Jørgensen Lemma

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BACHELORARBEIT

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Declaration

I hereby declare and confirm that this thesis is entirely the result of my own original work. Where other sources of information have been used, they have been indicated as such and properly acknowledged. I further declare that this or similar work has not been submitted for credit elsewhere.

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Abstract

In this thesis we will first do some classification of the elements in $PSL(2, \mathbb{R})$. After that we will introduce the notion of Fuchsian groups, i. e. a discrete subgroup of $PSL(2, \mathbb{R})$, and prove two criteria for discreteness in $PSL(2, \mathbb{R})$. One of them is the Jørgensen Inequality, which is the main theorem in this thesis. Finally we look at the special case of equality in the Jørgensen inequality.

Chapter 1

Introduction

1.1 Goals and Structure of this thesis

Our first goal in this thesis will be the following Theorem in Chapter 2.1:

Theorem (Jørgensen Inequality)**.** *Suppose that* $T, S \in \mathrm{PSL}(2,\mathbb{R})$ *and* $< T, S >$ *is a non-elementary Fuchsian group. Then*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| \geq 1. \tag{1.1}$$

The lower bound is best possible.

In order to prove that, we have to distinguish between three types of possible elements in $\mathrm{PSL}(2,\mathbb{R})$. In Chapter 1.3 we do this classification of $\mathrm{PSL}(2,\mathbb{R})$. After that, in Chapter 1.4 we introduce the notion of Fuchsian groups, i.e. discrete groups in $\mathrm{PSL}(2,\mathbb{R})$. We also prove some properties of Fuchsian groups, for example Theorem 5 which gives us a complete characterisation of elementary Fuchsian groups, which are Fuchsian groups with a finite orbit when acting on the upper half-plane $\mathbb{H}$. As from Chapter 2, we will only consider non-elementary Fuchsian groups, which occur in the Jørgensen Inequality. After that we will be able to prove our next criterion for discreteness in $\mathrm{PSL}(2,\mathbb{R})$ in Chapter 2.2:

Theorem. *A non-elementary subgroup* Γ *of* $\mathrm{PSL}(2,\mathbb{R})$ *is discrete if and only if, for each* T *and* S *in* Γ*, the group* $< T, S >$ *is discrete.*

Finally, in Chapter 2.3, we find that we have equality in (1.1) if and only if $< T, S >$ is a triangle group of order $(2, 3, p)$ with $p \in \{7, 8, 9, ..., \infty\}$.

1.2 Background

We start with the Classification of Möbius transformations of $\mathbb{H}$. For basic introduction of the hyperbolic plane in the upper half-plane $\mathbb{H} := \{z \in$

$\mathbb{C}|\,\mathrm{Im}(z) > 0\}$ with boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ and the unit disk model in the unit disk $\mathbb{D} := \{z \in \mathbb{C}||z| < 1\}$ with boundary $\partial\mathbb{D} = \{z \in \mathbb{C}||z| = 1\}$ see Walkden's script [6] or Katok's book [4].

Definition 1. *The set of* ***fractional linear (or Möbius) transformations of*** $\mathbb{H}$ *is defined as follows*

$$\mathrm{PSL}(2,\mathbb{R}) = \{\mathbb{H} \to \mathbb{H}, z \to \frac{az+b}{cz+d}|ad-bc = 1, a, b, c, d \in \mathbb{R}\}.$$

For $T \in \mathrm{PSL}(2,\mathbb{R})$ *we define the trace of* T $\mathrm{tr}(T) = |a+d|$ *and set* $Tr(T) := |tr(t)|$. *We call* $z \in \tilde{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ *a* ***fixed point*** *if* $T(z) = z$.

1.3 Classification of Möbius transformations of $\mathbb{H}$

In what follows the aim is to classify the types of behaviour that Möbius transformations of $\mathbb{H}$ exhibit. We will see that there are three different classes of Möbius transformation of $\mathbb{H}$.

We start with some Möbius transformation of $\mathbb{H}$ called T. Our initial classification of Möbius transformations of $\mathbb{H}$ is based on how many fixed points a given Möbius transformation of $\mathbb{H}$ has, and whether they lie in $\mathbb{H}$ or in $\partial\mathbb{H}$.

Clearly the identity map is a Möbius transformation of $\mathbb{H}$ which fixes every point. As from now, we will assume that T is not the identity.

Let us first consider the case when $\infty \in \partial\mathbb{H}$ is a fixed point. As

$$T(z) = \frac{az+b}{cz+d} = \frac{a+\frac{b}{z}}{c+\frac{d}{z}}$$

we note that as $z \to \infty$ we have $\frac{1}{z} \to 0$ and by that $T(\infty) = \frac{a}{c}$. Thus ∞ is a fixed point of T if and only if $T(\infty) = \infty$, and this happens if and only if $c = 0$.

Suppose that ∞ is a fixed point of T so that $c = 0$. What other fixed points z_0 can T have? Observe that now

$$T(z_0) = \frac{a}{d}z_0 + \frac{b}{d}.$$

Hence T also has a fixed point at $z_0 = \frac{b}{d-a}$. Note that if $a = d$ then this point may be ∞.

Thus if $\infty \in \partial\mathbb{H}$ is a fixed point for T then T has at most one other fixed point, and this fixed point also lies on $\partial\mathbb{H}$.

Now let us consider the case when ∞ is not a fixed point of T. In this case $c \neq 0$. Multiplying

$$T(z_0) = \frac{az_0+b}{cz_0+d} = z_0$$

by $cz_0 + d$ (which is non-zero as $z_0 \neq -\frac{d}{c}$) we see that z_0 is a fixed point if and only if

$$cz_0^2 + (d-a)z_0 - b = 0.$$

This is a quadratic in z_0 with real coefficients. Hence there are either one or two real solutions, or two complex conjugate solutions. In the latter case, only one solution lies in $\mathbb{H} \cup \partial\mathbb{H}$. Thus, we have proved:

Lemma 1. *Let T be a Möbius transformation of $\mathbb{H}$ and suppose that T is not the identity. Then T has either:*

1. *two fixed points in $\partial\mathbb{H}$ and none in $\mathbb{H}$.*
2. *one fixed point in $\partial\mathbb{H}$ and none in $\mathbb{H}$.*
3. *no fixed points in $\partial\mathbb{H}$ and one in $\mathbb{H}$.*

