $n - 1$.

Remark 2.3.1 The Legendre-Fenchel transformation induces a bijection between the set of proper lower semi-continuous convex functions and itself. It is continuous with respect to the epigraph topology[6].

[6] A sequence of functions f_n is said to converge in the epigraph topology if epi(f_n) converges in the Painlevé-Kuratowski topology.

2.4 Extreme Sets and Exposed Faces

We will follow [Roc70, Chapter 18] for an introduction on extreme sets and exposed faces.

Let C be a convex set. Generally speaking there are two kinds of representations of C: internal and external ones.
The first one is to express C as convex combinations of the elements of some point set S with $\operatorname{conv} S = C$. For a given convex set C there might be several point sets S with this property.
The external representation describes C as the intersection of some collection of half spaces[7].

Definition 2.4.1 Let $E \subseteq C$ be convex sets. Let $L_{a,b} := \{\lambda a + (1-\lambda)b \mid \lambda \in [0,1]\}$ with $a, b \in C$ be a straight line. Then E is called an *extreme set*[8] of C if and only if $\operatorname{ri} L_{a,b} \cap E \neq \emptyset$ implies that $a, b \in E$. *Extreme points* are extreme sets which consists of a single point.

The empty set and C itself are the trivial extreme sets of C. From this definition we can directly deduce the following lemma:

Lemma 2.4.2 ([Roc70, p.162]) *The point $x \in C$ is an extreme point of C if and only if we can not find any $a, b \in C$ with $a, b \neq x$ such that $x = (1-\lambda)a + \lambda b$ with $0 < \lambda < 1$.*

Lemma 2.4.3 ([Roc70, p.163]) *The property "extreme set" is transitive: if $F \subseteq E$ and $E \subseteq C$ are both extreme sets, then $F \subseteq C$ is an extreme set.*

Definition 2.4.4 Let C be a convex set. A non-empty proper subset $F \subseteq C$ is said to be an *exposed face* of C if there is a nontrivial supporting hyperplane H to C such that $F = C \cap H$. An *exposed point* of C is an exposed face which consists of a single point.

Remark 2.4.5 Let C be a convex set. An exposed point of C is a point through which there is a supporting hyperplane containing no other point of C.

Lemma 2.4.6 ([Roc70, p.163]) *Let E be an exposed face of C. Then E is also an extreme set of C.*

Remark 2.4.7 Consider the following example to see that not all extreme sets are exposed.

Example 2.4.8 Let $C \subseteq \mathbb{R}^2$ be the convex set given in figure 2.6.

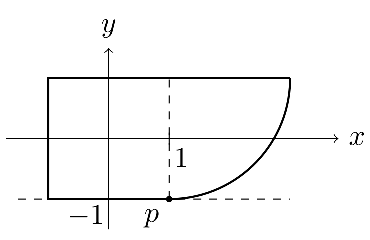


Figure 2.6: The point p is extreme but not exposed

The point $p = (1, -1)$ is an extreme but not an exposed point of C.

[7] A closed convex set C is the intersection of the closed half-spaces containing it. [Roc70, Thm. 11.5]

[8] Note that Rockafellar [Roc70, p.162] calls an extreme set a "face", but in my opinion this name is confusing because of the term "exposed faces". Other authors like Walsh use the name "extreme set".

2.5 The Dual Unit Ball

The content of this section, especially the geometrical construction and Lemmata 2.5.15 and 2.5.16, is of great technical importance and will be used in the examples (section 5).

Definition 2.5.1 Let $C \subseteq \mathbb{R}^n$ be convex. Then C is called a *polyhedral convex set* if C can be expressed as the intersection of finitely many closed half-spaces.
A *polytope* is the convex hull of a finite set of points. A polytope in three dimensions is often called a *polyhedron*.

Theorem 2.5.2 ([Roc70, Thm. 19.1]) *Let $C \subseteq \mathbb{R}^n$ be a convex set. Then the following statements are equivalent:*

(i) C is polyhedral.

(ii) C is closed and has only finitely many extreme sets.

(iii) C is finitely generated, that is C is the convex hull of finitely many points and directions (points at infinity).

Remark 2.5.3 The vertices, edges and faces of a polyhedron P are extreme sets of P.

Definition 2.5.4 Let X be a normed linear space and $C \subseteq X$. The *polar* C° of C is defined as

$$C^\circ := \{y^* \in X^* \mid \langle y^* | x\rangle \geq -1 \ \forall x \in C\} \subseteq X^*.$$

Remark 2.5.5 Several authors define the polar with the condition "$\langle y^*|x\rangle \leq 1$". For consistency with the rest of this work and [Wal07], we will use the above definition instead. If C is symmetric[9], both definitions amount to the same object.

Remark 2.5.6 ([Roc70, p.125]) The polar C° of $C \subseteq X$ is always closed and contains the origin.

Lemma 2.5.7 ([Roc70, Cor. 19.2.2]) *The polar of a polyhedral convex set is polyhedral.*

Theorem 2.5.8 ([Roc70, Thm. 14.5.1]) *Let $C \subseteq \mathbb{R}^n$.*

(i) $C^{\circ\circ} = \mathrm{cl}(\mathrm{conv}(C \cup \{0\}))$

(ii) If C is a closed convex set containing the origin $\{0\}$ then the polar C° is also closed convex, contains $\{0\}$ and $C^{\circ\circ} = C$.

Definition 2.5.9 Let $(X, \|\cdot\|)$ be a finite-dimensional normed space where the norm is not necessarily symmetric[10]. The *unit ball* B is defined as

$$B := \{x \in X \mid \|x\| \leq 1\}.$$

The *dual unit ball* is defined as the polar of B:

$$B^\circ :=\{y^* \in X^* \mid \langle y^*|x\rangle \geq -1 \ \forall x \in B\}.$$

[9] That is, symmetric with respect to the origin.
[10] That is, we have the condition $\|\lambda x\| = \lambda\|x\|$ for all $\lambda \geq 0$ but not for negative λ.

Definition 2.5.10 Let $B \subseteq X$ be an n-dimensional polytope containing the origin as an interior point. Then B defines a possibly non-symmetric norm on X by

$$\|x\|_B := \inf\{\alpha \geq 0 \mid x \in \alpha B\}$$

for any $x \in X$. We call such a B a *polyhedral unit ball*, especially if we consider its corresponding norm.

Remark 2.5.11 If B is symmetric, then so is $\|\cdot\|_B$.

Theorem 2.5.12 ([Roc70, Cor. 15.1.1 and Thm. 15.2]) *Let $\|\cdot\|_B$ be the norm defined by the polytope B. Then $\|\cdot\|_{B^\circ}$ defines a norm on X^*.*

Lemma 2.5.13 ([Roc70, Cor. 15.3.2]) *Let $V = \mathbb{R}^n$ and $\|\cdot\|_p$ be the p-norm*[11]*. If B is the unit ball with respect to this p-norm, then the dual unit ball B° is the unit ball with respect to the q-norm where $\frac{1}{p} + \frac{1}{q} = 1$.*

Lemma 2.5.14 *Let K, L be two subsets of a real vector space V, then:*

(i) $(K \cup L)^\circ = K^\circ \cap L^\circ$.

(ii) $K \subseteq L \Rightarrow L^\circ \subseteq K^\circ$.

If K and L are both closed convex and contain the origin $\{0\}$, then we also have:

(iii) $(K \cap L)^\circ = \operatorname{cl}\operatorname{conv}(K^\circ \cup L^\circ)$.

Proof.

(i) An easy calculation shows:

$$\begin{aligned}(K \cup L)^\circ &= \{y \in V^* \mid \langle x|y\rangle \geq -1 \ \forall x \in K \cup L\} \\ &= \{y \in V^* \mid \langle x|y\rangle \geq -1 \ \forall x \in K \text{ and } \langle x|y\rangle \geq -1 \ \forall x \in L\} \\ &= K^\circ \cap L^\circ\end{aligned}$$

(ii) Clear as $\{y \in V^* \mid \langle x|y\rangle \geq -1 \ \forall x \in L\} \subseteq \{y \in V^* \mid \langle x|y\rangle \geq -1 \ \forall x \in K \subseteq L\}$.

(iii) As K and L are both closed conxev and contain $\{0\}$, we know by Theorem 2.5.8 (ii) that $K^{\circ\circ} = K$ and $L^{\circ\circ} = L$. Therefore by using (i) of Theorem 2.5.8

$$\begin{aligned}(K \cap L)^\circ &= (K^{\circ\circ} \cap L^{\circ\circ})^\circ \\ &= [(K^\circ)^\circ \cap (L^\circ)^\circ]^\circ \\ &\overset{(i)}{=} [K^\circ \cup L^\circ]^{\circ\circ} \\ &= \operatorname{cl}\operatorname{conv}[(K^\circ \cup L^\circ) \cup \{0\}] \\ &= \operatorname{cl}\operatorname{conv}[K^\circ \cup L^\circ].\end{aligned}$$

In the last step we used that K° and L° already contain $\{0\}$. $\square$

Lemma 2.5.15 *Let $B \subseteq \mathbb{R}^n$ be a polyhedral unit ball with k vertices $a_1, \dots, a_k$ and l $n-1$-dimensional facets. Then there are unique $b_1, \dots, b_l \in (\mathbb{R}^n)^*$ such that*

$$B = \operatorname{conv}\{a_1, \dots, a_k\} \tag{2.1}$$

$$= \{x \in \mathbb{R}^n \mid \langle b_i|x\rangle \geq -1 \ \forall i = 1, \dots, l\}. \tag{2.2}$$

[11] $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

Proof. Definition of the b_j:
Let $a_{j_1}, \dots, a_{j_n}$ be n neighbouring vertices which span a facet, that is they determine the hyperplane L_j containing this facet. Determine b_j such that

$$\langle b_j | L_j \rangle = -1.$$

This is a uniquely solvable $n \times n$ linear system of equations as we have the condition that the vertices a_i span a facet of the convex polytope B. From this follows unique existence.

"$\subseteq$" Fix one arbitrary b_j . We defined the hyperplane L_j by

$$L_j = \{x \in \mathbb{R}^n \mid \langle b_j | x \rangle = -1\}.$$

Then $L_j = \operatorname{aff}(a_{j_1}, \dots, a_{j_n})$ and it contains the facets spanned by the a_{j_k} used to define b_j. L_j divides $\mathbb{R}^n$ into the two closed half-spaces $S_1 := \{x \mid \langle b_j | x \rangle \leq -1\}$ and $S_2 := \{x \mid \langle b_j | x \rangle \geq -1\}$. The origin $\{0\}$ is contained in S_2. As L_j is a supporting hyperplane to B, B lies completely in one of the two half-spaces and we know that this has to be S_2 because of $\{0\} \in B$. Therefore

$$\langle b_j | a_i \rangle \geq -1 \ \forall i = 1, \dots, k.$$

Now let $x \in \operatorname{conv}\{a_1, \dots, a_k\}$ be arbitrary. Then there are $t_i \in [0, 1]$ $(i = 1, \dots, k)$ with $\sum t_i = 1$ and $x = \sum_{i=1}^k t_i a_i$. Then

$$\langle b_j | x \rangle = \sum_{i=1}^k t_i \underbrace{\langle b_j | a_i \rangle}_{\geq -1} \geq - \sum_{i=1}^k t_i = -1$$

and as j was chosen arbitrarily, $x \in \{x \in \mathbb{R}^n \mid \langle b_i | x \rangle \geq -1 \ \forall i = 1, \dots, l\}$.

"$\supseteq$" Let $y \in \mathbb{R}^n$ with $\langle b_i | y \rangle \geq -1$ for all $i = 1, \dots, l$. As B is a polyhedral unit ball, there is an $x \in \partial B$ and an $s \geq 0$ such that $y = sx$. As x lies on the boundary of B, it is contained in an exposed face and therefore there is a b_i such that $\langle b_i | x \rangle = -1$. Then

$$-1 \leq \langle b_i | y \rangle = s \langle b_i | x \rangle = -s$$

and so $s \leq 1$ which means that $y \in B$ because B is convex. □

We are now prepared to formulate a useful lemma for calculating the dual unit ball. As its proof is similar to the proof of the previous lemma, we will give it in the appendix on page 91.

Lemma 2.5.16 *Let B be as above. Then*

$$B^\circ = \operatorname{conv}\{b_1, \dots, b_l\}.$$

Geometrical Construction of B° in $\mathbb{R}^2$

Based on our calculation above, there is an easy way to construct and draw the dual unit ball B° of a given unit ball B in the two-dimensional case. Let

$$\begin{aligned} B &= \operatorname{conv}\{a_1, \dots, a_k\} \\ &= \{x \in \mathbb{R}^2 \mid \langle b_j | x \rangle \geq -1 \ \forall j = 1, \dots, k\} \end{aligned}$$

be a unit ball with vertices $a_i \in \mathbb{R}^2$ and $b_j \in (\mathbb{R}^2)^* \cong \mathbb{R}^2$ $(i, j = 1, \ldots, k)$. We already know that

$$B^\circ = \mathrm{conv}\{b_1, \ldots, b_k\}.$$

Hence an easy way to draw the dual unit ball is to draw the k points b_i and connect them. If the b_i are not known (and you do not want to calculate them), there is another, rather constructive way to obtain the dual unit ball: Let l_i be the straight line through the origin and the vertex a_i. For each $i = 1, \ldots, k$ let $q_i \in \mathbb{R}^2$ be such that $\langle q_i | a_i \rangle = -1$. q_i is not uniquely determined and we can choose it as simple as possible. For $a_i = (a_i^{(1)}, a_i^{(2)})$ define

$$c_i := \begin{pmatrix} -a_i^{(2)} \\ a_i^{(1)} \end{pmatrix}.$$

Then $\langle a_i | c_i \rangle = a_i^{(1)} a_i^{(2)} - a_i^{(2)} a_i^{(1)} = 0 \ \forall i = 1, \ldots, k$ [12]. Define the hyperplane[13]

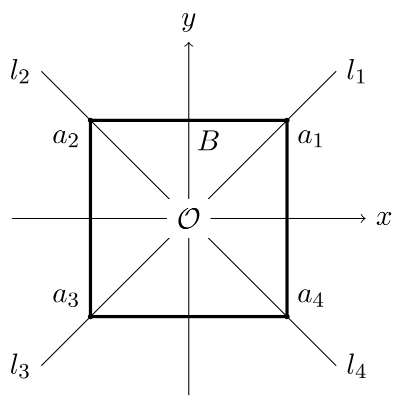
$$h_i := \langle c_i \rangle = \mathbb{R} c_i.$$

This is a hyperplane through the origin perpendicular to the line l_i. As we need the line for which the dual pairing is -1, we have to shift h_i by q_i and therefore define

$$H_i := h_i + q_i.$$

Then the dual unit ball is the area surrounded by these hyperplanes.

For drawing this, we first draw the lines l_i through the origin and the vertices. We show here as an example the construction of the dual unit ball of the L^∞-norm:



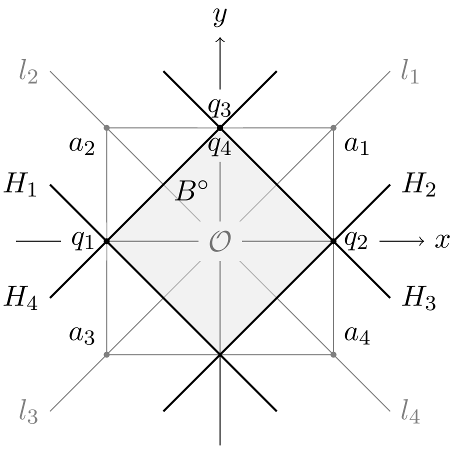
$$a_1 = (1, 1)$$
$$a_2 = (-1, 1)$$
$$a_3 = (-1, -1)$$
$$a_4 = (1, -1)$$

Figure 2.7: Step 1 of the construction: drawing the lines l_i passing through the vertices a_i

After calculating and drawing the points q_i, we draw the shifted hyperplanes H_i through the points q_i perpendicular to the lines l_i, which have been extended into the negative direction.

[12] For $n > 2$ and without loss of generality $a_i^{(1)} \neq 0$ we have $a_i = (a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)})$ and we then have to define $c_i^j := (-a_i^{(j)}, 0, \ldots, 0, a_i^{(1)}, 0, \ldots) \ \forall j = 2, \ldots, n$. Then we have $\langle a_i | c_i^{(j)} \rangle = 0 \ \forall j = 2, \ldots, n$.

[13] In higher dimensions we actually have a hyperplane, namely $h_i := \langle c_i^{(2)}, \ldots, c_i^{(n)} \rangle$. In two dimensions it is only a straight line.



$$q_1 = (-1, 0)$$
$$q_2 = (1, 0)$$
$$q_3 = (0, 1)$$
$$q_4 = (0, 1)$$

Figure 2.8: Step 2 of the construction: the hyperplanes H_i enclose the dual unit ball B°

The dual unit ball, here grey-shaded, is the area surrounded by the hyperplanes H_i. We see that in this example $\|\cdot\|_{B^\circ} = \|\cdot\|_1$. This fits with Lemma 2.5.13. We will need this result later in section 5.1.

3 Introduction to the Horofunction Boundary

In the next sections we want to explore the horofunction compactification, especially for finite-dimensional normed spaces. We will strongly follow the second section of the paper *The horoboundary and isometry group of Thurstons Lipschitz metric* [Wal10]. We will introduce horofunctions and show how to use them to compactify a metric space under several conditions. In the end we will also define Busemann points.

Let (X, d) be a possibly non-symmetric metric space, which means that d has all properties of a metric except maybe symmetry. The topology on X shall be induced by the symmetric metric

$$d_{sym}(x, y) := d(x, y) + d(y, x) \ \forall x, y \in X.$$

For a point $z \in X$ and some basepoint $p_0 \in X$ consider the map

$$\psi_z : \ X \longrightarrow \mathbb{R} \tag{3.1}$$

$$x \longmapsto \psi_z(x) := d(x, z) - d(p_0, z). \tag{3.2}$$

Let $C(X)$ be the space of continuous real valued functions on X endowed with the topology of uniform convergence on bounded sets with respect to d_{sym}. Now consider the map

$$\begin{aligned} \psi : \ X &\longrightarrow C(X); \\ z &\longmapsto \psi_z. \end{aligned}$$

Proposition 3.0.17 *The map ψ is continuous and injective.*

Proof. The proof of is lemma this based on the triangle inequality and is shown in the appendix. □

This is already enough preparation to define the horoboundary.

Definition 3.0.18 The *horofunction boundary* is defined as

$$X(\infty) := (\mathrm{cl}\{\psi_z | z \in X\}) \setminus \{\psi_z | z \in X\} \subseteq C(X).$$

Its elements are called *horofunctions*.

Lemma 3.0.19 *If we choose an alternative base point $p_0' \in X$, then the corresponding horofunction boundaries are homeomorphic.*

Proof. The maps referring to the alternative base point p_0' will be denoted by ψ_z' and ψ'. Then

$$\psi_z'(\cdot) = d(\cdot, z) - d(p_0', z) = d(\cdot, z) - d(p_0, z) - d(p_0', z) + d(p_0, z) = \psi_z(\cdot) - \psi_z(p_0').$$

The map $\psi_z \mapsto \psi_z'$ is a homeomorphism as can be seen with the help of (8.1) on page 92. Therefore $X_{p_0}(\infty) \simeq X_{p_0'}(\infty)$. □

Remark 3.0.20 The relation shown for ψ_z also holds for the horofunctions. Let $\xi \in X_{p_0}(\infty)$ be a horofunction with respect to the base point p_0 and $\xi' \in X_{p_0'}(\infty)$. Then

$$\xi'(\cdot) = \xi(\cdot) - \xi(p_0')$$

by a standard convergence argument.

Definition 3.0.21 A metric d is called *proper* if every closed ball is compact.

Lemma 3.0.22 *If the metric d_{sym} is proper, then uniform convergence on bounded sets is equivalent to uniform convergence on compact sets.*

Lemma 3.0.23 *Let $z \in X$. Then it holds:*

$$\psi_z(x) \leq d(x,y) + \psi_z(y) \quad \forall x, y \in X$$

and for all horofunctions $\xi \in X(\infty)$ we have

$$\xi(x) \leq d(x,y) + \xi(y) \quad \forall x, y \in X.$$

From this follows that all elements of $\mathrm{cl}\{\psi_z | z \in X\}$ are 1-Lipschitz with respect to d_{sym}. Hence uniform convergence on bounded sets is equivalent to pointwise convergence.

The main aim of this work is to study the horofunction compactification, which is also the subject of this important lemma:

Lemma 3.0.24 *If the metric d_{sym} is proper, then cl$\{\psi_z | z \in X\}$ is compact and is called the* horofunction compactification.

The proof, explicitly shown in the appendix, is based on the Theorem of Ascoli-Arzelà, see page 93.

Definition 3.0.25 Let (X, d) be a possibly non-symmetric metric space. A map $\gamma : I \longrightarrow X$ from a closed interval $I \subseteq \mathbb{R}$ is called a *geodesic*, if

$$d(\gamma(s), \gamma(t)) = t - s \quad \forall s, t \in I \text{ with } s < t.$$

We now want to make some useful assumptions we partially already used before:

(A) The metric d_{sym} is proper.

(B) Between any pair of points in X, there exists a geodesic with respect to d.

(C) For any point $x \in X$ and any sequence x_n in X we have

$$d(x_n, x) \longrightarrow 0 \Longleftrightarrow d(x, x_n) \longrightarrow 0.$$

Proposition 3.0.26 *Assume (A), (B) and (C) hold. Then ψ is an embedding of X into $C(X)$. In other words, there is a homeomorphism $X \simeq \psi(X)$.*

Outline of the proof[1]. By Proposition 3.0.17 it remains to show that ψ^{-1} is also continuous. So we have to show that if $\psi_{z_n} \longrightarrow \psi_y$ for some sequence (ψ_{z_n}), then $z_n \longrightarrow y$. This will be shown by contraposition: if $z_n \longrightarrow \infty$, then there is no subsequence of (ψ_{z_n}) converging to some ψ_y with $y \in X$.
We assume that $\psi_{z_n} \longrightarrow \xi \in \mathrm{cl}\{\psi_z | z \in X\}$ as $n \to \infty$. Let $y \in X$ be arbitrary.

- The first step is to define a geodesic segment $\gamma_n : [0, b_n] \longrightarrow X$ connecting y and z_n for each $n \in \mathbb{N}$.
- We then show with assumption (C) that the function

$$\begin{aligned} h : [0, b_n] &\longrightarrow \mathbb{R} \\ t &\longmapsto h(t) := d_{sym}(y, \gamma_n(t)) \end{aligned}$$

  is continuous for every $n \in \mathbb{N}$.
- The third step is to show that there is an $x \in X$ with $\gamma_n(t_n) \longrightarrow x$ as $n \longrightarrow \infty$ where $t_n \in \mathbb{R}_+$ such that

$$d_{sym}(y, \gamma_n(t_n)) = r$$

  for some fixed $r > d(p_0, y) + \xi(y)$. Then $d_{sym}(y, x) = r$ and we set $x_n := \gamma_n(t_n)$.
- As γ_n is geodesic, $\psi_{z_n}(x_n) - \psi_{z_n}(y) = -d(y, x_n)$ and by taking the limit we obtain $\xi(x) = \xi(y) - d(y, x)$.
- The last and most important step of the proof is to show that

$$\psi_y \neq \xi \ \forall y \in X,$$

  which can be deduced from $\psi_y(x) - \xi(x) = r - d(p, y) - \xi(y) > 0$. □

From now on, we will identify

$$X \sim \psi(X).$$

Proposition 3.0.27 *Assume (A), (B) and (C) hold. Let x_n be a sequence in X converging to a horofunction. Then only finitely many points of x_n lie in any closed ball of d_{sym}.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that some subsequence of $d_{sym}(p_0, x_n)$ is bounded (if necessary, take a subsequence) and assume that $x_n \longrightarrow x \in X$. Then by Proposition 3.0.26 $\psi_{x_n} \longrightarrow \psi_x$ and therefore x_n does not converge to a horofunction. □

We will now define Busemann points, which are special horofunctions. It is an interesting question whether a horofunction is also a Busemann point or not and we will give an answer to this question in the case where X is a finite-dimensional normed space in the next section.

Definition 3.0.28 Let $T \subseteq \mathbb{R}_+$ be unbounded and $0 \in T$. A map $\gamma : T \longrightarrow X$ is called an *almost geodesic*, if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall s, t \in T, t \geq s \geq N : \ |d(\gamma(0), \gamma(s)) + d(\gamma(s), \gamma(t)) - t| < \varepsilon.$$

Rieffel [Rie02] showed that every almost geodesic converges to a limit in $X(\infty)$.

Definition 3.0.29 A horofunction is called a *Busemann point* if there is an almost geodesic converging to it. We set

$$X_B(\infty) := \{\text{Busemann points in } X(\infty)\}.$$

Remark 3.0.30 An isometry $f : X \longrightarrow X'$ of (possibly non-symmetric) metric spaces induces a homeomorphism $f : \psi(X) \longrightarrow \psi(X')$ which extends to a homeomorphism of the horofunction compactifications.

Proposition 3.0.31 *Let (X, d) and (X', d') be possibly non-symmetric metric spaces with base points p_0, p_0'. Let $f : X \longrightarrow X'$ be an isometry. Then for all horofunctions ξ of X and $x' \in X'$ the extension is given by:*

$$f(\xi)(x') = \xi(f^{-1}(x')) - \xi(f^{-1}(p_0'))$$

with $f(\xi)(x') := \lim_{n\to\infty}[d'(x', f(x_n)) - d'(p_0', f(x_n))]$

Proof. Let x_n be a sequence in X with $\psi_{x_n} \longrightarrow \xi$ as $n \longrightarrow \infty$. Then

$$\begin{aligned} f(\xi)(x') &= \lim_{n\to\infty}[d'(x', f(x_n)) - d'(p_0', f(x_n))] \\ &\overset{f \text{ isometry}}{=} \lim_{n\to\infty}[\underbrace{d(f^{-1}(x'), x_n) - d(p_0, x_n)}_{\psi_{x_n}(f^{-1}(x'))} + \underbrace{d(p_0, x_n) - d(f^{-1}(p_0'), x_n)}_{-\psi_{x_n}(f^{-1}(p_0'))}] \\ &= \xi(f^{-1}(x')) - \xi(f^{-1}(p_0')). \end{aligned}$$

□

4 The Case of Finite-Dimensional Normed Spaces

In his paper *The horofunction boundary of finite-dimensional normed spaces*, [Wal07], Walsh proves two helpful theorems concerning Busemann points and horofunctions. We will present this paper here in a slightly different order and with some more elaborate proofs. In the section afterwards we will consider several examples. The norms considered are not necessarily symmetric.
Before discussing these theorems, we have to introduce the following map: Let V be a finite-dimensional normed vector space, B the unit ball and B° the dual unit ball. For some extreme set E of B° and $p \in V$ define

$$f_{E,p}: \ V^* \longrightarrow [0,\infty] \tag{4.1}$$
$$q \longmapsto f_{E,p}(q) := I_E(q) + \langle q|p\rangle - \inf_{y\in E}\langle y|p\rangle \tag{4.2}$$

with the indicator function

$$I_E(q) := \begin{cases} 0 & \text{if } q \in E; \\ \infty & \text{if } q \notin E. \end{cases}$$

Now we are prepared to formulate the theorems:

Theorem 4.0.32 *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. Then the set of Busemann points of V is the set $\{f^*_{E,p} | E$ is a proper extreme set of $B^\circ, p \in V\}$, where f^* denotes the Legendre-Fenchel transform of f.*

Theorem 4.0.33 *Let $(V, \|\cdot\|)$ be a finite-dimensional normed space. Then every horofunction of V is a Busemann point if and only if the set of extreme sets of the dual unit ball B° is closed in the Painlevé-Kuratowski topology.*

These theorems are very useful when dealing with horofunction or Busemann compactifications. The first one allows us to calculate the horofunctions explicitly and we will see in the next sections that based on this theorem we only have to know the dual unit ball to determine the Busemann compactification. The second theorem gives us a criterion to distinguish between Busemann points and horofunctions or to see when there is no difference between them.

4.1 Preparation

In the last section we used the metric d to define the horofunction boundary with respect to some base point p_0. As we are now dealing with normed spaces, we can use the norm instead and we will choose the origin $\mathcal{O} = 0$ as base point. By Lemma 3.0.19 we know that the resulting horofunction compactifications will be homeomorphic. Therefore we define:

$$\psi_z(y) := \|z - y\| - \|z\| \qquad \forall y, z \in V \tag{4.3}$$

We will need the following sets of maps and their Legendre-Fenchel transforms:

$$\begin{aligned} D &:= \{\psi_z \mid z \in V\} = \{\|z - \cdot\| - \|z\| \mid z \in V\} \\ {}^*D &:= \{f^* \mid f \in D\} = \{\psi_z^* | z \in V\} \\ {}^*A &:= \{f_{E,p} \mid E \text{ is an extreme set of } B^\circ, p \in V\} \\ A &:= \{f^* \mid f \in {}^*A\} = \{f_{E,p}^* \mid E \text{ is an extreme set of } B^\circ, p \in V\} \end{aligned} \tag{4.4}$$

${}^*A$ is the set of functions that are affine on some extreme set of B° and take the value $+\infty$ outside the extreme set and have infimum 0.

Some Technical Lemmata

Lemma 4.1.1 *For $y \in V^*$ we have*

$$\psi_0^*(y) = I_{B^\circ}(y).$$

Proof. Let $y \in V^*$. Then with the definition of the dual unit ball (page 11) and of the Legendre-Fenchel transformation (page 9) we obtain

$$\begin{aligned} \psi_0^*(y) &= (\|-\cdot\|^*)(y) \\ &= \sup_{x \in V}(\langle y|x\rangle - \|-x\|) \\ &= \sup_{x \in V}(-[\langle y| - x\rangle + \|-x\|]) \\ &= -\inf_{x \in V}(\|-x\|(\langle y|e_{-x}\rangle + 1)), \end{aligned}$$

where e_{-x} denotes the unit vector in the direction of $-x$. Therefore $e_{-x} \in B$ for all $x \in V$ and we have

$$c := \langle y|e_{-x}\rangle = \begin{cases} < -1 & \text{if } y \notin B^\circ \text{ and for some } x \in V \\ \geq -1 & \text{if } y \in B^\circ \text{ and for every } x \in V \end{cases}$$

and thus

$$c + 1 = \begin{cases} < 0 & \text{if } y \notin B^\circ \text{ and for some } x \in V \\ \geq 0 & \text{if } y \in B^\circ \text{ and for every } x \in V. \end{cases}$$

So we get

$$\begin{aligned} \psi_0^*(y) &= -\inf_{x \in V}(\|-x\| \cdot (c+1)) = -\begin{cases} -\infty & \text{if } y \notin B^\circ \\ 0 & \text{if } y \in B^\circ \end{cases} \\ &= I_{B^\circ}(y). \end{aligned}$$

□

Lemma 4.1.2 *For all $z \in V$ there holds:*

$$\|z\| = -\inf_{y \in B^\circ}\langle y|z\rangle.$$

Proof. Let $z \in V$ be arbitrary. Then

$$\begin{aligned} \|z\| &= (\|\cdot\|^*)^*(z) = (\psi_0^*(-\cdot))^*(z) = (I_{B^\circ}(-\cdot))^*(z) \\ &= \sup_{y \in V}(\langle -z|y\rangle - I_{B^\circ}(y)) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{y\in B^\circ} \langle -z|y\rangle \\
&= -\inf_{y\in B^\circ} \langle y|z\rangle.
\end{aligned}
$$

To achieve the supremum in the second line, $I_{B^\circ}(y)$ has to be 0. From this follows $y \in B^\circ$ in the third line. □

Lemma 4.1.3 *Let $y \in V^*$ and $z \in V$. Then*

$$
\begin{aligned}
\psi_z^*(y) &= I_{B^\circ}(y) + \langle y|z\rangle - \inf_{x\in B^\circ} \langle x|z\rangle \\
&= f_{B^\circ,z}(y).
\end{aligned}
$$

Proof. We have $\psi_z(\cdot) = \psi_0(\cdot - z) - \|z\|$, because $\psi_0(x-z) = \|0-(x-z)\| = \|z-x\|$. We use this to calculate

$$
\begin{aligned}
\psi_z^*(y) &= (\psi_0(\cdot - z) - \|z\|)^*(y) \\
&= \sup_{x\in V}[\langle y|x\rangle - \psi_0(x-z) + \|z\|] \\
&= \sup_{x\in V}[\langle y|x-z\rangle - \psi_0(x-z)] + \langle y|z\rangle + \|z\| \\
&= \underbrace{\sup_{h\in V}[\langle y|h\rangle - \psi_0(h)]}_{=\psi_0^*(y)=I_{B^\circ}(y)} + \langle y|z\rangle + \underbrace{\|z\|}_{-\inf_{x\in B^\circ}\langle x|z\rangle} \\
&= I_{B^\circ}(y) + \langle y|z\rangle - \inf_{x\in B^\circ} \langle x|z\rangle.
\end{aligned}
$$
□

From the relation $\psi_z^* = f_{B^\circ,z}$ shown in the last lemma follows

$$
\psi_z^* \in {}^*A \quad \forall z \in V
$$

because B° is an extreme set of itself. Or in other words with $\psi_z = f_{B^\circ,z}^*$:

$$
D \subseteq A.
$$

As defined above, the horofunction compactification of V is exactly the closure of D in the topology of uniform convergence on compact sets. Our aim now is to have a closer look at this closure. The elements of $\mathrm{cl}D$ are limits of sequences of convex 1-Lipschitz functions and therefore they are also convex and 1-Lipschitz. Since the functions in D are equi-Lipschitzian, uniform convergence of such functions on compact sets is equivalent to convergence in the epigraph topology[1] [2]. So $\mathrm{cl}D$ in the topology of uniform convergence on compact sets is also $\mathrm{cl}D$ in the epigraph topology.

[1] Let (X,d) be a metric space. Suppose $g_1, g_2, \dots$ is a sequence is $LSC(X)$, the space of proper lower semi-continuous extended real functions, and let $g \in LSC(X)$ such that g is real valued. If $(g_n)_n \longrightarrow g$ uniformly on bounded subsets of X, then $g = \tau_{AW_\rho}$-lim g_n, where τ_{AW_ρ} denotes the Attouch-Wets-topology ([Bee93, Lemma 7.1.2]). As we are dealing with finite-dimensional normed linear spaces, the metric induced by the norm is proper and therefore the Attouch-Wets-topology coincides with the Painlevé-Kuratowski topology, see also Lemma 2.2.24 and [Bee93, p. 235].

[2] Let(X,d) be a metric space and $f_0, f_1, f_2, \dots$ be a sequence of lower semi-continuous real valued functions on X such that f_0 is finite-valued and Lipschitz continuous on bounded subsets of X and such that $(f_n)_n$ is eventually equi-Lipschitz continuous on bounded subsets of X. Suppose that $f_0 = \tau_{AW_\rho}$-lim f_n. Then $(f_n)_n \longrightarrow f_0$ uniformly on bounded subsets of X ([Bee93, Prop. 7.1.3]).

4.2 Proof of Theorem 4.0.32

For this proof, we will use the following result from [AGW05] without proof:

Lemma 4.2.1 *A horofunction is a Busemann point if and only if it is not the minimum of two 1-Lipschitz functions each different from it.*

Lemma 4.2.2 *Each Busemann point is contained in $A \setminus D$*

Proof. Let $g \in (\mathrm{cl}D) \setminus A$. Then g is convex and 1-Lipschitz, as noted above. From this follows with Lemma 4.1.3 that the Legendre-Fenchel transform g^* takes the value $+\infty$ outside of B°.
As $g \notin A$, it is $g^* \notin {}^*A$.

We claim: $g^* \notin {}^*A$ if and only if we can find $x, y, z \in B^\circ, \lambda \in (0, 1)$ such that

$$y = (1 - \lambda)x + \lambda z \tag{4.5}$$

and

$$g^*(y) < (1 - \lambda)g^*(x) + \lambda g^*(z).$$

Proof of the claim:

"$\Longrightarrow$" Because g^* is convex, it is $g^*(y) \leq (1 - \lambda)g^*(x) + \lambda g^*(z)$ for any $x, y, z \in B^\circ$ and $\lambda \in (0, 1)$ with $y = (1 - \lambda)x + \lambda z$. Let $x, z \in B$ and $y_\lambda = (1 - \lambda)x + \lambda z$ for some $\lambda \in (0, 1)$. Assume there is no $\lambda \in (0, 1)$ satisfying the condition "$<$", then $g^*(y_\lambda) = (1 - \lambda)g^*(x) + \lambda g^*(z)$ for all $\lambda \in (0, 1)$. We show that this implies $g^* \in {}^*A$ which is a contradiction. For this we have to show that g^* is an affine function, that the set $E := \{y \in V^* \mid g^*(y) < \infty\}$ is an extreme set of B° and that $\inf g^* = 0$. Affinity follows just by definition and as g^* takes the value $+\infty$ outside of B°, $E \subseteq B^\circ$. For extremality of E we show that whenever an interior point of a straight line lies in E, then the endpoints also do: let $y \in E$ and $x, z \in B^\circ, \lambda \in (0, 1)$ such that $y = (1-\lambda)x+\lambda z$. Then $g^*(y) = (1-\lambda)g^*(x)+\lambda g^*(z)$ and therefore g^* is finite on x and z, that is $x, z \in E$. At last we show $\inf g^* = 0$. As $g \in \mathrm{cl}\, D = \mathrm{cl}\{\psi_z | z \in V\}$ and $\psi_z(0) = \|z-0\|-\|z\| = 0$ for all $z \in V$ we conclude that $g(0) = 0$. Because g is convex $\mathrm{epi}\, g$ is a convex set with a boundary point in 0. Therefore there is a supporting hyperplane to $\mathrm{epi}\, g$ through 0. By definition $g^*(m) = \sup_{x \in V}(\langle x|m\rangle - g(x))$ and as $g(0) = 0$ we see that $\inf g^* \geq 0$ and therefore the supporting hyperplane lies below $\mathrm{epi}\, g$ and touches it at the origin. All in all $\inf g^* = 0$.

"$\Longleftarrow$" We show that we cannot find such points $x, y, z \in B$ or a parameter $\lambda \in (0, 1)$ satisfying the condition if $g^* \in {}^*A$. Assume $g^* \in {}^*A$. Then there is an extreme set E of $B^\circ, p \in V$ such that $g^* = f_{E,p}$.
Let $y \notin E$. Then $g^*(y) = \underbrace{I_E(y)}_{\infty} + \langle y|p\rangle - \underbrace{\inf_{q \in E} \langle q|p\rangle}_{>-\infty} = \infty$ which would make the inequality impossible.
Let $y \in E$. Then also $x, z \in E$, because E is an extreme set. Thus we have

$$\begin{aligned} g^*(y) = f_{E,p}(y) &= \underbrace{I_E(y)}_{0} + \langle y|p\rangle - \inf_{q \in E} \langle q|p\rangle \\ &= (1 - \lambda)\langle x|p\rangle + \lambda \langle z|p\rangle - \inf_{q \in E} \langle q|p\rangle \end{aligned}$$

$$
\begin{aligned}
&= (1-\lambda)I_E(x) + (1-\lambda)\langle x|p\rangle - (1-\lambda)\inf_{q\in E}\langle q|p\rangle \\
&\quad + \lambda I_E(z) + \lambda\langle z|p\rangle - \lambda\inf_{q\in E}\langle q|p\rangle \\
&= (1-\lambda)g^*(x) + \lambda g^*(z),
\end{aligned}
$$

so we would have $g^*(y) = (1-\lambda)g^*(x) + \lambda g^*(z)$ which contradicts the inequality too.