*In the first case we call T **hyperbolic**, in the second **parabolic** and in the third **elliptic**.*

Corollary 1. *Suppose T is a Möbius transformation of $\mathbb{H}$ with three or more fixed points. Then T is the identity (and so fixes every point).*

Now we go on in our classification by looking at the absolute value Tr of the trace of some Möbius transformation of $\mathbb{H}$ called T. Suppose for simplicity that ∞ is not a fixed point (it follows that $c \neq 0$). So we know z_0 is a fixed point of T if and only if

$$z_0 = \frac{a - d \pm \sqrt{(a-d)^2 + 4bc}}{2c}.$$

Using

$$ad - bc = 1, \qquad\qquad (a+d)^2 = \mathrm{Tr}^2(T)$$

it is easy to see that

$$(a-d)^2 + 4bc = \mathrm{Tr}^2(T) - 4.$$

When $c = 0$, we must have that ∞ is a fixed point. The other fixed point is given by $\frac{b}{d-a}$. Hence ∞ is the only fixed point if $a = d$ (in which case we must have that $a = 1$, $d = 1$ or $a = -1$, $d = -1$ as $ad - bc = ad = 1$); hence $\mathrm{Tr}(T) = |\pm(1+1)| = 2$. If $a \neq d$ then there are two fixed points on $\partial\mathbb{H}$ if $\mathrm{Tr}(T) > 2$ and one fixed point in $\mathbb{H}$ if $\mathrm{Tr}(T) < 2$.

Thus, we have proved:

Lemma 2. *Let T be a Möbius transformation of $\mathbb{H}$ and suppose that T is not the identity. Then:*

- *T is hyperbolic if and only if $\mathrm{Tr}(T) > 2$.*
- *T is parabolic if and only if $\mathrm{Tr}(T) = 2$.*

- *T is elliptic if and only if* $\mathrm{Tr}(T) < 2$.

Definition 2. *Let T_1, T_2 be two Möbius transformations of $\mathbb{H}$. We say that T_1 and T_2 are* ***conjugate*** *if there exists another Möbius transformation of $\mathbb{H}$ called S such that $T_1 = S^{-1} \circ T_2 \circ S$.*

Remark 1.
- Conjugacy between Möbius transformations of $\mathbb{H}$ is an equivalence relation.
- It is easy to see that if T_1 and T_2 are conjugate then they have the same number of fixed points, hence they are of the same type of transformation (elliptic, parabolic, hyperbolic).
- If T_2 has matrix $A_2 \in SL(2, \mathbb{R})$ and S has matrix $A \in SL(2, \mathbb{R})$ then T_1 has matrix $\pm A^{-1} A_2 A$.
- Geometrically, if T_1 and T_2 are conjugate then the action of T_1 on $\mathbb{H} \cup \partial\mathbb{H}$ is the same as the action of T_2 on $S(\mathbb{H} \cup \partial\mathbb{H})$. Thus conjugacy reflects a change in coordinates of $\mathbb{H} \cup \partial\mathbb{H}$.

Lemma 3. *Let T be a Möbius transformation of $\mathbb{H}$ and suppose that T is not the identity. Then the following are equivalent:*

(i) *T is parabolic.*

(ii) $\mathrm{Tr}(T) = 2$.

(iii) *T is conjugate to a translation, i.e. T is conjugate to a Möbius transformation of $\mathbb{H}$ of the form $z \mapsto z + b$ for some $b \in \mathbb{R}$.*

(iv) *T is conjugate either to the translation $z \mapsto z + 1$ or to the translation $z \mapsto z - 1$.*

(v) *The matrix of T is conjugate to* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Proof. We already know that (i) and (ii) are equivalent. Clearly (iv) implies (iii) and (iv) and (v) are equivalent.

In order to prove that (iii) implies (iv) we have to choose $S(z) = \frac{z}{b}$ in Definition 2 and then $z \mapsto z + b$ is conjugate to $z \mapsto z + 1$ or $z \mapsto z - 1$ if $b > 0$ or $b < 0$. Notice that S actually is an element of $PSL(2, \mathbb{R})$ since $S(z) = \frac{z}{\sqrt{b} \cdot \sqrt{b}}$ and thus $ad - bc = \frac{1}{\sqrt{b}} \cdot \sqrt{b} - 0 = 1$.

Suppose now that (iv) holds. Recall that $z \mapsto z + 1$ has a unique fixed point at ∞. Hence if T is conjugate to $z \mapsto z + 1$ then T has a unique fixed point in $\partial\mathbb{H}$, and is therefore parabolic. The same argument holds for $z \mapsto z - 1$.

Finally we show that (i) implies (iii). Suppose that T is parabolic and has a unique fixed point at $\zeta \in \partial\mathbb{H}$. Let S be a Möbius transformation of $\mathbb{H}$ that maps ζ to ∞. Then STS^{-1} is a Möbius transformation of $\mathbb{H}$ with a unique fixed point at ∞. We claim that STS^{-1} is a translation. Write

$$STS^{-1}(z) = \frac{az + b}{cz + d}.$$

As ∞ is a fixed point of STS^{-1}, we must have that $c = 0$. Hence

$$STS^{-1}(z) = \frac{a}{d}z + \frac{b}{d}$$

and it follows that STS^{-1} has a fixed point at $\frac{b}{d-a}$. As STS^{-1} has only one fixed point and the fixed point is at ∞ we must have that $d = a$. Thus $STS^{-1}(z) = z + b'$ for some $b' \in \mathbb{R}$. Hence T is conjugate to a translation. □

Lemma 4. *Let T be a Möbius transformation of $\mathbb{H}$ and suppose that T is not the identity. Then the following are equivalent:*

(i) *T is hyperbolic.*
(ii) $\mathrm{Tr}(T) > 2$.
(iii) *T is conjugate to a dilation, i.e. T is conjugate to a Möbius transformation of $\mathbb{H}$ of the form $z \mapsto kz$, for some $k > 0$.*
(iv) *The matrix of T is conjugate to $\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$, for some $u \in \mathbb{R}$.*

Proof. We have already seen that (i) is equivalent to (ii). Obviously (iii) is equivalent to (iv) with $u^2 = k$.

Suppose (iii) holds. Then T is conjugate to a dilation which has 0 and ∞ as fixed points in $\partial\mathbb{H}$, namely 0 and ∞. Hence T also has exactly two fixed points in $\partial\mathbb{H}$. Hence (i) holds.

Finally, we prove that (i) implies (iii). We first make the remark that if T fixes both 0 and ∞ then T is a dilation. To see this, write

$$T(z) = \frac{az + b}{cz + d}$$

where $ad - bc = 1$. As ∞ is a fixed point of T, we must have that $c = 0$. Hence $T(z) = \frac{az+b}{d}$. As 0 is fixed, we must have that $b = 0$. Hence $T(z) = \frac{a}{d}z$ so that T is a dilation.

Suppose that T is a hyperbolic Möbius transformation of $\mathbb{H}$. Then T has two fixed points in $\partial\mathbb{H}$; denote them by ζ_1, ζ_2.

First suppose that $\zeta_1 = \infty$ and $\zeta_2 \in \mathbb{R}$. Let $S(z) = z - \zeta_2$. Then the transformation STS^{-1} is conjugate to T and has fixed points at 0 and ∞. By that above remark STS^{-1} is a dilation.

Now suppose that $\zeta_1 \in \mathbb{R}$ and $\zeta_2 \in \mathbb{R}$. We may assume that $\zeta_1 < \zeta_2$. Let S be the transformation

$$S(z) = \frac{z - \zeta_2}{z - \zeta_1}.$$

As $-\zeta_1 + \zeta_2 > 0$, this is a Möbius transformation of $\mathbb{H}$. Moreover, as $S(\zeta_1) = \infty$ and $S(\zeta_2) = 0$, we see that STS^{-1} has fixed points at 0 and ∞ and is therefore a dilation. Hence T is conjugate to a dilation. □

Sometimes it will be more convenient to look at elliptic Möbius transformations of $\mathbb{H}$ in the unit disk model. We know $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Aut}(\mathbb{H}) = w\,\mathrm{Aut}(\mathbb{D})w^{-1}$ where $w(z) = \frac{z-i}{z+i}$ is the map which maps $\mathbb{H}$ bijectively to $\mathbb{D}$ and $\partial\mathbb{H}$ bijectively to $\partial\mathbb{D}$ (but w is not a Möbius transformation of $\mathbb{H}$). See Walkden's Script [6] for details. So the corresponding Möbius transformation of $\mathbb{H}$ in the unit disk model of some Möbius transformation of $\mathbb{H}$ called γ in the upper half plane is given by

$$z \mapsto w\gamma w^{-1}. \tag{1.2}$$

By that, it is possible to calculate that a Möbius transformation of $\mathbb{H}$ in $\mathbb{D}$ is a map of the form

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1.$$

Since w is bijective, one can classify $\mathrm{PSL}(2,\mathbb{R})$ in $\mathbb{D}$ exactly the same as in $\mathbb{H}$ and a transformation T of $\mathbb{D}$ is hyperbolic for example if and only if T has two fixed points in $\partial\mathbb{D}$ or if and only if $\mathrm{Tr}(T) > 2$.

Lemma 5. *Let T be a Möbius transformation of $\mathbb{H}$ and suppose that T is not the identity. Then the following are equivalent:*

(i) *T is elliptic.*

(ii) $\mathrm{Tr}(T) < 2$.

(iii) *T is conjugate in $\mathbb{H}$ to a rotation* $z \mapsto \frac{\cos(\theta)z+\sin(\theta)}{-\sin(\theta)z+\cos(\theta)}$.

(iv) *T is conjugate in $\mathbb{D}$ to a rotation* $z \mapsto \exp(i\theta)z$.

(v) *The matrix of T in $\mathbb{H}$ is conjugate to* $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$.

(vi) *The matrix of T in $\mathbb{D}$ is conjugate to* $\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}$ *with* $|u| = 1$ *and* $u \neq 1$.

Proof. We have already seen that (i) is equivalent to (ii). Again it is obvious, that (iii) and (v) are equivalent and (iv) and (vi) are equivalent ($u^2 = \exp(i\theta)$).

(iii) and (iv) is equivalent. To see that we need again the fact (1.2). See Theorem 8.19 of [5] for detailed calculation.

Suppose that (iv) holds. A rotation has a unique fixed point (at the origin). If T is conjugate to a rotation then it must also have a unique fixed point, and so is elliptic.

Finally, we prove that (i) implies (iv). Suppose that T is elliptic and has a unique fixed point at $\zeta \in \mathbb{D}$. Let S be a Möbius transformation of $\mathbb{D}$ that maps ζ to the origin 0. Then STS^{-1} is a Möbius transformation of $\mathbb{H}$ that is conjugate to T and has a unique fixed point at 0. Suppose that

$$STS^{-1}(z) = \frac{\alpha z + \beta}{\bar{\beta} z - \bar{\alpha}}$$

where $|\alpha|^2 - |\beta|^2 > 0$. As 0 is a fixed point, we must have that $\beta = 0$. Write α in polar form as $\alpha = r\exp(i\theta)$. Then

$$STS^{-1}(z) = \frac{\alpha}{\bar{\alpha}}z = \frac{r\exp(i\theta)}{r\exp(-i\theta)}z = \exp(2i\theta)z$$

so that T is conjugate to a rotation. □

1.4 Fuchsian Groups

Besides being a group, $\mathrm{PSL}(2,\mathbb{R})$ is also a topological space in which a transformation $z \mapsto \frac{az+b}{cz+d}$ can be identified with the point $(a,b,c,d) \in \mathbb{R}^4$. More precisely, as a topological space, $SL(2,\mathbb{R})$ can be identified with the subset of $\mathbb{R}^4$

$$X = \{(a,b,c,d) \in \mathbb{R}^4 | ad - bc = 1\}.$$

We define $\delta : X \to X, (a,b,c,d) \mapsto (-a,-b,-c,-d)$ and topologize

$$PSL(2,\mathbb{R}) \simeq X/\{id, \delta\}$$

where id, δ is a cyclic group of order 2 acting on X. One can prove that the group multiplication and taking of inverse are actually continuous with respect to the topology on $\mathrm{PSL}(2,\mathbb{R})$. A norm on $\mathrm{PSL}(2,\mathbb{R})$ is induced from $\mathbb{R}^4$: for $T(z) = \frac{az+b}{cz+d}$ in $\mathrm{PSL}(2,\mathbb{R})$, we define

$$\|T\| = (a^2 + b^2 + c^2 + d^2)^{\frac{1}{2}}.$$

Hence $\mathrm{PSL}(2,\mathbb{R})$ is a topological group with respect to the metric $d(T,S) := \|T - S\|$ for $T, S \in \mathrm{PSL}(2,\mathbb{R})$.