Back to the original proof:

By completeness of $\mathbb{R}$, we can also say that there are $a, b \in \mathbb{R}$ with $a < g^*(x), b < g^*(z)$ such that $g^*(y) < (1-\lambda)a + \lambda b$. From $g^*(y) = \sup_{q\in V}[\langle y|q\rangle - g(q)] \geq \langle y|p\rangle - g(p)\ \forall p \in V$ we get

$$\langle y|p\rangle - g(p) \leq g^*(y) < (1-\lambda)a + \lambda b \quad \forall p \in V \tag{4.6}$$

Define

$$
\begin{aligned}
\Pi_1 &:= \{p \in V | \langle x|p\rangle - a \geq g(p)\} \\
\Pi_2 &:= \{p \in V | \langle z|p\rangle - b \geq g(p)\}
\end{aligned}
$$

We now want to show, that $\Pi_1 \cap \Pi_2 = \emptyset$. Let $p \in \Pi_1$. Equation (4.5) yields $x = \frac{y-\lambda z}{1-\lambda}$. Hence

$$
\begin{array}{rrcl}
p \in \Pi_1 \iff & \langle x|p\rangle - a & \geq & g(p) \\
\iff & \frac{1}{1-\lambda}\langle y|p\rangle - \frac{\lambda}{1-\lambda}\langle z|p\rangle - a & \geq & g(p) \\
\overset{(4.6)}{\Longrightarrow} & \frac{1}{1-\lambda}\langle y|p\rangle - \frac{\lambda}{1-\lambda}\langle z|p\rangle - a & > & \langle y|p\rangle - (1-\lambda)a - \lambda b \\
\iff & \langle y|p\rangle - (1-\lambda)\langle y|p\rangle - \lambda\langle z|p\rangle & > & (1-\lambda)a - (1-\lambda)^2 a - \lambda(a-\lambda)b \\
\iff & \lambda\langle y|p\rangle - \lambda\langle z|p\rangle & > & \lambda(-a + 2a + -\lambda a - b + \lambda b) \\
\overset{\lambda\neq 0}{\Longrightarrow} & \langle y|p\rangle - \langle z|p\rangle & > & (1-\lambda)(a-b) \\
\Longrightarrow & (1-\lambda)a + \lambda b + g(p) - \langle z|p\rangle & > & \langle y|p\rangle - \langle z|p\rangle > (1-\lambda)(a-b) \\
\overset{(4.6)}{\iff} & \lambda b + g(p) - \langle z|p\rangle & > & (\lambda - 1)b \\
\iff & g(p) & > & \langle z|p\rangle - b \\
\iff & p & \notin & \Pi_2
\end{array}
$$

Thus $\Pi_1 \cap \Pi_2 = \emptyset$.
We now define

$$
\begin{aligned}
g_1 &:= \max(g, \langle x|\cdot\rangle - a) \\
g_2 &:= \max(g, \langle z|\cdot\rangle - b)
\end{aligned}
$$

Both functions are 1-Lipschitz, because g is 1-Lipschitz and $x, z \in B^\circ$.

We claim: $g = \min(g_1, g_2)$

Proof of this claim:
Let $p \in \Pi_1$:

$$
\left.
\begin{array}{l}
g_1(p) = \max(g(p), \langle x|p\rangle - a) = \langle x|p\rangle - a \\
g_2(p) = \max(g(p), \langle z|p\rangle - b) = g(p)
\end{array}
\right\} \Rightarrow g(p) = g_2(p) \leq g_1(p).
$$

Now let $p \in \Pi_2$:

$$
\left.
\begin{array}{l}
g_1(p) = \max(g(p), \langle x|p\rangle - a) = g(p) \\
g_2(p) = \max(g(p), \langle z|p\rangle - b) = \langle z|p\rangle - b
\end{array}
\right\} \Rightarrow g(p) = g_1(p) \leq g_2(p).
$$

Together we have $g = \min(g_1, g_2)$.

Back to the original proof:

We want to apply Lemma 4.2.1 to show that g cannot be a Busemann point. We already found two functions g_1 and g_2 whose minimum is g. We now have to show that $g \neq g_1$ and $g \neq g_2$.

Let p be in the sub-differential of g^* at x [3]. This means that $\langle q - x|p\rangle + g^*(x) \leq g^*(q)$ $\forall q \in V^*$. If we look at the Legendre-Fenchel transform, we get $g(s) \leq I_{\{p\}}(s) + \langle x|p\rangle - g^*(x)$ $\forall s \in V$. Evaluation at $s = p$ results in $g(p) \leq \langle x|p\rangle - g^*(p)$ and therefore $g(p) < \langle x|p\rangle - a$ where we used that $-g^*(x) < a$ (a was just defined by this). So

$$g_1(p) = \max(g(p), \langle x|p\rangle - a) = \langle x|p\rangle - a \neq g(p)$$

and therefore $g_1 \neq g$. In the same way, one can show that $g_2 \neq g$. All together we have

$$g = \min(g_1.g_2),\ g \neq g_1,\ g \neq g_2.$$

and g_1, g_2 are 1-Lipschitz.

So with the criterion Lemma 4.2.1 from [AGW05], g cannot be a Busemann point. □

There is a rather elegant and convenient way express the function $f^*_{E,p}$ by some kind of a “pseudo-norm”:

Definition 4.2.3 Let $C \subseteq V^*$ be a convex subset and $p \in V$. Define

$$|p|_C := -\inf_{q \in C} \langle q|p\rangle. \tag{4.7}$$

Remark 4.2.4 In general, this is not a norm. But it is

$$|\cdot|_{B^\circ} = -\inf_{q \in B^\circ} \langle q|\cdot\rangle = \|\cdot\|$$

as shown in 4.1.2.

Lemma 4.2.5 *Let E be an extreme set of B° and $p \in V$. Then*

$$f^*_{E,p}(\cdot) = |p - \cdot|_E - |p|_E.$$

Proof. With $f_{E,p}(\cdot) = I_E(\cdot) + \langle \cdot|p\rangle - \inf_{q \in E}\langle q|p\rangle$ we obtain for all $y \in V$:

$$\begin{aligned} f^*_{E,p}(y) &= \sup_{x \in V^*} (\langle x|y\rangle - f_{E,p}(x)) \\ &= \sup_{x \in V^*} (\langle x|y\rangle - I_E(x) - \langle x|p\rangle + \inf_{q \in E}\langle q|p\rangle) \\ &= \sup_{x \in E}(\langle x|y - p\rangle) + \inf_{q \in E}\langle q|p\rangle \\ &= -\inf_{x \in E}(\langle x|p - y\rangle) + \inf_{q \in E}\langle q|p\rangle \\ &= |p - y|_E - |p|_E \end{aligned}$$

□

[3] The subdifferential of a convex function $f : X \longrightarrow \mathbb{R}$ (X a Banach space) at a point $x \in X$ is the set $\underline{\partial f(x)} := \{x^* \in X^* \mid f(q) - f(x) \geq \langle x^*|q - x\rangle\ \forall q \in X\}$. By the theory of supporting hyperplanes, it is not empty if f is continuous and convex.

Corollary 4.2.6 $D = \{f^*_{B^\circ,z} \mid z \in V\}$

Proof. Follows from Lemma 4.1.3. □

Lemma 4.2.7 *Let $C \subseteq V$ be a convex subset of a finite-dimensional vector space V. A set E is an extreme set of C if and only if there is a finite sequence of convex sets $F_0, \dots, F_n$, such that $F_0 = C, F_n = E$ and F_{i+1} is an exposed face of F_i for all $i = 0, \dots, n-1$.*

Proof.

"$\Longrightarrow$" Let E be an extreme set of C.
If E contains a relative interior point of C, then $E = C$ by extremality of E.
If E is contained entirely within the relative boundary of C, then $E \cap \mathrm{ri}C = \emptyset$. So by the second separation Theorem 2.2.18, there is a supporting hyperplane H_1 to C containing E. Define $F_1 := H_1 \cap C$, then by definition, F_1 is an exposed face of C containing E. E is an extreme set of F_1, since $E \subseteq F_1 \subseteq C$ and E is an extreme set of C. If we apply this procedure several times, we receive the required sequence of sets. As an exposed face is the intersection of the set with a hyperplane, we are losing one dimension in each step which guarantees that our sequence will be finite.

"$\Longleftarrow$" We now assume that such a sequence exists. Because of the transitivity of the property of being an extreme set (Lemma 2.4.3) and because every exposed face is also an extreme set (Lemma 2.4.6), $F_n = E$ is an extreme set of $F_0 = C$. □

Lemma 4.2.8 *Let $C \subseteq V^*$ be a compact convex set and F an exposed face of C. Suppose there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in V and an $\varepsilon > 0$ such that*

(i) $\sum_{i=0}^{n-1} |p_{i+1} - p_i|_F \leq |p_n - p_0|_F + \varepsilon \quad \forall n \in \mathbb{N}$

(ii) $|p_n - \cdot|_F - |p_n|_F \stackrel{n\to\infty}{\longrightarrow} g$ *pointwise*

where g is a lower semi-continuous convex function.
Then there is a sequence $(q_n)_{n\in\mathbb{N}}$ in V and an $\varepsilon' > 0$ such that

(I) $\sum_{i=0}^{n-1} |q_{i+1} - q_i|_C \leq |q_n - q_0|_C + \varepsilon' \quad \forall n \in \mathbb{N}$

(II) $|q_n - \cdot|_C - |q_n|_C \stackrel{n\to\infty}{\longrightarrow} g$ *pointwise*

Proof. The proof of this lemma is quite technical and long so it will be shown in the appendix in detail. □

Lemma 4.2.9 *Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in V satisfying*

$$\sum_{i=0}^{n-1} ||q_{i+1} - q_i|| \leq ||q_n - q_0|| + \varepsilon$$

for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. Then the sequence $(q_n)_{n\in\mathbb{N}}$ is an almost geodesic.

Proof. In [AGW05] an almost geodesic is defined in a very different way, but it it shown in Corollary 7.12 of the same paper that this definition and the one given in this thesis lead to the same Busemann points. A sequence as stated in the lemma satisfies the conditions of an almost geodesic in the sense of [AGW05]. □

Lemma 4.2.10 *Every function in $A \setminus D$ is a Busemann point.*

Proof. Let $g \in A \setminus D$. That means that there exists an extreme set E of B° and a point $p \in V$ such that $g = f^*_{E,p} = |p - \cdot|_E - |p|_E$, but that there is no point $z \in V$ such that $g = \|z - \cdot\| - \|z\| = |z - \cdot|_{B^\circ} - |z|_{B^\circ}$. (This results in the condition of E being a proper extreme set of B° in Theorem 4.0.32.) As E is an extreme set, we know from Lemma 4.2.7 that there is a finite sequence $F_0, \dots, F_n$ of convex sets with $F_0 = B^\circ$, $F_n = E$ and F_{i+1} is an exposed face of F_i for every $i \in \{0, \dots, n-1\}$. Take $F = E$, $C = F_{n-1}$, $\varepsilon = 0$ and $g = f^*_{E,p}$ with the sequence $p_n = p \ \forall n \in \mathbb{N}$. Then all conditions of Lemma 4.2.8 are satisfied and by applying this lemma several times, we obtain a sequence $(q_n)_{n\in\mathbb{N}}$ in V and an $\varepsilon' > 0$ satisfying the assertion of the lemma with $C = B^\circ$. Thus we also have

$$|q_n - \cdot|_{B^\circ} - |q_n|_{B^\circ} \longrightarrow f^*_{E,p} = g \quad \text{as } n \longrightarrow \infty$$

which means that $f^*_{E,p}$ is a horofunction. Because $f^*_{E,p} \neq \|z - \cdot\| - \|z\|$ for all $z \in V$, g really lies only in the boundary of the compactification. With Lemma 4.2.9 we see that (q_n) is an almost geodesic and therefore g is a Busemann point. □

Proof of Theorem 4.0.32. The theorem now follows directly from Lemma 4.2.2, Lemma 4.2.10 and Corollary 4.2.6. □

4.3 Proof of Theorem 4.0.33

We now come to the proof of the second theorem about the distinction between Busemann points and horofunctions.

Lemma 4.3.1 *If A is closed, then the set of extreme subsets of B° is closed in the Painlevé-Kuratowski topology.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of extreme sets of B° converging to E. We have to show that E also is an extreme set of B°. As $E_n \longrightarrow E$ we also have $I_{E_n} \longrightarrow I_E$ as $n \longrightarrow \infty$[4]. $I_{E_n} \in {}^*A \ \forall n \in \mathbb{N}$ as $f_{E_n,0}(q) = I_{E_n}(q) + 0 + 0 = I_{E_n}(q) \ \forall q \in V^*$. From this and because A is closed, it follows that also $I_E \in {}^*A$, so E is an extreme set of B°. □

Lemma 4.3.2 *If the set of extreme sets of B° is closed in the Painlevé-Kuratowski topology, then A is closed.*

Proof. We know from Lemma 4.2.10 that every function in $A \setminus D$ is a Busemann point and consequently also a horofunction. By definition the horofunctions are $\mathrm{cl}D \setminus D$, from which we obtain

$$D \subseteq A \subseteq \mathrm{cl}D.$$

To show that A is closed, it is enough to prove that if $(f_n)_{n\in\mathbb{N}}$ is a sequence in ${}^*D$, then $f := \lim_{n\to\infty} f_n \in {}^*A$.
For this we will use the following criterion shown as a claim in the proof of Lemma 4.2.2. If $f \in \mathrm{cl}(D)$ then:

$$f \in {}^*A \iff \text{if } y = (1-\lambda)x + \lambda z \text{ then } f(y) = (1-\lambda)f(x) + \lambda f(z).$$

Let $x, z \in B^\circ, x \neq z$ and $y = (1-\lambda)x + \lambda z$ with $\lambda \in (0,1)$. With the remark above, we have to show that $f(y) = (1-\lambda)f(x) + \lambda f(z)$. As f is convex, meaning $f(y) \leq (1-\lambda)f(x) + \lambda f(z)$, we have nothing to show if $f(y) = \infty$.

[4]See also Lemma 2.2.24.

Let $f(y) < \infty$ and $|\cdot|$ be any norm on V^*.
We *claim*[5] that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of points in B° and a constant $\delta > 0$ such that:

(i) $y_n \longrightarrow y$

(ii) $f_n(y_n) \longrightarrow f(y)$

(iii) $\forall n \in \mathbb{N}$ the point y_n is in some extreme set E_n and $|y_n - \partial_{\text{rel}} E_n| \geq \delta$

where $\partial_{\text{rel}} E_n$ denotes the relative boundary of E_n.

With this claim and the fact that $f_n \geq 0\ \forall n \in \mathbb{N}$, it follows that f_n is Lipschitz-continuous on E_n with Lipschitz constant $\frac{f_n(y_n)}{\delta_n}$. Because $\frac{f_n(y_n)}{\delta_n} \longrightarrow \frac{f(y)}{\delta}$ as n goes to ∞, we can find a constant l such that for each $n \in \mathbb{N}$ the function f_n is l-Lipschitzian.
Let F be a limit point of the sequence E_n, so by our assumption that the set of extreme sets is closed in the Painlevé-Kuratowski topology, F is an extreme set. Now assume that $E_n \longrightarrow F$ as $n \longrightarrow \infty$, if necessary by taking a subsequence. From $y_n \in E_n \forall n \in \mathbb{N}$ and $y_n \longrightarrow y$ we see that $y \in F$. So by extremality of F also x and z are elements of F. Therefore we can find sequences $(x_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ with $x_n \longrightarrow x$ and $z_n \longrightarrow z$ and $x_n, z_n \in E_n$ for each $n \in \mathbb{N}$, satisfying $f_n(x_n) \longrightarrow f(x)$ and $f_n(z_n) \longrightarrow f(z)$ as $n \longrightarrow \infty$.
Define

$$y'_n := (1-\lambda)x_n + \lambda z_n \quad \forall n \in \mathbb{N}.$$

By extremality of E_n and the fact that $x_n, z_n \in E_n$, we know that

$$y'_n \in E_n\ \forall n \in \mathbb{N}$$

and

$$y'_n \longrightarrow (1-\lambda)x + \lambda z = y \text{ as } n \longrightarrow \infty.$$

So the f_n are all l-Lipschitzian in E_n and y_n and y'_n have the same limit, namely y. Therefore

$$\lim_{n\to\infty} f_n(y'_n) = \lim_{n\to\infty} f_n(y_n) = f(y).$$

With this we have

$$\begin{aligned} f(y) &= \lim_{n\to\infty} f_n(y'_n) \\ &= \lim_{n\to\infty} ((1-\lambda)f(x_n) + \lambda f_n(z_n)) \\ &= (1-\lambda)x - \lambda z. \end{aligned}$$

We conclude that the set $\{x \in V^* \mid f \text{ is finite on } x\}$ is an extreme set and f is affine on it. Therefore $f \in^* A$. □

Lemma 4.3.3 *The set A is closed in the epigraph topology if and only if the set of extreme subsets of B° is closed in the Painlevé-Kuratowski topology.*

Proof. The proof of this lemma is a direct consequence of the two lemmata before which show one direction each. □

[5]The proof can be found in the appendix on page 95.

Proof of Theorem 4.0.33.

"$\Longleftarrow$" Let the set of extreme sets of B° be closed in the Painlevé-Kuratowski topology. Then by Lemma 4.3.3 A is closed. From the proof of Lemma 4.3.2 we know that $D \subseteq A \subseteq \mathrm{cl}D$. So A being closed means that $A = \mathrm{cl}D$. Therefore $A \setminus D = \mathrm{cl}D \setminus D$ which is equivalent to every horofunction being a Busemann point.

"$\Longrightarrow$" If every Busemann point is a horofunction, then $A \setminus D = \mathrm{cl}D \setminus D$. Therefore $A = \mathrm{cl}D$ which tell us that A is closed and with Lemma 4.3.3 the assertion follows. $\square$

5 Examples: Horocompactifications of $\mathbb{R}^m$

In this section we treat some examples to illuminate the findings of the previous section. As the examples are computationally extensive, we will only show four of them here in the main part, the others are given in the appendix. The reader is nevertheless invited to have a look at them to get an intuition for the horofunction compactification.
The structure of the examples is always the same. After determining the dual unit ball B° of B we calculate the functions $f_{E,p}$ for each extreme set E of B°. Afterwards we compute their Legendre-Fenchel transform $f^*_{E,p}$. In the last step we find sequences z_n in X, such that $\psi_{z_n} \longrightarrow f^*_{E,p}$ as $n \longrightarrow \infty$. The interesting part then is to see which sequences converge to the same functions, that is, which sequences determine the same point in the horofunction compactification, and to find out the geometrical meaning of the unit and the dual unit ball. In the examples we will only consider sequences $(z_n)_{n\in\mathbb{N}}$ which follow a straight line. We will give a characterisation of the converging sequences at the end of this section and explain why we are allowed to limit ourselves to straight lines.
In most of the examples we will consider polyhedral norms. For these the set $\mathcal{E}$ of extreme sets of B° is finite and hence closed in the Painlevé-Kuratowski topology. So we know by Theorem 4.0.33 that every horofunction is a Busemann point and sometimes speak of the Busemann compactification. In the examples with a curved unit sphere (section 5.4 and sections 8.4.3 and 8.4.4 in the appendix) $\mathcal{E}$ is not finite, but nevertheless, in the considered cases it is closed in this topology, so here we also have a Busemann compactification.

Before we start with the examples, we will show two simple lemmata concerning extreme sets, which contain exactly one point or all of B°.

Lemma 5.0.4 *Let X be a finite-dimensional normed space with unit ball B and let $E = \{e\}$ be an extreme point of the dual unit ball B°. Then for all $p \in X$:*

$$f_{E,p} = I_E \tag{5.1}$$

and

$$f^*_{E,p}(\cdot) = \langle e|\cdot\rangle. \tag{5.2}$$

Proof. Let $E = \{e\}$ and $p \in X$. Then we have for all $q \in X^*$

$$\begin{aligned} f_{E,p}(q) &= I_{\{e\}}(q) + \langle q|p\rangle - \inf_{y\in\{e\}} \langle y|p\rangle \\ &= \begin{cases} 0 & \text{if } q = e \\ \infty & \text{if } q \neq e \end{cases} + \langle q|p\rangle - \langle e|p\rangle \\ &= \begin{cases} 0 + \langle e|p\rangle - \langle e|p\rangle & \text{if } q = e \\ \infty + \langle q|p\rangle - \langle e|p\rangle & \text{if } q \neq e \end{cases} \\ &= \begin{cases} 0 & \text{if } q = e \\ \infty & \text{if } q \neq e \end{cases} \\ &= I_E(q). \end{aligned}$$

For all $y \in X^{**} \cong X$ the transform of $f_{E,p} = I_E$ is

$$\begin{aligned} f^*_{E,p}(y) &= \sup_{x\in\mathbb{R}^*} (\langle x|y\rangle - f_{E,p}(x)) \\ &= \sup_{x\in\mathbb{R}^*} (\langle x|y\rangle - \underbrace{I_{\{e\}}(x)}_{\in\{0,\infty\}}) \\ &= \langle e|y\rangle. \end{aligned}$$

□

Corollary 5.0.5 If the extreme set $E \subseteq B^\circ$ consists of a single point, then $f^*_{E,p}$ is independent of p.

Lemma 5.0.6 *Let X be a finite-dimensional normed space with unit ball B and let E be a one-dimensional extreme set of the dual unit ball B°. Let $a, b \in \partial B^\circ$ be the two vertices in the relative boundary of E. Then for $p \in V$ arbitrary:*

$$f_{E,p}(q) = I_E(q) + \langle q|p\rangle - \min\{\langle a|p\rangle, \langle b|p\rangle\} \tag{5.3}$$

and

$$f^*_{E,p}(y) = \max\{\langle a|y-p\rangle, \langle b|y-p\rangle\} + \min\{\langle a|p\rangle, \langle b|p\rangle\} \tag{5.4}$$

for all $q \in V^*$ and $y \in V$.

Proof. Let $x \in E$. Then there is a $\lambda \in [0,1]$ such that $x = \lambda a + (1-\lambda)b$. Therefore

$$\begin{aligned} \inf_{x\in E} \langle x|p\rangle &= \inf_{\lambda\in[0,1]} \langle \lambda a + (1-\lambda)b|p\rangle \\ &= \inf_{\lambda\in[0,1]} (\lambda\langle a-b|p\rangle + \langle b|p\rangle) \\ &= \langle b|p\rangle + \inf_{\lambda\in[0,1]} \lambda\langle a-b|p\rangle \\ &= \langle b|p\rangle + \begin{cases} 0 & \text{if } \langle a-b|p\rangle \geq 0 \\ \langle a-b|p\rangle & \text{if } \langle a-b|p\rangle < 0 \end{cases} \\ &= \begin{cases} \langle b|p\rangle & \text{if } \langle a|p\rangle \geq \langle b|p\rangle \\ \langle a|p\rangle & \text{if } \langle a|p\rangle < \langle b|p\rangle \end{cases} \\ &= \min\{\langle a|p\rangle, \langle b|p\rangle\}. \end{aligned}$$

So all together we have for $q \in V^*$:

$$\begin{aligned} f_{E,p}(q) &= I_E(q) + \langle q|p\rangle - \inf_{x\in E} \langle x|p\rangle \\ &= I_E(q) + \langle q|p\rangle - \min\{\langle a|p\rangle, \langle b|p\rangle\}. \end{aligned}$$

For the Legendre-Fenchel transform we obtain by a similar calculation:

$$\begin{aligned} f^*_{E,p}(y) &= \sup_{x\in V^*} (\langle x|y\rangle - f_{E,p}(x)) \\ &= \sup_{x\in V^*} (\langle x|y\rangle - I_E(x) - \langle x|p\rangle + \min\{\langle a|p\rangle, \langle b|p\rangle\}) \\ &= \sup_{x\in E} (\langle x|y-p\rangle) + \min\{\langle a|p\rangle, \langle b|p\rangle\} \\ &= \sup_{\lambda\in[0,1]} (\langle \lambda a + (1-\lambda)b|y-p\rangle) + \min\{\langle a|p\rangle, \langle b|p\rangle\} \\ &= \max\{\langle a|y-p\rangle, \langle b|y-p\rangle\} + \min\{\langle a|p\rangle, \langle b|p\rangle\} \end{aligned}$$

where we used that we can write every $x \in E$ as $x = \lambda a + (1-\lambda)b$ for some $\lambda \in [0,1]$. □

Actually we are only interested in the proper extreme sets of B°. But here is a nice little result we have for B° as an extreme set of itself.

Lemma 5.0.7 *Let B° be the dual unit ball of a finite-dimensional normed space X. Then for some $p \in X$ it is*

$$f^*_{B^\circ,p} = \psi_p$$

with ψ_p as defined in (4.4) on page 22.

Proof. Let $y \in X$. By Lemma 4.2.5 we already know that $f^*_{E,p}(y) = |p - y|_E - |p|_E$ for every extreme set E of B° and $p \in V$. So now we have

$$f^*_{B^\circ,p}(y) = |p - y|_{B^\circ} - |p|_{B^\circ} = \|p - y\| - \|p\| = \psi_p(y).$$

□

5.1 $X = \mathbb{R}^m$ with L^1-Metric

We will start with the example of $\mathbb{R}^m$ equipped with the L^1-norm[1]. This example is rather basic, as we already know the dual unit ball B° and the norm induced by B which saves ourselves several calculations. Another advantage of the L^1-norm is its symmetry which allows us to generalise it easily to higher dimensions. We will show the two-dimensional and the general case here in the main part. Although the three-dimensional case is contained in the general result, it is shown in the appendix on page 102 for those who would like to see another concrete example.

The Case of $m = 2$

The dual of the unit ball $B = \{x \in \mathbb{R}^2 \mid \ \|x\|_1 \leq 1\}$ is

$$\begin{aligned} B^\circ &= \{y \in \mathbb{R}^2 \mid \langle y|x\rangle \geq -1 \ \forall x \in B\} \\ &= \{y \in \mathbb{R}^2 \mid \langle y|x\rangle \leq 1 \forall x \in B\} \\ &= \{y \in \mathbb{R}^2 \mid \ \max(|y_1|, |y_2|) \leq 1\} \\ &= \{y \in \mathbb{R}^2 \mid \|y\|_\infty \leq 1\} \end{aligned}$$

by Lemma 2.5.13. A picture of B and B° is given in figure 5.1.

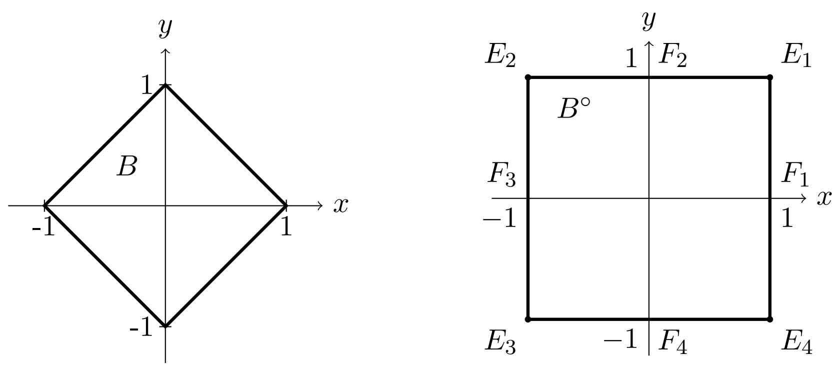


Figure 5.1: B and B° of the L^1-norm

[1]For $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ it is $\|x\|_1 := |x_1| + \ldots + |x_m|$.

The proper extreme sets of B° are:

$E_1 := \{e_1\} := \{(1,1)\}$ $\quad E_2 := \{e_2\} := \{(-1,1)\}$ $\quad E_3 := \{e_3\} := \{(-1,-1)\}$ $\quad E_4 := \{e_4\} := \{(1,-1)\}$

$$F_1 := \left\{ \begin{pmatrix} 1 \\ t \end{pmatrix} \middle| \, |t| \le 1 \right\} = \mathrm{conv}(E_1, E_4) \qquad F_2 := \left\{ \begin{pmatrix} t \\ 1 \end{pmatrix} \middle| \, |t| \le 1 \right\} = \mathrm{conv}(E_1, E_2)$$
$$F_3 := \left\{ \begin{pmatrix} -1 \\ t \end{pmatrix} \middle| \, |t| \le 1 \right\} = \mathrm{conv}(E_2, E_3) \qquad F_4 := \left\{ \begin{pmatrix} t \\ -1 \end{pmatrix} \middle| \, |t| \le 1 \right\} = \mathrm{conv}(E_3, E_4)$$

Let $\mathcal{E}$ be the set of extreme sets of B°,

$$\mathcal{E} := \{E_i, F_i \mid i = 1, \dots, 4\}.$$

Let $p = (p_1, p_2) \in \mathbb{R}^2$ be an arbitrary point.

From Lemma 5.0.4 we already know the form of the $f_{E,p}$-functions of the extreme points, namely

$$f_{E_i,p}(q) = I_{E_i}(q) \quad \forall i \in \{1,2,3,4\}, q \in (\mathbb{R}^2)^*.$$

Now to the one-dimensional extreme sets F_i:
We will calculate the deduce the expression for the $f_{E,p}$-function for the extreme set F_1 using equation (5.3) on page 32. As F_1 is the convex hull of the points $e_1 = (1,1)$ and $e_4 = (1,-1)$ we first have to calculate for an arbitrary $p \in \mathbb{R}^2$

$$\begin{aligned} \min\{\langle e_1|p\rangle, \langle e_4|p\rangle\} &= \min\{\langle (1,1)|\, p\rangle, \langle (1,-1)|\, p\rangle\} \\ &= \min\{p_1 + p_2, p_1 - p_2\} \\ &= \min\{p_2, -p_2\} + p_1 \\ &= -|p_2| + p_1. \end{aligned}$$

With this we get easily

$$\begin{aligned} f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - \min\{\langle e_1|p\rangle, \langle e_4|p\rangle\} \\ &= I_{F_1}(q) + \langle q|p\rangle + |p_2| - p_1 \\ &= \begin{cases} p_1 + tp_2 - p_1 + |p_2| & \text{if } q = (1,t) \in F_1 \text{ for some } |t| \le 1 \\ \infty & \text{if } q \notin F_1 \end{cases} \\ &= \begin{cases} tp_2 + |p_2| & \text{if } q = (1,t) \in F_1 \text{ for some } |t| \le 1 \\ \infty & \text{if } q \notin F_1 \end{cases} \end{aligned}$$

In the same way, we get:

$$\begin{aligned} f_{F_2,p}(q) &= \langle q|p\rangle - p_2 + |p_1| + I_{F_2}(q) \\ f_{F_3,p}(q) &= \langle q|p\rangle + p_1 + |p_2| + I_{F_3}(q) \\ f_{F_4,p}(q) &= \langle q|p\rangle + p_2 + |p_1| + I_{F_4}(q). \end{aligned}$$

Let $y = (y_1, y_2) \in \mathbb{R}^2$. Then for the Legendre-Fenchel transform of the extreme sets consisting of only one point we have by Lemma 5.0.4:

$$f^*_{E_1,p}(y) = y_1 + y_2 = \langle (1,1) \mid y\rangle,$$

and in the same way

$$f^*_{E_2,p}(y) = -y_1 + y_2 = \langle (-1,1) \mid y\rangle = \langle e_2|y\rangle$$

$$f^*_{E_3,p}(y) = -y_1 - y_2 = \langle(-1,-1) \mid y\rangle = \langle e_3|y\rangle$$
$$f^*_{E_4,p}(y) = y_1 - y_2 = \langle(1,-1) \mid y\rangle = \langle e_4|y\rangle.$$

As expected, $f^*_{E_i,p}$ is independent of p.
Now to the one-dimensional extreme sets. We use the notation $e_1 = (1,1)$ and $e_4 = (1,-1)$ and equation (5.4) to derive:

$$\begin{aligned} f^*_{F_1,p}(y) &= \max\{\langle e_1|y-p\rangle, \langle e_4|y-p\rangle\} + \min\{\langle e_1|p\rangle, \langle e_4|p\rangle\} \\ &= \max\left\{\left\langle (1,1)\middle|\, y-p\right\rangle, \left\langle (1,-1)\middle|\, y-p\right\rangle\right\} + \min\left\{\left\langle (1,1)\middle|\, p\right\rangle, \left\langle (1,-1)\middle|\, p\right\rangle\right\} \\ &= \max\{y_1 - p_1 + (y_2 - p_2), y_1 - p_1 - (y_2 - p_2)\} + \min\{p_1 + p_2, p_1 - p_2\} \\ &= y_1 - p_1 + \max\{y_2 - p_2, -(y_2 - p_2)\} + p_1 + \min\{p_2, -p_2\} \\ &= y_1 + |y_2 - p_2| - |p_2|. \end{aligned}$$

And for the extreme sets we get:

$$\begin{aligned} f^*_{F_2,p}(y) &= |y_1 - p_1| + y_2 - |p_1| \\ f^*_{F_3,p}(y) &= |y_2 - p_2| - y_1 - |p_2| \\ f^*_{F_4,p}(y) &= |y_1 - p_1| - y_2 - |p_1|. \end{aligned}$$

We see that $f^*_{F_i,p}$ is either dependent on p_1 or on p_2, a fact that will fall into context after we found sequences converging to these functions.

The Geometric Interpretation We now need a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $\psi_{z_n} \longrightarrow f^*_{E,p}$ as $n \longrightarrow \infty$. It is

$$\psi_{z_n}(y) = \|z_n - y\|_1 - \|z_n\|_1 = |z_{n,1} - y_1| + |z_{n,2} - y_2| - |z_{n,1}| - |z_{n,2}|,$$

where the second index denotes the component.
If $z_{n,1}, z_{n,2} \longrightarrow -\infty$ then there is an $N \in \mathbb{N}$ such that $z_{n,1} < y_1$ and $z_{n,2} < y_2$ $\forall n \geq N$. For these n we have $|z_{n,1} - y_1| = y_1 - z_{n,1}$ and $|z_{n,2} - y_2| = y_2 - z_{n,2}$. Therefore

$$\begin{aligned} |z_{n,1} - y_1| + |z_{n,2} - y_2| - |z_{n,1}| - |z_{n,2}| &= y_1 - z_{n,1} + y_2 - z_{n,2} + z_{n,1} + z_{n,2} \\ &= y_1 + y_2. \end{aligned}$$

If our sequence is following a straight line which is shifted to pass through a point $p = (p_1, p_2)$, we have the sequence $z_n = (-k, -l) \cdot n + (p_1, p_2) \longrightarrow (-\infty, -\infty)$ with $k, l > 0$ and we obtain

$$\begin{aligned} \|z_n - y\|_1 - \|z_n\|_1 &= |-kn + p_1 - y_1| + |-ln + p_2 - y_2| - |-kn + p_1| - |-ln + p_2| \\ &\overset{n \gneqq 0}{=} kn - p_1 + y_1 + ln - p_2 + y_2 - kn + p_1 - ln + p_2 \\ &= y_1 + y_2 \end{aligned}$$

and therefore

$$\psi_{z_n} \longrightarrow f^*_{E_1,p},$$

independent of the point p.
In the same way we get

$$z_n = \begin{pmatrix} -k \\ -l \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_1}$$

$$z_n = \begin{pmatrix} k \\ -l \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_2,p}$$
$$z_n = \begin{pmatrix} k \\ l \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_3,p}$$
$$z_n = \begin{pmatrix} -k \\ l \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_4,p}.$$

For the one-dimensional extreme sets we need one of the two components to remain constant. This will give us the point $p = (p_1, p_2)$ of $f^*_{E,p}$.
Let $z_n = (-1, 0) \cdot n + (p_1, p_2) \longrightarrow (-\infty, p_2)$ be a sequence in $\mathbb{R}^2$. Then

$$\begin{aligned} \psi_{z_n}(y) &= \|z_n - y\| - \|z_n\| \\ &= |-n + p_1 - y_1| + |p_2 - y_2| - |-n + p_1| - |p_2| \\ &\overset{n \geqslant 0}{=} n - p_1 + y_1 + |p_2 - y_2| - n + p_1 - |p_2| \\ &= |p_2 - y_2| - |p_y| + y_1 = f^*_{F_1,p}(y). \end{aligned}$$

And similarly for the other extreme sets. So in the end we have

$$z_n = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ p_2 \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{F_1,p}$$
$$z_n = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} p_1 \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{F_2,p}$$
$$z_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ p_2 \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{F_3,p}$$
$$z_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} p_1 \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{F_4,p}.$$

We already noticed that $f^*_{F_i,p}$ depends on one component of p only and now we see that it is the one which defines the left-right respectively the up-down shift of the straight line with respect to the origin. These four lines here are exactly those going through an extreme point of B if they are not shifted. So the directions of the straight line gives us the extreme set E whereas the parallel shift determines the p of $f^*_{E,p}$.

The General Case

For simplicity we say that a sequence $(z_n)_{n\in\mathbb{N}}$ following a straight line is converging to the extreme set $E \subseteq \partial B^\circ$ if $\psi_{z_n} \longrightarrow f^*_{E,p}$ for some $p \in \mathbb{R}^m$. The treatment of the three dimensional case can be found in the appendix and I will refer to it in the following. Its main result can be split into two aspects which fit with our deductions from the two-dimensional case. The first is the fact that the extremal set a sequence is converging to lies in some manner on the opposite side (with respect to the origin $\mathcal{O}$) of the intersection point of ∂B with the straight line the sequence is following. The second result is that if the extremal set E is not only a single point, then parallel lines lead to different horofunctions (different p), but of the same extremal set E. Only the direction of the sequence has influence on the E of $f^*_{E,p}$.
The last step now is to generalise our calculations to m dimensions. In both the two- and three-dimensional examples, especially in the three-dimensional one, we remarked that there are three types of sequences for each component of z_n to consider, those going to

either $+\infty$ or $-\infty$ and those remaining constant.
Let $D = \{1, \dots, m\}$. Let $\{P, M, T\}$[2] be a partition of D, that is $P \cup M \cup T = D$ and P, M, T are pairwise disjoint. Define the following set:

$$E_{PMT} := \{x \in \mathbb{R}^m \mid x_i = 1 \ \forall i \in P;\ x_j = -1 \ \forall j \in M;\ x_k = t_k \text{ with } |t_k| \leq 1 \ \forall k \in T\}.$$

For example, if $m = 3$ and $P = \{2\}$, $M = \{3\}$ and $T = \{1\}$, then (cf. page 104)

$$E_{PMT} = \{(t, , 1, -1) | |t| \leq 1\} = F_4.$$

With this definition, the set of extreme sets of B° is

$$\mathcal{E} := \{E_{PMT} \mid D = P \dot\cup M \dot\cup T\}.$$

Now one can easily conclude that $\#\mathrm{Ext}(B^\circ) = 3^m$.
Let $E := E_{PMT}$ be an extreme set of B° for some sets P, M and T and $p \in \mathbb{R}^m$. Then

$$\begin{aligned}
f_{E,p}(q) &= I_E(q) + \langle q|p\rangle - \inf_{y \in E} \langle y|p\rangle \\
&= I_E(q) + \langle q|p\rangle - \sum_{i \in P} p_i + \sum_{j \in M} p_j - \inf \left\{ \sum_{k \in T} t_k p_k \mid |t_k| \leq 1 \right\} \\
&= I_E(q) + \langle q|p\rangle - \sum_{i \in P} p_i + \sum_{j \in M} p_j - \sum_{k \in T} \inf_{|t_k| \leq 1} (t_k p_k) \\
&= I_E(q) + \langle q|p\rangle - \sum_{i \in P} p_i + \sum_{j \in M} p_j + \sum_{k \in T} |p_k|,
\end{aligned}$$

which is in accordance with the 2- and the 3- dimensional case.
The Legendre-Fenchel transform then is:

$$\begin{aligned}
f^*_{E,p}(y) &= \sup_{x \in \mathbb{R}^m} (\langle x|y\rangle - f_{E,p}(x)) \\
&= \sup_{x \in \mathbb{R}^m} \left[\langle x|y\rangle - I_E(x) - \langle x|p\rangle + \sum_{i \in P} p_i - \sum_{j \in M} p_j - \sum_{k \in T} |p_k| \right] \\
&= \sup_{x \in E} [\langle x|y - p\rangle] + \sum_{i \in P} p_i - \sum_{j \in M} p_j - \sum_{k \in T} |p_k| \\
&= \sum_{i \in P} (y_i - p_i) - \sum_{j \in M} (y_j - p_j) + \sup \left\{ \sum_{k \in T} t_k (y_k - p_k) \middle| |t_k| \leq 1 \right\} \\
&\quad + \sum_{i \in P} p_i - \sum_{j \in M} p_j - \sum_{k \in T} |p_k| \\
&= \sum_{k \in T} \sup_{|t_k| \leq 1} [t_k (y_k - p_k)] - \sum_{k \in T} |p_k| + \sum_{i \in P} (y_i - p_i + p_i) - \sum_{j \in M} (y_j - p_j + p_j) \\
&= \sum_{i \in P} y_i - \sum_{j \in M} y_j + \sum_{k \in T} (|y_k - p_k| - |p_k|).
\end{aligned}$$

For the geometrical interpretation, let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^m$ with its components defined as:

$$z_{n,i} = \begin{cases} -k_i n & \text{if } i \in P \\ k_i n & \text{if } i \in M \\ p_i & \text{if } i \in T, \end{cases}$$

[2]P stands for "plus", M stands for "minus" and T for the parameter "t".

where $k_i > 0$ for all $i \in P \cup M$. Then we have for a fixed $y \in \mathbb{R}^m$ and n large enough such that $k_i n > |y_i|$ for all $i \in P \cup M$:

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_1 - \|z_n\|_1 \\
&= \sum_{i=1}^{n}(|z_{n,i} - y_i| - |z_{n,i}|) \\
&= \sum_{i\in P}(|-k_i n - y_i| - |-k_i n|) + \sum_{j\in M}(|k_j n - y_j| - |k_j n|) + \sum_{k\in T}(|p_k - y_k| - |p_k|) \\
&= \sum_{i\in P}(k_i n + y_i - k_i n) + \sum_{j\in M}(k_j n - y_j - k_j n) + \sum_{k\in T}(|p_k - y_k| - |p_k|) \\
&= \sum_{i\in P} y_i - \sum_{j\in M} y_j + \sum_{k\in T}(|y_k - p_k| - |p_k|),
\end{aligned}$$

which shows that

$$\psi_{z_n} \longrightarrow f^*_{E,p} \quad \text{as } n \longrightarrow \infty.$$

So independent of the dimension, the result is the same: the direction of the sequence determines the extreme set E and if E contains more than one point, the parallel shift of the line determines p. The only thing not really clear yet is how to find E. We already know that this has to do with the extreme sets opposite of the intersection point of the sequence with B°, but the precise result will be presented later in section 5.6.

5.2 $X = \mathbb{R}^2$ with a Symmetric Polyhedral Unit Ball

In this example we will limit ourselves to two dimensions and consider the Busemann compactification of $\mathbb{R}^2$ equipped with a norm induced by an arbitrary polytope B. We recall that the norm induced by B for $x \in \mathbb{R}^2$ was defined by:

$$\|x\|_B = \inf\{\alpha > 0 \mid x \in \alpha B\}.$$

By definition, B is the unit ball of this norm.

Consider the following convex set in $\mathbb{R}^2$ (compare figure 5.2):

$$B := \operatorname{conv}\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$$

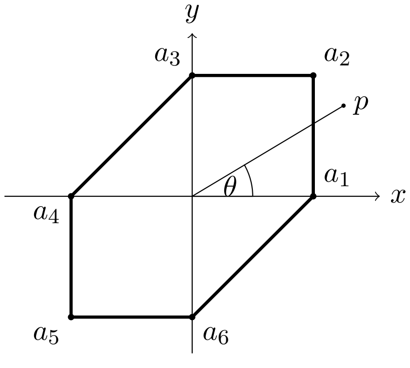


Figure 5.2: B as a hexagonal polytope

Then B is a bounded, convex, open and centrally symmetric set and therefore it defines a norm. The first difference to the example before is that we have to find an expression

for the norm $\|p\|_B$ of some point $p \in \mathbb{R}^2$. We will use that the shape of B is a mixture of the unit balls of the L^1- and the L^∞-norm.
Let $p \in \mathbb{R}^2$ be a point and let $\theta \in [-\pi, \pi]$ be the angle between $p \neq 0$ and the positive x-axis, namely as seen in figure 5.2. Then the norm of p is symmetric and given by

$$\|p\|_B = \begin{cases} \max\{|p_1|, |p_2|\} & \text{if } \theta \in [0, \frac{\pi}{2}) \\ |p_1| + |p_2| & \text{if } \theta \in [\frac{\pi}{2}, \pi] \\ \|-p\|_B & \text{if } \theta \in (-\pi, 0). \end{cases}$$

We now follow [Wal07] to calculate the horofunctions of $\mathbb{R}^2$ with this metric. So we first have to determine the extremal sets of the dual unit ball B° and therefore we have to find B°.

The Dual Unit Ball The vertices of B are:

$$\begin{array}{lll} a_1 = a_7 = (1,0) & a_2 = (1,1) & a_3 = (0,1) \\ a_4 = (-1,0) & a_5 = (-1,-1) & a_6 = (0,-1). \end{array}$$

For the dual unit ball B°, defined by the condition $B^\circ = \{y \in \mathbb{R}^2 \mid \langle y|x\rangle \geq -1 \ \forall x \in B\}$, we have to find points $b_i \in (\mathbb{R}^2)^*$ for $i = 1, \dots, 6$ such that

$$\langle b_i|a_i\rangle = \langle b_i|a_{i+1}\rangle = -1 \ \forall i = 1, \dots, 6.$$

Therefore we get:

$$\begin{array}{lll} b_1 = (-1,0) & b_2 = (0,-1) & b_3 = (1,-1) \\ b_4 = (1,0) & b_5 = (0,1) & b_6 = (-1,1). \end{array}$$

And so by Lemma 2.5.16

$$B^\circ = \text{conv}\{(1,0), (0,1), (-1,1), (-1,0), (0,-1), (1,-1)\}.$$

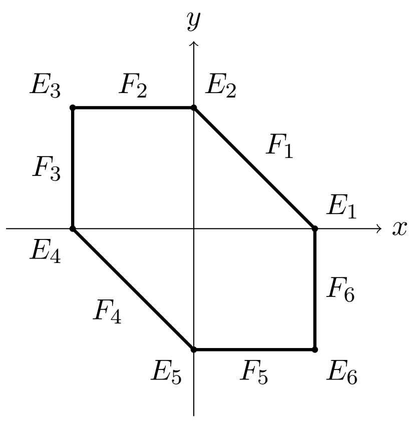


Figure 5.3: B° of our polyhedral B

The proper extreme sets of B° are (see figure 5.3):
Points:

$$\begin{array}{lll} E_1 := \{e_1\} := \{(1,0)\} & E_2 := \{e_2\} := \{(0,1)\} & E_3 := \{e_3\} := \{(-1,1)\} \\ E_4 := \{e_4\} := \{(-1,0)\} & E_5 := \{e_5\} := \{(0,-1)\} & E_6 := \{e_6\} := \{(1,-1)\} \end{array}$$

Facets:

$$F_1 := \left\{ \begin{pmatrix} t \\ -t+1 \end{pmatrix} \middle| \, 0 \leq t \leq 1 \right\} = \operatorname{conv}(e_1, e_2) \quad F_2 := \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| -1 \leq x \leq 0 \right\} = \operatorname{conv}(e_2, e_3)$$

$$F_3 := \left\{ \begin{pmatrix} -1 \\ y \end{pmatrix} \middle| \, 0 \leq y \leq 1 \right\} = \operatorname{conv}(e_3, e_4) \quad F_4 := \left\{ \begin{pmatrix} t \\ -t-1 \end{pmatrix} \middle| -1 \leq t \leq 0 \right\} = \operatorname{conv}(e_4, e_5)$$

$$F_5 := \left\{ \begin{pmatrix} x \\ -1 \end{pmatrix} \middle| \, 0 \leq x \leq 1 \right\} = \operatorname{conv}(e_5, e_6) \quad F_6 := \left\{ \begin{pmatrix} 1 \\ y \end{pmatrix} \middle| -1 \leq y \leq 0 \right\} = \operatorname{conv}(e_1, e_6)$$

Second Step: the $f_{E,p}$-Functions From Lemma 5.1, we already know the form of $f_{E,p}$ if E is a set consisting of a single point. So we have:

$$f_{E_i,p} = I_{E_i} \qquad \forall i \in \{1, \dots, 6\}.$$

For the facets we calculate for some point $p = (p_1, p_2) \in \mathbb{R}^2$ using equation (5.3):

$$\begin{aligned} f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\} \\ &= I_{F_1}(q) + \langle q|p\rangle - \min\{\langle (1,0)|\, p\rangle, \langle (0,1)|\, p\rangle\} \\ &= I_{F_1}(q) + \langle q|p\rangle - \min\{p_1, p_2\}. \end{aligned}$$

We find the functions for the other extreme sets in the same way. So in the end we have:

$$\begin{aligned} f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - \min\{p_1, p_2\} \\ f_{F_2,p}(q) &= I_{F_2}(q) + \langle q|p\rangle - p_2 + \max\{p_1, 0\} \\ f_{F_3,p}(q) &= I_{F_3}(q) + \langle q|p\rangle + p_1 - \min\{0, p_2\} \\ f_{F_4,p}(q) &= I_{F_4}(q) + \langle q|p\rangle + \max\{p_1, p_2\} \\ f_{F_5,p}(q) &= I_{F_5}(q) + \langle q|p\rangle + p_2 - \min\{p_1, 0\} \\ f_{F_6,p}(q) &= I_{F_6}(q) + \langle q|p\rangle - p_1 + \max\{0, p_2\} \end{aligned}$$

Comparing these functions, we notice that for each $i \in \{1, 2, 3\}$ the expressions of $f_{F_i,p}(q)$ and $f_{F_{i+3},p}(q)$ are quite similar. Looking at the picture we see that these are exactly the pairs for which the facets are parallel.

Third Step: the $f^*_{E,p}$-Functions In this step, we have to calculate the Legendre-Fenchel transforms of our functions above. For the extremal points $E_i = \{e_i\}$ ($i \in \{1, \dots, 6\}$) we know by Lemma 5.2

$$f^*_{E_i,p}(y) = \langle e_i|y\rangle,$$

and therefore

$$\begin{aligned} f^*_{E_1,p}(y) &= y_1 \\ f^*_{E_2,p}(y) &= y_2 \\ f^*_{E_3,p}(y) &= -y_1 + y_2 \\ f^*_{E_4,p}(y) &= -y_1 \\ f^*_{E_5,p}(y) &= -y_2 \\ f^*_{E_6,p}(y) &= y_1 - y_2. \end{aligned}$$

Now to the one-dimensional extremal sets, the F_i, $i \in \{1, \dots, 6\}$. The calculations by using equation (5.4) are as follows:

$$\begin{aligned} f^*_{F_1,p}(y) &= \max\{\langle e_1|y-p\rangle, \langle e_2|y-p\rangle\} + \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\} \\ &= \max\{\langle (1,0)|\, y-p\rangle, \langle (0,1)|\, y-p\rangle\} + \min\{p_1, p_2\} \\ &= \max\{y_1 - p_1, y_2 - p_2\} + \min\{p_1, p_2\}. \end{aligned}$$

For the other functions we get by similar calculations:

$$\begin{aligned} f^*_{F_2,p}(y) &= -\min\{y_1 - p_1, 0\} - \max\{p_1, 0\} + y_2 \\ f^*_{F_3,p}(y) &= \max\{0, y_2 - p_2\} + \min\{0, p_2\} - y_1 \\ f^*_{F_4,p}(y) &= -\min\{y_1 - p_1, y_2 - p_2\} - \max\{p-1, p_2\} \\ f^*_{F_5,p}(y) &= \max\{y_1 - p_1, 0\} + \min\{p_1, 0\} - y_2 \\ f^*_{F_6,p}(y) &= -\min\{0, y_2 - p_2\} - \max\{0, p_2\} + y_1 \end{aligned}$$

Here again the transforms of the sets parallel to each other are very similar.
As $\max\{x\} = -\min\{-x\}$, the relation between these functions is

$$f^*_{F_i,p}(y) = f^*_{F_{i+3},-p}(-y) \; \forall i \in \{1, 2, 3\}.$$

The Geometrical Part As in the examples above, we now want to find sequences $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^2$, such that

$$\psi_{z_n}(y) = \|z_n - y\|_B - \|z_n\|_B \longrightarrow f^*_{E,p}(y) \text{ as } n \longrightarrow \infty$$

where E denotes a proper extreme set of B°.

Just like before, we start with sequences which converge to some $f^*_{E_i,p}$. Inspired by the first example we will therefore consider sequences $(z_n)_{n\in\mathbb{N}}$ along a straight line in the direction of a facet of B, possibly shifted by p. Let $z_n = (k, 1) \cdot n + (p_1, p_2)$ with $k > 0, k \neq 1$. Then $(z_n) \to (\infty, \infty)$ as $n \longrightarrow \infty$. So

$$\theta(z_n) = \arctan\left(\frac{1}{k}\right) \in \left(0, \frac{\pi}{2}\right)$$

for n large enough such that z_n lies in the first quadrant. Then

$$\|z_n\|_B = \max\{|kn + p_1|, |n + p_2|\} = \max\{kn + p_1, n + p_2\} = \begin{cases} kn + p_1 & \text{if } k \geq 1 \\ n + p_2 & \text{if } k < 1 \end{cases}$$

and for n large enough (such that $z_n - y$ lies in the first quadrant)

$$\begin{aligned} \|z_n - y\|_B &= \max\{|kn + p_1 - y_1|, |n + p_2 - y_2|\} \\ &= \max\{kn + p_1 - y_1, n + p_2 - y_2\} \\ &= \begin{cases} kn + p_1 - y_1 & \text{if } k \geq 1 \\ n + p_2 - y_2 & \text{if } k < 1. \end{cases} \end{aligned}$$

Hence for $k > 1$ and n large enough we obtain

$$\begin{aligned} \psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\ &= kn + p_1 - y_1 - kn - p_1 \\ &= -y_1 = f^*_{E_4,p}(y). \end{aligned}$$

For $k < 1$ and again n large enough it is

$$\begin{aligned}\psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\ &= n + p_2 - y_2 - n - p_2 \\ &= -y_2 = f^*_{E_5,p}(y).\end{aligned}$$

So for $k > 1$

$$\begin{aligned}z_n &= \begin{pmatrix} k \\ 1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_4,p} \\ z_n &= \begin{pmatrix} 1 \\ k \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_5,p}.\end{aligned}$$

In contrast to the examples before, the angle of the straight line the sequence is following within the first quadrant is important now. The straight line through the origin and a_2 is just the line dividing the two areas in which each sequence converges to the same Busemann point.
With similar calculations we find for the other extreme sets:

$$\begin{aligned}z_n &= \begin{pmatrix} -k \\ 1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ \infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_6,p} \text{ for } k > 0 \\ z_n &= \begin{pmatrix} -k \\ -1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_1,p} \text{ for } k > 1 \\ z_n &= \begin{pmatrix} -1 \\ -k \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_2,p} \text{ for } k > 1 \\ z_n &= \begin{pmatrix} k \\ -1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ -\infty \end{pmatrix} \text{ yields } \psi_{z_n} \longrightarrow f^*_{E_3,p} \text{ for } k > 0.\end{aligned}$$

We notice that all these sequences are independent of p and depend only on the angle θ as the pictures 5.4 and 5.5 illustrate.

We now fix a point $p = (p_1, p_2) \in \mathbb{R}^2$ and consider sequences parallel to those straight lines through the vertices of B, for example the sequence $(z_n)_{n \in \mathbb{N}}$ with

$$z_n = (1, 0) \cdot n + (p_1, p_2).$$

Then for $p_2 \geq 0$, z_n lies in the first quadrant and for $p_2 < 0$ it lies in the forth. Therefore

$$\begin{aligned}\|z_n\|_B &= \begin{cases} \max\{|n|, |p_2|\} & \text{if } p_2 \geq 0 \\ |n| + |p_2| & \text{if } p_2 < 0 \end{cases} \\ &= \begin{cases} n & \text{if } p_2 \geq 0 \text{ and for } n \text{ big enough} \\ n - p_2 & \text{if } p_2 < 0 \end{cases} \\ &= n - \min\{0, p_2\}.\end{aligned}$$

And we have

$$\begin{aligned}\|z_n - y\|_B &= \left\| \begin{pmatrix} n - y_1 \\ p_2 - y_2 \end{pmatrix} \right\|_B \\ &= \begin{cases} \max\{|n - y_1|, |p_2 - y_2|\} & \text{if } p_2 - y_2 \geq 0 \\ |n - y_1| + |p_2 - y_2| & \text{if } p_2 - y_2 < 0 \end{cases}\end{aligned}$$

$$= \left\{ \begin{array}{ll} n - y_1 & \text{if } p_2 - y_2 \geq 0 \\ n - y_1 + y_2 - p_2 & \text{if } p_2 - y_2 < 0 \end{array} \right\} \text{for } n \gg 0.$$

So together we get for $n \gg 1$

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\
&= \left\{ \begin{array}{ll} n - y_1 - n + \min\{0, p_2\} & \text{if } y_2 - p_2 \leq 0 \\ n - y_1 + y_2 - p_2 - n + \min\{0, p_2\} & \text{if } y_2 - p_2 > 0 \end{array} \right. \\
&= -y_1 + \max\{0, y_2 - p_2\} + \min\{0, p_2\} \\
&= f^*_{F_3,p}(y).
\end{aligned}$$

With similar calculations we obtain the following result:

$$\begin{aligned}
z_n &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} p_1 \\ \infty \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_5,p} \\
z_n &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} p_1 \\ -\infty \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_2,p} \\
z_n &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ p_2 \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_3,p} \\
z_n &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ p_2 \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_6,p} \\
z_n &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_4,p} \\
z_n &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\infty \\ -\infty \end{pmatrix} && \text{yields } \psi_{z_n} \longrightarrow f^*_{F_1,p}.
\end{aligned}$$

The following picture illustrates the result. Straight lines in a coloured area of the left picture converge to a Busemann point $f^*_{E,p}$ where E can be found in the right picture with the same colour.

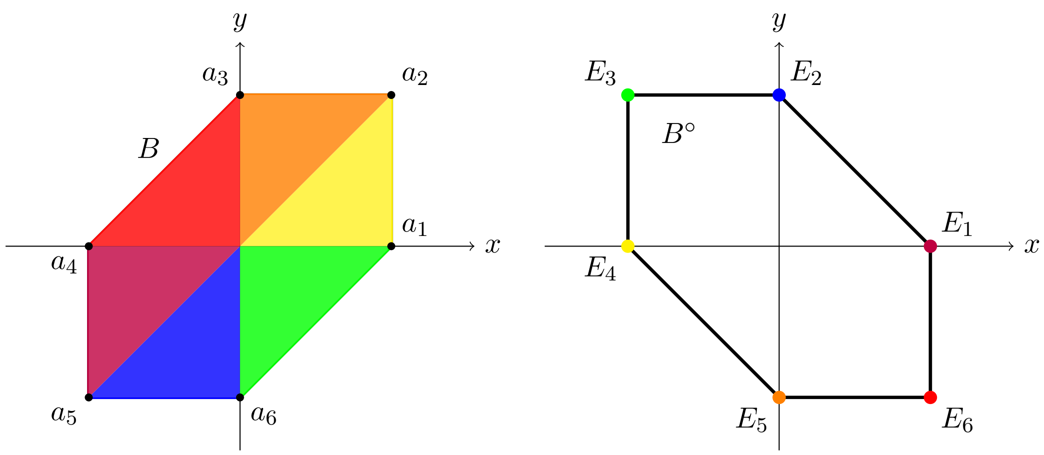


Figure 5.4: Sequences along a straight line in the direction of a facet of B converge to a Busemann point $f^*_{E,p}$ where E is a single point.

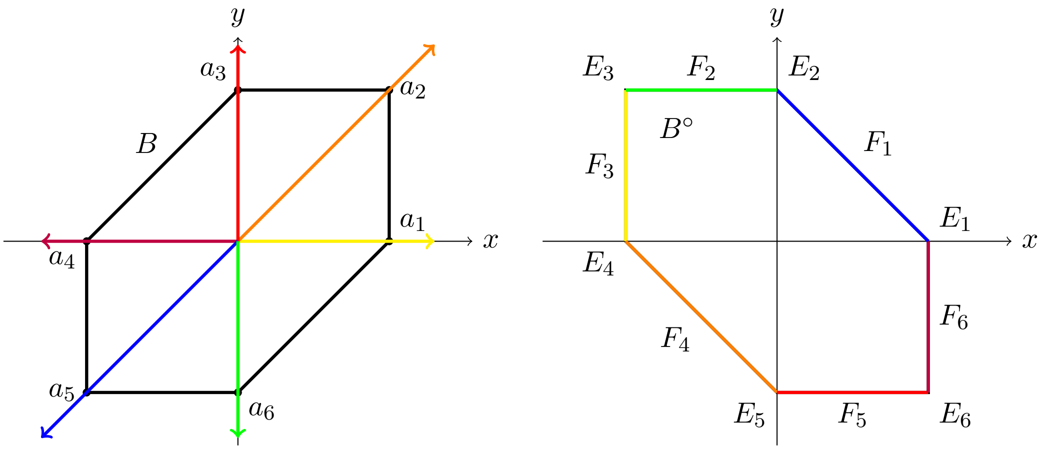


Figure 5.5: Sequences along straight lines through vertices of B converge to Busemann points $f^*_{E,p}$ where E is one-dimensional.

Lines through an extreme set F of B but not through an extreme point converge all to the same Busemann point $f^*_{E,p}$ where E is an extreme point of B° lying in some way on the opposite side of F and orthogonal to it. The point p has no influence in this case, only the direction counts. If the sequence runs parallel to a straight line through a vertex of B, then p denotes this parallel shift and the sequence converges to a Busemann point belonging to a one-dimensional extreme set of B°.

5.3 $X = \mathbb{R}^2$ with a Non-Symmetric Polyhedral Unit Ball

In the examples before we always considered a polyhedral symmetric convex set as unit ball. We will now look at an example where the unit ball is still convex and polyhedral but not symmetric. The introduction to horofunction compactification in section 3 deals with a possibly non-symmetric metric and symmetry is not required in section 4 to determine the Busemann points.
We will need the result of this example later in section 7.1 when we examine the horofunction compactification of $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$.

Let B be the convex hull of

$$a_1 = (1,0); \quad a_2 = (-\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}); \quad a_3 = (-\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{3}).$$

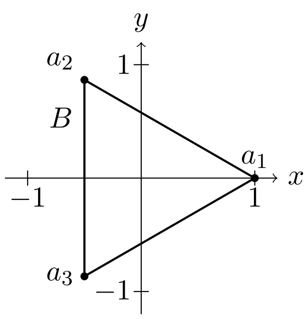


Figure 5.6: Unit ball B as a triangle

The Norm Induced by B Let $C_{i,j} := K_{\mathrm{conv}(a_i,a_j)}$ be the cone generated by the line between a_i and a_j. We first calculate the norm $\|x\|_B$ for every x in the cone $C_{1,2}$. The line l, on which every point has norm 1, can be described as

$$l = \left\{ \begin{pmatrix} s \\ -\frac{1}{\sqrt{3}}s + \frac{1}{\sqrt{3}} \end{pmatrix} \in \mathbb{R}^2 \middle| s \in \left[-\frac{1}{2}, 1\right] \right\}.$$

Let $0 \neq x = (a,b) \in C_{1,2}$. Then there is a $k > 0$ and an $s \in \left[-\frac{1}{2}, \sqrt{3}\right]$ such that

$$\begin{pmatrix} a \\ b \end{pmatrix} = k \cdot \begin{pmatrix} s \\ -\frac{1}{\sqrt{3}}s + \frac{1}{\sqrt{3}} \end{pmatrix}.$$

From this it follows $a = ks$ and therefore from the second component

$$\sqrt{3}b = -ks + k = -a + k.$$

So

$$\|x\|_B = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_B = a + \sqrt{3}b \;\forall x \in C_{1,2}.$$

Because of the symmetry of B with respect to the x-axis we know that

$$\|x\|_B = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_B = a - \sqrt{3}b \;\forall x \in C_{3,1}.$$

The norm of the point in the last cone $C_{2,3}$ is just twice the negative first component $\|x\|_B = -2a$. All together we have for some $x = (a,b) \in \mathbb{R}^2$

$$\|x\|_B = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_B = \begin{cases} a + \sqrt{3}|b| & \text{if } a \geq 0 \text{ or } |b| \geq -\sqrt{3}a; \\ -2a & \text{if } a < 0 \text{ and } |b| < -\sqrt{3}a. \end{cases} \tag{5.5}$$

The Dual Unit Ball We did not require B to be symmetric in the proofs of Lemma 2.5.15 and Lemma 2.5.16, so we can use them now to determine the dual unit ball B°. At first we need $b_{i,j} \in \mathbb{R}^2$ such that $\langle b_{i,j} | a_i \rangle = \langle b_{i,j} | a_j \rangle = -1$. We calculate

$$b_{1,2} = (-1, -\sqrt{3}); \quad b_{2,3} = (-2, 0); \quad b_{3,1} = (-1, \sqrt{3}),$$

and so we know

$$B^\circ = \mathrm{conv} \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}, \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \right\}.$$

In this special case the dual unit ball has the same shape as B but twice as big as illustrated in figure 5.7.

The extreme sets of B° are:

$$E_1 := \{e_1\} := \{(2,0)\}; \quad E_2 := \{e_2\} := \{(-1, \sqrt{3})\}; \quad E_3 := \{e_3\} := \{(-1, -\sqrt{3})\};$$

and

$$\begin{aligned} F_1 &:= \left\{ \begin{pmatrix} t \\ -\frac{1}{\sqrt{3}}t + \frac{2}{\sqrt{3}} \end{pmatrix} \middle| -1 \leq t \leq 2 \right\} = \mathrm{conv}(e_1, e_2); \\ F_2 &:= \left\{ \begin{pmatrix} -1 \\ t \end{pmatrix} \middle| -\sqrt{3} \leq t \leq \sqrt{3} \right\} = \mathrm{conv}(e_2, e_3); \\ F_3 &:= \left\{ \begin{pmatrix} t \\ \frac{1}{\sqrt{3}}t - \frac{2}{\sqrt{3}} \end{pmatrix} \middle| -1 \leq t \leq 2 \right\} = \mathrm{conv}(e_1, e_3). \end{aligned}$$

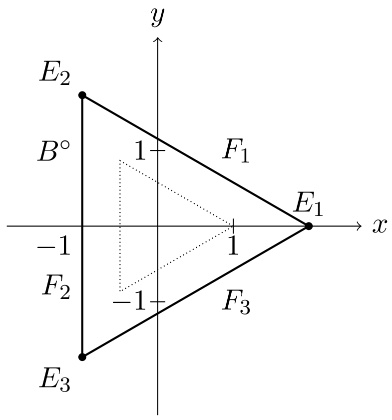


Figure 5.7: B° of our triangular B

Calculation of $f_{E,p}$ As we already know the result for the extreme points of B° we will start directly with the one-dimensional extreme sets of B°. For some $p \in \mathbb{R}^2$ and $q \in (\mathbb{R}^2)^*$ we have with the notations above:

$$\begin{aligned}\min\{\langle e_1|p\rangle, \langle e_2|p\rangle\} &= \min\{\langle (2,0)|p\rangle, \langle (-1,\sqrt{3})|p\rangle\}\\ &= \min\{2p_1, -p_1 + \sqrt{3}p_2\}\\ &= 2p_1 + \min\{0, -3p_1 + \sqrt{3}p_2\}\\ &= 2p_1 - \max\{3p_1 - \sqrt{3}p_2, 0\}\end{aligned}$$

and therefore

$$\begin{aligned}f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\}\\ &= I_{F_1}(q) + \langle q|p\rangle - 2p_1 + \max\{3p_1 - \sqrt{3}p_2, 0\}.\end{aligned}$$

For the other facets we get by similar calculations:

$$\begin{aligned}f_{F_2,p}(q) &= I_{F_2}(q) + \langle q|p\rangle + p_1 + \sqrt{3}|p_2|\\ f_{F_3,p}(q) &= I_{F_3}(q) + \langle q|p\rangle - 2p_1 + \max\{3p_1 + \sqrt{3}p_2, 0\}.\end{aligned}$$

The Legendre-Fenchel Transform $f^*_{E,p}$ We know by Lemma 5.0.4 that for $E_i = \{e_i\}$, $i = 1, \ldots, 3$, the Legendre-Fenchel transform at $y \in \mathbb{R}$ is

$$f^*_{E_i,p}(y) = \langle e_i|y\rangle,$$

independent of the point p.
For the other extreme sets we calculate using Lemma 5.0.6

$$\begin{aligned}f^*_{F_1,p}(y) &= \max\{\langle e_1|y-p\rangle, \langle e_2|y-p\rangle\} + \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\}\\ &= \max\{\langle (2,0)|y-p\rangle, \langle (-1,\sqrt{3})|y-p\rangle\} + 2p_1 - \max\{3p_1 - \sqrt{3}p_2, 0\}\\ &= 2(y_1 - p_1) + \max\{0, -3(y_1-p_1) + \sqrt{3}(y_2 - p_2)\} + 2p_1 - \max\{3p_1 - \sqrt{3}p_2, 0\}\\ &= 2y_1 + \max\{3(p_1 - y_1) - \sqrt{3}(p_2 - y_2), 0\} + \max\{3p_1 - \sqrt{3}p_2, 0\}.\end{aligned}$$

In the same way we get

$$\begin{aligned}f^*_{F_2,p}(y) &= \sqrt{3}|y_2 - p_2| - \sqrt{3}|p_2| - y_1\\ f^*_{F_3,p}(y) &= \max\{3(p_1 - y_1) + \sqrt{3}(p_2 - y_2), 0\} - \max\{3p_1 + \sqrt{3}p_2, 0\} + 2y_1.\end{aligned}$$

The Geometrical Part Just as in the previous examples, we want to find sequences $(z_n)_{n\in\mathbb{N}}$ such that $\psi_{z_n} \longrightarrow f^*_{E,p}$ $(n \to \infty)$ for some extreme set $E \subseteq \partial B^\circ$ and $p \in \mathbb{R}^2$. We already saw (for polyhedral unit balls) that if a sequence following a parallel to a straight line passing through the origin and an extreme set F of B with $\dim F = k$ converges to a Busemann point $f^*_{E,p}$, then $\dim E = n - 1 - k$. Furthermore, the extreme set the sequence is converging to is perpendicular to the straight line. We will check whether it is also like this in the non-symmetric case.
At first we take a straight line of direction a_1, say $z_n = (1,0)\cdot n + (p_1,p_2)$. Then with $y \in \mathbb{R}^2$ and n large, we calculate

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\
&= \left\|\begin{pmatrix} n + p_1 - y_1 \\ p_2 - y_2 \end{pmatrix}\right\|_B - \left\|\begin{pmatrix} n + p_1 \\ p_2 \end{pmatrix}\right\|_B \\
&= n + p_1 - y_1 + \sqrt{3}|p_2 - y_2| - n - p_1 - \sqrt{3}|p_2| \\
&= \sqrt{3}|p_2 - y_2| - \sqrt{3}|p_2| - y_1 = f^*_{F_2,p}(y).
\end{aligned}$$

If we take a sequence parallel to a straight line through a_2 we have to be careful which norm to take. So we obtain with $z_n = (-1, \sqrt{3})\cdot n + (p_1,p_2)$:

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\
&= \left\|\begin{pmatrix} -n + p_1 - y_1 \\ \sqrt{3}n + p_2 - y_2 \end{pmatrix}\right\|_B - \left\|\begin{pmatrix} -n + p_1 \\ \sqrt{3}n + p_2 \end{pmatrix}\right\|_B \\
&= \begin{cases} -n + p_1 - y_1 + \sqrt{3}(\sqrt{3}n + p_2 - y_2) & \text{if } p_1 - y_1 \geq -\frac{1}{\sqrt{3}}(p_2 - y_2); \\ 2n - 2p_1 + 2y_1 & \text{if } p_1 - y_1 < -\frac{1}{\sqrt{3}}(p_2 - y_2) \end{cases} \\
&\quad - \begin{cases} -n + p_1 + \sqrt{3}(\sqrt{3}n + p_2) & \text{if } p_1 \geq -\frac{1}{\sqrt{3}}p_2; \\ 2n - 2p_1 & \text{if } p_1 < -\frac{1}{\sqrt{3}}p_2 \end{cases} \\
&= \begin{cases} 2n + p_1 - y_1 + \sqrt{3}(p_2 - y_2) & \text{if } 3(p_1 - y_1) + \sqrt{3}(p_2 - y_2) \geq 0; \\ 2n - 2p_1 + 2y_1 & \text{if } 3(p_1 - y_1) + \sqrt{3}(p_2 - y_2) < 0 \end{cases} \\
&\quad - \begin{cases} 2n + p_1 + \sqrt{3}p_2 & \text{if } 3p_1 + \sqrt{3}p_2 \geq 0; \\ 2n - 2p_1 & \text{if } 3p_1 + \sqrt{3}p_2 < 0 \end{cases} \\
&= \max\{3(p_1 - y_1) + \sqrt{3}(p_2 - y_2), 0\} - \max\{3p_1 + \sqrt{3}p_2, 0\} - 2p_1 + 2y_1 + 2p_1 \\
&= f^*_{F_3,p}(y).
\end{aligned}$$

The calculation for the straight line trough a_3 goes similarly and we obtain for a sequence $z_n = (-1, -\sqrt{3})\cdot n + (p_1,p_2)$:

$$\psi_{z_n}(y) = f^*_{F_1,p}(y)$$

for n large enough.
If we take a sequence in the direction of an one-dimensional extreme set of B but not in the direction of a vertex, then we know that this sequence will converge to a Busemann point associated to an extreme point of B° and therefore we don't have to care about the point p. If n is large enough, then z_n and $z_n - y$ will be in the same cone $C_{i,j}$ and therefore we don't have to distinguish between the norms within one calculation. So we get for $z_n = (k,l)\cdot n$ and n large enough independent of p:

$$\psi_{z_n}(y) = \left\|\begin{pmatrix} kn - y_1 \\ ln - y_2 \end{pmatrix}\right\|_B - \left\|\begin{pmatrix} kn \\ ln \end{pmatrix}\right\|_B$$

$$= \begin{cases} kn - y_1 + \sqrt{3}(ln - y_2) - k_n - \sqrt{3}ln & \text{if } (k,l) \in C_{1,2}; \\ -2(kn - y_1) + 2kn & \text{if } (k,l) \in C_{2,3}; \\ kn - y_1 - \sqrt{3}(ln - y_2) - kn + \sqrt{3}ln & \text{if } (k,l) \in C_{3,1} \end{cases}$$

$$= \begin{cases} -y_1 - \sqrt{3}y_2 & \text{if } (k,l) \in C_{1,2}; \\ 2y_1 & \text{if } (k,l) \in C_{2,3}; \\ -y_1 + \sqrt{3}y_2 & \text{if } (k,l) \in C_{3,1} \end{cases}$$

$$= \begin{cases} \langle (-1, -\sqrt{3}) | y \rangle & \text{if } (k,l) \in C_{1,2}; \\ \langle (2, 0) | y \rangle & \text{if } (k,l) \in C_{2,3}; \\ \langle (-1, \sqrt{3}) | y \rangle & \text{if } (k,l) \in C_{3,1} \end{cases}$$

$$= \begin{cases} f^*_{E_3,p}(y) & \text{if } (k,l) \in C_{1,2}; \\ f^*_{E_1,p}(y) & \text{if } (k,l) \in C_{2,3}; \\ f^*_{E_2,p}(y) & \text{if } (k,l) \in C_{3,1}. \end{cases}$$

We see that it makes no difference whether the unit ball is symmetric or not, the connection between the shape of B and B° and the Busemann points is always the same.

5.4 $\mathbb{R}^2$ Equipped with a Lens-Shaped Norm

We come now to norms whose unit spheres are curved. The cases of $\mathbb{R}^2$ equipped with the usual Euclidean norm as well as with the $L^{\frac{3}{2}}$-norm can be found in the appendix[3]. There the boundary of the unit ball is differentiable everywhere. Conversely we consider now the following lens-shaped region B, whose boundary is differentiable everywhere except for two points. B can be described as an intersection of two discs with radius 2 centred at $\pm\sqrt{3}$.

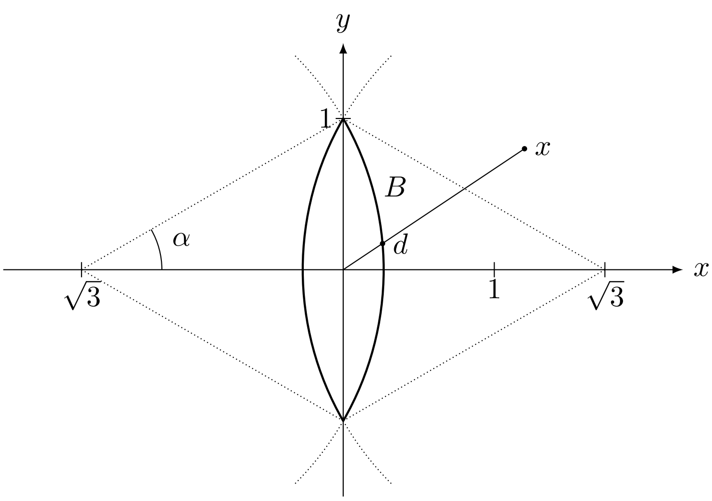


Figure 5.8: Lens-shaped unit ball B

There are two ways to describe B. On the one hand we can describe it as the intersection of the two circles as mentioned above:

$$B = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, (x - \sqrt{3})^2 + y^2 \le 4 \right\} \cap \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, (x + \sqrt{3})^2 + y^2 \le 4 \right\}.$$

[3]See page 107 and 109.