Definition 3. *A set S in a topological space X is called **discrete** if every point $x \in S$ has a neighbourhood U such that $S \cap U = \{x\}$.*

Definition 4. *A discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *is called **Fuchsian group**.*

Examples. (i) The subgroup of integer translations $\{\gamma_n(z) = z+n | n \in \mathbb{Z}\}$ is a Fuchsian group. For example here, for $n, m \in \mathbb{Z}$ we have

$$\begin{aligned} d(\gamma_n, \gamma_m) = \|\gamma_n - \gamma_m\| &= \big((1-1)^2 + (n-m)^2 + (0-0)^2 + (1-1)^2\big)^{\frac{1}{2}} \\ &= n - m. \end{aligned}$$

The subgroup of all translations $\{\gamma_b(z) = z+b | b \in \mathbb{R}\}$ is not a Fuchsian group, as it is not discrete.

(ii) Any finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group. This is because any finite subset of any metric space is discrete.

(iii) As a specific example, let

$$\gamma_\theta(z) = \frac{\cos(\theta)z + \sin(\theta)}{-\sin(\theta)z + \cos(\theta)}$$

be a rotation around i. Let $q \in \mathbb{N}$. Then $\{\gamma_{\frac{\pi j}{q}} | 0 \leq j \leq q-1\}$ is a finite subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

(iv) The *modular group* $PSL(2, \mathbb{Z})$ is Fuchsian. This is the group given by Möbius transformation of $\mathbb{H}$s of the form

$$\gamma(z) = \frac{az+b}{cz+d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1.$$

(v) Let $q \in \mathbb{N}$. Define

$$\Gamma_q = \{\gamma(z) = \frac{az+b}{cz+d} | a, b, c, d \in \mathbb{Z}, ad - bc = 1, b, c \text{ are divisible by } q\}.$$

This is called the *level q modular group* which is also a Fuchsian group.

Definition 5. *A group G is called* ***cyclic*** *if*

$$G =< g >= \{g^n | n \textit{ is an integer}\}$$

for some $g \in G$.

Definition 6. *Let X be a metric space, and let G be a group of homeomorphisms of X. For $x \in X$, we call*

$$Gx = \{g(x) | g \in G\}$$

the ***G-orbit of the point x****.*

Definition 7. *A subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ is called* ***elementary*** *if there exists a finite Γ-orbit in $\tilde{\mathbb{H}} := \mathbb{H} \cup \partial\mathbb{H}$.*

Definition 8. *Let G be a group. The* ***centralizer*** *of $g \in G$ is defined by*

$$C_G(g) = \{g \in G | hg = gh\}.$$

Lemma 6. *If $ST = TS$ then S maps the fixed-point set of T to itself.*

Proof. Suppose $T(p) = p$ for some p. Then

$$S(p) = ST(p) = TS(p).$$

So T fixes $S(p)$. □

Lemma 7. *(i) Any non-trivial discrete subgroup of $\mathbb{R}$ is infinite cyclic.*
(ii) Any discrete subgroup of S^1 is finite cyclic.

Proof. (i) Let Γ be a discrete subgroup of $\mathbb{R}$. We have $0 \in \Gamma$ and since Γ is discrete there exists a smallest positive $x \in \Gamma$. Then $\{nx|n \in \mathbb{Z}\}$ is a subgroup of Γ. Suppose there is a $y \in \Gamma$, $y \neq nx$. We may assume $y > 0$, otherwise we take $-y$ which also belongs to Γ. There exists an integer $k \geq 0$ such that $kx < y < (k+1)x$. and $y - kx < x$, and $(y - kx) \in \Gamma$ which contradicts the choice of x.

(ii) Let Γ now be a discrete subgroup of S^1. By discreteness there exists $z = \exp^{i\phi_0} \in \Gamma$, with the smallest argument ϕ_0, and for some $m \in \mathbb{Z}$, $m\phi_0 = 2\pi$, otherwise we get a contradiction with the choice of ϕ_0. □

Theorem 1. *Two non-identity elements of* $\mathrm{PSL}(2, \mathbb{R})$ *commute if and only if they have the same fixed-point set.*

Proof. To prove that, let us look at the centralizer of parabolic, elliptic and hyperbolic elements in $\mathrm{PSL}(2, \mathbb{R})$. Suppose that $T(z) = z + 1$. If $S \in C_{\mathrm{PSL}(2,\mathbb{R})}(T)$ then $S(\infty) = \infty$. Therefore, $S(z) = az + b$. $ST = TS$ gives us $a = 1$. Hence

$$C_{\mathrm{PSL}(2,\mathbb{R})}(T) = \{z \mapsto z + k|k \in \mathbb{R}\}.$$

The centralizer of an elliptic transformation of the unit disk $\mathbb{D}$ fixing 0 (i.e. $z \mapsto \exp(i\varphi)z$) consists of all transformations of the form $z \mapsto \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}$. fixing 0, i.e. of the form $z \mapsto \exp(i\theta)z$ $(0 \leq \theta < 2\pi)$. Let $T(z) = \lambda z$ $(\lambda > 0, \lambda \neq 1)$ and $S \in C_{\mathrm{PSL}(2,\mathbb{R})}(T)$. After some direct calculation we find out that S is given by a diagonal matrix and hence $S(z) = \gamma z$ $(\gamma > 0)$. □

Theorem 2. *Let* Γ *be a Fuchsian group. If all non-identity elements of* Γ *have the same fixed-point set, then* Γ *is cyclic.*

Proof. Suppose all elements of Γ are hyperbolic, so they have two fixed points in $\mathbb{R} \cup \{\infty\}$. By choosing a conjugate group we may assume that each $S \in \Gamma$ fixes 0 and ∞. Thus Γ is a discrete subgroup of $H = \{z \to \lambda z|\lambda > 0\}$ which is isomorphic to $(\mathbb{R}^*, \cdot)$. As a topological group $(\mathbb{R}^*, \cdot)$ is isomorphic to $\mathbb{R}$ via $x \mapsto \ln x$. By Lemma 7, Γ is infinite cyclic. If Γ contains a parabolic element, then Γ is an infinite cyclic group containing only parabolic elements.