On the other hand we can use polar coordinates. For this we need the angle α as in the picture above. From $\tan\alpha = \frac{1}{\sqrt{3}}$ we obtain $\alpha = \frac{\pi}{6}$. With this we can write

$$B = \left\{ \begin{pmatrix} r\cos\alpha - \sqrt{3} \\ r\sin\alpha \end{pmatrix} \middle| \alpha \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right];\ r \in [0,2] \right\} \cap \left\{ \begin{pmatrix} -r\cos\alpha + \sqrt{3} \\ r\sin\alpha \end{pmatrix} \middle| \alpha \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right];\ r \in [0,2] \right\}$$

The boundary of B is

$$\partial B = \left\{ \begin{pmatrix} \pm(2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \middle| -\frac{\pi}{6} \le \alpha \le \frac{\pi}{6} \right\}. \tag{5.6}$$

The Norm Induced by B We now want to calculate the norm on $\mathbb{R}^2$ induced by B. For an arbitrary point $x = (a,b) \in \mathbb{R}^2$ let $d = (d_1, d_2) \in \partial B$ be the intersection point of ∂B with the straight line from the origin to x. Then the distance

$$k := \|x\|_B = \left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_B$$

is defined by the equation

$$\begin{pmatrix} a \\ b \end{pmatrix} = k \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

We already know that d is of the form $\begin{pmatrix} \pm(2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix}$ for some $\alpha \in [-\frac{\pi}{6}, \frac{\pi}{6}]$. As B is symmetric, it suffices to consider either the left or the right part of the lens. We will deal with the right part. This means we have to solve

$$\begin{pmatrix} a \\ b \end{pmatrix} = k \begin{pmatrix} 2\cos\alpha - \sqrt{3} \\ 2\sin\alpha \end{pmatrix}$$

for k. From the first component we obtain $a = 2k\cos\alpha - \sqrt{3}k$, and from this

$$\cos^2\alpha = \frac{(a + \sqrt{3}k)^2}{4k^2}.$$

If we insert this in the equation from the second component using that $\sin^2\alpha = 1 - \cos^2\alpha$, we obtain

$$k^2 - 2a\sqrt{3}k - (a^2 + b^2) = 0,$$

an therefore

$$k = a\sqrt{3} \pm \sqrt{4a^2 + b^2}.$$

We have to choose the positive sign for the square root so that the boundary of B really has norm 1. For the norm of some $x = (a,b) \in \mathbb{R}^2$ with $a > 0$ this yields

$$\left\| \begin{pmatrix} a \\ b \end{pmatrix} \right\|_B := |a|\sqrt{3} + \sqrt{4a^2 + b^2}. \tag{5.7}$$

Because of the symmetry of the lens, this also holds for $a \le 0$ and we have found the norm of all $x = (a,b) \in \mathbb{R}^2$.

The Dual Unit Ball The most difficult part of this example is to calculate the dual unit ball B° of B as we don't have a polytope and therefore can't use the results of section 2.5. We claim that B° is the region bounded by

$$H := \left\{ \begin{pmatrix} z \\ \pm 1 \end{pmatrix} \middle| -\sqrt{3} \le z \le \sqrt{3} \right\} \cup \left\{ \begin{pmatrix} \pm(z+\sqrt{3}) \\ \pm\frac{1}{2}\sqrt{4-z^2} \end{pmatrix} \middle| 0 \le z \le 2 \right\}. \tag{5.8}$$

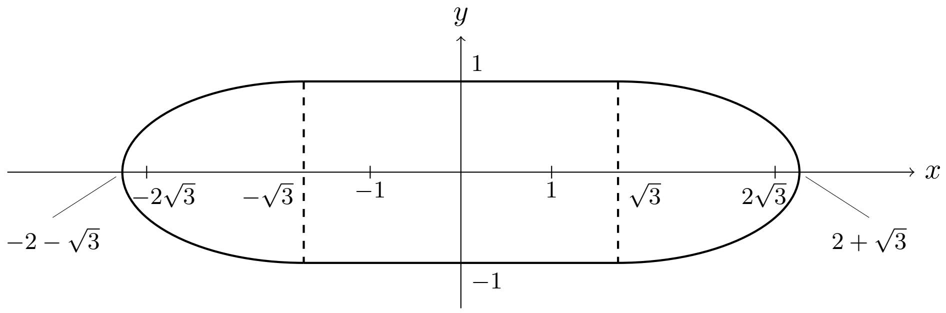


Figure 5.9: B° of the lens shaped norm: a ellipse pulled apart symmetrically

As long as we have not proven this fact, define for notational reasons

$$D := \left\{ \begin{pmatrix} z \\ \pm h \end{pmatrix} \middle| -\sqrt{3} \le z \le \sqrt{3};\ 0 \le h \le 1 \right\} \cup \left\{ \begin{pmatrix} \pm(z+\sqrt{3}) \\ \pm l\sqrt{4-z^2} \end{pmatrix} \middle| 0 \le z \le 2;\ 0 \le l \le \frac{1}{2} \right\}.$$

Then $\partial D = H$. D is an ellipse cut in the middle and pulled apart symmetrically to accommodate a rectangle of height 2 and length $2\sqrt{3}$ in the middle. We now want to prove, that $D = B^\circ$. For this we have to show two things:

1. $D \subseteq B^\circ : \langle x|y\rangle \ge -1$ for all $x \in B,\ y \in D$
2. $B^\circ \subseteq D : \forall y \notin D\ \exists\, x \in B : \langle x|y\rangle < -1$.

Proof of 1.
Strategy of the proof: we prove that the minimum of $\langle\cdot|\cdot\rangle : \partial D \times \partial B \longrightarrow \mathbb{R}; (d,b) \longmapsto \langle d|b\rangle$ is ≥ -1. Then by bilinearity of the dual pairing this is also true on $D \times B$.
Let $x = \begin{pmatrix} \pm\ (2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \in \partial B$ for some $\alpha \in [-\frac{\pi}{6}, \frac{\pi}{6}]$.
We first show the assumption for the rectangle part. Let therefore $y = (z,\ \pm\ 1)$ for some $z \in [-\sqrt{3}, \sqrt{3}]$. The different colours of the signs mark their relations. Signs of the same colour are linked, which means that you can choose either the upper or the lower sign for all of them. The dual pairing of x and y then is

$$\begin{aligned} \langle x|y\rangle &= \begin{pmatrix} \pm\ (2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \begin{pmatrix} z \\ \pm\ 1 \end{pmatrix} \\ &= \ \pm\ z(2\cos\alpha - \sqrt{3})\ \pm\ 2\sin\alpha \\ &\ge -\sqrt{3}(2\cos\alpha - \sqrt{3})\ \pm\ 2\sin\alpha \end{aligned} \tag{5.9}$$

as $z \in [-\sqrt{3}, \sqrt{3}]$. We now want to find the minimum of this expression with respect to α, so we set the partial derivative to zero:

$$\partial_\alpha(\langle x|y\rangle)|_{z=-\sqrt{3}} = 2\sqrt{3}\sin\alpha\ \pm\ 2\cos\alpha \stackrel{!}{=} 0$$

This condition is fulfilled for $\tan(\alpha) = \mp \frac{1}{\sqrt{3}}$ from which $\alpha = \mp \frac{\pi}{6}$ follows. If we insert this into (5.9) we obtain

$$\langle x|y\rangle \geq -\sqrt{3}(\sqrt{3}-\sqrt{3}) \pm \mp 1 = -1.$$

As $\{\pm\frac{\pi}{6}\}$ are exactly the boundary points of $[-\frac{\pi}{6}, \frac{\pi}{6}]$ the first case is shown.

Now we make a similar calculation for the elliptical part of D. Let x be as above and $y = \begin{pmatrix} \pm (z+\sqrt{3}) \\ \pm \frac{1}{2}\sqrt{4-z^2} \end{pmatrix}$ for some $z \in [0,2]$. Then

$$\begin{aligned}
\langle x|y\rangle &= \begin{pmatrix} \pm (2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \begin{pmatrix} \pm (z+\sqrt{3}) \\ \pm \frac{1}{2}\sqrt{4-z^2} \end{pmatrix} \\
&= \underbrace{\pm 2z\cos\alpha \mp \sqrt{3}z \pm 2\sqrt{3}\cos\alpha \mp 3}_{=:A} \pm \sqrt{4-z^2}\sin\alpha \\
&= A \pm \sqrt{4-z^2}\sin\alpha.
\end{aligned} \tag{5.10}$$

The red sign stems from the combination of the orange and the blue one where we either choose the same ($+$) or different ($-$) signs.
The partial derivatives are

$$\begin{aligned}
\partial_\alpha(\langle x|y\rangle) &= \mp 2z\sin\alpha \mp 2\sqrt{3}\sin\alpha \pm \sqrt{4-z^2}\cos\alpha \overset{!}{=} 0 \\
\partial_z(\langle x|y\rangle) &= \pm 2\cos\alpha \mp \sqrt{3} \mp \frac{z\sin\alpha}{\sqrt{4-z^2}} \overset{!}{=} 0.
\end{aligned} \tag{5.11}$$

The second equation yields the condition $\pm \cos\alpha\sqrt{4-z^2} = \pm \frac{\sqrt{3}}{2}\sqrt{4-z^2} \pm \frac{1}{2}z\sin\alpha$ and from this follows

$$\sqrt{4-z^2} = \pm \pm \frac{\frac{1}{2}z\sin\alpha}{(\cos\alpha - \frac{\sqrt{3}}{2})}.$$

If we insert this into the derivative with respect to α and rewrite the resulting equation, it is independent of the choice of signs:

$$z = 2\sqrt{3}\frac{2\cos\alpha - \sqrt{3}}{2\sqrt{3} - 3\cos\alpha}. \tag{5.12}$$

This means that for every $x = \begin{pmatrix} \pm(2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix}$ there is one choice for the parameter z in the first component of y such that the dual pairing $\langle x|y\rangle$ is extremal. With z as in (5.12) we calculate

$$\begin{aligned}
4 - z^2 &= 4 - 12\frac{4\cos^2\alpha + 3 - 4\sqrt{3}\cos\alpha}{12 - 12\sqrt{3}\cos\alpha + 9\cos^2\alpha} \\
&= \frac{4\cdot 3}{(2\sqrt{3} - 3\cos\alpha)^2}(-\cos^2\alpha + 1) \\
&= \frac{12\sin^2\alpha}{(2\sqrt{3} - 3\cos\alpha)^2}.
\end{aligned}$$

If we insert expression (5.12) for z into (5.11), we see that the condition of being 0 is fulfilled independently of α. So we have to determine α another way. For this we calculate

$$A = \pm 2z\cos\alpha \mp \sqrt{3}z \pm 2\sqrt{3}\cos\alpha \mp 3$$

$$
\begin{aligned}
&= \mp 2\cos\alpha \cdot 2\sqrt{3}\frac{2\cos\alpha-\sqrt{3}}{3\cos\alpha-2\sqrt{3}} \pm \sqrt{3}\cdot 2\sqrt{3}\frac{2\cos\alpha-\sqrt{3}}{3\cos\alpha-2\sqrt{3}} \pm 2\sqrt{3}\cos\alpha \mp 3\\
&= \pm \frac{-8\sqrt{3}\cos^2\alpha+12\cos\alpha+12\cos\alpha-6\sqrt{3}+6\sqrt{3}\cos^2\alpha-12\cos\alpha-9\cos\alpha+6\sqrt{3}}{3\cos\alpha-2\sqrt{3}}\\
&= \pm \frac{1}{N}(-2\sqrt{3}\cos^2\alpha+3\cos\alpha)
\end{aligned}
$$

where we set

$$N := 3\cos\alpha - 2\sqrt{3}.$$

$N \neq 0$ because $\frac{2\sqrt{3}}{3} > 1$.
Using these results we obtain

$$
\begin{aligned}
\langle x|y\rangle &= \pm \frac{3\cos\alpha-2\sqrt{3}\cos^2\alpha}{3\cos\alpha-2\sqrt{3}} \pm \sin\alpha\frac{2\sqrt{3}\sin\alpha}{3\cos\alpha-2\sqrt{3}}\\
&= \frac{1}{N}(\pm 3\cos\alpha \mp 2\sqrt{3}\cos^2\alpha \pm 2\sqrt{3}(1-\cos^2\alpha))\\
&= \frac{1}{N}[(\mp 1 \mp 1)2\sqrt{3}\cos^2\alpha \pm 3\cos\alpha \pm 2\sqrt{3}].
\end{aligned}
$$

Now there are two cases to distinguish. The first one is that we choose the upper or the lower sign for both red and green. We will mark this case with a subscript "1". This leads to

$$\langle x|y\rangle_1 = \pm \frac{1}{N}(-4\sqrt{3}\cos^2\alpha+3\cos\alpha+2\sqrt{3}).$$

We want to find the minimum of this function with respect to α. Therefore we take the partial derivative with respect to α and set this equal to zero:

$$\partial_\alpha(\langle x|y\rangle_1) = \pm \frac{4\sqrt{3}\sin\alpha}{N^2}[(\sqrt{3}\cos\alpha-2)^2-1] \stackrel{!}{=} 0.$$

The expression in squared brackets is never equal to zero, because this would lead either to the contradiction $\cos\alpha = \sqrt{3} > 1$ or to $\cos\alpha = \frac{1}{\sqrt{3}}$. From this would follow $\alpha > \frac{\pi}{4} > \frac{\pi}{6}$ which is a contradiction as well. So the only possibility for this term to be zero is for $\alpha = 0$. The second derivative [4] of $\langle x|y\rangle_1$ at $\alpha = 0$ is negative for $+$ and positive for $-$. So in the first case we have a maximum and a minimum in the second case. The two values of M at these extrema are

$$
\begin{aligned}
\langle x|y\rangle_1|_{\alpha=0} &= \pm \frac{-4\sqrt{3}+3+2\sqrt{3}}{3-2\sqrt{3}}.\\
&= \pm 1
\end{aligned}
$$

Now let us look at the other case (which gets a subscript "2") where we choose the upper sign for one and the lower sign for the other coloured sign. This choice gives us a new blue sign. We then have:

$$
\begin{aligned}
\langle x|y\rangle_2 &= \pm \frac{3\cos\alpha-2\sqrt{3}}{3\cos\alpha-2\sqrt{3}}\\
&= \pm 1.
\end{aligned}
$$

[4] $\partial_\alpha^2(\langle x|y\rangle_1) = \pm \frac{12\sqrt{3}}{(3\cos\alpha-2\sqrt{3})^3}(15\cos^4\alpha - 22\sqrt{3}\cos^3\alpha + 13\cos^2\alpha + 14\sqrt{3}\cos\alpha - 14)$.

It remains to check the dual pairing on the boundary points $z \in \{0,2\}$ and $\alpha = \pm\frac{\pi}{6}$. Easy calculations show that in all these cases $\langle x|y\rangle \geq -1$.

So in both cases $\langle x|y\rangle \geq -1$, which is exactly what we wanted to show.

Proof of 2.
Let $y \notin D$. Then y can be written as

$$y = \begin{pmatrix} \pm\ (z+\sqrt{3}+a) \\ \pm\ (\frac{1}{2}\sqrt{4-z^2}+b) \end{pmatrix}$$

with $a, b \geq 0$, not $a = b = 0$, both not uniquely determined and for some $z \in [0,2]$. From the calculation above we know that there is an $x = \begin{pmatrix} \pm\ (2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \in \partial B$ (that is $\alpha \in [-\frac{\pi}{6}, \frac{\pi}{6}]$) where $\pm\ = -\ \pm$ and $\operatorname{sign}(\sin\alpha) = \ \mp$ with

$$\left\langle x \middle| \begin{pmatrix} \pm\ (z+\sqrt{3}) \\ \pm\ \frac{1}{2}\sqrt{4-z^2} \end{pmatrix} \right\rangle = -1.$$

Then

$$\begin{aligned} \langle x|y\rangle &= \begin{pmatrix} \mp\ (2\cos\alpha - \sqrt{3}) \\ 2\sin\alpha \end{pmatrix} \begin{pmatrix} \pm\ (z+\sqrt{3}+a) \\ \pm\ \frac{1}{2}(\sqrt{4-z^2}+b) \end{pmatrix} \\ &= -1 \mp\ \pm\ (2\cos\alpha - \sqrt{3})a \pm\ \mp 2|\sin\alpha|b \\ &< -1. \end{aligned}$$

This shows that we can find an $x \in B$ for every $y \notin D$ such that $\langle x|y\rangle < -1$.
With 1. and 2. together we have shown, that

$$D = B^\circ.$$

Extreme Sets of B° Let B° be as described above. We have three different kinds of extreme sets:

$$\begin{aligned} E_1 &:= B^\circ \\ F_\pm &:= \left\{ \begin{pmatrix} z \\ \pm 1 \end{pmatrix} \middle| -\sqrt{3} \leq z \leq \sqrt{3} \right\} = \operatorname{conv}\left(\begin{pmatrix} \sqrt{3} \\ \pm 1 \end{pmatrix}, \begin{pmatrix} -\sqrt{3} \\ \pm 1 \end{pmatrix} \right) \\ E_s &:= \{s\} \text{ where } s = \begin{pmatrix} \pm\ (z+\sqrt{3}) \\ \pm\ \frac{1}{2}\sqrt{4-z^2} \end{pmatrix} \text{ for some } 0 \leq z \leq 2 \end{aligned}$$

Computation of the $f_{E,p}$ -Functions As in our examples before, the next step is to calculate the $f_{E,p}$-functions[5].
As we only need the proper extreme sets of B°, we don't have to make the calculation for E_1. So for $F_\pm$ let $p = (p_1, p_2) \in \mathbb{R}^2$.

$$\begin{aligned} \min\left\{ \langle(\sqrt{3}, \pm 1)|p\rangle, \langle(-\sqrt{3}, \pm 1)|p\rangle \right\} &= \min\left\{ \sqrt{3}p_1 \pm p_2, -\sqrt{3}p_1 \pm p_2 \right\} \\ &= \sqrt{3}\min\{p_1, -p_1\} \pm p_2 \\ &= -\sqrt{3}|p_1| \pm p_2 \end{aligned}$$

[5]Remember that $f_{E,p}(q) = I_E(q) + \langle q|p\rangle - \inf_{y\in E}\langle y|p\rangle$, where E is an extreme set of B° and $p \in V$.

and therefore by Lemma 5.0.6:

$$\begin{aligned} f_{F_\pm,p}(q) &= I_{F_\pm}(q) + \langle q|p\rangle - \min\left\{\langle(\sqrt{3},\pm 1)|p\rangle, \langle(-\sqrt{3},\pm 1)|p\rangle\right\} \\ &= I_{F_\pm}(q) + \langle q|p\rangle + \sqrt{3}|p_1| \mp p_2 \end{aligned}$$

The last type of extreme sets are extreme sets consisting of a single point. We already know the result for this case from Lemma 5.0.4:

$$f_{E_s,p}(q) = I_{E_s}(q).$$

The Legendre-Fenchel Transforms Using equation (5.4) we get for $y \in \mathbb{R}^2$

$$\begin{aligned} f^*_{F_\pm,p}(y) &= \max\left\{\langle(\sqrt{3},\pm 1)|y-p\rangle, \langle(-\sqrt{3},\pm 1)|y-p\rangle\right\} + \min\left\{\langle(\sqrt{3},\pm 1)|p\rangle, \langle(-\sqrt{3},\pm 1)|p\rangle\right\} \\ &= \max\left\{\sqrt{3}(y_1-p_1) \pm (y_2-p_2), -\sqrt{3}(y_1-p_1) \pm (y_2-p_2)\right\} - \sqrt{3}|p_1| \pm p_2 \\ &= \sqrt{3}|y_1-p_1| \pm y_2 \mp p_2 - \sqrt{3}|p_1| \pm p_2 \\ &= \sqrt{3}|y_1-p_1| - \sqrt{3}|p_1| \pm y_2 \end{aligned}$$

and by (4.2.5)

$$f^*_{E_s,p}(y) = \langle y|s\rangle.$$

The Geometrical Part The last step is to find sequences z_n in $\mathbb{R}^2$, such that

$$\psi_{z_n}(y) = \|z_n - y\|_B - \|z_n\|_B \longrightarrow f^*_{E,p}(y) \text{ as } n \to \infty.$$

Recall that the norm in our example is

$$\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\|_B = \sqrt{3}|a| + \sqrt{4a^2+b^2}$$

as calculated in (5.7). Let us first look at sequences following straight lines parallel to the axes. Let $p = (p_1, p_2)$ be a point in $\mathbb{R}^2$.
Consider the sequence $z_n = (1,0)\cdot n + (p_1,p_2) \longrightarrow (\infty, p_2)$ with $n \in \mathbb{N}$. Then

$$\begin{aligned} \psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\ &= \sqrt{3}|n-y_1| + \sqrt{4(n-y_1)^2 + (p_2-y_2)^2} - \sqrt{3}|n| - \sqrt{4n^2+p_2^2} \\ &\overset{n \gg y_1}{=} \sqrt{3}n - \sqrt{3}y_1 - \sqrt{3}n + 2|n-y_1|\sqrt{1 + \frac{(p_2-y_2)^2}{4(n-y_1)^2}} - 2n\sqrt{1+\frac{p_2^2}{4n^2}} \\ &= -\sqrt{3}y_1 + 2(n-y_1)\left[1 + \frac{(p_2-y_2)^2}{4(n-y_2)^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] - 2n\left[1 + \frac{p_2^2}{4n^2} + \mathcal{O}\left(\frac{1}{n^3}\right)\right] \\ &= -\sqrt{3}y_1 + 2(n-y_1) - 2n + \mathcal{O}\left(\frac{1}{n}\right) \\ &\longrightarrow -\sqrt{3}y_1 - 2y_1 = -\left\langle \begin{pmatrix} 2+\sqrt{3} \\ 0 \end{pmatrix} \middle| y \right\rangle \end{aligned}$$

If we go along the same line to the other direction, we have to be careful with the minus signs, and obtain after a similar calculation:

$$\psi_{z_{-n}}(y) = \|z_{-n} - y\|_B - \|z_{-n}\|_B$$

$$\begin{aligned}
&\overset{n\geq y_1}{=} \sqrt{3}n + \sqrt{3}y_1 - \sqrt{3}n + 2(n+y_1)\sqrt{1+\frac{(p_2-y_2)^2}{4(n+y_1)^2}} - 2n\sqrt{1+\frac{p_2^2}{4n^2}} \\
&= \sqrt{3}y_1 + 2(n+y_1) - 2n + \mathcal{O}\left(\frac{1}{n}\right) \\
&\longrightarrow \sqrt{3}y_1 + 2y_1 = -\left\langle \begin{pmatrix} -(2+\sqrt{3}) \\ 0 \end{pmatrix} \middle| y \right\rangle = f^*_{E_{s'},p} \qquad \text{for } s' = \begin{pmatrix} -2+\sqrt{3} \\ 0 \end{pmatrix}.
\end{aligned} \tag{5.13}$$

We now consider a sequence following a straight line parallel to the y-axis.
Let $z_n = (0,1)\cdot n + (p_1,p_2) \longrightarrow (p_1,\infty)$ and again let $p=(p_1,p_2)$ be any point. Then

$$\begin{aligned}
\psi_{z_n}(y) &= \left\| \begin{pmatrix} p_1 - y_1 \\ z_n - y_2 \end{pmatrix} \right\|_B - \left\| \begin{pmatrix} p_1 \\ z_n \end{pmatrix} \right\|_B \\
&= \sqrt{3}|p_1-y_1| + \sqrt{4(p_1-y_1)^2 + (z-n-y_2)^2} - \sqrt{3}|p_1| - \sqrt{4p_1^2 + z_n^2} \\
&= \sqrt{3}|p_1-y_1| - \sqrt{3}|p_1| + |z_n - y_2| \cdot \left[1 + \frac{4(p_1-y_1)^2}{(z_n-y_2)^2} + \mathcal{O}\left(\frac{1}{z_n^3}\right)\right] \\
&\quad - |z_n| \cdot \left[1 + \frac{4p_1^2}{z_n^2} + \mathcal{O}\left(\frac{1}{z_n^3}\right)\right] \\
&= \sqrt{3}|p_1-y_1| - \sqrt{3}|p_1| + z_n - y_2 - z_n + \mathcal{O}\left(\frac{1}{z_n}\right) \\
&\longrightarrow \sqrt{3}|p_1-y_1| - \sqrt{3}|p_1| - y_2 = f^*_{E_{2-},p}(y),
\end{aligned}$$

and similarly for a sequence in the other direction

$$\psi_{z_n}(y) \longrightarrow f^*_{E_{2+},p}(y).$$

The next sequence follows a straight line not parallel to one of the axes. Let therefore $z_n = (k,l)\cdot t + (p_1,p_2) \longrightarrow (\infty,\infty)$ as $t \longrightarrow \infty$ with $k,l\in\mathbb{R}, k,l>0$. Then by a long calculation, carried out in the appendix, we get

$$\begin{aligned}
\psi_{z_n}(y) &= \left\| \begin{pmatrix} kt + p_1 - y_1 \\ lt + p_2 - y_2 \end{pmatrix} \right\|_B - \left\| \begin{pmatrix} kt + p_1 \\ lt + p_2 \end{pmatrix} \right\|_B \\
&\longrightarrow \left(-\sqrt{3} - \frac{4k}{\sqrt{4k^2+l^2}}\right) y_1 - \frac{l}{\sqrt{4k^2+l^2}} y_2 = -\left\langle \underbrace{\begin{pmatrix} \sqrt{3} + \frac{4k}{\sqrt{4k^2+l^2}} \\ \frac{l}{\sqrt{4k^2+l^2}} \end{pmatrix}}_{:=q} \middle| y \right\rangle.
\end{aligned}$$

As

$$\partial B^\circ_{\text{elliptic}} = \left\{ \begin{pmatrix} \pm(z+\sqrt{3}) \\ \pm\frac{1}{2}\sqrt{4-z^2} \end{pmatrix} \middle| 0 \leq z \leq 2 \right\},$$

we see by choosing

$$z_q = \frac{4k}{\sqrt{4k^2+l^2}}$$

that $q \in \partial B^\circ_{\text{elliptic}}$. So for every sequence z_n following a line not parallel to the y-axis, ψ_{z_n} converges to the dual pairing of a point in the elliptic boundary part of B° with y.

Geometrical Construction There is an easy way to find the point q geometrically without any calculation.

Claim

We use the notations from above. For a sequence z_t following the straight line

$$h_t = \begin{pmatrix} k \\ l \end{pmatrix} \cdot t \text{ with } k, l \in \mathbb{R}^+,\ t \longrightarrow \infty,$$

let g be the line given by

$$g_t = \begin{pmatrix} l \\ -k \end{pmatrix} \cdot t + b,$$

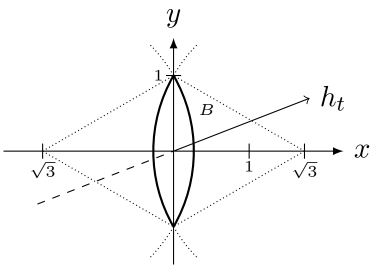
that is the vertical line to $-h$ with $b \in \mathbb{R}^2$ such that the image of g is a supporting hyperplane, namely a tangent line, to B°. Then

$$\psi_{z_t} \longrightarrow f^*_{E_s,p},$$

where

$$E_s = \{s\} = g_t \ \cap\ \partial B^\circ$$

is the point at which g touches B°. Here $s = -\begin{pmatrix} \sqrt{3} + \frac{4k}{\sqrt{4k^2+l^2}} \\ \frac{l}{\sqrt{4k^2+l^2}} \end{pmatrix}$.



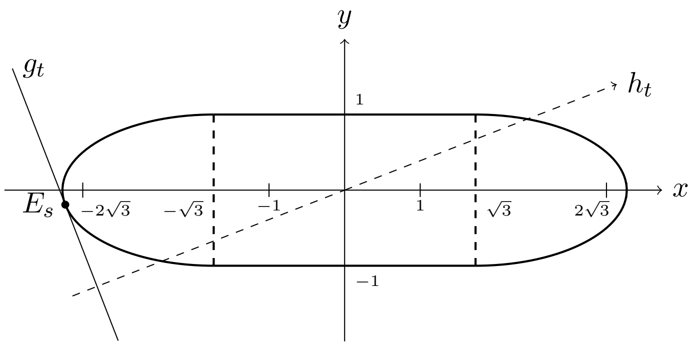


Figure 5.10: Constructing the hyperplane g_t perpendicular to h_t to find the extreme set E_s as the intersection of g_t with ∂B°

Proof of the claim

We already know that $\psi_{z_t} \longrightarrow f^*_{E_s,p}$. Let $q_{k,l} = g_t \cap \partial B^\circ$ be the intersection point of the vertical line g_t with the dual unit ball. It remains to prove that $s = q_{k,l}$. The line $g_t = \begin{pmatrix} l \\ -k \end{pmatrix} \cdot t$ has slope $-\frac{k}{l}$. g_t is tangent to $B^\circ_{\text{elliptic}}$ at $x \in \mathbb{R}^2$ if $f'(x) = -\frac{k}{l}$, where $f(x) = -\frac{1}{2}\sqrt{4 - x^2}$ describes the boundary of the ellipse in the section of the negative straight line $-h_t$ our sequence is following[6]. Hence

$$\begin{array}{rrcl} f'(x) & = \quad -\frac{-2x}{4\sqrt{4-x^2}} = \frac{x}{2\sqrt{4-x^2}} & \overset{!}{=} & -\frac{k}{l} \\ \Longleftrightarrow & lx & = & -2k\sqrt{4-x^2} \\ \Longleftrightarrow & l^2x^2 & = & 16k^2 - 4k^2x^2 \\ \Longleftrightarrow & (4k^2 + l^2)x^2 & = & 16k^2 \\ \Longleftrightarrow & x & = & \pm\frac{4k}{\sqrt{4k^2+l^2}} = \pm z_q. \end{array}$$

[6]In the strict sense we are considering the ellipse without the rectangle in the middle here, because it makes no difference in calculating the derivative and the slope. We must not forget to add $\pm\sqrt{3}$ to the x-value in the end.

Because of the direction of the line h_t we have to choose the minus sign in the last equation and get for the intersection point:

$$q_{k,l} = \begin{pmatrix} x - \sqrt{3} \\ -\frac{1}{2}\sqrt{4 - x^2} \end{pmatrix} = -\begin{pmatrix} \sqrt{3} + \frac{4k}{\sqrt{4k^2+l^2}} \\ \frac{l}{\sqrt{4k^2+l^2}} \end{pmatrix} = s.$$

A special case occurs when our sequence is following a straight line parallel to the x-axis, then $\psi_{z_t}(y) \longrightarrow \langle y \mid \pm(2+\sqrt{3},0)\rangle$ as calculated above. The points $\pm(2+\sqrt{3},0)$ are the only two tangent intersection points of vertical lines parallel to the y-axis with ∂B°.
A similar result holds for the one-dimensional extreme set. The only difference is that we have to consider the point p now. Remember that if we take a sequence parallel to the positive y-axis, $z_n = (p_1, n)$, we obtained $\psi_{z_n} \longrightarrow f^*_{E_{2-},p}$. If we take the vertical to the negative straight line, namely to the negative y-axis, such that it is tangent to B°, we get $g_t = (t, -1)$ and therefore

$$g_t \cap \partial B^\circ = E_{2-}.$$

We will come back to this in section 5.6.

5.5 Horofunctions $\neq$ Busemann Points

If we want to get an example[7] of a horofunction that is not a Busemann point, we have to go to the three dimensional space. There we define the norm

$$\|(x,y,z)\| := \max\left(|x| + |z|, \sqrt{x^2+y^2}\right).$$

Then the unit ball is

$$B = \left\{(x,y,z) \in \mathbb{R}^3 \middle| \, |x| + |z| \leq 1 \text{ and } x^2 + y^2 \leq 1\right\}.$$

As the dual unit ball B° is the polar of B and the polar of an intersection is the convex hull of the polars[8], the dual unit ball B° is the convex hull of the square with vertices $(\pm 1, 0, \pm 1)$ and the unit circle in the $x-y$-plane.

We define the sequence

$$p_n := \begin{pmatrix} \cos\frac{1}{n} \\ \sin\frac{1}{n} \\ 0 \end{pmatrix} \quad \text{for all } n \in \mathbb{N}.$$

Then every point $\{p_n\}$ is an extreme point of B°, but the limit $(1,0,0)$ for $n \longrightarrow \infty$ is not extreme because it can be written as

$$(1,0,0) = \frac{1}{2}(1,0,1) + \frac{1}{2}(1,0,-1)$$

with $(1,0,\pm 1) \in B^\circ$, see also Lemma 2.4.2 on page 10. So the set $\mathcal{E}$ of extreme sets of B° is not closed in the Painlevé-Kuratowski topology and from Theorem 4.0.32 we now know that there must be a horofunction that is not a Busemann point.
One example is the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}; \quad (x,y,z) \longmapsto -x$$

[7] Following [Wal07].
[8] See Lemma 2.5.14.

We first show that f is a horofunction:
For every $n \in \mathbb{N}$ the function $\|mp_n - \cdot\| - \|mp_n\|$ converges to the function

$$\begin{aligned}\xi_n : \mathbb{R}^3 &\longrightarrow \mathbb{R}\\ q &\longmapsto -\langle p_n|q\rangle = -q_1 \cos\frac{1}{n} - q_2 \sin\frac{1}{n}\end{aligned}$$

as $m \longrightarrow \infty$. Therefore ξ_n is a horofunction for every $n \in \mathbb{N}$. Furthermore $\xi_n \longrightarrow f$ as $n \longrightarrow \infty$, so f is also a horofunction.
To show that f is not a Busemann point, we use the criterion of Lemma 4.2.1. That means that we have to find two 1-Lipschitz functions each different from f whose minimum is f. The following functions $f_1, f_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}$ sufice:

$$f_1(x,y,z) := \begin{cases} -x+z & \text{if } z \geq 0 \\ -x & \text{if } z < 0 \end{cases}$$

and

$$f_2(x,y,z) := \begin{cases} -x & \text{if } z \geq 0 \\ -x-z & \text{if } z < 0 \end{cases}$$

Both functions are 1-Lipschitz and $f = \min(f_1, f_2)$. Thus f is a horofunction but not a Busemann point.

5.6 Deductions from the Examples

In this section we want to find a characterisation of the sequences in $\mathbb{R}^m$ defining the Busemann compactification of $\mathbb{R}^m$ equipped with a polyhedral norm.
In the following let B always be a convex polyhedral unit ball in $\mathbb{R}^m$ and B° its dual.
Before we come to the main result, we have to prove some useful lemmata.

Lemma 5.6.1 *For each $x \in \mathbb{R}^m$ there is a proper extreme set $E \subseteq B^\circ$ of the dual unit ball, such that*

$$\|x\|_B = |x|_E,$$

where $|x|_C = -\inf_{q\in C}\langle q|x\rangle$ for a convex set C (see (4.7) on page 26).

Proof. Let $x \in \mathbb{R}^m$ be arbitrary. Then there is a unique $\tilde{x} \in \partial B$ such that

$$x = k\tilde{x}$$

for some $k \geq 0$. Clearly $\|\tilde{x}\|_B = 1$ and therefore $\|x\|_B = k$. By construction of B° there is a $y \in \partial B^\circ$ such that $\langle y|\tilde{x}\rangle = -1$. This means that there is a minimal extreme set $E \subseteq \partial B^\circ$ containing y. Hence

$$\begin{aligned}\|x\|_B = k\|\tilde{x}\|_B &= -k \inf_{q\in B^\circ}\langle q|\tilde{x}\rangle \\ &= -k \inf_{q\in E}\langle q|\tilde{x}\rangle = -\inf_{q\in E}\langle q|x\rangle \\ &= |x|_E.\end{aligned}$$

□

The important point of this proof is to see how the extreme set E depends on the point x and which role the dual unit ball plays.

Let $F \subseteq B$ be an extreme set of B. As B is polyhedral, F is polyhedral too. We will denote by F° the extreme set of B° for which $\langle x|y\rangle = -1$ for all $x \in F^\circ$ and $y \in F$. Then F° is a polyhedral convex extreme set of B°.

Lemma 5.6.2 *Let F be a non-empty proper extreme set of B and $E := F^\circ$. Let $x \in \mathbb{R}^m$ be such that $\frac{x}{\|x\|_B} \in \operatorname{ri} F$. Let $E_1, \ldots, E_k$ be the vertices of the convex set $E \subseteq B^\circ$, that is extreme points of B°, $E_i =: \{e_i\}$ and let $F_i := E_i^\circ$ for all $i = 1, \ldots, k$. Let $p \in \mathbb{R}^m$ be small enough such that there is a $j \in \{1, \ldots, k\}$ with $\frac{x+p}{\|x+p\|_B} \in F_j$. Then*

$$|x + p|_E = |x + p|_{E_j}.$$

Remark 5.6.3 If $F \subseteq B$ is not a hyperplane, the F_i are facets of B with F in their relative boundary. If F is a hyperplane, E consists of a single point and $E = E_i$ for all i. In this case the lemma is trivial.

Proof of the lemma. Define the function $f : E \longrightarrow \mathbb{R}$ via $f(q) = \langle q|x+p\rangle$. As E is compact and f is affine, f takes its maximum and its minimum on the boundary of E[9]. As the boundary of E is the finite union of several polyhedral convex sets, we can conclude that f takes its minimum and maximum on the vertices $E_1, \ldots, E_j$ of E. Because of the duality $F_j = E_j^\circ = \{e_j\}^\circ$ and as $\frac{x+p}{\|x+p\|_B} \in F_j$ we know that

$$\left\langle e_j \,\middle|\, \frac{x+p}{\|x+p\|_B} \right\rangle = -1$$

and

$$\left\langle e_i \,\middle|\, \frac{x+p}{\|x+p\|_B} \right\rangle \geq -1 \ \forall i \neq j.$$

Therefore we have

$$\begin{aligned}
-|x+p|_E &= \inf_{q \in E} \langle q|x+p\rangle \\
&= \inf_{i=1,\ldots,k} \langle e_i|x+p\rangle \\
&= \|x+p\|_B \inf_{i=1,\ldots,k} \left\langle e_i \,\middle|\, \frac{x+p}{\|x+p\|_B} \right\rangle \\
&= \|x+p\|_B \left\langle e_j \,\middle|\, \frac{x+p}{\|x+p\|_B} \right\rangle \\
&= \langle e_j|x+p\rangle \\
&= \inf_{q \in E_j} \langle q|x+p\rangle \\
&= -|x+p|_{E_j}.
\end{aligned}$$

□

Lemma 5.6.4 *Let F be a proper extreme set of B and $x \in \mathbb{R}^m$ such that $\frac{x}{\|x\|_B} \in \operatorname{ri} F$. Let $E = F^\circ$. Then for all $p \in \mathbb{R}^m$*

$$|x+p|_E = \|x\|_B + |p|_E.$$

Proof. As $\frac{x}{\|x\|_B} \in F$, we know that for all $q \in E$ there holds $\langle q|\frac{x}{\|x\|_B}\rangle = -1$ and therefore $\langle q|x\rangle = -\|x\|_B$. With this we obtain

$$\begin{aligned}
|x+p|_E &= -\inf_{q \in E} \langle q|x+p\rangle \\
&= -\inf_{q \in E} [\langle q|x\rangle + \langle q|p\rangle] \\
&= \|x\|_B - \inf_{q \in E} \langle q|p\rangle \\
&= \|x\|_B + |p|_E.
\end{aligned}$$

□

[9] As f is continuous and E is compact, we can conclude that it takes its minimum and maximum on E. If they would lie only in the interior of E, the derivative would be 0 at that point. As f is affine, it would be constant in contradiction to the assumption that it takes its extrema not on the boundary.

We remind the reader of the following definition:

Definition Let $C \subseteq \mathbb{R}^m$ be a convex set. The smallest cone containing C was defined by

$$K_C := \{x \in \mathbb{R}^m \mid x = \alpha c \text{ for some } \alpha > 0, c \in C\}.$$

Definition 5.6.5 Let $F \subseteq \partial B$ be an extreme set of B. Let $V(F)$ be the subspace generated by the cone K_F, that is, the smallest subspace of $\mathbb{R}^m$ containing K_F and let $V(F)^\perp$ be its orthogonal complement with respect to the Euclidean scalar product on $\mathbb{R}^m$. Let Π_F denote the projection onto the subspace $V(F)$. Let $\mathcal{F}$ be the set of extreme sets of B.