Suppose Γ contains an elliptic element. In $\mathbb{D}$, Γ is a discrete subgroup of orientation-preserving isometries of $\mathbb{D}$. Again by choosing a conjugate group we may assume that all elements of Γ have 0 as a fixed point, and so all elements of Γ are of the form $z \mapsto \exp^{i\phi} z$. Thus Γ is isomorphic to a subgroup of S^1, and it is discrete if and only if the corresponding subgroup of S^1 is discrete. The rest follows from Lemma 7. □

Theorem 3. *Every Abelian Fuchsian group is cyclic.*

Proof. By Theorem 1, all non-identity elements in an Abelian Fuchsian group have the same fixed-point set. The theorem follows immediately from Theorem 2. □

Theorem 4. *Let Γ be a Fuchsian subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *containing besides the identity only elliptic elements. Then all elements of Γ have the same fixed point, and hence Γ is a cyclic group, Abelian and elementary.*

Proof. We shall prove that all elliptic elements in Γ must have the same fixed point. In the unit disk let us conjugate Γ in such a way that an element $\mathrm{id} \neq g \in \Gamma$ fixes 0, so $g = \left(\begin{smallmatrix} u & 0 \\ 0 & \overline{u} \end{smallmatrix}\right)$. Let $h = \left(\begin{smallmatrix} a & \overline{c} \\ c & \overline{a} \end{smallmatrix}\right) \in \Gamma$, $h \neq g$. We have $\mathrm{tr}[g,h] = \mathrm{tr}(g \circ h \circ g^{-1} \circ h^{-1}) = 2 + 4|c|^2(\mathrm{Im}(u))^2$. Since Γ does not contain hyperbolic elements, $|\mathrm{tr}[g,h]| \leq 2$. So either $\mathrm{Im}(u) = 0$ or $c = 0$. If $\mathrm{Im}(u) = 0$ then $u = \overline{u}$ and hence $g = \mathrm{id}$, a contradiction. Hence $c = 0$, and so $h = \left(\begin{smallmatrix} a & 0 \\ 0 & \overline{a} \end{smallmatrix}\right)$ also fixes 0. Thus all elements of Γ have the same fixed point. By Theorem 2, Γ is a cyclic group, and hence Abelian. The set $\{0\}$ is a Γ-orbit, and so Γ is elementary. $\square$

So we have the obvious

Corollary 2. *Any Fuchsian group containing besides the identity only elliptic elements is a finite cyclic group.*

We can now state a theorem which describes all elementary Fuchsian groups.

Theorem 5. *Any elementary Fuchsian group is either cyclic or is conjugate in* $\mathrm{PSL}(2,\mathbb{R})$ *to a group generated by $h(z) = -\frac{1}{z}$ and $g(z) = kz$ for some $k > 1$.*

Proof. Case 1. Suppose Γ fixes a single point $a \in \tilde{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$. If $a \in \mathbb{H}$, then all elements of Γ are elliptic; by Corollary 2, Γ is a finite cyclic group.

Suppose $a \in \mathbb{R} \cup \{\infty\}$.Then Γ cannot have elliptic elements. We are going to show that hyperbolic and parabolic elements cannot have a common fixed point. Assume the opposite, and suppose this point is ∞, and $g(z) = \lambda z$ for some $\lambda > 1$ and $h(z) = z + k$ (since g and h have only one common point, $k \neq 0$). Then

$$g^{-n} \circ h \circ g^n(z) = z + \lambda^{-n}k.$$

As $\lambda > 1$ the sequence $\|g^{-n} \circ h \circ g^n\|$ is bounded and so, $\{g^{-n} \circ h \circ g^n\}$ contains a convergent subsequence of distinct terms which contradicts the discreteness of Γ. So, Γ must contain only elements of one type. If Γ only contains only parabolic elements, we know from Theorem 2 it is an infinite cyclic group. Now consider the case in which Γ contains only hyperbolic elements. We are going to prove that the second fixed point of these hyperbolic elements must also coincide, and so Γ will fix two points in $\mathbb{R} \cup \{\infty\}$. Suppose $f(z) = \lambda^2 z$ where $\lambda > 1$ so it fixes 0 and ∞ and suppose $g(z) = \frac{az+b}{cz+d}$ which fixes 0 but not ∞. Then $b = 0$, $c \neq 0$, $a \neq 0$ and $d = \frac{1}{a}$. Then

$$[f,g] = f \circ g \circ f^{-1} \circ g^{-1} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

with $t = \frac{c}{a}\left(\frac{1}{\lambda^2} - 1\right)$. Since $c \neq 0$, $[f, g]$ is a parabolic element in Γ, a contradiction.

Case 2. Suppose Γ has an orbit in $\mathbb{R} \cup \{\infty\}$ consisting of two points. An element of Γ either fixes each of them or interchanges them. A parabolic element cannot fix two points. Since each orbit (except for a single point of a parabolic transformation) is infinite, a parabolic element cannot interchange these points; hence Γ does not contain any parabolic elements. All hyperbolic elements must have the same fixed point set. If Γ contains only hyperbolic elements, then it is cyclic by Theorem 3. If it contains only elliptic elements, it is finite cyclic by Corollary 2. If Γ contains both hyperbolic and elliptic elements, it must contain an elliptic element of order 2 interchanging the common fixed points of the hyperbolic elements; and then Γ is conjugate to a group generated by $g(z) = kz$ $(k > 1)$ and $h(z) = -\frac{1}{z}$.