Lemma 5.6.6 *Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$ with $\|y_n\|_B \longrightarrow \infty$ as $n \longrightarrow \infty$. Then y_n has a subsequence y_{n_k} which satisfies the following conditions:*
$\exists F \in \mathcal{F}, p \in V(F)^\perp$ such that:

(i) $\exists N \in \mathbb{N}\ \forall n_k \geq N : \Pi_F(y_{n_k}) \in K_F$.

(ii) $d\left(\Pi_F(y_{n_k}), \partial_{\mathrm{rel}} K_F\right) \longrightarrow \infty$ *as* $n \longrightarrow \infty$.

(iii) $\|y_{n_k} - \Pi_F(y_{n_k}) - p\|_B \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. [10] Every $y \in \mathbb{R}^m$ can be uniquely written as $y = y_F + y^F$ with $y_F \in V(F)$ and $y^F \in V(F)^\perp$. So a lower index denotes the projection onto the space and an upper index the projection onto the complementary space.
We will use an induction over the dimension m of the space $\mathbb{R}^m$. As the proof gets a bit complicated in the end, where we have to consider several projections of extreme sets, we give a picture (see figure 5.11 and 5.12) as an example to get an image for the procedure.

Let $m = 1$. Then we have two extreme sets F_1, F_2 of the unit ball, one on each side of the origin and $(y_n)_n$ has a subsequence $(y_{n_k})_{n_k}$ going to infinity in one of the two cones (that is in one of the two half-spaces), say in K_{F_1}. Then $\Pi_{F_1}(y_{n_k}) = y_{n_k} \in K_{F_1}$ which shows the first and third condition. As the relative boundary of K_{F_1} is the origin, we see that the second condition is also true.

Let now $m > 1$ and (y_n) a sequence in $\mathbb{R}^m$ with $\|y_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$. Then it has a subsequence $(y_{n_k})_{n_k}$ such that the sequence $\left(\frac{y_{n_k}}{\|y_{n_k}\|_B}\right)_{n_k}$ converges to some point $b \in B$. Let $F \in \mathcal{F}$ be the smallest extreme set such that $b \in F$ and write $y_{n_k} = y_{n_k,F} + y^F_{n_k}$. Then $y_{n_k,F} = \Pi_F(y_{n_k}) \in K_F$ for all n_k large enough and therefore (i) is satisfied. The second condition is also satisfied because $\frac{y_{n_k}}{\|y_{n_k}\|_B} \longrightarrow b \in F$ and if the distance $d(\Pi_F(y_{n_k}), \partial_{\mathrm{rel}} K_F)$ would not go to infinity, we would have found another $F \in \mathcal{F}$.
If condition (iii) is not already satisfied we have to distinguish two cases. Either the subsequence $(y^F_{n_k})_{n_k} \subseteq V(F)^\perp$ is bounded. Then it has a converging subsequence in $V(F)^\perp$ which fulfils the last condition.
Alternatively, if $(y^F_{n_k})$ is not bounded, we have to make use of the induction. Consider $W := V(F)^\perp$ as our new vector space equipped with the unit norm obtained by projecting those F_i having F in their relative boundary. Then the norm on W is also polyhedral and $\dim W < m$. Let a prime $'$ at some object denote the object projected onto $W = V(F)^\perp$

[10] We will follow the idea of the proof of Proposition 3.25 (2) of [GJT98] but with different notations. Our conditions are equivalent to those in [GJT98] where the polyhedral compactification of a flat is explained. The polyhedral compactification is isomorphic to the Busemann compactification, see [Bri06, Beh. 2.16].

(for example, $\mathcal{F}'$ denotes the set of extreme sets of B' in W). By induction we know that $(y^F_{n_k})_{n_k}$ has a subsequence $(y^F_{n_{k_j}})_{n_{k_j}}$ which satisfies all conditions for some $F'_1 \in \mathcal{F}'$ and $p_1 \in V(F'_1)^\perp$, the complement taken in W. For the corresponding sequence $(y_{n_{k_j}})_{n_{k_j}}$ in $\mathbb{R}^m$ with $y_{n_{k_j}} = y_{n_{k_j},F} + y^F_{n_{k_j}}$ we set for convenience $x_j := y_{n_{j_k}}$. Then $(x^F_j)_j = (y^F_{n_{k_j}})_{n_{k_j}}$ satisfies

(i') $(x^F_j)_{F'_1} = \Pi_{F'_1}(x^F_j) \in K_{F'_1}$

(ii') $d\left((x^F_j)_{F'_1}, \partial_{\text{rel}} K_{F'_1}\right) \longrightarrow \infty$

(iii') $\|x^F_j - (x^F_j)_{F'_1} - p_1\|_{B'} \longrightarrow 0$

Let $F_1 \in \mathcal{F}$ be such that $F \subseteq \partial_{\text{rel}} K_{F_1}$ and F'_1 the projection of F_1 onto W.
We will show, that x_j satisfies all conditions for the extreme set F_1.
We can split x_j in

$$\begin{aligned} x_j &= x_{j,F} + x^F_j \\ &= \underbrace{x_{j,F} + (x^F_j)_{F'_1}} + (x^F_j)^{F'_1} \\ &= \qquad x_{j,F_1} \qquad + \quad x^{F_1}_j \end{aligned}$$

because $(x^F_j)^{F'_1} = x^{F_1}_j$, where the orthogonal complement of $V(F_1)$ is taken in W. This follows from the fact that $F \subseteq F_1$ and $F'_1 \subseteq V(F)^\perp$. For the same reasons $x_{j,F} + (x^F_j)_{F'_1} = (x_{j,F})_{F_1} + (x^F_j)_{F_1} = (x_j)_{F_1} = x_{j,F_1}$ (see also figure 5.11 and 5.12 for an illustration).

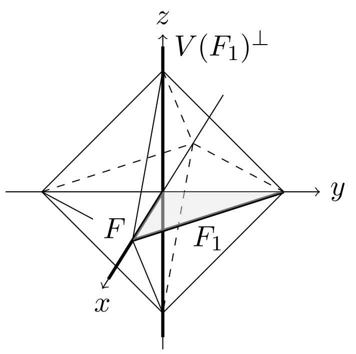


Figure 5.11: F, F_1 and $V(F_1)^\perp$

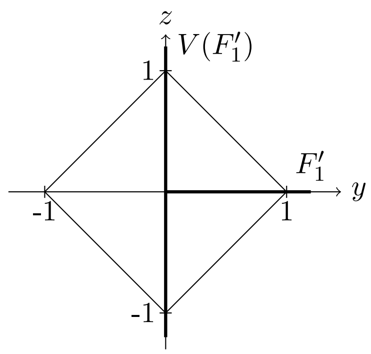


Figure 5.12: $V(F)^\perp$ with F'_1 and $V(F'_1)^\perp$

We will now show that (x_j) satisfies the conditions $(i) - (iii)$ with respect to F_1.

(i) We have to show that the projection of x_j on $V(F_1)$ lies in the cone K_{F_1}. We know that $x_{j,F_1} = x_{j,F} + (x^F_j)_{F_1}$ and because of the choice of the subsequence at the beginning, $x_{j,F} \in K_F$ and $K_F \subseteq \partial_{\text{rel}} K_{F_1}$. So x_{j,F_1} lies in one or more of the cones in $V(F_1)$ having F in their relative boundary. If the sequence did lie in another cone than K_{F_1}, the projection to W and $V(F'_1)$ would not lie in $K_{F'_1}$ which would be a contradiction to $(2')(i)$.

(ii) $d(x_{j,F_1}, \partial_{\text{rel}} K_{F_1}) \longrightarrow \infty$ because x_{j,F_1} lies in K_{F_1} and as $\frac{x_j}{\|x_j\|_B} \longrightarrow b \in F$, the sequences goes to infinite distance from all parts of the relative boundary of K_{F_1} aside from maybe those lying next to F. Projecting them to W they become part of the relative boundary of F'_1 so condition $(2')(ii)$ guarantees, that these distances are also unbounded.

(iii) It is

$$\|x_j - x_{j,F_1} - p_1\|_B = \|x_j^{F_1} - p_1\|_B = \|(x_j^F)^{F_1'} - p_1\|_{B'} \longrightarrow 0$$

by the last condition of x_j^F.

So we finally found a subsequence $(x_j)_j = (y_{n_{k_j}})_{n_{k_j}}$ of $(y_n)_n$ satisfying all conditions. $\square$

We are now approaching the main theorem of this section. We will use the notations of the chapters before and start with an overview of the maps needed.
Let

$$\begin{aligned} f_{E,p} :\ (\mathbb{R}^m)^* &\longrightarrow [0,\infty) \\ q &\longmapsto f_{E,p}(q) = I_E(q) + \langle q|p\rangle - \inf_{y\in E}\langle y|p\rangle \end{aligned}$$

be the first of our two considered functions, where $I_E(q)$ denotes the indicator function for the extreme set $E \subseteq B^\circ$ (see also (4.1) on page 21). We will need the Legendre-Fenchel transform of $f_{E,p}$. In (4.2.5) we already calculated that

$$\begin{aligned} f_{E,p}^*(y) &= \sup_{x\in(\mathbb{R}^m)^*} \left(\langle y|x\rangle - f_{E,p}(x)\right) \\ &= -\inf_{q\in E}\langle q|p-y\rangle + \inf_{q\in E}\langle q|p\rangle \\ &= |p-y|_E - |p|_E. \end{aligned}$$

The other map we need is

$$\begin{aligned} \psi_z : \mathbb{R}^m &\longrightarrow \mathbb{R}_{\geq 0} \\ y &\longmapsto \psi_z(y) = \|z-y\|_B - \|z\|_B \end{aligned}$$

for each $z \in \mathbb{R}^m$. By Lemma 4.1.2 we know that

$$\begin{aligned} \psi_z(y) &= \|z-y\|_B - \|z\|_B \\ &= -\inf_{q\in B^\circ}\langle q|z-y\rangle + \inf_{q\in B^\circ}\langle q|z\rangle \end{aligned}$$

Theorem 5.6.7 *Let $B \subseteq \mathbb{R}^m$ be a convex polyhedral unit ball and B° its dual. Let $\mathcal{F}$ denote the set of proper extreme sets of B and $\mathcal{E}$ those of B°. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^m$.*
Then $\psi_{z_n}(\cdot) = \|z_n - \cdot\|_B - \|z_n\|_B$ converges to the Busemann point $f_{E,p}^*$ with $p \in \mathbb{R}^m$ and $E \in \mathcal{E}$ if and only if the following conditions are satisfied:

(1) $\|z_n\|_B \longrightarrow \infty$ as $n \longrightarrow \infty$.

(2) $\exists F \in \mathcal{F}, p \in V(F)^\perp$ such that:

(i) $\exists N \in \mathbb{N}\ \forall n \geq N : \Pi_F(z_n) \in K_F$.

(ii) $d\left(\Pi_F(z_n), \partial_{rel}K_F\right) \longrightarrow \infty$ as $n \longrightarrow \infty$.

(iii) $\|z_n - \Pi_F(z_n) - p\|_B \longrightarrow 0$ as $n \longrightarrow \infty$.

If ψ_{z_n} converges, then $E = F^\circ$.

Proof.

"$\Longleftarrow$" We first show that ψ_{z_n} converges if the conditions are satisfied, so let $(z_n)_{n\in\mathbb{N}}$ be a sequences satisfying (1) and (2). Define the sequence

$$k_n := \Pi_F(z_n)$$

whose distance from the relative boundary of K_F goes to infinity by $2(ii)$. Let $y \in \mathbb{R}^m$ be arbitrary.

I. We first show that $\psi_{k_n+p} \longrightarrow f^*_{E,p}$ with $E = F^\circ$. Let n be large enough such that there are $F_i, F_j \in \mathcal{F}$ with $\dim F_i = \dim F_j = m-1$ satisfying

$$\frac{k_n + p - y}{\|k_n + p - y\|_B} \in F_i; \qquad \frac{k_n + p}{\|k_n + p\|_B} \in F_j. \tag{5.14}$$

As B is polyhedral, each extreme set of B lies in the relative boundary of an $m-1$-dimensional extreme set of B and if n is large enough then the above mentioned conditions can always be satisfied.
Let $E_i = F_i^\circ \in \mathcal{E}$ and $E_j = F_j^\circ \in \mathcal{E}$. Then

$$\begin{aligned}
\psi_{k_n+p}(y) &= \|k_n + p - y\|_B - \|k_n + p\|_B \\
&\stackrel{1}{=} |k_n + p - y|_{E_i} - |k_n + p|_{E_j} \\
&\stackrel{2}{=} |k_n + p - y|_E - |k_n + p|_E \\
&\stackrel{3}{=} \|k_n\|_B + |p - y|_E - \|k_n\|_B - |p|_E \\
&= |p - y|_E - |p|_E = f^*_{E,p}(y).
\end{aligned}$$

Step 1 follows by Lemma 5.6.1 and with equation (5.14). The second step is a consequence of Lemma 5.6.2 and the third one of Lemma 5.6.4. The sets F_i, F_j are chosen precisely such that all these lemmata can be applied.

II. We now show the statement for z_n where we will use (I.) and the fact that $\|z_n - k_n - p\|_B \longrightarrow 0$. Then we have

$$\begin{aligned}
(\psi_{z_n} - f^*_{E,p})(y) &= \|z_n - y\|_B - \|z_n\|_B - f^*_{E,p}(y) \\
&= \|z_n - k_n - p + k_n + p - y\|_B - \|z_n - k_n - p + k_n + p\|_B - f^*_{E,p}(y) \\
&\leq \|z_n - k_n - p\|_B + \|k_n + p - y\|_B + \|z_n - k_n - p\|_B - \|k_n + p\|_B - f^*_{E,p}(y) \\
&\longrightarrow 0
\end{aligned}$$

by the usual and the reverse triangle inequality. Similarly we get

$$\begin{aligned}
(\psi_{z_n} - f^*_{E,p})(y) &= \|z_n - k_n - p + k_n + p - y\|_B - \|z_n - k_n - p + k_n + p\|_B - f^*_{E,p}(y) \\
&\geq -\|z_n - k_n - p\|_B + \|k_n + p - y\|_B - \|z_n - k_n - p\|_B - \|k_n + p\|_B - f^*_{E,p}(y) \\
&\longrightarrow 0,
\end{aligned}$$

so we have shown that $\psi_{z_n}(y) \longrightarrow f^*_{E,p}(y)$. By section 3 (page 18) we know that pointwise convergence of ψ_{z_n} is equivalent to uniform convergence on bounded sets, which again is equivalent to uniform convergence on compact sets in $C(\mathbb{R}^m)$. Therefore $\psi_{z_n} \longrightarrow f^*_{E,p}$.

"$\Longrightarrow$" We have to show that every sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^m$ with $\psi_{z_n} \longrightarrow f^*_{E,p}$ for some $E \in \mathcal{E}$ and $p \in V(E^\circ)^\perp$ satisfies the conditions of the theorem. The proof is based on Lemma 5.6.6, where we have shown that every sequence converging to infinity has a subsequence fulfilling conditions $(2)(i) - (iii)$ for some $F \in \mathcal{F}$.
Let $(z_n)_n$ be a sequence with $\psi_{z_n} \longrightarrow f^*_{E,p}$ and let $F := E^\circ$. Assume a subsequence (z_{n_k}) of (z_n) satisfies all conditions for some $F_1 \in \mathcal{F}$ different from F. Then by the first part of the proof we would have $\psi_{z_{n_k}} \longrightarrow f^*_{E_1,p} \neq f^*_{E,p}$ as $E_1 \neq E$ which is a contradiction. Thus (z_n) has a subsequence satisfying all conditions with respect to F. We have to show that this subsequence can be chosen as the whole sequence.
If not the entire sequence (z_n) fulfils condition (1), there are two possibilities. If (z_n) converges to some point in x we get functions in the closure but not in the boundary of $\psi(X)$, that is, no Busemann points. The other possibility is that (z_n) has at least two subsequences converging to different Busemann points. In both versions ψ_{z_n} does not converge. Therefore we can conclude that (z_n) satisfies (1) and use Lemma 5.6.6.
If $(2)(i)$ is not fulfilled, we can find two different subsequences, the first one, (z_{n_i}), with $\Pi_f(z_{n_i}) \in K_F$ and another one, (z_{n_j}), with $\Pi_F(z_{n_j}) \notin K_F$ for all n_j. Then (n_j) and (z_{n_j}) are unbounded and there is a subsequence, also denoted by (z_{n_j}), such that there is an $n_j \in \mathbb{N}$ with $\Pi_{F_1}(z_{n_j}) \in K_{F_1}$ and $F \neq F_1$. (z_{n_j}) also satisfies the other conditions and thus $\psi_{z_{n_j}} \longrightarrow f^*_{E_1,p_1}$ where $E_1 := F_1^\circ$. Then ψ_{z_n} would have subsequences with different limits and would not converge any more.
If one of the other conditions is not fulfilled, the proof goes similarly. We always have a subsequence satisfying the conditions and one not satisfying it and get two different limits in the end. The last condition depends also on the point p. Therefore the difference between the Busemann points may occur because of different $p \in V(F)^\perp$, for example if we have parallel subsequences.
This shows that a sequence converging to some Busemann point fulfils all these conditions. $\square$

Remark 5.6.8

1. We saw that both sequences (k_n) in the cone K_F (generated by $F \in \mathcal{F}$) whose distance to $\partial_{\text{rel}} K_F$ is unbounded, and sequences (z_n) converging to such a sequence (k_n) converge to the same Busemann point. A parallel shift by a constant $p \in V(F)^\perp$ determines the point p of $f^*_{E,p}$. But the only property having influence on the proper extreme set $E \subseteq \partial B^\circ$ of $f^*_{E,p}$ is the direction of the sequence. This is the reason why we only considered sequences along straight lines in our examples. If z_n follows a straight line h, we can easily determine F: first shift h so that it passes through the origin. Then F is the smallest extreme set in which h intersects the boundary of the dual unit ball.

2. It is remarkable that the sequences $\psi_{k_n}(y)$ become constant if B is polyhedral, that is $\psi_{k_n}(y) = f^*_{E,p}(y)$ for n large enough (dependent on y). If B is not polyhedral but has a curved boundary, we do have sequences that do not become constant but converge to some $f^*_{E,p}$, see for example (5.13) on page 55. The reason is that we cannot find m-dimensional extreme sets $F_i, F_j \in \mathcal{F}$ such that equation (5.14) holds for all $n \geq N$ for some $N \in \mathbb{N}$.

Remark 5.6.9 We gave the proof here only for polyhedral norms B. We saw that the geometrical construction also works for the lens-shaped unit ball and the Euclidean metric. That is a reason why we only considered sequences following a straight line also in those examples. I suppose that a result similar to the theorem also holds for some of the non-polyhedral norms, especially if every horofunction is a Busemann point, and that it can be shown by using the continuity of the norm.

Examples to Illustrate the Conditions in Theorem 5.6.7

We will consider three examples of sequences not following a straight line to see what the conditions given in the theorem mean. Let $X = \mathbb{R}^2$ be equipped with the L^1-norm (see also section 5.1 on page 33). The unit ball and its dual with the notations of the extreme sets used in the following are illustrated in figure 5.13.

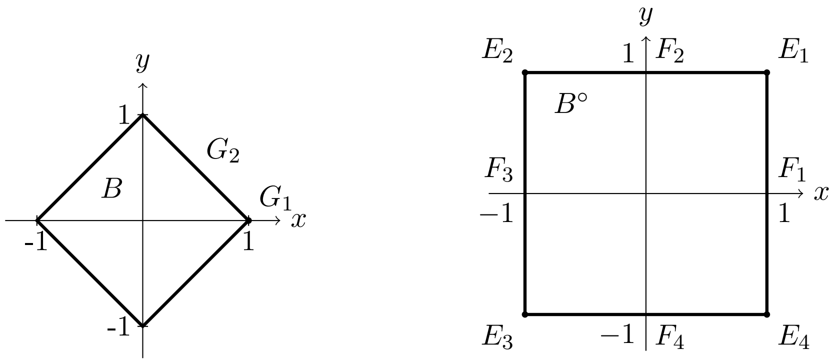


Figure 5.13: B and B° of the L^1-norm in $\mathbb{R}^2$

1) At first we consider the sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n = \left(n, \frac{1}{2}\sin(n) + 1\right)$. Then the first condition is readily verified. The other conditions require the choice of an extreme set $F \in \mathcal{F}$. The only reasonable choices are $F = G_1$, the extreme point, or $F = G_2$, the extreme facet. If $F = G_1$ then the third condition is not fulfilled, because $\|z_n - \Pi_{G_1}(z_n) - p\|_B = \left\| \begin{pmatrix} n \\ \frac{1}{2}\sin(n)+1 \end{pmatrix} - \begin{pmatrix} n \\ 0 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|_B$ does not go to 0 as $n \longrightarrow \infty$ for any $p \in \mathbb{R}^2$. If we take $F = G_2$, then the third condition is satisfied because $\Pi_{G_2}(z_n) = z_n$, but z_n remains in finite distance to $\partial_{\mathrm{rel}} K_{G_2}$ and therefore the second condition is not satisfied.Indeed, when we compute ψ_{z_n}, we get

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_1 - \|z_n\|_1 \\
&= \left\| \begin{pmatrix} n \\ \frac{1}{2}\sin(n)+1 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_1 - \left\| \begin{pmatrix} n \\ \frac{1}{2}\sin(n)+1 \end{pmatrix} \right\|_1 \\
&= |n - y_1| + \left|\frac{1}{2}\sin(n) + 1 - y_2\right| - |n| - \left|\frac{1}{2}\sin(n)+1\right| \\
&\overset{n \gg 0}{=} n - y_1 + \left|\frac{1}{2}\sin(n) + 1 - y_2\right| - n - \left|\frac{1}{2}\sin(n)+1\right| \\
&= -y_1 + \left|\frac{1}{2}\sin(n) + 1 - y_2\right| - \left|\frac{1}{2}\sin(n)+1\right|
\end{aligned}$$

which does not converge at all, in particular not to a Busemann point.

2) Let us consider the sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n = (n, \log(n))$. Then all conditions are fulfilled for $F = G_2$ and we have

$$\psi_{z_n}(y) = \|z_n - y\|_1 - \|z_n\|_1$$

$$\begin{aligned}
&= |n - y_1| + |\log(n) - y_2| - |n| - |\log(n)| \\
&\overset{n \gg 0}{=} n - y_1 + \log(n) - y_2 - n - \log(n) \\
&= -y_1 - y_2 = f^*_{E_3,p}(y)
\end{aligned}$$

as expected (with an arbitrary $p \in \mathbb{R}^2$). Rather remarkable is, that $\frac{z_n}{\|z_n\|_B}$ converges to G_1 while E_3 is the dual of G_2.

3) At last we consider $(z_n)_{n\in\mathbb{N}}$ with $z_n = (n, \frac{1}{n})$. Then z_n lies completely in K_{G_2} but converges to $K_{G_1} \subseteq \partial_{\mathrm{rel}} K_{G_2}$ which is the relevant information as we will see. If $F = G_2$, then z_n converges to the relative boundary of F and therefore the second condition is not fulfilled. If $F = G_1$ all conditions are satisfied and we suppose that $\psi_{z_n} \longrightarrow f^*_{F_3,p}$ with $p = 0$. Indeed

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\|_1 - \|z_n\|_1 \\
&= |n - y_1| + \left|\frac{1}{n} - y_2\right| - |n| - \left|\frac{1}{n}\right| \\
&\overset{n \gg 0}{=} n - y_1 + \left|y_2 - \frac{1}{n}\right| - n - \left|\frac{1}{n}\right| \\
&\overset{n \gg 0}{=} -y_1 + |y_2| = f^*_{F_3,0}(y)
\end{aligned}$$

for $y \in \mathbb{R}^2$.

The next question is how to determine E geometrically, once the direction of the sequence is fixed. We already saw one geometrical construction for the lens-shaped norm in section 5.4 on page 56.

Geometrical Construction

Based on the proof of the theorem above and the lemmata at the beginning of this section, there is an easy way to find the extreme set E needed for the horofunction $f^*_{E,p}$ by construction. We will again only consider sequences $(z_n)_{n\in\mathbb{N}}$ along straight lines. Let h be the straight line our sequence is following. Now draw (in the picture of B°) a hyperplane g [11] perpendicular to $-h$, such that it is a supporting hyperplane to B°. Then the extreme set, which is the intersection of this hyperplane and ∂B°, is our extreme set E (see also figure 5.14).

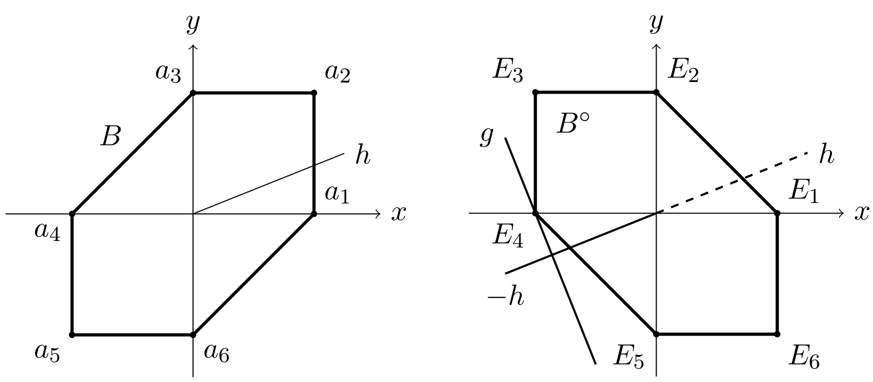


Figure 5.14: Finding the associated extreme set E_4 for h geometrically

[11] That is a straight line in $\mathbb{R}^2$.

If the line h is passing through the extreme set F of B, then we know by Lemma 5.6.1 that we have to find F°. The construction of the dual unit ball in section 2.5 tell us that we obtain exactly E by drawing that perpendicular hyperplane.
Now we can draw such an extreme set for every kind of sequence and we will get the picture of B° in the end. This is clear by comparing this construction with the way we constructed the dual unit ball.

Remark 5.6.10 Another way to construct the horoboundary geometrically is the "blow up and shift" technique, explained in [KMN06].

Remark 5.6.11 If we understand the p in $f_{E,p}$ as a coordinate of a point on E, we get a bijection between B° and the horoboundary.

6 Symmetric Spaces

6.1 Some Basic Facts and the Diffeomorphism $M \cong G/K$

We start with a definition:

Definition 6.1.1 A *symmetric space* M is a Riemannian manifold (M, g) such that for every point $p \in M$ there is an isometry

$$s_p : M \longrightarrow M$$

with

- $s_p(p) = p$
- $ds_{p|p} = -\operatorname{id}_{T_pM}$.

Remark 6.1.2 There holds $s_p^2 = \operatorname{id}_M$.

From now on let M denote a symmetric space if not stated otherwise.

Equipped with the compact-open-topology [1] $\operatorname{Isom}(M, g)$ becomes a locally compact topological group and acts as a transformation group on M[2]. Now fix a $\phi \in O_pM$, the orthogonal frame bundle at some $p \in M$. By the embedding

$$\begin{aligned} \operatorname{Isom}(M, g) &\longrightarrow OM \\ f &\longmapsto f_{*p}\phi \end{aligned}$$

of $\operatorname{Isom}(M, g)$ into the orthogonal frame bundle, $\operatorname{Isom}(M, g)$ gets a smooth structure independent of the choices of ϕ and p. This smooth structure is compatible with the group structure and thus

$$G := \operatorname{Isom}(M, g)$$

carries the structure of a Lie group (see also [Hel78, Ch. IV, Lem. 3.2]). We set

$$K := G_{p_0} = \{f \in G \mid f(p_0) = p_0\}$$

the stabiliser of some point $p_0 \in M$ in G.

Lemma 6.1.3 ([Hel78, Ch.IV, Thm. 2.5]) *With the notations above, K is a compact subgroup of G.*

A very important fact is that we can identify our symmetric space M with the space of left cosets G/K:

[1] This topology is generated by the open sets $W(U, C) := \{f \in \operatorname{Isom}(M, g) \mid f(C) \subseteq U\}$, where $U \subseteq M$ is open and $C \subseteq M$ is compact .

[2] A topological group G is called a transformation group of M, if there is group homomorphism $G \times M \longrightarrow M$ and if the actions of G on M is continuous.

Theorem 6.1.4 ([Hel78, Ch. IV, Thm. 3.3]) *Let M be a symmetric space. Then with the notations from above we have*

$$G/K \cong M$$

by the analytic diffeomorphism $gK \mapsto gp_0$.

Now we will assign a pair of Lie groups to each symmetric space and vice versa. This correspondence allows us to work with Lie groups and compact subgroups when talking about symmetric spaces. Therefore we need the following definition.

Definition 6.1.5 Let G be a connected Lie group and $H \leq G$ a closed subgroup. We call (G, H) a *symmetric pair*, if there is an involutive automorphism $\sigma : G \longrightarrow G$ such that

$$(G^\sigma)^\circ \subseteq H \subseteq G^\sigma,$$

where $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is the set of fixed points of G and $(G^\sigma)^\circ$ is the connected component of the identity.
If $Ad_G(H)$ [3] is compact, (G, H) is called a *Riemannian symmetric pair*.

This definition is motivated by the following theorem:

Theorem 6.1.6 ([Hel78, Ch. IV, Thm. 3.3]) *Let $M = G/K$ be a symmetric space with $G = \mathrm{Isom}(M, g)$ and $K = G_{p_0}$ for some $p_0 \in M$. Then the mapping*

$$\begin{aligned} \sigma_{p_0} : G &\longrightarrow G \\ g &\longmapsto s_{p_0} \circ g \circ s_{p_0} \end{aligned}$$

is an involutive automorphism of G such that

$$(G^{\sigma_{p_0}})^\circ \subseteq K \subseteq G^{\sigma_{p_0}}$$

Furthermore K has no normal subgroup of G apart from $\{id\}$.

Proposition 6.1.7 ([Ji05, Prop. 4.4]) *If G is a Lie group and K a compact subgroup, then the homogeneous space G/K admits a left G-invariant Riemannian metric.*

The action of G on M is given by the diffeomorphism

$$\begin{aligned} \tau(g) : G/K &\longrightarrow G/K \\ xK &\longmapsto gxK. \end{aligned}$$

The opposite of the proposition is also true as the following theorem shows.

Theorem 6.1.8 ([Hel78, Ch. IV, Prop. 3.4]) *Let (G, K) be a symmetric pair with involution σ and $\pi : G \longrightarrow G/K$ the usual projection. Denote $p_0 := \pi(e)$ the image of the identity element of G. Then with any G-invariant Riemannian metric h on G/K, the manifold G/K is a symmetric space and the geodesic symmetry s_{p_0} is independent of the choice of h and fulfils*

$$\begin{aligned} s_{p_0} \circ \pi &= \pi \circ \sigma \\ \tau(\sigma(g)) &= s_{p_0} \tau(g) s_{p_0}. \end{aligned}$$

[3] The adjoint map is defined in (6.4) on page 71.

As G is a Lie group, we can describe M not only in terms of Lie groups but also by the associated Lie algebras.
Let therefore σ_{p_0} be the involutive automorphism as in Theorem 6.1.6 and let $\mathfrak{g}$ be the Lie algebra of G. By the identification $\mathfrak{g} = T_e G$ we obtain the involution

$$\theta_{p_0} : \mathfrak{g} \longrightarrow \mathfrak{g};\ \theta_{p_0} = (d\sigma_{p_0})_e.$$

Then there holds $\sigma_{p_0}(e^{tX}) = e^{t\theta_{p_0}(X)}$, $\forall X \in \mathfrak{g}$.
As $\theta_{p_0}^2 = \mathrm{id}$, θ_{p_0} is diagonalisable and the only possible eigenspaces are those to the eigenvalues 1 and -1. Then[4] the Lie algebra of K is given by the positive eigenspace, namely

$$\mathfrak{k} := \mathcal{L}(K) = \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = X\}. \tag{6.1}$$

If we set

$$\mathfrak{p} := \{X \in \mathfrak{g} \mid \theta_{p_0}(X) = -X\} \tag{6.2}$$

we can write $\mathfrak{g}$ as the direct sum of vectorspaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

This decomposition is called the *Cartan decomposition* of $\mathfrak{g}$ with *Cartan involution* θ_{p_0}.
As θ_{p_0} preserves the Lie bracket, that is $\theta_{p_0}[X,Y] = [\theta_{p_0}(X), \theta_{p_0}(Y)]\ \forall X, Y \in \mathfrak{g}$, we have the following relations:

$$\begin{aligned}[]
[\mathfrak{k},\mathfrak{k}] &\subseteq \mathfrak{k} \\
[\mathfrak{k},\mathfrak{p}] &\subseteq \mathfrak{p} \\
[\mathfrak{p},\mathfrak{p}] &\subseteq \mathfrak{k}.
\end{aligned}$$

The usual projection coincides with

$$\begin{aligned}
\pi : G &\longrightarrow M \\
g &\longmapsto g.p_0,
\end{aligned}$$

the natural mapping induced by the action of G on M. Then by the differential $(\mathrm{d}\pi)_e : \mathfrak{k} \mapsto \{0\}$ we obtain the isomorphism

$$\mathfrak{p} \cong T_{p_0} M.$$

We now define some maps of G and $\mathfrak{g}$, we will need in the following chapters.

Definition 6.1.9

- For $h \in G$ we have the *conjugation*

$$\begin{aligned}
Int(h) :\ G &\longrightarrow G \\
g &\longmapsto hgh^{-1}.
\end{aligned} \tag{6.3}$$

- By taking the differential of the conjugation at the identity element e we get

$$\begin{aligned}
Ad : G &\longrightarrow Gl(\mathfrak{g}) \\
h &\longmapsto Ad(h)
\end{aligned} \tag{6.4}$$

  with

$$Ad(h) := dInt(h)_{|_e} :\ \mathfrak{g} \longrightarrow \mathfrak{g}.$$

[4]See [Hel78, Ch. IV, Thm. 3.3].

- Another differential leads to

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ X &\longmapsto ad(X) \end{aligned} \tag{6.5}$$

with

$$\begin{aligned} ad(X) : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\longmapsto ad(X)(Y) = [X, Y], \end{aligned} \tag{6.6}$$

where $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$.

- At last we have the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ defined by

$$\exp(X) := \gamma_X(1)$$

where $\gamma_X : \mathbb{R} \longrightarrow G$ is the unique analytic homomorphism such that $\dot{\gamma}_X(0) = X$ (see also [Hel78, Ch. II, Cor. 1.5]).

6.2 Root Space Decomposition of $\mathfrak{g}$

In this section we will follow [Ebe96, p.71ff].
As we are interested in compactifications of symmetric spaces, we will only consider symmetric spaces of non-compact type.

Definition 6.2.1 Let M be a symmetric space. M is called a symmetric space of *non-compact type*, if M is of non-positive sectional curvature, simply connected and not the Riemannian product of an Euclidean space $\mathbb{R}^k$, $k \geq 1$, and another manifold N.

Lemma 6.2.2 ([Ebe96, Prop. 2.1.1]) *Let $M = G/K$ be a symmetric space of non-compact type. Then G is a semisimple Lie group with trivial center.*

We already had the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ was the eigenspace of θ_{p_0} to -1 and $\mathfrak{p} \cong T_{p_0}M$. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra[5].

Lemma 6.2.3 ([Hel78, Ch. V, Lemma 6.3])

(i) All maximal abelian subalgebras of $\mathfrak{p}$ are conjugate to each other, that is for all $\mathfrak{a}, \mathfrak{a}' \subseteq \mathfrak{p}$ maximal abelian there is a $k \in K$ such that $Ad(k)\mathfrak{a} = \mathfrak{a}'$.

(ii) Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. Then $\mathfrak{p} = Ad(K)\mathfrak{a} = \bigcup_{k \in K} Ad(k)\mathfrak{a}$.

The Cartan decomposition is respected by adjunction, so for $g \in G$ and $q = g(p_0)$ the Cartan decomposition of $\mathfrak{g}$ with respect to q is given by $\mathfrak{g} = Ad(g)\mathfrak{k} + Ad(g)\mathfrak{p}$. It follows that the dimension of a maximal abelian subspace of $\mathfrak{p}$ is independent of our choice of $\mathfrak{a}$. So we will give it a name.

Definition 6.2.4 Let M be a symmetric space and $\mathfrak{p}$ as above. The *rank* of M is the dimension of some maximal abelian subspace of $\mathfrak{p}$.

Let $\kappa(X, Y) = tr(ad(X) \circ ad(Y))$ be the Killing form of $\mathfrak{g}$.

Lemma 6.2.5 ([Ebe96, p. 77]) *If M is of non-compact type, the Killing form satisfies*

[5] $\mathfrak{p}$ is not an algebra itself, so when we say that $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{p}$, we mean that it is a subspace of $\mathfrak{p}$ and a subalgebra of $\mathfrak{g}$. As $[\mathfrak{p}, \mathfrak{p}] \cap \mathfrak{p} = \{0\}$, a subalgebra of $\mathfrak{p}$ is automatically abelian.

(i) $\kappa_{|\mathfrak{k}}$ *is negative definite,*

(ii) $\kappa_{|\mathfrak{p}}$ *is positive definite,*

(iii) $\kappa(\mathfrak{k}, \mathfrak{p}) = 0$.

We set

$$Q(X, Y) := -\kappa(X, \theta_p(Y)).$$

Proposition 6.2.6 *Q is a positive definite bilinear form on $\mathfrak{g}$.*

Proof. It is obvious that Q is bilinear, the positive definiteness follows from the previous lemma. □

On $\mathfrak{p}$ we have $Q = \kappa$.

Lemma 6.2.7 ([Ebe96, 2.7.1])

(i) $Q(\theta_p(X), \theta_p(Y)) = Q(X, Y)\ \forall X, Y \in \mathfrak{g}$.

(ii) If $X \in \mathfrak{p}$, then the map $adX : \mathfrak{g} \longrightarrow \mathfrak{g}$ is symmetric with respect to Q[6].

(iii) $Ad(K)$ preserves Q on $\mathfrak{g}$ and $Ad(\phi) \circ \theta_p = \theta_p \circ\ Ad(\phi)\ \forall \phi \in K$.

Now let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Then by (ii) of the previous lemma, we know that the maps $ad(X)$ are symmetric with respect to κ for all X in $\mathfrak{a}$. Additionally $\mathfrak{a}$ is abelian and so we have

$$ad(X) \circ ad(Y) = ad(Y) \circ ad(X)\ \forall X, Y \in \mathfrak{a}.$$

By linear algebra, we know now that all maps $ad(X)$, $X \in \mathfrak{a}$, are simultaneously diagonalisable with a κ-orthogonal transformation and their eigenvalues depend on $X \in \mathfrak{a}$. So we define for each $\alpha \in \mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a}, \mathbb{R})$:

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H, X] = ad(H)X = \alpha(H)X\ \forall H \in \mathfrak{a}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, then $\alpha \in \mathfrak{a}^*$ is called a *root*. The set of all roots is denoted by Σ:

$$\begin{aligned} \Sigma &:= \{\alpha \in \mathfrak{a}^* \mid \alpha \text{ is a root }\} \\ &= \{\alpha : \mathfrak{a} \to \mathbb{R} \text{ linear} \mid \exists 0 \neq X \in \mathfrak{g} : ad(H)X = \alpha(H)X\ \forall H \in \mathfrak{a}\} \subseteq \mathfrak{g}^*. \end{aligned}$$

Σ is non-empty and furthermore we get the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \alpha} \mathfrak{g}_\alpha.$$

Remark 6.2.8 As $\mathfrak{a}$ is abelian, $\mathfrak{a} \subseteq \mathfrak{g}_0$.

For $\alpha \in \Sigma \subseteq \mathfrak{a}^*$ its kernel

$$\ker(\alpha) = \{H \in \mathfrak{a} \mid \alpha(H) = 0\},$$

is a hyperplane which divides the vector space $\mathfrak{a}$ into several connected components.

[6] That is $Q(\mathrm{ad}(X)Y, Z) = Q(Y, \mathrm{ad}(X)Z)\ \forall Y, Z \in \mathfrak{g}$.

Definition 6.2.9 The connected components of $\mathfrak{a}\backslash\bigcup_{\alpha\in\Sigma}\ker(\alpha)$ are called *Weyl chambers*. An element of a Weyl chamber, that is some $H \in \mathfrak{a}$ with $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, is called *regular*. Otherwise it is called *singular*.
Fix one Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Then a root α is called *positive* ($\alpha > 0$) if $\alpha(H) > 0$ for all $H \in \mathfrak{a}^+$. The set of positive roots is denoted by Σ^+.

Definition 6.2.10 A positive root $\alpha \in \Sigma^+$ is called *simple*, if α is not the sum of two positive roots. The set of simple roots is denoted by Δ.

We now come to the Weyl group. It acts on the Weyl chambers by permutation. For its definition we need the following two subgroups of K.

Definition 6.2.11 The *normaliser* of $\mathfrak{a}$ in K is defined as

$$N_K(\mathfrak{a}) := \{k \in K \mid Ad(k)\mathfrak{a} \subseteq \mathfrak{a}\}. \tag{6.7}$$

It contains the *centraliser*

$$C_K(\mathfrak{a}) := \{k \in K \mid Ad(k)H = H \ \forall H \in \mathfrak{a}\}$$

which is a normal subgroup of $N_K(\mathfrak{a})$.

Definition 6.2.12 The *Weyl group* is the quotient group

$$W := N_K(\mathfrak{a})/C_K(\mathfrak{a}).$$

Lemma 6.2.13 ([Ebe96, section 2.9]) *The Weyl group W satisfies:*

(i) W is discrete and finite.

(ii) W is generated by reflections at the hyperplanes $\ker(\alpha)$.

(iii) The action of W on the set of Weyl chambers of $\mathfrak{a}$ is simply transitive.

Let k be the rank of M. Corresponding to maximal abelian subalgebras $\mathfrak{a}$ of $\mathfrak{p}$ we have certain submanifolds of M, so-called k-flats:

Definition 6.2.14 A *k-flat* F in M is a complete, totally geodesic k-dimensional submanifold of M.
Totally geodesic means that every geodesic in F is also a geodesic in M.

Theorem 6.2.15 ([Ji05, Prop. 4.70]) *Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and $p_0 \in M$ a chosen basepoint. Let $A := \exp(\mathfrak{a})$ be the corresponding subgroup of G.*

(i) The orbit $F := A.p_0$ is a k-flat in M.

(ii) Any k-flat of M passing through the basepoint p_0 is of the form $F = \exp(\mathfrak{a})$ for some maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$.

There is a close relation between maximal abelian subalgebras and k-flats. Thus it is not surprising that k-flats are also conjugate to each other:

Theorem 6.2.16 ([Ebe96, Prop. 2.10, p.85]) *Let F_1 and F_2 be k-flats in M and $p_1 \in F_1$, $p_2 \in F_2$ points. Then there is a $g \in G$ such that $g(p_1) = p_2$ and $g(F_1) = F_2$.*

Lemma 6.2.17 ([Ji05, Lem. 4.80]) *With the notations as before, $A = \exp(\mathfrak{a})$ is a closed subgroups of G.*

Lemma 6.2.18 ([Hel78, Ch. V, Thm. 6.7 and Ch. IX, Thm. 1.1]) *Let $A = \exp(\mathfrak{a})$, $A^+ = \exp(\mathfrak{a}^+)$ the exponential of the positive Weyl chamber and $\overline{A^+}$ its closure in G. Then*

$$G = KAK.$$

Moreover, as the Weyl group acts simply transitive on the set of Weyl chambers and permutes them, we also have

$$G = K\overline{A^+}K.$$

That is, for every $g \in G$ there are $k_1, k_2, k_3, k_4 \in K$ and an $a \in A$ or an $a^+ \in \overline{A^+}$ such that $g = k_1 a k_2$ respectively $g = k_3 a^+ k_4$.
In the second case $\mathfrak{a}^+ \in \overline{A^+}$ is unique.

6.3 Finsler Geometry of Symmetric Spaces

Again Some Convex Analysis

We will follow [Pla95] and [Hol04] for an introduction on Finsler metrics on symmetric spaces.
We begin this section with a theorem of Kostant, followed by some convex analysis.

Theorem 6.3.1 (Kostant) *With the notations from above, let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace and $\pi : \mathfrak{p} \longrightarrow \mathfrak{a}$ the orthogonal projection with respect to the Killing form κ. Let W denote the Weyl group and let $\eta \in \mathfrak{a}$. Then*

$$\pi(Ad(K)\eta) = \operatorname{conv}(W\eta).$$

Let $B \subseteq \mathfrak{a}$ be convex and W-invariant. Set

$$C := Ad(K)B.$$

Proposition 6.3.2 *$C \subseteq \mathfrak{p}$ is convex.*

Proof. Let $\xi \in \operatorname{conv}(C)$ be a point. Then $\exists \xi_1, \dots, \xi_m \in C$ and $\exists t_1, \dots, t_m \in [0, 1]$ with $\sum_{i=1}^m t_i = 1$ such that

$$\xi = \sum_{i=1}^m t_i \xi_i.$$

By the definition of C, we can choose $k_i \in K, \eta_i \in B$ such that

$$\xi_i = Ad(k_i)\eta_i \quad \forall i = 1, \dots, m.$$

It is $\mathfrak{p} = Ad(K)\mathfrak{a}$ and therefore $\exists k \in K : Ad(k)\xi \in \mathfrak{a}$. So we have

$$\begin{aligned} Ad(k)\xi &= Ad(k)\sum_{i=1}^m t_i Ad(k_i)\eta_i \\ &= \sum_{i=1}^m t_i Ad(kk_i)\eta_i \in \mathfrak{a} \\ &= \sum_{i=1}^m t_i \pi(Ad(kk_i)\eta_i) \end{aligned}$$

because $\pi_{|\mathfrak{a}} = id$.
As $\eta_i \in B \subseteq \mathfrak{a}\ \forall i$ we can deduce with the Theorem of Kostant:

$$\pi(Ad(kk_i)\eta_i) \in \operatorname{conv}(W\eta_i) \quad \forall i.$$

From this it follows that

$$Ad(k)\xi = \sum_{i=1}^{m} t_i \underbrace{\pi(Ad(kk_i)\eta_i)}_{\in \mathrm{conv}(W\eta_i)\subseteq B} \in B$$

because B is assumed to be W-invariant and convex. There is a $b \in B$ with

$$Ad(k)\xi = b.$$

Applying $Ad(k^{-1})$ to this equation gives us $\xi = Ad(k^{-1})b \in Ad(K)B = C$. □

Corollary 6.3.3

(i) $C \cap \mathfrak{a} = B$ and $\pi(C) = B$

(ii) Let $B \subseteq \mathfrak{a}$ be a W-invariant convex ball, that is a neighbourhood of the origin. Then $C := Ad(K)B \subseteq \mathfrak{p}$ is also a convex ball, that is it contains the origin and is n-dimensional.

Proof.

(i) Let $\xi \in C \cap \mathfrak{a}$. Then $\pi(\xi) = \xi = Ad(k)b$ for some $k \in K, b \in B$ and so $\xi = \pi(\xi) = \pi(Ad(k)b) \in \mathrm{conv}(Wb) \subseteq B$ by the Theorem of Kostant and because B is W-invariant and convex.
$\pi(C) = B$ follows in the same way.

(ii) We already know that C is convex so it remains to show that C is a ball. As B is a W-invariant convex ball of $\mathfrak{a}$ it defines a norm on $\mathfrak{a}$. Let $p \in \mathfrak{p}$ be a point. By Theorem 6.2.3 there is a $k \in K$ and an $a \in \mathfrak{a}$ with $p = Ad(k)a$. Since B is the unit ball of a norm on $\mathfrak{a}$ there is a $b \in B$ and a $r \in \mathbb{R}$ such that $a = rb$. Together we have

$$x = Ad(k)rb = rAd(k)b$$

where $Ad(k)b \in C$. As p was arbitrary, C is a ball. □

Finsler Metrics

We provide two equivalent definitions for a Finsler structure on a smooth manifold. See for example [BCS00, 1.2.B] where the equivalence is shown.
The more elegant way is to define the Finsler metric like [Pla95] by a norm on each tangent space together with some conditions on the variation of the norm.

Definition 6.3.4 Let N be a differentiable manifold. $\|\cdot\| : TN \longrightarrow \mathbb{R}_{\geq 0}$ is called a *Finsler metric* on N, if for each vector field $X \in \mathcal{V}(N)$ the following map is continuous:

$$\begin{aligned} N &\longrightarrow \mathbb{R}_+ \\ p &\longmapsto \|X(p)\|_p. \end{aligned}$$

The norm $\|\cdot\|_p$ on T_pN does not have to be symmetric.

It is also common, and we will do so in the following, to choose differentiable variations like the smoothness of the above map on the slit tangent bundle $TN - 0 = \bigcup_{p\in N}(T_pN - 0)$.

The other definition following [Hol04] is not as beautiful but sometimes easier to deal with because it clearly lists the conditions on the metric.

Definition 6.3.5 Let N be a smooth manifold. A *Finsler metric* on N is a continuous function

$$F: \; TN \longrightarrow [0,\infty)$$

such that

(1) smoothness: F is smooth on $TN - 0$.

(2) homogeneity: $F(p, \lambda X) = \lambda F(p,X) \; \forall \lambda > 0$.

(3) strong convexity: let (x^i) be a chart of N centered at p such that $(x^i, X_j = \frac{\partial}{\partial x^j})$ is a chart of TN near $(p, 0)$. Then the matrix

$$g_{ij}(p,X) := \frac{\partial}{\partial X_i \partial X_j}\left(\frac{1}{2}F^2\right)(p,X)$$

is positive definite for all $(p,X) \in TN - 0$.

Lemma 6.3.6 ([Pla95, Ex. 6.1.2]) *Let N be homogeneous, that is, there is some topological group G which acts transitively on N by diffeomorphisms. Let $p_0 \in N$ be a point and $C \subseteq T_{p_0}N$ a convex G_{p_0}-invariant ball. Then there is exactly one G-invariant Finsler metric on N with C as unit ball of this norm $\|\cdot\|_{p_0}$.*

So the Finsler structure on a smooth manifold can be seen as a generalisation of a Riemannian metric. Instead of a scalar product on each tangent space, we now have a (not necessarily symmetric) norm.

In the same way as in Riemannian geometry, we can now define the *length of a curve* $\gamma : I \longrightarrow N$ by

$$L(\gamma) := \int_I F(\gamma(t), \dot{\gamma}(t))dt$$

and the *forward distance* between two points p and q by

$$d_F(p,q) := \inf_{\gamma \in \Omega_q^p} L(\gamma),$$

where Ω_q^p is the space of continuously differentiable paths $\gamma : I \longrightarrow N$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Is is important to distinguish between the forward and the backward distance of two points, because the homogeneity condition of the norms is only true for positive λ and hence in general $d_F(p,q) \neq d_F(q,p)$.

A ball with radius r around the point p

$$B_F(p;r) := \{q \in N \mid d_F(p,q) < r\}$$

is a differentiable subset of N with continuous boundary. The *unit ball* is

$$B_p^F := \{X \in T_pN \mid F(p,X) < 1\}, \tag{6.8}$$

defined separately in each tangent space. The *indicatrix* is defined to be

$$I_p^F := \{X \in T_pN \mid F(p,X) = 1\}. \tag{6.9}$$

Lemma 6.3.7 ([Run59, p. 11]) *The closure $\overline{B_p^F}$ of the unit ball is*

$$\{X \in T_pN \mid F(p,X) \leq 1\},$$

which is a convex body, and its boundary is given by the indicatrix.

A Useful Characterisation

Theorem 6.3.8 *Let M be a symmetric space of non-compact type and let $p_0 \in M$ be a point, $G = \mathrm{Isom}(M, g)$ be the group of isometries of M and $K = G_{p_0}$. Let $\mathfrak{p} \cong T_{p_0}M$ be defined as in section 6.2 and $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subspace. Let W be the Weyl group.*
Then there is a bijection between

(1) the W-invariant convex balls B of $\mathfrak{a}$

(2) the $Ad(K)$-invariant convex balls C of $\mathfrak{p}$ and

(3) the G-invariant Finsler metrices on M.

Proof.

(1) "⇔" (2) Let $B \subseteq \mathfrak{a}$ be a W-invariant convex ball. $Ad(K)B \subseteq \mathfrak{p}$ is an $Ad(K)$-invariant convex ball by Corollary 6.3.3.
Let $C \subseteq \mathfrak{p}$ be given. Then by the same corollary $C \cap \mathfrak{a}$ is a convex ball and as the action of $W = N_K(\mathfrak{a})/C_K(\mathfrak{a})$ on $\mathfrak{a}$ is induced by the action of K, we know that B is W-invariant. As the maps $B \mapsto Ad(k)B$ and $C \mapsto C \cap \mathfrak{a}$ are inverse to each other the equivalence is shown.

(2) "⇔" (3) Let $C \subseteq \mathfrak{p}$ be given. By Lemma 6.3.6 there is exactly one Finsler metric $\|\cdot\|$ on M, which is G-invariant such that C is the convex unit ball of $\|\cdot\|_{p_0}$. Conversely every Finsler metric has a uniquely defined unit ball C. □

Remark 6.3.9 If the rank of the symmetric space is one, that is $\dim \mathfrak{a} = 1$, then there is only one G-invariant Finsler-metric on M.

Lemma 6.3.10 *If we choose the Euclidean unit sphere with respect to the norm induced by the Killing form κ as the W-invariant convex ball in $\mathfrak{a}$, then the corresponding Finsler structure on G/K induces a Riemannian metric on $\mathfrak{g}$ for all $V, W \in T_{p_0}M$ by*

$$g_{p_0}(V, W) := \frac{1}{2}\left[F(p_0, V + W)^2 - F(p_0, V)^2 - F(p_0, W)^2\right].$$

The other way round we have

$$F(V) := \sqrt{g_{p_0}(V, V)}.$$

6.4 Horofunction Compactification of Symmetric Spaces

In this section we want to examine the Busemann compactification of a symmetric space equipped with a Finsler metric. We will need everything we did up to now and as usual we will use the notations introduced in the chapters before.
Let $M = G/K$ be a symmetric space, $G = \mathrm{Isom}^\circ(M, g)$ the identity component of its group of isometries[7] and $K = G_{p_0}$ for some base point $p_0 \in M$. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition, where $\mathfrak{p} \cong T_{p_0}M$ is the eigenspace of the Cartan involution θ_{p_0} to the eigenvalue -1. Let $\mathfrak{a} \subseteq \mathfrak{p}$ be a maximal abelian subalgebra

[7] Just like the entire $\mathrm{Isom}(M, g)$ also the identitiy component of the isometry group acts transitively on M (see [Ji05, Lem. 4.2]). So it suffices to use the identity component only.

and $A := \exp(\mathfrak{a})$ the connected subgroup of G.
The obvious action of G on $M = G/K$, given by left multiplication,

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, x) &\longmapsto g.x \end{aligned}$$

is transitive. Therefore

$$G.p_0 = \{g.p_0 \mid g \in G\} = M.$$

We can now use the decomposition $G = KAK$ (see Lemma 6.2.18) to see

$$G.p_0 = KAK.p_0 = KA.p_0 = M. \tag{6.10}$$

Let now H be a group acting on a metric space (X, d). Then H acts on $C(X)$ via

$$(h.f)(x) := f(h^{-1}.x)$$

for all $f \in C(X),\ h \in H$ and $x \in X$.
This is applicable to our symmetric space M and the group G. Consider the map ψ defined in (3.1) on page 17:

$$\begin{array}{llll} \psi: & M & \longrightarrow & C(M) \\ & z & \longmapsto & \psi_z \end{array} \quad \text{with} \quad \begin{array}{llll} \psi_z: & M & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \psi_z(x) = d(x, z) - d(p_0, z) \end{array} \tag{6.11}$$

Lemma 6.4.1 *ψ is K-equivariant:*

$$\psi_{k.z} = k.\psi_z \quad \forall k \in K, z \in M.$$

Proof. Let $z \in M, k \in K$ and $x \in M$. By Lemma 6.1.7 we know that the distance function d is left invariant and thus we obtain with $K = G_{p_0}$:

$$\begin{aligned} \psi_{k.z}(x) &= d(x, k.\ z) - d(p_0, k.\ z) \\ &= d(k^{-1}.x,\ z) - d(k^{-1}.p_0,\ z) \\ &= d(k^{-1}.x,\ z) - d(p_0,\ z) \\ &= \psi_z(k^{-1}.x) = k.\psi_z(x). \end{aligned}$$ □

The horofunction compactification was based on the embedding $M \hookrightarrow C(M)$ via[8]

$$\overline{\{\psi_z \mid z \in M\}} = \overline{\psi(M)} \subseteq C(M).$$

The horofunction boundary was defined by

$$\overline{\{\psi_z \mid z \in M\}} \setminus \{\psi_z \mid z \in M\}.$$

Lemma 6.4.2 *With the notations from above*

$$\overline{\psi(M)} = K.\overline{\psi(A.p_0)}.$$

[8]For better readability we will denote the closure of a space Y by $\overline{Y}$ from now on. If needed, the surrounding space is indicated.

Proof. G acts on M and therefore $g_1.(g_2.x) = (g_1 g_2).x \ \forall g_1, g_2 \in G$ and $x \in M$. Because $K = G_{p_0}$ is a subgroup, this also holds for elements of K and we conclude with Lemma 6.4.1

$$\psi(M) = \psi(K.A.p_0) = K.\psi(A.p_0)$$

We now have to show that K can be taken out of the closure:

$$\overline{K.\psi(A.p_0)} \overset{!}{=} K.\overline{\psi(A.p_0)}.$$

"$\subseteq$" Let $f \in \overline{K.\psi(A.p_0)}$. Then there is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq K.\psi(A.p_0)$ such that $f_n \longrightarrow f$, so there are $k_n \in K$, $a_n \in A$ for each $n \in \mathbb{N}$ with

$$f_n = k_n.\psi(a_n.p_0).$$

As K is compact, we can find a converging subsequence $(k_{n_j}) \subseteq (k_n)$ such that

$$k_{n_j} \longrightarrow k$$

for some $k \in K$. Define

$$g_{n_j} := \psi(a_{n_j}.p_0).$$

Again because of compactness (now of $\overline{\psi(M)}$) we take another subsequence, denoted by n again, such that $g_n \longrightarrow g \in \overline{\psi(M)}$ for some g. Then

$$\begin{aligned} f &= \lim_{n_j\to\infty} f_{n_j} = \lim_{n_j\to\infty} (k_{n_j}.\psi(a_{n_j}.p_0)) \\ &= \lim_{n\to\infty} (k_n.g_n) \\ &= k.g = k.\lim_{n\to\infty} \underbrace{\psi(a_n.p_0)}_{\in\overline{\psi(A.p_0)}} \in K.\overline{\psi(A.p_0)}. \end{aligned}$$

We can calculate the limits separately in the second line, because the action is continuous.

"$\supseteq$" Let $f \in K.\overline{\psi(A.p_0)}$, that is $\exists k \in K$ and a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \psi(A.p_0)$ such that

$$k.\lim_{n\to\infty} f_n = f.$$

This means that there is a sequence $(a_n)_n \subseteq A$ such that $f_n = \psi(a_n.p_0)$. Then $k.f_n = k.\psi(a_n.p_0) = \psi(ka_n.p_0)$ and therefore because of the continuity of the action

$$\begin{aligned} f &= k.\lim_{n\to\infty} f_n \\ &= \lim_{n\to\infty} k.f_n \\ &= lim_{n\to\infty}(k.\psi(a_n.p_0)) \in \overline{K.\psi(A.p_0)}. \end{aligned}$$ □

Now we would like to use our knowledge about horofunction compactifications of finite-dimensional normed spaces to compactify our symmetric space M. This will be done in the following way.
Take $M = G/K$ as before and find a maximal Abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$. Then $\mathfrak{a}$ is a finite-dimensional normed space. When M is endowed with a G-invariant Finsler metric, there is a unique W-invariant convex ball B of $\mathfrak{a}$ corresponding to our metric via Theorem 6.3.8. This ball is the unit ball of our norm induced on $\mathfrak{a}$ and we can find the compactification of $\mathfrak{a}$ with respect to B by determining B° and calculating the $f^*_{E,p}$-functions as detailed in Chapter 5.

Theorem 6.4.3 ([Pla95, Prop. 6.2.2]) *The flat $A.p_0$ equipped with the induced Finsler structure is a normed real space. If $\mathfrak{a}$ is equipped with the corresponding norm induced by the convex unit ball B, the restriction*

$$\exp_{p_0}|_{\mathfrak{a}} : \mathfrak{a} \longrightarrow A.p_0$$

is an isometry.

So we know that

$$\psi(\mathfrak{a}) \overset{homeo}{\cong} \mathfrak{a} \overset{isometry}{\cong} A.p_0 \overset{homeo}{\cong} \psi(A.p_0).$$

Therefore $C(\mathfrak{a})$ and $C(A.p_0)$ are homeomorphic and we obtain the homeomorphism

$$\overline{\psi(\mathfrak{a})}^{\,C(\mathfrak{a})} \simeq \overline{\psi(A.p_0)}^{\,C(A.p_0)},$$

where the second closure is taken in $C(A.p_0)$ but not in $C(M)$ as needed. But as each function f in $\psi(A.p_0)$ is a linear combination of the distance function d which is defined on M, f is also defined on M. Among the extensions to M there is a unique one lying in $\psi(M)$, namely the obvious one given by defining ψ on M. This can be seen by (8.2) on page 92. Thus

$$\overline{\psi(M)} \simeq K.\overline{\psi(A.p_0)}^{\,C(M)} \simeq K.\overline{\psi(A.p_0)}^{\,C(A.p_0)} \simeq K.\overline{\psi(\mathfrak{a})}^{\,C(\mathfrak{a})} \subseteq C(M).$$

Although we take the closure with respect to $C(A.p_0)$ or $C(\mathfrak{a})$, the action of K can only be understood by treating the closures as subsets of $C(M)$. The point is that there is no meaningful action of K on $\psi(\mathfrak{a})$ because normally (that is if $K \neq N_k(\mathfrak{a})$) there are $k \in K$ with $Ad(k^{-1}).H \notin \mathfrak{a}$ for some $H \in \mathfrak{a}$. But there is an action of $N_K(\mathfrak{a})$ on $\psi(\mathfrak{a})$ by definition of $N_K(\mathfrak{a})$, see also (6.7) on page 74. In fact $K.\overline{\psi(\mathfrak{a})}$ is homeomorphic to $\left(K \times \overline{\psi(\mathfrak{a})}\right)/N_K(\mathfrak{a})$.
All together we deduced the following theorem:

Theorem 6.4.4 *Let $M = G/K$ be a symmetric space of non-compact type equipped with a Finsler metric F. Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$. Then the horofunction compactification of M is homeomorphic to the orbit of the horofunction compactification of $\mathfrak{a}$ under K:*

$$\overline{\psi(M)} \simeq K.\overline{\psi(\mathfrak{a})}.$$

7 Examples: Horoboundaries of Symmetric Spaces

7.1 $X = SL(3,\mathbb{R})/SO(3)$

We will now examine the Busemann compactification of the symmetric space $M = SL(3,\mathbb{R})/SO(3)$ equipped with a Finsler metric. It is

$$G = SL(3,\mathbb{R})$$

and

$$K = SO(3) = \{A \in SL(3,\mathbb{R}) \mid A^T = A^{-1}\}.$$

The Lie algebra $\mathfrak{g}$ of G is

$$\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R}) = \{X \in \text{ Mat}(3,\mathbb{R}) \mid trX = 0\}.$$

With the Cartan involution

$$\begin{aligned} \theta_p : \mathfrak{g} &\longrightarrow \mathfrak{g}; \\ X &\longmapsto -X^T \end{aligned}$$

we get the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{g} \mid \theta_p(X) = X\} \\ &= \{X \in \mathfrak{g} \mid X^T = -X\} \\ &= \mathfrak{so}(3), \end{aligned}$$

is the Lie algebra of K, and

$$\begin{aligned} \mathfrak{p} &= \{X \in \mathfrak{g} \mid \theta_p(X) = -X\} \\ &= \{X \in \mathfrak{g} \mid X^T = X \text{ and } trX = 0\}. \end{aligned}$$

The set

$$\mathfrak{a} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in \text{ Mat}(3,\mathbb{R}) \middle| \sum_{i=1}^{3} \lambda_i = 0 \right\} \subseteq \mathfrak{p}$$

is a maximal Abelian subalgebra of $\mathfrak{p}$ of dimension 2. Therefore the rank of our symmetric space is 2.

Root space decomposition The next step is to determine the root spaces of $\mathfrak{g}$. We recall the definition of such a root space for some root $\alpha \in \mathfrak{a}^* = \text{Hom}(\mathfrak{a},\mathbb{R})$:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H,X] = ad(H)X = \alpha(H)X \ \forall H \in \mathfrak{a}\}.$$

Let therefore $H = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \in \mathfrak{a}$ and $X = (x_{ij})_{i,j} \in \mathrm{Mat}(3,\mathbb{R})$ some matrix with $\mathrm{tr}X = 0$. Then

$$\begin{aligned} ad(H)X &= [H,X] = HX - XH \\ &= \begin{pmatrix} \lambda_1 x_{11} & \lambda_1 x_{12} & \lambda_1 x_{13} \\ \lambda_2 x_{21} & \lambda_2 x_{22} & \lambda_2 x_{23} \\ \lambda_3 x_{31} & \lambda_3 x_{32} & \lambda_3 x_{33} \end{pmatrix} - \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} & \lambda_3 x_{13} \\ \lambda_1 x_{21} & \lambda_2 x_{22} & \lambda_3 x_{23} \\ \lambda_1 x_{31} & \lambda_2 x_{32} & \lambda_3 x_{33} \end{pmatrix} \\ &= \begin{pmatrix} 0 & (\lambda_1-\lambda_2)x_{12} & (\lambda_1-\lambda_3)x_{13} \\ (\lambda_2-\lambda_1)x_{21} & 0 & (\lambda_2-\lambda_3)x_{23} \\ (\lambda_3-\lambda_1)x_{31} & (\lambda_3-\lambda_2)x_{32} & 0 \end{pmatrix} \stackrel{!}{=} cX \end{aligned}$$

for some $c := \alpha(H) \in \mathbb{R}$.
To get a general result, we take a basis $\mathcal{B}$ of $\mathfrak{g}$ consisting of elemenatry matrices, namely $\mathcal{B} = \{E_{i,j} \mid i,j = 1,\dots,3\}$ where $E_{i,j}$ is given by $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$, the matrix whose only non-zero component is indicated by its name. Then

$$ad(H)E_{i,j} = (\lambda_i - \lambda_j)E_{i,j}.$$

Therefore it is obvious what the roots look like. We define $\alpha_{ij} \in \mathfrak{a}^*$ via

$$\alpha_{ij}\left(\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}\right) = \lambda_i - \lambda_j.$$

Then $ad(H)E_{i,j} = \alpha_{ij}E_{i,j}$ as required and for the root spaces we get

$$\mathfrak{g}_{\alpha_{ij}} = \mathbb{R}E_{i,j}$$

and

$$\mathfrak{g}_0 = \mathfrak{a}.$$

Therefore $\ker(\alpha_{ij}) = \{H \in \mathfrak{a} \mid \lambda_i = \lambda_j\}$ and with this

$$\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} ker(\alpha) = \{H \in \mathfrak{a} \mid \lambda_i \neq \lambda_j \ \forall i,j = 1,\dots,3\}.$$

We fix the Weyl chamber

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} \mid \lambda_1 > \lambda_2 > \lambda_3\}.$$

Then the positive roots are

$$\Sigma^+ = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}.$$

As $\alpha_{12} + \alpha_{23} = \alpha_{13}$, the two simple roots are

$$\Delta = \{\sigma_1 := \alpha_{12}, \sigma_2 := \alpha_{23}\}$$

and they form a basis of $\mathfrak{a}^*$. As the Weyl group consists of all reflections of the Weyl chambers, we can calculate the reflections of the simple roots, the norm of them and the angle between them as shown in [TY05, Chapter 18]. $\mathfrak{a}^*$ with the positive roots and the Weyl chambers of $\mathfrak{a}$ are illustrated in figure 7.1.

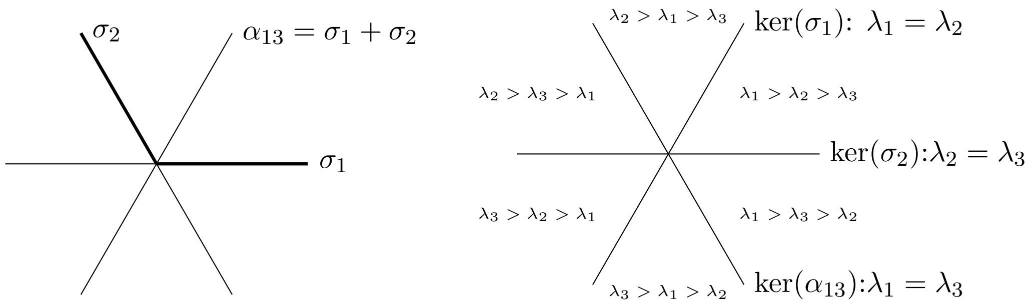


Figure 7.1: The root lattice of $\mathfrak{a}^*$ and the Weyl-Chamber system of $\mathfrak{a} \subseteq \mathfrak{sl}(3, \mathbb{R})$.

Compactification We know by Theorem 6.3.8 that every G-invariant Finsler structure on $SL(3, \mathbb{R})/SO(3)$ induces a W-invariant convex unit ball in $\mathfrak{a}$ and vice versa. W-invariant in this setting means, that the unit ball has to be symmetric with respect to the three hyperplanes. One possibility is to take the Finsler structure on M which is induced by the Riemannian metric on M, that induces the Killing form on $\mathfrak{a}$, which gives the Euclidean norm on $\mathfrak{a}$, which is clearly symmetric. If we start on $\mathfrak{a}$ there are three possibilities to choose a unit ball as simple as possible. These choices are to take the extremal points of the unit ball only on the Weyl chamber walls, that is the hyperplanes, whichare the kernels of the simple roots. The first option is to take six points on the walls such that their convex hull is hexagonal, for example take all points to have the same distance[1] from the origin and define the unit ball to be the convex hull of these points. The other two unit ball possibilities are to take equilateral triangles. Because of the action of the Weyl group W, if we fix one point on one of the hyperplanes, the other two points are determined in this case. So there are up to reparametrisation two possibilities for the triangulars, which are shown in figure 7.2.

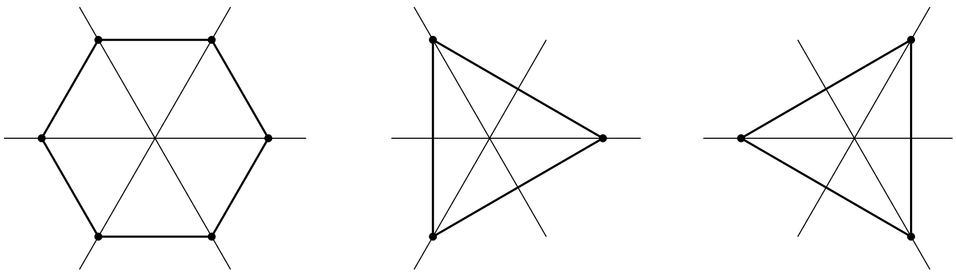


Figure 7.2: Hexagon or triangles as W-invariant norms on $\mathfrak{a}$.

As $\mathfrak{a}$ is a finite-dimensional normed space, we know by chapter 5.6 what the Busemann compactifications of these poytopes look like, namely they look like the dual unit ball of the norm. For the hexagonal norm it is a hexagon of the same shape rotated by 30° such that the facets are perpendicular to the hyperplanes. The dual unit ball of the triangle was already calculated in 5.3 and had the same shape as the unit ball, just reparametrised.

Thus all horofunction compactifications of $SL(3, \mathbb{R})/SO(3)$ have the structure $SO(3).\overline{\mathbb{R}^2}$ with some horofunction compactification of $\mathbb{R}^2$ compatible with Weyl-group invariance.

[1]This is the usual distance of $\mathbb{R}^2$.It coincides with the distance coming from the Killing form.

7.2 $X = Sp(4,\mathbb{R})/U(2)$

The next example is the Busemann compactification of the symmetric space

$$M := Sp(4,\mathbb{R})/U(2).$$

It is

$$G = Sp(4,\mathbb{R}) = \left\{ A \in \mathrm{Mat}(4,\mathbb{R}) \,\middle|\, A^T\Omega A = \Omega \text{ with } \Omega = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} \right\}$$

and[2]

$$K = U(2) = O(4) \cap GL(2,\mathbb{C}) \cap Sp(4,\mathbb{R})$$

where $GL(2,\mathbb{C}) = \{A \in GL(4,\mathbb{R}) \mid AJ = JA\}$ is the group of complex, invertible matrices corresponding to the complex structure

$$J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

The Lie algebra of G is

$$\begin{aligned} \mathfrak{sp}(4,\mathbb{R}) &= \{A \in \mathfrak{gl}(4,\mathbb{R}) \mid \Omega A + A^T\Omega = 0\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix} \in \ \mathrm{Mat}(4,\mathbb{R}) \,\middle|\, a,b,c \in \ \mathrm{Mat}(2,\mathbb{R});\ b^T = b;\ c^T = c \right\} \end{aligned}$$

and the Cartan involution is just as in the last example $\theta_p(A) = -A^T$. By some computation we get the following decomposition of $\mathfrak{sp}(4,\mathbb{R})$:

$$\begin{aligned} \mathfrak{k} &= \{A \in \mathfrak{sp}(4,\mathbb{R}) \mid A^T = -A\} \\ &= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \ \mathrm{Mat}(4,\mathbb{R}) \,\middle|\, a,b \in \ \mathrm{Mat}(2,\mathbb{R});\ a^T = -a;\ b^T = b \right\} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{p} &= \{A \in \mathfrak{sp}(4,\mathbb{R}) \mid A^T = A\} \\ &= \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \in \ \mathrm{Mat}(4,\mathbb{R}) \,\middle|\, a,b \in \ \mathrm{Mat}(2,\mathbb{R});\ a^T = a;\ b^T = b \right\}. \end{aligned}$$

A maximal Abelian subalgebra of $\mathfrak{p}$ is

$$\mathfrak{a} := \left\{ \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & -\lambda_1 & \\ & & & -\lambda_2 \end{pmatrix} \in \ \mathrm{Mat}(4,\mathbb{R}) \,\middle|\, \lambda_1, \lambda_2 \in \mathbb{R} \right\} \subseteq \mathfrak{p}.$$

[2]Actually K is the intersection of any two groups among these three.

Root space decomposition of $\mathfrak{g}$ We have to find roots $\alpha \in \mathfrak{a}^*$ and $X \in \mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}$. This means that we have to solve the eigenvalue problem for $ad(H)$. Consider the following basis of $\mathfrak{g}$:

$$\mathcal{B} := \{A_{ij} \mid i, j \leq 2\} \cup \{B_{ij} \mid i \leq j \leq 2\} \cup \{C_{ij} \mid i \leq j \leq 2\}$$

with

$$\begin{array}{llll} A_{ij} & = (a_{pq})_{p,q} & \text{and} & a_{pq} & = \delta_{pi}\delta_{qj} - \delta_{p,j+2}\delta_{q,i+2} \\ B_{ij} & = (b_{pq})_{p,q} & \text{and} & b_{pq} & = \delta_{pi}\delta_{q,j+2} + \delta_{pj}\delta_{q,i+2} - \delta_{ij} \\ C_{ij} & = (c_{pq})_{p,q} & \text{and} & c_{pq} & = \delta_{p,i+2}\delta_{qj} + \delta_{p,j+2}\delta_{qi} - \delta_{ij}. \end{array}$$

Then for some $H = \mathrm{diag}[\lambda_1, \lambda_2, -\lambda_1, -\lambda_2] \in \mathfrak{a}$ it is

$$\begin{aligned} [H, A_{ij}] &= (\lambda_i - \lambda_j)A_{ij} \\ [H, B_{ij}] &= (\lambda_i + \lambda_j)B_{ij} \\ [H, C_{ij}] &= -(\lambda_i + \lambda_j)C_{ij}. \end{aligned}$$

Therefore the roots are

$$\Sigma = \{\alpha_{ij} \mid , i \neq j; i, j \leq 2\} \cup \{\beta_{ij} \mid i \leq j \leq 2\} \cup \{\gamma_{ij} \mid i \leq j \leq 2\}$$

with

$$\begin{aligned} \alpha_{ij}(H) &= \lambda_i - \lambda_j \\ \beta_{ij}(H) &= \lambda_i + \lambda_j \\ \gamma_{ij}(H) &= -(\lambda_i + \lambda_j). \end{aligned}$$

For the Weyl chambers we get

$$\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} ker(\alpha) = \{H \in \mathfrak{a} \mid \lambda_1 \neq \pm\lambda_2; \lambda_1 \neq 0 \neq \lambda_2\}$$

and we fix the Weyl chamber

$$\mathfrak{a}^+ := \left\{ \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & -\lambda_1 & \\ & & & -\lambda_2 \end{pmatrix} \in \mathfrak{a} \,\middle|\, \lambda_1 > \lambda_2 > 0 \right\}.$$

The positive roots then are

$$\Sigma^+ = \{\alpha_{12}, \beta_{11}, \beta_{12}, \beta_{22}\}$$

and the simple ones are only

$$\Delta = \{\alpha := \alpha_{12}, \beta := \beta_{22}\}$$

because $\alpha_{12} + \beta_{22} = \beta_{12}$ and $2\alpha_{12} + \beta_{22} = \beta_{11}$.
Again by [TY05, Chapter 18] we conclude that $||\beta|| = \sqrt{2}||\alpha||$ and that the angle between these two roots is $135°$. So we get the following picture of $\mathfrak{a}$ and its Weyl chambers (see figure 7.3).