Case 3. Suppose now Γ has an orbit in $\mathbb{H}$ consisting of $k = 2$ points or an orbit in $\tilde{\mathbb{H}}$ consisting of $k \geq 3$ points. Γ must contain only elliptic elements, since the parabolic and hyperbolic elements can have only either fixed points at infinite or infinite orbits. So Γ is a finite cyclic group and it is conjugate to a group generated by $z \to \exp^{\frac{2\pi i}{k}} z$. $\square$

Chapter 2

Jørgensen Inequality

2.1 Proof of Jørgensen Inequality

Theorem 6. *A non-elementary subgroup* Γ *of* $\mathrm{PSL}(2,\mathbb{R})$ *must contain a hyperbolic element.*

Proof. Suppose Γ does not contain hyperbolic elements. If Γ contains only elliptic elements (and id), then by Theorem 4 it is elementary. Hence Γ contains a parabolic element, say $f(z) = z + 1$, which fixes ∞. Let $g(z) := \frac{az+b}{cz+d}$ be an element in Γ. Then $f^n \circ g = \frac{(a+nc)z+(b+nd)}{cz+d}$. So we have

$$\mathrm{tr}^2(f^n \circ g) = (a + d + nc)^2.$$

Since all elements in the group are either elliptic or parabolic, we have $0 \leq (a + d + nc)^2 \leq 4$ for all n, so $c = 0$. But then g fixes ∞ as well, so that ∞ is fixed by all elements in Γ; hence Γ is elementary, a contradiction. $\square$

Let $< T, S >$ be the group generated by Möbius transformations of $\mathbb{H}$ called T and S. So $< T, S >= \{\prod_{n,m=1}^{r} T^{l_n} S^{l_m} \mid l_n, l_m, r \in \mathbb{N}\}$.

The Jørgensen Inequality which follows now, states that if a discrete group generated by two elements in $\mathrm{PSL}(2,\mathbb{R})$ is non-elementary, then at least one of this elements must differ considerably from the identity.

Theorem 7 (Jørgensen Inequality)**.** *Suppose that* $T, S \in \mathrm{PSL}(2,\mathbb{R})$ *and* $< T, S >$ *is a discrete non-elementary group. Then*

$$|\,\mathrm{tr}^2(T) - 4| + |\,\mathrm{tr}(TST^{-1}S^{-1}) - 2| \geq 1. \tag{2.1}$$

The lower bound is best possible.

Before we are able to prove that, we need three Lemmas:

Lemma 8. *Suppose* $T, S \in \mathrm{PSL}(2,\mathbb{R})$ *and* $T \neq \mathrm{id}$. *Define* $S_0 = S$, $S_1 = S_0 \circ T \circ S_0^{-1}$, *...,* $S_r \circ T \circ S_r^{-1}$, *... . If, for some* n, $S_n = T$, *then* $< T, S >$ *is elementary and* $S_2 = T$.

Proof. Suppose the case where T has one fixed point α, so T is either parabolic or elliptic. S_r has one fixed point as well, since it is conjugate to T. Because of

$$S_{r+1} \circ S_r(\alpha) = S_r \circ T \circ S_r^{-1} \circ S_r(\alpha) = S_r(\alpha)$$

we have that S_{r+1} fixes $S_r(\alpha)$. So S_{r+1} fixes the same point as S_r. Now by the fact that $S_n(= T)$ fixes α and that each S_r has one fixed point it follows that S_r fixes α for every $r \geq 0$.

As a consequence we know that all elements in $< T, S >$ fix α. We conclude that all elements in $< T, S >$ are parabolic and by Theorem 6. If T is elliptic, all elements in $< T, S >$ are elliptic and by Theorem 4 $< T, S >$ is elementary.

Suppose now that T has exactly two fixed points. We may assume then that $T(z) = kz$. With he same argument as above $S_1, ..., S_n$ have exactly two fixed points and for $0 \leq r \leq n$ we have $\{S_r(0), S_r(\infty)\} = \{0, \infty\}$. Since S_r $(r \geq 1)$ is conjugate to T, it cannot interchange two points (all orbits of a hyperbolic transformation are infinite with the exception of two fixed points). Thus $S_1, ...S_n$ fix 0 and ∞, and both $S = S_0$ and T leave the set $\{0, \infty\}$ invariant. Therefore $< T, S >$ is elementary. □

Lemma 9. *If* $T^2 = \mathrm{id}$ *for some* $T \in \mathrm{PSL}(2, \mathbb{R})$*,* $T \neq \mathrm{id}$ *then* $\mathrm{tr}(T) = 0$

Proof. It is

$$T^2 = \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So either $a, d = 0$ and $b, c = 1$ or $a, d = 1$ and $b, c = 0$. The second case is not possible since $T \neq \mathrm{id}$. So $\mathrm{tr}(T) = a + d = 0$. □

Lemma 10. *Let* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$*,* $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be two matrices in* $SL(2, \mathbb{R})$*. Prove that the Jørgensen inequality for* T *and* S *holds if and only if* $|c| \geq 1$*.*

Proof.

$$|\,\mathrm{tr}^2(T) - 4| + |\,\mathrm{tr}(TST^{-1}S^{-1}) - 2| = 0 + |2(ad - bc) + c^2 - 2| = c^2$$

So (2.4) is hold if and only if $|c| \geq 1$. □

Proof of Theorem 7. We know that $< T, S >$ is discrete and non-elementary. Now (2.4) holds if T is of order two (because then we know from Lemma 9, $\mathrm{tr}^2(T) = 0$) so we may assume that if T is not of order two. We define

$$S_0 = S, \qquad S_{n+1} = S_n T S_n^{-1}. \tag{2.2}$$

By Lemma 8 we know $S_n \neq T$ for any n. It remains only to show that if (2.4) fails, then for some n we have

$$S_n = T \tag{2.3}$$

and we consider two cases.

Case 1. T is parabolic.

We first assume that

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $c \neq 0$ (else $< T, S >$ was elementary). We are assuming that (2.4) fails and this is by Lemma 10 the assumption that

$$|c| < 1.$$

The relation (2.2) yields

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_n & -b_n \\ -c_n & a_n \end{pmatrix} = \begin{pmatrix} 1 - a_n c_n & a_n^2 \\ -c_n^2 & 1 + a_n c_n \end{pmatrix}$$

So by induction $c_n = -(-c)^{2n} = -c^{2n}$ for $n > 0$ and thus $c_n \to 0$ as $|c| < 1$. Since we have $|c_n| < 1$, by induction we see that $a_n \leq n + |a_0|$, so $a_n c_n \to 0$ and $a_{n+1} \to 1$. This proves that

$$S_{n+1} \to T$$

which, by discreteness, yields (2.3) for large n.