Compactification We did not choose a Finsler structure yet on $M = Sp(4, \mathbb{R})/U(2)$ and so we can start from the other side and fix a W-invariant norm on $\mathfrak{a}$ which stands in 1-to-1 correspondence to some G-invariant Finsler structure on M. If we want to take a polyhedral unit ball, there are again three simple choices where the extremal points lie only on the hyperplanes. These three possibilities are to take an octagon or one of two possible squares. These possibilities are illustrated in figure 7.4

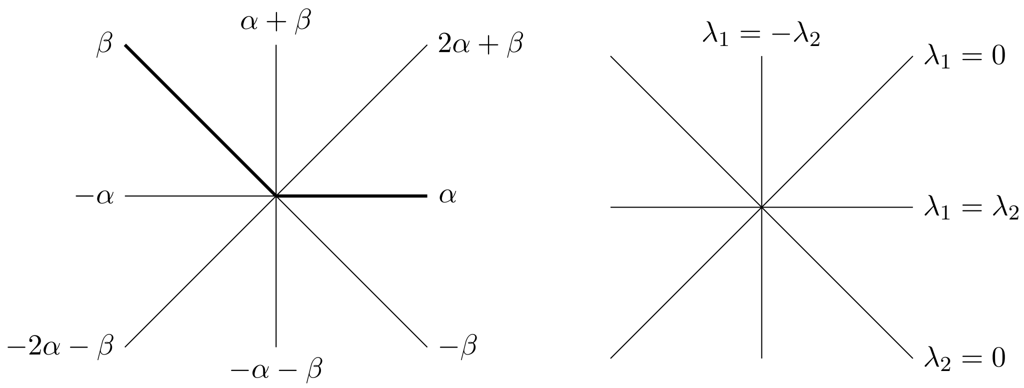


Figure 7.3: The root lattice of $\mathfrak{a}^*$ and the Weyl chamber system of $\mathfrak{a} \subseteq \mathfrak{sp}(4,\mathbb{R})$

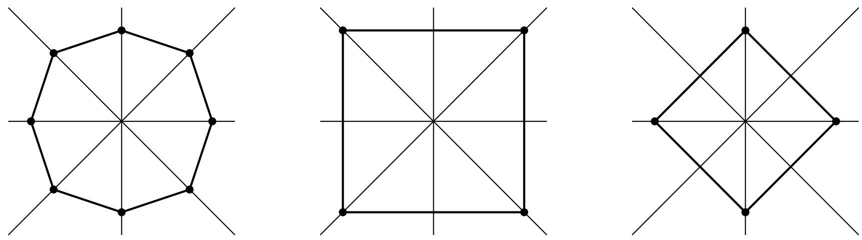


Figure 7.4: W-invariant polytopes as norms on $\mathfrak{a}$

Possible Further Projects: Comparison with the Furstenberg Compactification When we chose a Finsler norm on X by fixing the convex unit ball on $\mathfrak{a}$ in the examples above, we took polytopes with extreme points on the hyperplanes only. These hyperplanes were determined as those subspaces, on which one of the positive roots vanishes.

The Furstenberg compactification of a symmetric space $X = G/K$ is one of the many other ways to compactify X which are isomorphic in some cases. For detailed explanation and characterisation of the isomorphic compactifications, see [GJT98]. We will also follow [GJT98] now to introduce the Furstenberg compactification and to motivate the isomorphism between the Furstenberg and the horofunction compactification of the three symmetric spaces with Finsler metric determined by the choice of the three unit balls in figure 7.4.

Besides the Cartan decomposition $G = KAK$ as in Lemma 6.2.18, there is also the Iwasawa decomposition:

$$G = KAN \tag{7.1}$$

with $A = \exp(\mathfrak{a})$ and N the nilpotent Lie group with Lie algebra $\mathfrak{n} := \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$.
Additionally we have

Proposition 7.2.1 ([GJT98, Prop. 2.4]) *$P := MAN$ is a closed subgroup of G with Lie algebra $\mathfrak{g}_0 + \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ where $M := C_K(\mathfrak{a})$ is the centraliser of $\mathfrak{a}$ in K and $\mathfrak{m}$ the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$.*

Definition 7.2.2 We call a closed subgroup P' of G *parabolic*, if there is a $g \in G$ such that $gPg^{-1} \subseteq P'$. If $P \subseteq P'$, then P' is called *standard parabolic.*

The concept of Furstenberg was to use the natural affine action of G on the compact convex set of probability measures on the so-called Furstenberg boundary to define a compactification of X.

Definition 7.2.3 A compact homogeneous space Y of G is called a *boundary of* G, if there is a sequence (g_n) for every probability measure μ on Y such that $g_n \cdot \mu$ converges to a measure supported by a point.

Lemma 7.2.4 ([GJT98, Thm. 2.8]) *The standard parabolic subgroups of G are in one-to-one-corespondence with the subsets $I \subseteq \Delta$.*

Therefore we make the following definition.

Definition 7.2.5 Let P^I be the parabolic subgroup of G corresponding to $I \subseteq \Delta$.

Proposition 7.2.6 ([GJT98, Thm. 9.37]) *Let G be a semi-simple Lie group with finite center. Then $\mathcal{F} = G/P$ is a boundary. Furthermore, E is a boundary of G if and only if there exists a parabolic subgroup Q such that $E = G/Q$.*

Let $\mathcal{M}_1(G/P^I)$ denote the set of probability measures on the boundary $E = G/P^I$. With help of the Iwasawa decomposition (7.1) one can conclude that K acts transitively on G/P^I. So there is a unique K-invariant probability measure on G/P^I and we will denote this by $\overline{m}$. The map $\nu : G \longrightarrow \mathcal{M}_1(G/P^I); g \longmapsto g \cdot \overline{m}$ induces a continuous map

$$\begin{aligned} \varphi^I : G/K &\longrightarrow \mathcal{M}_1(G/P^I) \\ gK &\longmapsto g \cdot \overline{m} \end{aligned}$$

which is injective if and only if K is the stabiliser of $\overline{m}$ in G (see also [GJT98, p.68]). In this case, X can be identified, as a set, with the G-orbit of $\overline{m}$ under the map φ^I and we call such a boundary *faithful.*

There is a useful characterisation for such faithful boundaries:

Lemma 7.2.7 ([GJT98, Lem. 4.50]) *If G is simple, then G/P^I is a faithful boundary if and only if P^I is a proper parabolic subgroup of G, that is I is a proper subset of Δ.*

We are now prepared to define the Furstenberg compactification:

Definition 7.2.8 Let G/P^I be a faithful boundary. Then the closure of the image $\varphi^I(M)$ in $\mathcal{M}_1(G/P^I)$ is called a *Furstenberg compactification* of X.

We will illustrate now the isomorphism between the two compactifications by showing that each of the three convex balls determines uniquely a parabolic subgroup of G and vice versa.

Definition 7.2.9 Let I be a proper subset of Δ. The Weyl chamber face C_I is defined to be

$$C_I := \{H \in \overline{\mathfrak{a}^+} \mid \alpha_i(H) = 0 \text{ if and only if } \alpha_i \in I\}.$$

Considering picture 7.3, we see that the Weyl chamber faces are exactly the hyperplanes deviding $\mathfrak{a}$, that is the relative boundaries of a Weyl chamber.

Lemma 7.2.10 ([GJT98, Prop. 3.9]) *The standard parabolic subgroup P^I is the stabilizer of $[\gamma]$ if $\gamma(t) = exp(tH).p_0$, with $H \in C_I$ of unit length.*

The equivalence relation on the set of directed unit speed geodesics is given by

$$\gamma_1 \sim \gamma_2 :\Longleftrightarrow \lim_{t \to +\infty} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

Therefore two geodesics in $\mathbb{R}^2$ are equivalent if and only if they are parallel.

Example 7.2.11 In order to use the lemmata just presented, we have to complexify our Lie algebra and consider from now on $X = Sp(4,\mathbb{C})/SU(4)$, which has the same rootspace decompostion. So the simple roots are $\Delta = \{\alpha := \alpha_{12}, \beta := \beta_{22}\}$. $G = Sp(4,\mathbb{C})$ is simple Lie group and we know by Lemma 7.2.7 and Definition 7.2.8 that we have three compactifications of X, namely those corresponding to the proper subsets $I_0 := \emptyset$, $I_1 := \{\alpha\}$ and $I_3 := \{\beta\}$ of Δ.
In the first case when $I = \emptyset$, $C_I = \mathfrak{a}^+$ and P^I is the stabilizer of all sequences with H in $\mathfrak{a}^+$. That means that they all converge to the same Busemann point, in accordance to our result of section 5.6 and therefore this corresponds to the horofunction compactification of X with the Finsler metric induced by the octagon in figure 7.4.
In the second case when $I = \{\alpha\}$ we obtain the unit ball in the middle of picture 7.4, because C_I is the hyperplane where α vanishes, so all parallel sequences in that direction have to converge to the same boundary point. If $I = \{\beta\}$ we obtain the compactification of X with a Finsler metric determined by the square on the right in picture 7.3.

8 Appendix

8.1 Proof of Lemma 2.5.16

Proof of Lemma 2.5.16

Lemma *Let $B \subseteq \mathbb{R}^n$ be a polyhedral unit ball with k vertices $a_1, \dots, a_k$ and l $(n-1)$-dimensional facets. Let $b_1, \dots, b_l \in (\mathbb{R}^n)^*$ such that $B = \{x \in \mathbb{R}^n \mid \langle b_i | x \rangle \geq -1 \ \forall i = 1, \dots, l\}$ (as defined in Lemma 2.5.15). Then*

$$B^\circ = \operatorname{conv}\{b_1, \dots, b_l\} =: \tilde{B}.$$

Proof We recall the definition of the dual unit ball:

$$B^\circ = \{y \in (\mathbb{R}^n)^* \mid \langle y | x \rangle \geq -1 \ \forall x \in B\}.$$

"$\subseteq$" Let $y \in B^\circ$, that is $\langle y | x \rangle \geq -1 \ \forall x \in B$. We have to show that y can be written as a linear combination of the b_i with coefficients $s_i \in [0,1]$ and $\sum_{i=1}^{l} s_i = 1$.
Consider the straight line $g = \mathbb{R}_{\geq 0} y$ through the origin and y. Let $x := g \cap \partial \tilde{\mathrm{B}}$ be the intersection point of this line with the boundary of $\tilde{\mathrm{B}}$. As $\tilde{\mathrm{B}}$ is convex, x is unique. Then there is an $s \in \mathbb{R}_{\geq 0}$ with

$$y = sx.$$

We claim that $s \leq 1$. Indeed, assume $s > 1$. Then as $x \in \partial \tilde{\mathrm{B}}$, there are boundary points $\{b_{i_1}, \dots, b_{i_n}\} \subseteq \{b_1, \dots b_l\} \subseteq \partial \tilde{\mathrm{B}}$ all lying in one supporting hyperplane to $\tilde{\mathrm{B}}$ and there are $t_{i_1}, \dots, t_{i_n} \in [0,1]$ such that $\sum_{k=1}^{n} t_{i_k} = 1$ and

$$x = \sum_{k=1}^{n} t_{i_k} b_{i_k}.$$

Then there is an $m \in \{i_1, \dots, i_n\}$ such that a_m is the intersection point of the n hyperplanes defined by $b_{i_1}, \dots, b_{i_n}$. Then $\langle b_{i_k} | a_m \rangle = -1$ for all $k = 1, \dots, n$ by construction of the b_{i_k}. Therefore we get

$$\begin{aligned}
\langle y | a_m \rangle = \langle sx | a_m \rangle &= s \langle \sum_{k=1}^{n} t_{i_k} b_{i_k} | a_m \rangle \\
&= s \sum_{k=1}^{n} t_{i_k} \langle b_{i_k} | a_m \rangle \\
&= -s \sum_{k=1}^{n} t_{i_k} \\
&= -s < -1
\end{aligned}$$

which is a contradiction to $y \in B^\circ$.
So $s \leq 1$ and therefore $y \in \operatorname{conv}\{b_1, \dots, b_l\}$.

"$\supseteq$" Let $y \in \tilde{\mathrm{B}}$ =conv$\{b_1, \ldots, b_l\}$. Then there are $t_i \in [0,1]$ $(i = 1, \ldots, l)$ with $\sum_{i=1}^{l} t_i = 1$ such that

$$y = \sum_{i=1}^{l} t_i b_i.$$

Let $x \in B$ be arbitrary. Then

$$\begin{aligned} \langle y|x\rangle &= \sum_{i=1}^{j} t_i \langle b_i|x\rangle \\ &\geq -\sum_{i=1}^{l} t_i = -1. \end{aligned}$$

This shows that $\langle y|x\rangle \geq -1$ $\forall x \in B$, which means that $y \in B^\circ$.

8.2 Proof of Propositions 3.0.17, 3.0.26 and of Lemma 3.0.24

Proof of Proposition 3.0.17

Proposition *The map ψ is continuous and injective.*

Proof. We first show the continuity of ψ. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X converging to some $x \in X$. We show continuity of ψ by showing that $\psi_{x_n} \longrightarrow \psi_x$ uniformly on X and therefore also on bounded sets.
Let $\varepsilon > 0$ and $N \in \mathbb{N}$ large enough with $d_{sym}(x, x_n) \leq \varepsilon$ for all $n \geq N$. By the triangle inequality, we get

$$\begin{aligned} \psi_x(z) - \psi_{x_n}(z) &= d(z,x) - d(p_0,x) - d(z,x_n) + d(p_0,x_n) \\ &\leq d(z,x_n) + d(x_n,x) - d(p_0,x) - d(z,x_n) + d(p_0,x) + d(x,x_n) \\ &= d(x_n,x) + d(x,x_n) \\ &= d_{sym}(x,x_n) \end{aligned} \tag{8.1}$$

In the same way, we also get $\psi_x(z) - \psi_{x_n}(z) \geq -d_{sym}(x, x_n)$ and together

$$|\psi_x(z) - \psi_{x_n}(z)| \leq d_{sym}(x,x_n) \leq \varepsilon$$

independent of z. Therefore ψ_{x_n} converges uniformly on X from which continuity follows. Next we show that ψ is injective. Therefore let $x, y \in X$, $x \neq y$ and labelled such that $d(p_0, x) \geq d(p_0, y)$. Then

$$\begin{aligned} \psi_y(x) - \psi_x(x) &= d(x,y) - d(p_0,y) - d(x,x) + d(p_0,x) \\ &= \underbrace{d(p_0,x) - d(p_0,y)}_{\geq 0} + d(x,y) \\ &\geq d(x,y) \end{aligned} \tag{8.2}$$

So $\psi_y \neq \psi_x$ and ψ is injective. $\square$

Proof of Lemma 3.0.24

For this proof we need the famous Theorem of Ascoli-Arzelà:

Theorem of Ascoli-Arzelà *Let X be a locally compact*[1] *topological space and let Y be a metric space*[2]*. Then a subset $H \subseteq C(X,Y)$ is relatively compact if and only if H is equicontinuous and $H(x) \subseteq Y$ is relatively compact for all $x \in X$.*

Now we can prove the lemma:

Lemma *If our metric d_{sym} is proper, then $\mathrm{cl}\{\psi_z | z \in X\}$ is compact.*

Proof. In our case we have $Y = \mathbb{R}$, which is a metric space. As X is a metric space and d_{sym} is proper, any closed ball is a compact neighbourhood for its inner points, so X is locally compact. Let $H = \psi(X) \subseteq C(X, \mathbb{R}) = C(X)$. We have to show, that H is equicontinuous and $H(x) \subseteq \mathbb{R}$ is relatively compact. Then $\overline{H} = \mathrm{cl}\{\psi_z | z \in X\}$ is compact by the Theorem of Ascoli-Arzelà.
We first show the equicontinuity: Let $z \in X$ and $\psi_z = \psi(z) \in \psi(X)$. Then

$$|\psi_z(x) - \psi_z(y)| = |d(x,z) - d(p_0,z) - d(y,z) + d(p_0,z)| \leq d_{sym}(x,y) < \varepsilon$$

independently from z and equicontinuity follows.
Now we show, that $H(x)$ is relatively compact in $\mathbb{R}$ for all $x \in X$. Let $x, z \in X$. Then

$$\psi_z(x) = d(x,z) - d(p_0,z) \leq d(x,p_0) + d(p_0,z) - d(p_0,z) \leq d_{sym}(x,p_0)$$
$$\psi_z(x) = d(x,z) - d(p_0,z) \geq d(x,z) - d(p_0,x) - d(x,z) \geq -d_{sym}(x,p_0)$$

and therefore $|\psi_z(x)| \leq d_{sym}(x,p_0)$ for all $z \in X$ and so $H(x)$ is bounded. With the theorem of Heine-Borel we conclude, that $H(x)$ is relatively compact for every $x \in X$ and the assertion follows. $\square$

Proof of Proposition 3.0.26

Proposition *Assume (A), (B) and (C) (see page 18) hold. Then ψ is an embedding of X into $C(X)$. In other words, ψ is a homeomorphism from X to $\psi(X)$.*

Proof. In Proposition 3.0.17 we already showed that ψ is continuous and injective. It remains to show that ψ^{-1} is also continuous. This means, that we have to show that $\psi_{z_n} \longrightarrow \psi_y$ for some sequence (ψ_{z_n}) implies $z_n \longrightarrow y$.

We will show the assertion by contraposition: If $z_n \longrightarrow \infty$ [3], we show that there is no subsequence of (ψ_{z_n}) converging to some ψ_y with $y \in X$.

Therefore let (z_n) be a sequence in X with $z_n \longrightarrow \infty$. Without loss of generality, we assume that $\psi_{z_n} \overset{n\to\infty}{\longrightarrow} \xi \in \mathrm{cl}\{\psi_z | z \in X\}$.
Because the metric d_{sym} is proper, $d_{sym}(y, z_n) \overset{n\to\infty}{\longrightarrow} \infty$ for all $y \in X$.

Let $y \in X$ be arbitrary. We define the following geodesic segments with respect to the metric d:

$$\gamma_n : [0, b_n] \longrightarrow X$$

with

$$\gamma_n(0) = y, \text{ and } \gamma_n(b_n) = z_n \; \forall n \in \mathbb{N}.$$

[1] A space X is called locally compact, if each point of X has a compact neighbourhood.
[2] Or a uniform space in general.
[3] Which means that it eventually leaves and never returns to every compact set.

Choose and fix $r \in \mathbb{R}$ with

$$r > d(p_0, y) + \xi(y).$$

The reason for this choice will become clear later on.
From assumption (C) it follows, that the function

$$\begin{aligned} h : [0, b_n] &\longrightarrow \mathbb{R} \\ t &\longmapsto h(t) := d_{sym}(y, \gamma_n(t)) \end{aligned}$$

is continuous for every $n \in \mathbb{N}$.
Proof of this little assertion: Let $\varepsilon > 0$. Assumption (C) tells us that for all $\delta > 0$ there is an $M \in \mathbb{N}$ such that for all $m \geq M$ there holds

$$d(\gamma_n(t), \gamma_n(t_m)) < \delta \Longleftrightarrow d(\gamma_n(t_m), \gamma_n(t)) < \delta,$$

where t_m is a sequence in $\mathbb{R}$ converging to t.
Now let m be appropriate such that $d(\gamma_n(t), \gamma_n(t_m)) = t_m - t < \delta := \frac{\varepsilon}{2}$. Then

$$\begin{aligned} h(t_m) - h(t) &= d_{sym}(y, \gamma_n(t_m)) - d_{sym}(y, \gamma_n(t)) \\ &= d(y, \gamma_n(t_m)) + d(\gamma_n(t_m), y) - d(y, \gamma_n(t)) - d(\gamma_n(t), y) \\ &\leq d(y, \gamma_n(t)) + d(\gamma_n(t), \gamma_n(t_m)) + d(\gamma_n(t_m), \gamma_n(t)) + d(\gamma_n(t), y) \\ &\quad - d(y, \gamma_n(t)) - d(\gamma_n(t), y) \\ &= d(\gamma_n(t), \gamma_n(t_m)) + d(\gamma_n(t_m), \gamma_n(t)) \\ &< 2\delta = \varepsilon. \end{aligned}$$

In the same way, one can show that $h(t_m) - h(t) > -\varepsilon$. So together we have

$$|h(t_m) - h(t)| < \varepsilon$$

from which continuity of h follows.
Back to the main proof. It is

$$\begin{aligned} h(0) &= d_{sym}(y, \gamma_n(0)) = d_{sym}(y, y) = 0 \\ h(b_n) &= d_{sym}(y, \gamma_n(b_n)) = d_{sym}(y, z_n), \end{aligned}$$

so for n large enough we can find a $t_n \in \mathbb{R}_+$ with $d_{sym}(y, x_n) = r$ where $x_n := \gamma_n(t_n)$. By construction, all x_n lie in a closed ball of radius r around y. Since the metric d_{sym} is proper, this closed ball is also compact. Therefore there is an $x \in X$ such that $x_n \longrightarrow x$ if $n \longrightarrow \infty$ (if necessary we take a subsequence). In particular we also have $d_{sym}(y, x) = r$.

We come now to the last step of the proof. As γ_n was a geodesic segment with respect to d, there holds $d(y, x_n) + d(x_n, z_n) = d(y, z_n)$ and from this it follows

$$\begin{aligned} \psi_{z_n}(x_n) - \psi_{z_n}(y) &= d(x_n, z_n) - d(p_0, z_n) - d(y, z_n) + d(p_0, z_n) \\ &= d(x_n, z_n) - d(y, z_n) \\ &= -d(y, x_n). \end{aligned}$$

Written the other way round, it is $\psi_{z_n}(x_n) = \psi_{z_n}(y) - d(y, x_n)$ for every $n \in \mathbb{N}$. The ψ_{z_n} are 1-Lipschitz with respect to d_{sym} which yields $\psi_{z_n}(x_n) \longrightarrow \xi(x)$ and in the limit we obtain

$$\xi(x) = \xi(y) - d(y, x). \tag{8.3}$$

The essential step now is to proof, that $\psi_y \neq \xi$.
We know that $\psi_y(x) = d(x, y) - d(p_0, y)$. Together with the choice $r > d(p_0, y) + \xi(y)$ we get:

$$\begin{aligned}\psi_y(x) - \xi(x) &= d(x, y) - d(p_0, y) - \xi(y) + d(y, x) \\ &= d_{sym}(y, x) - d(p_0, y) - \xi(y) \\ &= r - d(p_0, y) - \xi(y) \\ &> 0\end{aligned}$$

As y was arbitrary, $\psi_y \neq \xi \ \forall y \in X$.

The conclusion of the proof is as following: Let q_n be a sequence in X such that $\psi_{q_n} \longrightarrow \psi_q$ in $\psi(X)$. As seen above, (q_n) can not have any subsequence escaping to infinity which means that (q_n) is bounded in the d_{sym}-metric. That is the sequence (q_n) remains in a closed and therefore compact ball, so we know that is has converging subsequences $(q_{n_k})_{n_k}$. As ψ is continuous and injective, we can conclude that the limit of each converging subsequence is q and therefore that $q_n \longrightarrow q$. □

8.3 Proof of Lemma 4.2.8 and of the Claim in Lemma 4.3.2

Proof of Lemma 4.2.8

For the proof of this lemma, we will need the following two properties of the epigraph topology:

(P1) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of proper lower semi-continuous functions $f_n : X \longrightarrow \mathbb{R} \cup \{\infty\}$ with $f_n \longrightarrow f$ in the epigraph topology and let each function take the value $+\infty$ outside a fixed bounded region common for all functions.
Then $\inf f_n \longrightarrow \inf f$. [Bee93, Lemma 7.5.3]

(P2) Let f_n and f be as above an let $g : X \longrightarrow \mathbb{R}$ be a real-valued lower semi-continuous convex function which is continuous at a point where f is finite. Then $f_n + g \longrightarrow f + g$. [Bee93, Lemma 7.4.5]

Lemma *Let $C \subseteq V^*$ be a compact convex set and F an exposed face of C. Suppose there exists a sequence $(p_n)_{n\in\mathbb{N}}$ in V and an $\varepsilon > 0$ such that*

(i) $\sum_{i=0}^{n-1} |p_{i+1} - p_i|_F \leq |p_n - p_0|_F + \varepsilon \quad \forall n \in \mathbb{N}$

(ii) $|p_n - \cdot|_F - |p_n|_F \overset{n\to\infty}{\longrightarrow} g$ *pointwise*

where g is a lower semi-continuous convex function.
Then there is a sequence $(q_n)_{n\in\mathbb{N}}$ in V and an $\varepsilon' > 0$ such that

(I) $\sum_{i=0}^{n-1} |q_{i+1} - q_i|_C \leq |q_n - q_0|_C + \varepsilon' \quad \forall n \in \mathbb{N}$

(II) $|q_n - \cdot|_C - |q_n|_C \overset{n\to\infty}{\longrightarrow} g$ *pointwise*

Proof. We first show (I).
There is an affine function[4] $f : V^* \longrightarrow \mathbb{R}$ with $\begin{cases} f(y) = 0 & \forall y \in F \\ f(x) > 0 & \forall x \in C \setminus F \end{cases}$ Let $\hat{f} := f - f(0)$

[4] As F is part of a hyperplane H, f can be constructed by setting $f(y) = \langle h|y\rangle - \alpha$ where h is the normal to H and $\alpha = \langle h|z\rangle \ \forall z \in H$.

be the linear functional on V^* with the same gradient. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of points in V such that

$$Z := \bigcup_{n\in\mathbb{N}} \{z_n\}$$

is dense in V and contains the origin.
<u>We claim</u>: It is possible to find a sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that if we define $q_n := p_n + \lambda_n \hat{f}$ $\forall n \in \mathbb{N}$, then there holds:

$$|q_{n+1} - q_n|_C - |q_{n+1} - q_n|_F < \frac{1}{2^n} \tag{8.4}$$

$$|q_n - z|_C - |q_n - z|_F < \frac{1}{n} \quad \forall z \in \{z_0, \ldots, z_n\}. \tag{8.5}$$

<u>Proof</u> of the claim: We show (8.5) first. Fix n, then we have to find $\lambda_n \in \mathbb{R}$. As for every $x \in V^*$ and $\lambda \in \mathbb{R}$ it is

$$\begin{aligned}
(\lambda f + I_C)(x) &= \lambda f(x) + I_C(x) \\
&= \begin{cases} \lambda f(x) & \text{if } x \in C \\ \infty & \text{if } x \notin C \end{cases} \\
&= \begin{cases} \lambda f(x) = 0 & \text{if } x \in F \subseteq C \\ \lambda f(x) > 0 & \text{if } x \in C \setminus F \\ \infty & \text{if } x \notin C \end{cases} \\
&\overset{\lambda\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } x \in F \\ \infty & \text{if } x \notin F \end{cases} = I_F(x),
\end{aligned}$$

we have in the epigraph topology

$$\lambda f + I_C \longrightarrow I_F \ (\lambda \longrightarrow \infty).$$

By the second property $(P2)$ we get $\lambda f + I_C + \langle \cdot | r \rangle \longrightarrow I_F + \langle \cdot | r \rangle$ $(\lambda \longrightarrow \infty)$ for every point $r \in V$. With the first one $(P1)$ we obtain

$$\inf(\lambda f - I_C + \langle \cdot | r \rangle) \longrightarrow \inf(I_F + \langle \cdot | r \rangle) \text{ as } \lambda \longrightarrow \infty.$$

It is

$$\inf(I_F + \langle \cdot | r \rangle) = \inf(\lambda f + I_F + \langle \cdot | r \rangle) \quad \forall \lambda \in \mathbb{R},$$

which follows from

$$\begin{aligned}
\inf_{x\in V^*}(\lambda f(x) + I_F(x) + \langle x | r \rangle) &= \inf_{x\in F}(\lambda f(x) + \langle x | r \rangle) \\
&= \inf_{x\in F}(\langle x | r \rangle) && \text{because } f(y) = 0 \ \forall y \in F \\
&= \inf_{x\in V^*}(\langle x | r \rangle + I_F(x)) && \forall \lambda \in \mathbb{R}, r \in V.
\end{aligned}$$

So together we get by the identification $\lambda \hat{f} \in V^{**} \cong V$

$$\begin{aligned}
\lim_{\lambda\to\infty}(|\lambda\hat{f} + r|_C - |\lambda\hat{f} + r|_F) &= \lim_{\lambda\to\infty}\left(-\inf_{x\in C}\langle x | \lambda\hat{f} + r \rangle + \inf_{y\in F}\langle y | \lambda\hat{f} + r \rangle\right) \\
&= \lim_{\lambda\to\infty}\left(-\inf_{x\in C}[\langle x | r \rangle + \lambda f(x) - \lambda f(0)] + \inf_{y\in F}[\langle y | r \rangle + \lambda \underbrace{f(y)}_{=0} - \lambda f(0)]\right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{\lambda\to\infty}\left(-\inf_{x\in V^*}[\langle x|r\rangle + \lambda f(x) + I_C(x)] + \inf_{y\in V^*}[\langle y|r\rangle + I_F(y)]\right)\\
&= \lim_{\lambda\to\infty}\left(-\inf_{x\in V^*}[\langle x|r\rangle + \lambda f(x) + I_C(x)]\right) + \inf_{y\in V^*}[\langle y|r\rangle + I_F(y)]\\
&= -\inf_{x\in V^*}[I_F(x) + \langle x|r\rangle] + \inf_{y\in V^*}[\langle y|r\rangle + I_F(y)]\\
&= 0
\end{aligned}$$

By using the equality

$$|q_n - z_i|_C - |q_n - z_i|_F = |\lambda_n^{(i)}\hat{f} + (p_n - z_i)|_C - |\lambda_n^{(i)}\hat{f} + (p_n - z_i)|_F \longrightarrow 0$$

as $\lambda_n^{(i)} \to \infty$ for each $n \in \mathbb{N}$, we see that for a fixed p_n and a $z_i \in \{z_0, \dots, z_n\}$ we can find a $\lambda_n^{(i)}$ large enough such that (8.5) is fulfilled. As the set $\{z_0, \dots, z_n\}$ is finite we can take the maximum and set $\lambda_n := \max_{i=0,\dots,n}\lambda_n^{(i)}$.

For equation (8.4), we have to choose λ_{n+1} large enough once λ_n is fixed.

Back to the original proof. With the identity $f(y) = 0 \ \forall y \in F$ we get

$$\begin{aligned}
|q_{i+1} - q_i|_F &= -\inf_{y\in F}\langle y|q_{i+1} - q_i\rangle\\
&= -\inf_{y\in F}\langle y|p_{i+1} + \lambda_{i+1}\hat{f} - p_i - \lambda_i\hat{f}\rangle\\
&= -\inf_{y\in F}[\langle y|p_{i+1} - p_i\rangle + \lambda_{i+1}f(y) - \lambda_{i+1}f(0) - \lambda_i f(y) + \lambda_i f(0)]\\
&= -\inf_{y\in F}\langle y|p_{i+1} - p_i\rangle + f(0)(\lambda_{i+1} - \lambda_i)\\
&= |p_{i+1} - p_i|_F + f(0)(\lambda_{i+1} - \lambda_i) \qquad (8.6)
\end{aligned}$$

and

$$\begin{aligned}
|q_n - q_0|_F &= -\inf_{y\in F}(\langle y|p_n + \lambda_n\hat{f} - p_0 - \lambda_0\hat{f}\rangle)\\
&= -\inf_{y\in F}[\langle y|p_n - p_0\rangle + \lambda_n f(y) - \lambda_n f(0) - \lambda_0 f(y) + \lambda_0 f(0)]\\
&= -\inf_{y\in F}\langle y|p_n - p_0\rangle + f(0)(\lambda_n - \lambda_i)\\
&= |p_n - p_0|_F + f(0)(\lambda_n - \lambda_0). \qquad (8.7)
\end{aligned}$$

Because of $F \subseteq C$, we have

$$|\cdot|_F \leq |\cdot|_C. \qquad (8.8)$$

Before we calculate the next step, let's bring together the equations we need for it.

(8.4) $|q_{n+1} - q_n|_C - |q_{n+1} - q_n|_F < \frac{1}{2^n} \ \forall n \in \mathbb{N}$

(8.8) $|q_n - q_0|_C \geq |q_n - q_0|_F \ \forall n \in \mathbb{N}$

(8.6) $|q_{i+1} - q_i|_F = f(0)(\lambda_{i+1} - \lambda_i) + |p_{i+1} - p_i|_F$

(8.7) $|q_n - q_0|_F = f(0)(\lambda_n - \lambda_0) - |p_n - p_0|_F$

Lemma 4.2.8(ii) $\sum_{i=0}^{n-1}|p_{i+1} - p_i|_F \leq |p_n - p_0|_F + \varepsilon \ \forall n \in \mathbb{N}$

And with these we get

$$\begin{aligned}
\sum_{i=0}^{n-1} |q_{i+1}-q_i|_C - |q_n-q_0|_C &< \sum_{i=0}^{n-1}(|q_{i+1}-q_i|_F + \frac{1}{2^n}) - |q_n-q_0|_F \\
&= \sum_{i=0}^{n-1} |q_{i+1}-q_i|_F - |q_n-q_0|_F + \sum_{i=0}^{n-1}\frac{1}{2^n} \\
&= \sum_{i=0}^{n-1}[f(0)(\lambda_{i+1}-\lambda_i) - |p_{i+1}-p_i|_F] - f(0)(\lambda_n-\lambda_0) - |p_n-p_0|_F + \sum_{i=0}^{n-1}\frac{1}{2^n} \\
&= \sum_{i=0}^{n-1} |p_{i+1}-p_i|_F - |p_n-p_0|_F + f(0)(\lambda_n-\lambda_0) - f(0)(\lambda_n-\lambda_0) + \sum_{i=0}^{n-1}\frac{1}{2^n} \\
&< \varepsilon + 2 =: \varepsilon'.
\end{aligned}$$

The assertion follows with the sequence $(q_n)_{n\in\mathbb{N}}$ and with $\varepsilon' := \varepsilon + 2$.

We will now proof (II).
Let $|p_n - \cdot|_F - |p_n|_F \longrightarrow g$ pointwise. We have to show that $|q_n - \cdot|_C - |q_n|_C \longrightarrow g$ pointwise with the sequence (q_n) as defined above.
Let $u \in Z = \bigcup_{n\in\mathbb{N}}\{z_n\} \subseteq V$. If n is large enough, then $0, u \in \{z_0, \dots, z_n\}$. With (8.5) and $|\cdot|_C \geq |\cdot|_F$ we obtain:

$$\begin{aligned}
|q_n-u|_C - |q_n|_C - |q_n-u|_F + |q_n|_F &\leq \frac{1}{n} - |q_n|_C + |q_n|_C \\
&= \frac{1}{n} \qquad (8.9)\\
|q_n-u|_C - |q_n|_C - |q_n-u|_F + |q_n|_F &\geq |q_n-u|_F - |q_n-u|_F - (|q_n-0|_C - |q_n-0|_F) \\
&\geq -\frac{1}{n} \qquad (8.10)
\end{aligned}$$

for every n large enough. Furthermore, we have:

$$\begin{aligned}
|q_n-u|_F - |q_n|_F &= -\inf_{y\in F}\langle y|p_n + \lambda_n\hat{f} - u\rangle + \inf_{x\in F}\langle x|p_n + \lambda_n\hat{f}\rangle \\
&= -\inf_{y\in F}[\langle y|p_n - u\rangle - \lambda_n f(y) - \lambda_n f(0)] + \inf_{x\in F}[\langle x|p\rangle + \lambda_n f(x) - \lambda_n f(0)] \\
&= -\inf_{y\in F}\langle y|p_n - u\rangle + \inf_{x\in F}\langle x|p_n\rangle \\
&= |p_n-u|_F - |p_n|_F \longrightarrow g(u) \text{ as } n \longrightarrow \infty. \qquad (8.11)
\end{aligned}$$

The equations (8.9), (8.10) and (8.11) together lead to

$$\begin{aligned}
|q_n-u|_C - |q_n|_C &\leq \frac{1}{n} + |q_n-u|_F - |q_n|_F \longrightarrow g(u) \\
|q_n-u|_C - |q_n|_C &\geq -\frac{1}{n} + |q_n-u|_F - |q_n|_F \longrightarrow g(u)
\end{aligned}$$

and therefore

$$|q_n-u|_C - |q_n|_C \longrightarrow g(u) \text{ as } n \longrightarrow \infty\ \forall u \in Z.$$

From Lemma 4.2.5 we know that $|q_n - \cdot|_C - |q_n|_C = f^*_{C,q_n}$. If we take the Legendre-Fenchel transform, we obtain

$$(|q_n - \cdot|_C - |q_n|_C)^* = f_{C,q_n} = I_C + \langle\cdot|q_n\rangle + |q_n|_C.$$

Since this function takes the value $+\infty$ everywhere outside a compact set, $|q_n - \cdot|_C - |q_n|_C$ is 1-Lipschitz with respect to any norm on V. Therefore pointwise convergence on a dense subset (here: Z) of V implies convergence everywhere, which proofs the assertion. □

Proof of the claim in the proof of Lemma 4.3.2

Claim
There is a sequence $(y_n)_{n\in\mathbb{N}}$ of points in B° and a constant $\delta > 0$ such that:

(i) $y_n \longrightarrow y$

(ii) $f_n(y_n) \longrightarrow f(y)$

(iii) $\forall n \in \mathbb{N}$ the point y_n is in some extreme set E_n and $|y_n - \partial_{\text{rel}} E_n| \geq \delta$

where $\partial_{\text{rel}} E_n$ denotes the relative boundary of E_n.

Proof. By [Bee93, Thm. 5.3.5][5] we know that we can find a sequence $(a_n)_{n\in\mathbb{N}}$ in B° with $a_n \longrightarrow y$ and $\lim_{n\to\infty} f_n(a_n) = f(y)$. For all $n \in \mathbb{N}$ let E_n be smallest with $a_n \in E_n$[6] and define

$$\delta_n := \inf_{\omega \in \partial_{\text{rel}} E_n} |a_n - \omega|,$$

the distance of a_n and $\partial_{\text{rel}} E_n$. Now set

$$\delta := \limsup_{n\longrightarrow\infty} \delta_n.$$

If $\delta > 0$, then there is a subsequence of (a_n) satisfying the claim.
Let $\delta = \limsup_{n\longrightarrow\infty} \delta_n = 0$. For each $n \in \mathbb{N}$ choose $b_n \in \partial_{\text{rel}} E_n$ such that

$$|a_n - b_n| = \delta_n.$$

Such a b_n exists because $\partial_{\text{rel}} E_n$ is closed. If $n \longrightarrow \infty$, then $|a_n - b_n| = \delta_n \longrightarrow 0$ and therefore

$$b_n \longrightarrow y \text{ as } n \longrightarrow \infty.$$

Define c_n as the point different from b_n, which lies on the intersection of $\partial_{\text{rel}} E_n$ and the straight line through b_n and a_n[7]. We now define the following sequence $(s_n)_{n\in\mathbb{N}}$ of points:

$$s_n := \begin{cases} b_n & \text{if } f_n(b_n) \leq f_n(a_n) + \sqrt{\delta_n}; \\ c_n & \text{otherwise.} \end{cases}$$

Let $(n_i)_{i\in\mathbb{N}}$ be a sequence of indices for which the first case holds and let $(m_i)_{i\in\mathbb{N}}$ be one for which the second case holds.
As $\limsup_{i\to\infty} f_{n_i}(b_{n_i}) \leq \limsup_{i\to\infty}(f_{n_i}(a_{n_i}) + \sqrt{\delta_{n_i}}) = f(y)$, we also have

$$\limsup_{i\to\infty} f_{n_i}(b_{n_i}) \leq f(y). \tag{8.12}$$

Using $\lim_{n\to\infty} b_n = y$ and the fact that $f_n \longrightarrow f$ in the epigraph topology we conclude by [Bee93, Thm. 5.3.5] (see footnote 5)

$$\liminf_{i\longrightarrow\infty} f_{n_i}(b_{n_i}) \geq f(y)$$

[5] Let X be a first countable Hausdorff space and let $f, f_1, f_2, \ldots$ be a sequence in $\mathrm{L}(X)$, the space of lower semi-continuous functions. Then $f_n \xrightarrow{K} f$ if and only if the following two conditions hold:

- $\exists (x_n) \longrightarrow x$ with $f(x) = \lim_{n\to\infty} f_n(x_n)$
- if $(x_n) \longrightarrow x$ we have $f(x) \leq \liminf_{n\to\infty} f_n(x_n)$.