Actually we have to consider the case $T = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. But this works completely similar and yields the same result.

Case 2. T is hyperbolic or elliptic.

Without loss of generality,

$$T = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}.$$

In the hyperbolic case, T is the matrix for the transformation in $\mathbb{H}$, in the elliptic case, T is the matrix for the transformation in $\mathbb{D}$. S is as in Case 1 and $bc \neq 0$ (else $< T, S >$ is elementary). The assumption that (2.4) fails is

$$\mu := |\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| = (1 + |bc|)|u - \frac{1}{u}|^2 < 1.$$

Again we write $S_n = \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$ and obtain from $S_{n+1} = S_n \circ T \circ S_n^{-1}$

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} a_n d_n u - \frac{b_n c_n}{u} & a_n b_n \left(\frac{1}{u} - u\right) \\ c_n d_n \left(u - \frac{1}{u}\right) & \frac{a_n d_n}{u} - b_n c_n u \end{pmatrix}$$

so $b_{n+1} c_{n+1} = -b_n c_n (1 + b_n c_n)(u - \frac{1}{u})^2$. By induction

$$|b_n c_n| \leq \mu^n |bc| \leq |bc|.$$

So $b_n c_n \to 0$ and $a_n d_n = 1 + b_n c_n \to 1$. Also, we obtain $a_{n+1} \to u$ and $d_{n+1} \to \frac{1}{u}$. We have

$$|\frac{b_{n+1}}{b_n}| = |a_n \left(\frac{1}{u} - u\right)| \leq \mu^{\frac{1}{2}}|u|.$$

Thus $|\frac{b_{n+1}}{u^{n+1}}| < \mu^{\frac{1}{2}}|\frac{b_n}{u^n}|$ for some sufficiently large n. So $\frac{b_n}{u^n} \to 0$ and similarly $c_n u^n \to 0$. So

$$T^{-n} S_{2n} T^n = \begin{pmatrix} a_{2n} & \frac{b_{2n}}{u^{2n}} \\ c_{2n} u^{2n} & d_{2n} \end{pmatrix}$$

Since $< T, S >$ is discrete, for large n we have

$$T^{-n} S_{2n} T^n = T$$

and again we have $S_{2n} = T$.

Finally, we show that the lower bound (2.4) is best possible. Consider the group generated by $T(z) = z + 1$ and $S(z) = -\frac{1}{z}$. It is $< T, S >= \mathrm{PSL}(2, \mathbb{Z})$ (see Katok [4] Chapter 3.2, Example A) which is discrete and non-elementary. We have $T \circ S \circ T^{-1} \circ S^{-1}(z) = \frac{2z+1}{z+1}$ with trace 3, and hence the equality holds in (2.4). □

Remark 2. The Jørgensen Inequality also holds for non-elementary discrete groups in $\mathrm{PSL}(2, \mathbb{C})$.

2.2 A criterion for discreteness

In order to prove Theorem 8 we need the following two general results.

Lemma 11. *If Γ is elementary, for any $T, S \in \Gamma$, $< T, S >$ is elementary.*

Proof. Conversely, suppose Γ is not elementary. By Theorem 6 it contains a hyperbolic element T with fixed points α and β. Since Γ is not elementary, there exists $S \in \Gamma$ which does not leave the set $\{\alpha, \beta\}$ invariant. Hence $< T, S >$ is not elementary. □

Lemma 12. *Any non-elementary subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ must contain infinitely many hyperbolic elements, no two of which have a common fixed point.*

Proof. We choose a hyperbolic element T in Γ with fixed points $\{\alpha, \beta\}$ and an element S in Γ which does not leave $\{\alpha, \beta\}$ fixed. We can find such S like in the proof of Lemma 11. Suppose first that the sets $\{\alpha, \beta\}$ and $\{S(\alpha), S(\beta)\}$ do not intersect. In this case, the elements T and $T_1 = STS^{-1}$ both are hyperbolic and have no common fixed point ($S(\alpha)$ and $S(\beta)$ are the fixed points of T_1). The sequence $\{T^n T_1 T^{-n}\}$ consists of hyperbolic elements with fixed points $T^n S(\alpha)$ and $T^n S(\beta)$ which are pairwise different.

If the set $\{\alpha, \beta\}$ and $\{S(\alpha), S(\beta)\}$ have one point of intersection, say α, then we wish to show that $P = [T, T_1]$ is parabolic with α as the only fixed point. So we conjugate Γ so that the fixed points of T are 0 and ∞, and the fixed point it shares with T_1 is ∞. Then

$$T = \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}, \qquad T_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $u > 1$ and $a, b, c, d \in \mathbb{R}$. If T_1 is to fix ∞, then $c = 0$, and since 0 is not a fixed point, $b \neq 0$. This means T_1 has the form $T_1 = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. We can then compute

$$P = [T, T_1] = TT_1T^{-1}T_1^{-1} = \begin{pmatrix} 1 & b(u^2 - 1) \\ 0 & 1 \end{pmatrix},$$

so that its transform is of the form $T_1(z) = z + t$ for non-zero t, and other than ∞, every point is fixed.

Since $\{\alpha\}$ cannot be Γ-invariant there exists $U \in \Gamma$ not fixing α. So $Q = UPU^{-1}$ is parabolic and does not fix α. Therefore Q and T have no common fixed points. Then for large n, the elements T and Q^nTQ^{-n} are hyperbolic and have no common fixed point, and the problem is reduced to the first case. □

Theorem 8. *A non-elementary subgroup Γ of $\mathrm{PSL}(2, \mathbb{R})$ is discrete if and only if, for each T and S in Γ, the group $< T, S >$ is discrete.*

Proof. If Γ is discrete, then every subgroup of it is also discrete.