[6] If E_n is not a singleton, then $a_n \notin \partial_{\text{rel}} E_n$.
[7] That means that c_n lies on the other side of a_n.

and together with (8.12) that

$$\lim_{i\to\infty} f_{n_i}(b_{n_i}) = f(y).$$

Now to the second case. As a_m, b_m and c_m lie on a line, there is $\lambda \in (0,1)$ such that $a_m = \lambda b + (1-\lambda)c_m$. So it is

$$c_m = \frac{1}{1-\lambda}(a_m - \lambda b_m)$$

from which we obtain

$$|c_m - a_m| = |\frac{1}{1-\lambda}a_m - \frac{\lambda}{1-\lambda}b_m - a_m| = \frac{\lambda}{1-\lambda}|a_m - b_m|.$$

As $f_m \in {}^*D$ we have $f_m(c_m) \geq 0$. A small calculation, using that f_m is affine of B° yields:

$$\begin{aligned} f_m(a_m) &\geq f_m(a_m) - f_m(c_m) \\ &= f_m(a_m) - \frac{1}{1-\lambda}f_m(a_m) + \frac{\lambda}{1-\lambda}f_m(b_m) \\ &= \frac{\lambda}{1-\lambda}(f_m(b_m) - f_m(a_m)) \end{aligned}$$

Together we obtain

$$\begin{aligned} \frac{f_m(a_m)}{|c_m - a_m|} &= \frac{f_m(a_m)}{\frac{\lambda}{1-\lambda}|a_m - b_m|} \\ &\geq \frac{\frac{\lambda}{1-\lambda}(f_m(b_m) - f_m(a_m))}{\frac{\lambda}{1-\lambda}|a_m - b_m|} \\ &= \frac{f_m(b_m) - f_m(a_m)}{|a_m - b_m|} \\ &> \frac{\sqrt{\delta_m}}{\delta_m} = \frac{1}{\sqrt{\delta_m}}, \end{aligned}$$

where the last inequality follows from the fact that we are in the second case (that is $f_m(b_m) - f_m(a_m) > \sqrt{\delta_m}$) and from $|a_m - b_m| = \delta_m$ by definition. We know that $\lim_{n\to\infty} f_m(a_m) = f(y)$ and that $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$. These two equations lead to $\lim_{i\to\infty} |a_{m_i} - c_{m_i}| = 0$ from which we get

$$\lim_{i\to\infty} c_{m_i} = \lim_{i\to\infty} a_{m_i} = y.$$

So

$$\liminf_{i\longrightarrow\infty} f_{m_i}(c_{m_i}) \geq f(y).$$

Because $f_{m_i}(b_{m_i}) > f_{m_i}(a_{m_i})$ for all $i \in \mathbb{N}$, we have

$$\begin{aligned} f_{m_i}(c_{m_i}) &= \frac{1}{1-\lambda}f_{m_i}(a_{m_i}) - \frac{\lambda}{1-\lambda}f_{m_i}(b_{m_i}) \\ &\leq \frac{1}{1-\lambda}f_{m_i}(a_{m_i}) - \frac{\lambda}{1-\lambda}f_{m_i}(a_{m_i}) \\ &= f_{m_i}(a_{m_i}) \end{aligned}$$

and therefore

$$\limsup_{i\longrightarrow\infty} f_{m_i}(c_{m_i}) \leq f(y).$$

All together we obtain

$$f_m(c_m) \longrightarrow f(y)$$

and

$$s_n \longrightarrow y \text{ as well as } f_n(s_n) \longrightarrow f(y) \text{ as } n \longrightarrow \infty.$$

As $s_n \in \partial_{\mathrm{rel}} E_n$ for all $n \in \mathbb{N}$, unless E_n is a singleton, the smallest extreme set containing s_n has dimension less than the one of E_n. If we repeat this procedure of constructing (s_n) several times, we obtain a sequence $(y_n)_{n\in\mathbb{N}}$ either consisting entirely of extreme points or containing a subsequence satisfying the claim. If the former would hold, then $\{y\}$ would be a limit point of the extreme sets $\{y_n\}$. So by our assumption (that set of extreme sets of B° is closed in the Painlevé-Kuratowski topology) $\{y\}$ would also be an extreme set. But this would be a contradiction to $y = (1-\lambda)x + \lambda z$ as $y \neq x, z$. Therefore the other case holds and the claim is proofed. □

8.4 Further Examples

In this section of the appendix I will give some more examples of how to calculate the horofunctions following [Wal07]. Some of the examples are really additional ones, not mentioned in the main part, one of them, with the L^1-metric, contains the three dimensional case to make is easier to understand the general higher dimensional case dealt with in the main part. Some of the subsections, like for the lens shaped norm, are only some of the calculations which were just long but not essential for understanding.

8.4.1 $X = \mathbb{R}$ Equipped with the Standard Metric

In the first case let $X = \mathbb{R}$. Let the unit ball be

$$B = \{x \in \mathbb{R} \mid \ -3 \leq x \leq 1\},$$

then

$$\begin{aligned} B^\circ &= \{y \in \mathbb{R}^* \mid \langle y|x\rangle \geq -1 \ \forall x \in [-3,1]\} \\ &= \left\{y \in \mathbb{R}^* \mid \ -1 \leq y \leq \frac{1}{3}\right\}. \end{aligned}$$

As $B = B^\circ$ if we identify $\mathbb{R} \cong \mathbb{R}^*$, we can illustrate them both in the same picture:

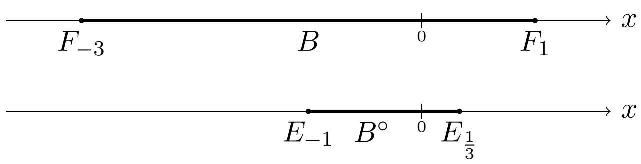


Figure 8.1: B° and B in $\mathbb{R}$

The norm defined by B is

$$\|x\|_B = \begin{cases} x & \text{if } x \geq 0 \\ -\frac{1}{3}x & \text{if } x < 0. \end{cases}$$

The proper extreme sets of B° are:

$$E_{-1} := \{-1\} \qquad \text{and} \qquad E_{\frac{1}{3}} := \left\{\frac{1}{3}\right\}.$$

The Functions $f_{E,p}$ and $f^*_{E,p}$ We now have to calculate the functions $f_{E,p}$ and $f^*_{E,p}$ for each extreme set E. From Lemma 5.0.4 it follows for all $q \in X$:

$$\begin{aligned} f_{E_{-1},p}(q) &= I_{E_{-1}}(q) \\ f_{E_{\frac{1}{3}},p}(q) &= I_{E_{\frac{1}{3}}}(q), \end{aligned}$$

and

$$\begin{aligned} f^*_{E_{-1},p}(\cdot) &= \langle -1|\cdot\rangle \\ f^*_{E_{\frac{1}{3}},p}(\cdot) &= \left\langle \frac{1}{3}\middle| \cdot \right\rangle. \end{aligned}$$

or in other words

$$\begin{aligned} f^*_{E_{-1},p} &= -\mathrm{id} \\ f^*_{E_{\frac{1}{3}},p} &= \frac{1}{3}\ \mathrm{id} \quad \forall p \in \mathbb{R}. \end{aligned}$$

From Theorem 4.0.32 we know, that the set of Busemann points of $\mathbb{R}$ is $\{\mathrm{id}, \frac{1}{3}\mathrm{id}\ \}$. If we take any sequence $(z_n^{\pm})_{n\in\mathbb{N}}$ in $\mathbb{R} \cup \{\pm\infty\}$ converging to $\pm\infty$, then

$$\begin{aligned} \psi_{z_n^+}(y) &\longrightarrow f^*_{E_{-1},p}(y) \\ \psi_{z_n^-}(y) &\longrightarrow f^*_{E_{\frac{1}{3}},p}(y) \end{aligned}$$

as $n \longrightarrow \infty$.
Indeed, let (z_n) be a sequence in $\mathbb{R}$ going to ∞. Then there is an $N \in \mathbb{N}$ such that $z_n \geq y$ and $z_n \geq 0$ for all $n \geq N$. With this we get

$$\begin{aligned} \psi_{z_n}(y) &= \|z_n - y\|_B - \|z_n\|_B \\ &= z_n - y - z_n \\ &= -y = f^*_{E_{-1},p}(y), \end{aligned}$$

for all $n \geq N$ and therefore $\psi_{z_n}(y) \longrightarrow f^*_{E_{-1},p}(y)$ as $n \longrightarrow \infty$.
The case $z_n \longrightarrow -\infty$ follows the same way.

So for $X = \mathbb{R}$ there are only two different types of sequences to consider. This gives us the horofunction compactification by adding $\{+\infty\}$ and $\{-\infty\}$ which in this case is a well known compactificatioin.

8.4.2 $X = \mathbb{R}^3$ with the L^1-Norm

We already saw this example in the case where $X = \mathbb{R}^2$. In the main part we also dealt with the general higher dimensional case. This example for $n = 3$ shall help us understand this general case because the calculations and notations are very similar and it is still imaginable and drawable.

B, B° and Their Extreme Sets In this case the unit ball is defined as

$$B := \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \middle| \sum_{i=1}^{3} |x_i| \leq 1 \right\}$$

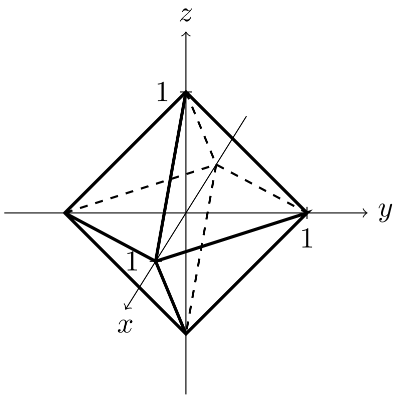


Figure 8.2: B of the L^1-norm in $\mathbb{R}^3$

$$= \operatorname{conv}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right\},$$

see also figure 8.3.

With an analogous calculations as in the two-dimensional case, we obtain for the dual unit ball

$$\begin{aligned} B^\circ &= \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 \mid \max(|y_1|, |y_2|, |y_3|) \le 1)\} \\ &= \operatorname{conv}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}\right\}. \end{aligned}$$

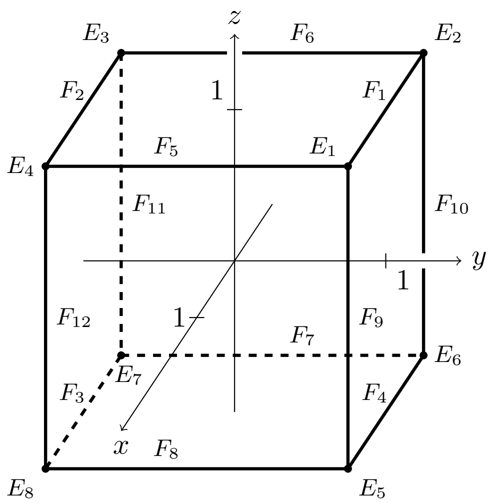


Figure 8.3: B° of the L^1-norm in $\mathbb{R}^3$

The extreme sets of B° are these:

1. Points: the points whose convex hull define the dual unit ball, let's enumerate them with $E_1, \ldots, E_8$ in the same order as above.

2. Lines:

$$F_1 := \left\{ \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_5 := \left\{ \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_9 := \left\{ \begin{pmatrix} 1 \\ 1 \\ t \end{pmatrix} \middle| |t| \le 1 \right\}$$

$$F_2 := \left\{ \begin{pmatrix} t \\ -1 \\ 1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_6 := \left\{ \begin{pmatrix} -1 \\ t \\ 1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_{10} := \left\{ \begin{pmatrix} -1 \\ 1 \\ t \end{pmatrix} \middle| |t| \le 1 \right\}$$

$$F_3 := \left\{ \begin{pmatrix} t \\ -1 \\ -1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_7 := \left\{ \begin{pmatrix} -1 \\ t \\ -1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_{11} := \left\{ \begin{pmatrix} -1 \\ -1 \\ t \end{pmatrix} \middle| |t| \le 1 \right\}$$

$$F_4 := \left\{ \begin{pmatrix} t \\ 1 \\ -1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_8 := \left\{ \begin{pmatrix} 1 \\ t \\ -1 \end{pmatrix} \middle| |t| \le 1 \right\} \quad F_{12} := \left\{ \begin{pmatrix} 1 \\ -1 \\ t \end{pmatrix} \middle| |t| \le 1 \right\}$$

The lines are numbered in such way, that the first four extreme sets are denoting parallel lines, as well as the other two groups of four.

3. Faces:

$$G_1 := \left\{ \begin{pmatrix} 1 \\ s \\ t \end{pmatrix} \middle| |s|, |t| \le 1 \right\} \quad G_3 := \left\{ \begin{pmatrix} s \\ 1 \\ t \end{pmatrix} \middle| |s|, |t| \le 1 \right\} \quad G_5 := \left\{ \begin{pmatrix} s \\ t \\ 1 \end{pmatrix} \middle| |s|, |t| \le 1 \right\}$$

$$G_2 := \left\{ \begin{pmatrix} -1 \\ s \\ t \end{pmatrix} \middle| |s|, |t| \le 1 \right\} \quad G_4 := \left\{ \begin{pmatrix} s \\ -1 \\ t \end{pmatrix} \middle| |s|, |t| \le 1 \right\} \quad G_6 := \left\{ \begin{pmatrix} s \\ t \\ -1 \end{pmatrix} \middle| |s|, |t| \le 1 \right\}$$

Here also, we number the sets according to parallelism.

4. The last extreme set is of course B° itself.

So all together we have 27 extreme sets of B°.

Calculation of the $f_{E,p}$-functions For the extreme points we have as usual

$$f_{E_i,p} = I_{E_i} \quad \forall i \in \{1, \dots, 8\},$$

independent of p.
For the one dimensional extreme sets, the edges of the cube, we have

$$\begin{aligned} f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\} \\ &= I_{F_1}(q) + \langle q|p\rangle - \min\{p_1 + p_2 + p_3, -p_1 + p_2 + p_3\} \\ &= I_{F_1}(q) + \langle q|p\rangle - p_2 - p_3 + |p_1| \end{aligned}$$

because $F_1 = \operatorname{conv}(E_1, E_2)$ with $E_1 = \{e_1\}$ and $E_2 = \{e_2\}$.
By similar calculations and symmetry arguments we receive

$$\begin{aligned} f_{F_1,p}(q) &= I_{F_1}(q) + \langle q|p\rangle - p_2 - p_3 + |p_1| \\ f_{F_2,p}(q) &= I_{F_2}(q) + \langle q|p\rangle + p_2 - p_3 + |p_1| \end{aligned}$$

$$
\begin{aligned}
f_{F_3,p}(q) &= I_{F_3}(q) + \langle q|p\rangle + p_2 + p_3 + |p_1| \\
f_{F_4,p}(q) &= I_{F_4}(q) + \langle q|p\rangle - p_2 + p_3 + |p_1| \\
f_{F_5,p}(q) &= I_{F_5}(q) + \langle q|p\rangle - p_1 - p_3 + |p_2| \\
f_{F_6,p}(q) &= I_{F_6}(q) + \langle q|p\rangle + p_1 - p_3 + |p_2| \\
f_{F_7,p}(q) &= I_{F_7}(q) + \langle q|p\rangle + p_1 + p_3 + |p_2| \\
f_{F_8,p}(q) &= I_{F_8}(q) + \langle q|p\rangle - p_1 + p_3 + |p_2| \\
f_{F_9,p}(q) &= I_{F_9}(q) + \langle q|p\rangle - p_1 - p_2 + |p_3| \\
f_{F_{10},p}(q) &= I_{F_{10}}(q) + \langle q|p\rangle + p_1 - p_2 + |p_3| \\
f_{F_{11},p}(q) &= I_{F_{11}}(q) + \langle q|p\rangle + p_1 + p_2 + |p_3| \\
f_{F_{12},p}(q) &= I_{F_{12}}(q) + \langle q|p\rangle - p_1 + p_2 + |p_3|
\end{aligned}
$$

and

$$
\begin{aligned}
f_{G_1,p}(q) &= I_{G_1}(q) + \langle q|p\rangle - p_1 + |p_2| + |p_3| \\
f_{G_2,p}(q) &= I_{G_2}(q) + \langle q|p\rangle + p_1 + |p_2| + |p_3| \\
f_{G_3,p}(q) &= I_{G_3}(q) + \langle q|p\rangle - p_2 + |p_1| + |p_3| \\
f_{G_4,p}(q) &= I_{G_4}(q) + \langle q|p\rangle + p_2 + |p_1| + |p_3| \\
f_{G_5,p}(q) &= I_{G_5}(q) + \langle q|p\rangle - p_3 + |p_1| + |p_2| \\
f_{G_6,p}(q) &= I_{G_6}(q) + \langle q|p\rangle + p_3 + |p_1| + |p_2|
\end{aligned}
$$

Now it is clear why we chose this numeration of the extreme sets. The sets, no matter if one or two dimensional, lying parallel to each other have yield to nearly the same functions, differing only by sign. We will come back to this observation when we calculate the general n-dimensional case.

The Legendre-Fenchel-transform The next step is to calculate the Legendre-Fenchel transforms of our functions above. The case of the extreme points is simple and well known by now. We have for $i \in \{1, \dots, 8\}$:

$$
f^*_{E_i,p}(y) = \langle e_i|y\rangle \quad \text{for } E_i = \{e_i\}.
$$

For the lines we have

$$
\begin{aligned}
f^*_{F_1,p}(y) &= \max\{\langle e_1|y-p\rangle, \langle e_2|y-p\rangle\} + \min\{\langle e_1|p\rangle, \langle e_2|p\rangle\} \\
&= \max\{(y_1-p_1)+(y_2-p_2)+(y_3-p_3), -(y_1-p_1)+(y_2-p_2)+(y_3-p_3)\} \\
&\quad - |p_1| + p_2 + p_3 \\
&= |y_1-p_1| + y_2 - p_2 + y_3 - p_3 - |p_1| + p_2 + p_3 \\
&= |y_1-p_1| - |p_1| + y_2 + y_3.
\end{aligned}
$$

For the calculation of the transforms of the other functions, we can use the similarity of them and obtain:

$$
\begin{aligned}
f^*_{F_1,p}(y) &= |y_1-p_1| - |p_1| + y_2 + y_3 \\
f^*_{F_2,p}(y) &= |y_1-p_1| - |p_1| - y_2 + y_3 \\
f^*_{F_3,p}(y) &= |y_1-p_1| - |p_1| - y_2 - y_3 \\
f^*_{F_4,p}(y) &= |y_1-p_1| - |p_1| + y_2 - y_3 \\
f^*_{F_5,p}(y) &= |y_2-p_2| - |p_2| + y_1 + y_3
\end{aligned}
$$

$$\begin{aligned}
f^*_{F_6,p}(y) &= |y_2 - p_2| - |p_2| - y_1 + y_3\\
f^*_{F_7,p}(y) &= |y_2 - p_2| - |p_2| - y_1 - y_3\\
f^*_{F_8,p}(y) &= |y_2 - p_2| - |p_2| + y_1 - y_3\\
f^*_{F_9,p}(y) &= |y_3 - p_3| - |p_3| + y_1 + y_2\\
f^*_{F_{10},p}(y) &= |y_3 - p_3| - |p_3| - y_1 + y_2\\
f^*_{F_{11},p}(y) &= |y_3 - p_3| - |p_3| - y_1 - y_2\\
f^*_{F_{12},p}(y) &= |y_3 - p_3| - |p_3| + y_1 - y_2.
\end{aligned}$$

For the faces we get:

$$\begin{aligned}
f^*_{G_1,p}(y) &= |y_2 - p_2| - |p_2| + |y_3 - p_3| - |p_3| + y_1\\
f^*_{G_2,p}(y) &= |y_2 - p_2| - |p_2| + |y_3 - p_3| - |p_3| - y_1\\
f^*_{G_3,p}(y) &= |y_1 - p_1| - |p_1| + |y_3 - p_3| - |p_3| + y_2\\
f^*_{G_4,p}(y) &= |y_1 - p_1| - |p_1| + |y_3 - p_3| - |p_3| - y_2\\
f^*_{G_5,p}(y) &= |y_1 - p_1| - |p_1| + |y_2 - p_2| - |p_2| + y_3\\
f^*_{G_6,p}(y) &= |y_1 - p_1| - |p_1| + |y_2 - p_2| - |p_2| - y_3.
\end{aligned}$$

We see that every transform consists of an index- and sign permutation of $|y_i - p_i| - |p_i| + |y_j - p_j| - |p_j| \pm y_k$ with $i, j, k \in \{1, 2, 3\}$ pairwise distinct.
Before we interpret these results in the next section, we will first have a look at the geometrical interpretation.

Geometrical part Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^3$. Then

$$\begin{aligned}
\psi_{z_n}(y) = \|z_n - y\|_1 - \|z_n\|_1 &= |z_{n,1} - y_1| + |z_{n,2} - y_2| + |z_{n,3} - y_3| - |z_{n,1}| - |z_{n,2}| - |z_{n,3}|\\
&= |z_{n,1} - y_1| - |z_{n,1}| + |z_{n,2} - y_2| - |z_{n,3}| + |z_{n,3} - y_3| - |z_{n,3}|
\end{aligned}$$

Let one of the components of this sequence, $z_{n,i},\ i \in \{1, 2, 3\}$, tend to ∞. Then there is an $N \in \mathbb{N}$, such that $z_{n,i} > \max(0, y_i)$ for all n bigger than N. Therefore

$$|z_{n,i} - y_i| - |z_{n,i}| = z_{n,i} - y_i - z_{n,i} = -y_i \ \forall n > N.$$

This means that this part of our sum converges to $-y_i$. Similarly, if $z_{n,j} \longrightarrow -\infty$, then this part of the sum converges to $+y_j$. The third case to consider is, what happens if $z_{n,k}$ converges to a constant component p_k. Then $|z_{n,k} - y_k| - |z_{n,k}| \longrightarrow |p_k - y_k| - |p_k|$. Compared with the end of the last paragraph, we see that each of the three types of sequences converges to one of the three possibilities we had for a component.

Summarised we have that all 8 variations of the signs of a sequence of the form

$$z_n = \begin{pmatrix} \pm kn \\ \pm ln \\ \pm n \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \\ \infty \end{pmatrix}$$

with $k, l > 0$ yield the Legendre-Fenchel transforms belonging to the 8 extremal points. They correspond to sequences through one of the 8 open octants of $\mathbb{R}^3$. If we would draw B and B° in the same picture, the extreme point of B° they are converging to lies directly on the opposite site of the open octant with respect to the origin. Here it makes no difference whether the sequence goes through the origin or not, the only important factor is the direction, that is the face of B it would intersect with if the sequence would follow

a line through the origin.
If we fix one of the components, we get a sequence belonging to one of the extremal edges. Let $(z_n)_n$ be a sequence with $z_{n,j} = p_j$ for one $j \in \{1, 2, 3\}$. Then $\psi_{z_n} \longrightarrow f^*_{F_j,p}$, where F_j is the extremal set with t in the j-th component and and the signs of the 1 in the other components are opposite to those of z_n. These are exactly the sequences parallel to a line passing through the origin and through one edge of the octagon B, but not going through an extreme point of it. The edge the sequence is converging to lies again on the opposite side with respect to the origin.
The last case, belonging to the faces, are sequences $(z_n)_n$ with two components fixed, say $z_{n,i} = p_i$ and $z_{n,j} = p_j$, $i \neq j \in \{1, 2, 3\}$. Here again we see the dependence on the point p the sequence is passing through. These 6 kinds of sequences (3 possibilities for choosing the fixed component and for each of them two possibilities for the sign of the third component) tend to one transform belonging to an extreme face of B°. These are the sequences following lines parallel to one of the straight lines going through an extreme face of B°.

8.4.3 $X = \mathbb{R}^2$ with the Euclidean Norm

As a first step to unit balls with a curved boundary, we have a look at the standard Euclidean norm in $\mathbb{R}^2$. If we identify $\mathbb{R}^2 = (\mathbb{R}^2)^*$, the unit ball and it's dual are the same, namely

$$B = B^\circ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \middle| x^2 + y^2 \leq 1 \right\}.$$

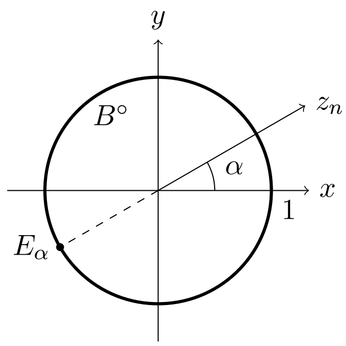


Figure 8.4: Euclidean unit ball B and B°

As the boundary of B° is a curved line, every point of the boundary is an extreme point and these are the only proper extreme sets of B°. So the set of proper extreme sets of B° is $\{E_\alpha \mid 0 \leq \alpha < 2\pi\}$ where

$$E_\alpha := \left\{ \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \right\}.$$

Let $E = E_\alpha$ for some $\alpha \in [0, 2\pi)$ and $p = (p_1, p_2) \in \mathbb{R}^2$ be a point. As E_α is a one pointed set, $f_{E,p} = I_E$. Therefore

$$f^*_{E,p}(y) = \sup_{x \in E} \langle x | y \rangle = y_1 \cos\alpha + y_2 \sin\alpha.$$

We now have to find a sequence (z_n) in $\mathbb{R}^2$ such that ψ_{z_n} converges to $f^*_{E,p}$ as n tends to infinity. Inspired by the examples above, we suppose, that for a given α there will be exactly one straight line emanating from the origin that converges to $f^*_{E_\alpha,p}$ and that this line has to run exactly in the opposite direction of E_α. Therefore we consider the sequence $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^2$ with $z_n = (-n\cos\alpha, -n\sin\alpha)$. Then

$$\begin{aligned}
\|z_n - y\| &= \sqrt{(-n\cos\alpha - y_1)^2 + (-n\sin\alpha - y_2)^2} \\
&= \sqrt{n^2\cos^2\alpha + n^2\sin^2\alpha + 2n(y_1\cos\alpha + y_2\sin\alpha) + y_1^2 + y_2^2} \\
&= \sqrt{n^2\left(1 + \frac{2}{n}(y_1\cos\alpha + y_2\sin\alpha) + \frac{1}{n^2}(y_1^2 + y_2^2)\right)} \\
&= n\sqrt{1 + \frac{2}{n}(y_1\cos\alpha + y_2\sin\alpha) + \frac{1}{n^2}(y_1^2 + y_2^2)} \\
&= n\left[1 + \frac{1}{2}(\frac{2}{n}(y_1\cos\alpha + y_2\sin\alpha) + \frac{1}{n^2}(y_1^2 + y_2^2)) + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&= n\left[1 + \frac{1}{n}(y_1\cos\alpha + y_2\sin\alpha) + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&= n + y_1\cos\alpha + y_2\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

and

$$\|z_n\| = n.$$

Here we used the first-order Taylor series approximation of $\sqrt{1+x}$. As there is an n in the denominator of each summand in the forth line, we are allowed to use this approximation by choosing n large enough. Together we obtain

$$\begin{aligned}
\psi_{z_n}(y) &= \|z_n - y\| - \|z_n\| \\
&= n + y_1\cos\alpha + y_2\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right) - n \\
&= y_1\cos\alpha + y_2\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right),
\end{aligned}$$

which leads to

$$\psi_{z_n}(y) \longrightarrow f^*_{E,p}(y) \quad \text{as } n \longrightarrow \infty.$$

The set $\{E_\alpha \mid \alpha \in [0, 2\pi)\}$ is closed in the Painlevé-Kuratowski topology, so we know that every horofunction is a Busemann point.
If we take a sequence (z'_n) following a line not emanating from the origin but going through the point $q = (q_1, q_2)$, we have to replace y_1 and y_2 by $y_1 - q_1$ and $y_2 - q_2$ respectively in the first calculation. With this we receive (with z_n as above)

$$\|z'_n - y\| = \|z_n + q - y\| = n + (y_1 - q_1)\cos\alpha + (y_2 - q_2)\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right)$$

and

$$\begin{aligned}
\|z'_n\| = \|z_n + q\| &= \left\|\begin{pmatrix} -n\cos\alpha + q_1 \\ -n\sin\alpha + q_2 \end{pmatrix}\right\| \\
&= \sqrt{n^2\cos^2\alpha + n^2\sin^2\alpha + q_1^2 + q_2^2 - 2n(q_1\cos\alpha + q_2\sin\alpha)} \\
&= n\sqrt{1 - \frac{2}{n}(q_1\cos\alpha + q_2\sin\alpha) + \frac{1}{n^2}(q_1^2 + q_2^2)} \\
&= n\left[1 - \frac{1}{2}\left(\frac{2}{n}(q_1\cos\alpha + q_2\sin\alpha) + \mathcal{O}\left(\frac{1}{n^2}\right)\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right]
\end{aligned}$$

$$= n - q_1 \cos\alpha - q_2 \sin\alpha + \mathcal{O}\left(\frac{1}{n}\right).$$

So in this case we get

$$\begin{aligned}
\psi_{z'_n}(y) &= \|z'_n - y\| - \|z'_n\| \\
&= n + (y_1 - q_1)\cos\alpha + (y_2 - q_2)\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right) - \left(n - q_1\cos\alpha - q_2\sin\alpha + \mathcal{O}\left(\frac{1}{n}\right)\right) \\
&= y_1 \cos\alpha + y_2 \sin\alpha + \mathcal{O}\left(\frac{1}{n}\right) \\
&= \|z_n - y\| - \|z_n\| + \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

so both converge to the same Busemann point $f^*_{E,p}$.

We see that it makes no difference whether we take a sequence through the origin or a parallel one, the only important information is the direction the sequence is following. As $B = B^\circ$, it is easy to draw them both in the same picture (see also figure 8.4) and we can find out easily, that (if we only consider sequences on lines through the origin) the extreme set a sequence is converging to is exactly the negative point of the intersection point of this sequence and B°.

8.4.4 The Case $X = \mathbb{R}^2$ with the $L^{\frac{3}{2}}$- Norm

As another example of a norm with curved unit sphere we consider the $\frac{3}{2}$-norm on $\mathbb{R}^2$. We consider this example because the unit sphere is curved everywhere, just as the euclidean one, but now $B \neq B^\circ$.
So for a point $x = (x_1, x_2) \in \mathbb{R}^2$ its distance to the origin is

$$\|x\|_{\frac{3}{2}} = \left(|x_1|^{\frac{3}{2}} + |x_2|^{\frac{3}{2}}\right)^{\frac{2}{3}}.$$

Therefore the unit ball is

$$B = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, |x|^{\frac{3}{2}} + |y|^{\frac{3}{2}} \leq 1 \right\},$$

and it's dual is by Lemma (2.5.13)

$$B^\circ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, |x|^3 + |y|^3 \leq 1 \right\}.$$

The proper extreme sets of the dual unit ball are the boundary points of B°. We define the set of an extreme point as

$$E_k^\pm := \left\{ \begin{pmatrix} k \\ \pm\sqrt[3]{1 - |k|^3} \end{pmatrix} \right\},$$

where $-1 \leq k \leq 1$.
Let $E := E_k^+$ for some $k \in [-1, 1]$. It is sufficient to consider only those points with non-negative second component because the other case follows just by changing the signs of the second component. As all extreme sets are extreme points, we always have

$$f_{E,p} = I_E$$

and

$$f^*_{E,p}(y) = \sup_{x\in E}\langle x|y\rangle = ky_1 + \sqrt[3]{1-|k|^3}y_2.$$

We now want to find a sequence $(z_n)_{n\in\mathbb{N}}$ such that ψ_{z_n} converges to $f^*_{E,p}$. Inspired by the examples above, we suppose that one such sequence should follow a straight line perpendicular to the tangent line at $-E_k$. Consider the following sequence (z_n) with

$$z_n = -\begin{pmatrix} k^2 \\ (1-k^3)^{\frac{2}{3}} \end{pmatrix} \cdot n$$

This sequence is indeed perpendicular to the tangent line because $\|z_1\|_{\frac{3}{2}} = 1$ and $\langle z_1|E_k\rangle = -1$, therefore orthogonality follows by the definition and the construction of B°. Then we have

$$\begin{aligned}
\|z_n\|_{\frac{3}{2}} &= \left\{|-nk^2|^{\frac{3}{2}} + |-n(1-k^3)^{\frac{2}{3}}|^{\frac{3}{2}}\right\}^{\frac{2}{3}} \\
&= \left(n^{\frac{3}{2}}k^3 + n^{\frac{3}{2}}(1-k^3)\right)^{\frac{2}{3}} \\
&= n.
\end{aligned}$$

For the calculation of $\|z_n - y\|_{\frac{3}{2}}$ we will need the following formulas for $a, b \in \mathbb{R}$ small enough:

$$(1+a)^{\frac{3}{2}} = 1 + \frac{3}{2}a + \mathcal{O}\left(a^2\right) \tag{8.13}$$

$$(1+b)^{\frac{2}{3}} = 1 + \frac{2}{3}b + \mathcal{O}\left(b^2\right) \tag{8.14}$$

Then we get:

$$\begin{aligned}
\|z_n - y\|_{\frac{3}{2}} &= \left\{\left|-nk^2 - y_1\right|^{\frac{3}{2}} + \left|-n(1-k^3)^{\frac{2}{3}} - y_2\right|^{\frac{3}{2}}\right\}^{\frac{2}{3}} \\
&= \left\{\left(nk^2 + y_1\right)^{\frac{3}{2}} + \left[n(1-k^3)^{\frac{2}{3}} + y_2\right]^{\frac{3}{2}}\right\}^{\frac{2}{3}} \\
&= \left\{n^{\frac{3}{2}}k^3\left(1 + \frac{y_1}{nk^2}\right)^{\frac{3}{2}} + n^{\frac{3}{2}}\left[\left(1-k^3\right)^{\frac{2}{3}} + \frac{y_2}{n}\right]^{\frac{3}{2}}\right\}^{\frac{2}{3}} \\
&= n\left\{k^3\left(1 + \frac{y_1}{nk^2}\right)^{\frac{3}{2}} + \left(1-k^3\right)\left[1 + \frac{y_2}{n(1-k^3)^{\frac{2}{3}}}\right]^{\frac{3}{2}}\right\}^{\frac{2}{3}} \\
&= n\left\{k^3\left(1 + \frac{3y_1}{2nk^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right) + \left(1-k^3\right)\left[1 + \frac{3y_2}{2n(1-k^3)^{\frac{2}{3}}} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]\right\}^{\frac{2}{3}} \\
&= n\left\{1 + \frac{3}{2n}\left(ky_1 + (1-k^3)^{\frac{1}{3}}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}^{\frac{2}{3}} \\
&= n\left\{1 + \frac{2}{3}\frac{3}{2n}\left(ky_1 + (1-k^3)^{\frac{1}{3}}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}
\end{aligned}$$

Together we have

$$\psi_{z_n}(y) = \|z_n - y\|_{\frac{3}{2}} - \|z_n\|_{\frac{3}{2}}$$

$$= n + ky_1 + \sqrt[3]{1-k^3}y_2 + \mathcal{O}\left(\frac{1}{n}\right) - n$$
$$= ky_1 + \sqrt[3]{1-k^3}y_2 + \mathcal{O}\left(\frac{1}{n}\right)$$

and therefore

$$\psi_{z_n} \longrightarrow f^*_{E,p} \quad \text{as } n \longrightarrow \infty.$$

If we would have taken E_k^- and the sequence $z_n = -\begin{pmatrix} k^2 \\ -(1-k^3)^{\frac{2}{3}} \end{pmatrix} \cdot n$, it would have been $\psi_{z_n} \longrightarrow f^*_{E_k^-,p}$ as $n \longrightarrow \infty$.
So even if B is not polyhedral and $B \neq B^\circ$, we see the same result as in the other examples. The starting point of the sequence has no influence on the extreme set the sequence is converging to. Only the direction matters and the extreme set lies in the opposite quadrant with respect to the origin.

8.4.5 $\mathbb{R}^2$ with a Lens-Shaped Norm - Calculation of ψ_{z_n}

We show here the calculation of $\psi_{z_n}(y)$ for the sequence $z_n = \begin{pmatrix} k \\ l \end{pmatrix} \cdot n + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \infty \\ \infty \end{pmatrix}$ as $n \longrightarrow \infty$ with $k, l \in \mathbb{R}, k, l > 0$ and arbitrary $y \in \mathbb{R}^2$.

$$\begin{aligned}
\psi_{z_n}(y) &= \left\| \begin{pmatrix} kn + p_1 - y_1 \\ ln + p_2 - y_2 \end{pmatrix} \right\|_B - \left\| \begin{pmatrix} kn + p_1 \\ ln + p_2 \end{pmatrix} \right\|_B \\
&= \sqrt{3}|kn + p_1 - y_1| - \sqrt{3}|kn + p_1| + \sqrt{4(kn + p_1 - y_1)^2 + (ln + p_2 - y_2)^2} \\
&\quad - \sqrt{4(kn + p_1)^2 + (ln + p_2)^2} \\
&= \sqrt{3}|kn + p_1 - y_1| - \sqrt{3}|kn + p_1| - \sqrt{(4k^2 + l^2)n^2 + 2(4kp_1 + lp_2)n + 4p_1^2 + p_2^2} \\
&\quad + \Big\{(4k^2 + l^2)n^2 + 2(4kp_1 - 4ky_1 + lp_2 - ly_2)n \\
&\quad + 4y_1^2 + y_2^2 + 4p_1^2 + p_2^2 - 8p_1y_1 - 2p_2y_2\Big\}^{\frac{1}{2}} \\
&\overset{n \gg 1}{=} -\sqrt{3}y_1 - n \cdot \sqrt{4k^2 + l^2}\sqrt{1 + \frac{2(4kp_1 + lp_2)}{(4k^2 + l^2)n} + \frac{4p_1^2 + p_2^2}{(4k^2 + l^2)n^2}} \\
&\quad + n \cdot \sqrt{4k^2 + l^2}\Big\{1 + \frac{2(4kp_1 - 4ky_1 + lp_2 - ly_2)}{(4k^2 + l^2)n} \\
&\quad + \frac{4y_1^2 + y_2^2 + 4p_1^2 + p_2^2 - 8p_1y_1 - 2p_2y_2}{(4k^2 + l^2)n^2}\Big\}^{\frac{1}{2}} \\
&= -\sqrt{3}y_1 - n \cdot \sqrt{4k^2 + l^2}\left[1 + \frac{1}{2}\left(\frac{2(4kp_1 + lp_2)}{(4k^2 + l^2)n} + \frac{4p_1^2 + p_2^2}{(4k^2 + l^2)n^2}\right) + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad + n \cdot \sqrt{4k^2 + l^2} \cdot \Bigg[1 + \frac{1}{2}\Bigg(\frac{2(4kp_1 - 4ky_1 + lp_2 - ly_2)}{(4k^2 + l^2)n} \\
&\quad + \frac{4y_1^2 + y_2^2 + 4p_1^2 + p_2^2 - 8p_1y_1 - 2p_2y_2}{(4k^2 + l^2)n^2}\Bigg) + \mathcal{O}\left(\frac{1}{n^2}\right)\Bigg] \\
&= -\sqrt{3}y_1 + n \cdot \sqrt{4k^2 + l^2} + \frac{4kp_1 - 4ky_1 + lp_2 - ly_2}{\sqrt{4k^2 + l^2}} + \mathcal{O}\left(\frac{1}{n}\right) \\
&\quad - t \cdot \sqrt{4k^2 + l^2} - \frac{4kp_1 + lp_2}{\sqrt{4k^2 + l^2}} + \mathcal{O}\left(\frac{1}{n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sqrt{3}y_1 - \frac{4ky_1 + ly_2}{\sqrt{4k^2+l^2}} + \mathcal{O}\left(\frac{1}{n}\right) \\
&\longrightarrow \left(-\sqrt{3} - \frac{4k}{\sqrt{4k^2+l^2}}\right) y_1 - \frac{l}{\sqrt{4k^2+l^2}} y_2 = -\left\langle \underbrace{\begin{pmatrix} \sqrt{3} + \frac{4k}{\sqrt{4k^2+l^2}} \\ \frac{l}{\sqrt{4k^2+l^2}} \end{pmatrix}}_{:=q} \middle| \, y \right\rangle .
\end{aligned}
$$

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