Suppose now that every subgroup $< T, S >$ in Γ is discrete, but Γ itself is not. So we find a sequence of distinct transformations in Γ, $T_1, T_2, ...$, such that $T_i \neq \mathrm{id}$ and $\lim_{n\to\infty} T_n = \mathrm{id}$. From Lemma 9 we know that $T^2 = \mathrm{id}$ implies $\mathrm{tr}(T) = 0$ and as tr is a continuous function on $\mathrm{PSL}(2, \mathbb{R})$, we may choose a subsequence that contains no elements of order 2. For any $S \in \Gamma$ we have

$$|\,\mathrm{tr}^2(T_n) - 4| + |\,\mathrm{tr}(T_nST_n^{-1}S^{-1}) - 2| \to 0$$

and so by Theorem 7, there is some $N := N(S) \in \mathbb{N}$ such that for $n \geq N$, the group $< T_n, S >$ is elementary. By Lemma 12 we know that Γ contains two hyperbolic elements S_1 and S_2 with no common fixed points. So for $n \geq \max(N(S_1), N(S_2))$ both $< T_n, S_1 >$ and $< T_n, S_2 >$ are elementary, discrete and according to Theorem 5, they leave the (distinct) fixed point pair of S_1 and that of S_2 invariant. Since T_n is not elliptic of order 2, it cannot interchange a pair of points, so T_n must fix four distinct points which implies $T_n = \mathrm{id}$ which is a contradiction. □

2.3 Extreme Fuchsian groups

In this section we will work with the notion of an Extreme Fuchsian group and a triangle group. We will write down the definitions first:

Definition 9. *Suppose that* $T, S \in \mathrm{PSL}(2,\mathbb{R})$ *and* $< T, S >$ *is a discrete non-elementary group. Then* $< T, S >$ *is called* ***extreme*** *if*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| = 1, \tag{2.4}$$

so if there is equality in the Jørgensen Inequality.

Definition 10. *A* ***Fuchsian triangle group of signature*** (n, m, p) *with* $n, m, p \in \mathbb{N} \cup \{\infty\}$ *is a subgroup of* $\mathrm{PSL}(2,\mathbb{R})$ *generated by three elliptic elements* $\delta_1, \delta_2, \delta_3$ *whith orders* n, m, p *respectively and* $\frac{1}{n} + \frac{1}{m} + \frac{1}{p} < 1$.

Our last result in this thesis Theorem 9 was proved by Jørgensen himself in [3]. We need two lemmas and the following identity for $T, S \in \mathrm{PSL}(2,\mathbb{R})$

$$\operatorname{tr}^2(T) + \operatorname{tr}^2(S) + \operatorname{tr}^2(TS) = \operatorname{tr}(TST^{-1}S^{-1}) + \operatorname{tr}(T)\operatorname{tr}(S)\operatorname{tr}(TS) + 2 \tag{2.5}$$

which can be proven by calculation.

Lemma 13. *Suppose that* $T, S \in \mathrm{PSL}(2,\mathbb{R})$ *and* $< T, S >$ *is a discrete non-elementary group and*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| = 1. \tag{2.6}$$

Then $< T, S_1 >$ *is a discrete non-elementary group where* $S_1 = STS^{-1}$ *and*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TS_1T^{-1}S_1^{-1}) - 2| = 1. \tag{2.7}$$

Proof. The group generated by T and S_1 is discrete because it is a subgroup of the discrete group generated by S and T.

Suppose T is parabolic. Then the group generated by T and S_1 were elementary if and only if the fixed point of T where fixed by S_1 and hence by S. As the group generated by S and T is non-elementary, this is not so. If $\operatorname{tr}(TST^{-1}S^{-1})$ where equal to 2, then T and S would have a common fixed point. This is not so, the group generated by T and S being non-elementary. Thus $|\operatorname{tr}(T) - 2|$ is strictly less than 1, by (2.6), and hence the order of T exceeds 6.

Therefore assuming now T is elliptic, if the group generated by T and S_1 were elementary, then either S_1 would keep the fixed points of T fixed or S_1 would interchange the fixed points of T. In the former case we deduce that also S would either keep the fixed points of T fixed or S would interchange the fixed points of T . In the latter case, S would have to interchange the fixed points of T. None of the two cases can thus occur since the group generated by T and S is non-elementary. We have proved that the group $< T, S_1 >$ is non-elementary.

To verify (2.6), we apply (2.5) to T and S_1^{-1}. Since the T and S_1^{-1} are conjugate we have $\operatorname{tr}(T) = \operatorname{tr}(S_1)$. After some elementary calculations we find out that $\operatorname{tr}(TS_1^{-1}T^{-1}S_1) = \operatorname{tr}(TS_1T^{-1}S_1^{-1})$ and

$$\operatorname{tr}(TS_1T^{-1}S_1^{-1}) - 2 = \left[\operatorname{tr}(TST^{-1}S^{-1}) - 2\right]\left[\operatorname{tr}(TST^{-1}S^{-1}) - \operatorname{tr}^2(T) + 2\right]. \tag{2.8}$$

Because of (2.6) and the triangle inequality, we get from (2.8)

$$|\operatorname{tr}(TS_1T^{-1}S_1^{-1}) - 2| \leq |\operatorname{tr}(TST^{-1}S^{-1}) - 2|. \tag{2.9}$$

Applying (2.6) once more, we get from (2.9)

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TS_1T^{-1}S_1^{-1}) - 2| \leq 1.$$

But here equality must hold good since otherwise the Jørgensen inequality would be violated. □

Lemma 14. *Suppose that $T, S \in \mathrm{PSL}(2, \mathbb{R})$ and $< T, S >$ is a discrete non-elementary group and*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| = 1.$$

Then T is elliptic of order at least 7 or T is parabolic. Furthermore, if T is elliptic, then $\operatorname{tr}(TSTS^{-1}) = 1$.

Proof. Suppose that T is not parabolic. Then we have $\operatorname{tr}(T) \neq 2$. Consider $S_1 = STS^{-1}$. As a corollary to the proof of Lemma 13, we have

$$|\operatorname{tr}(TST^{-1}S^{-1}) - \operatorname{tr}^2(T) + 2| = |\operatorname{tr}(TST^{-1}S^{-1}) - 2| + |\operatorname{tr}^2(T) - 4|.$$

this because (2.9) was seen to hold good with equality. Consequently, the ratio between $\operatorname{tr}(TST^{-1}S^{-1})$ and $4 - \operatorname{tr}^2(T)$ must be a positive real number. Notice here that $\operatorname{tr}(TST^{-1}S^{-1}) \neq 2$ since T and S generate a non-elementary group.

Repeating the argument, now with $S_2 = S_1TS_1^{-1}$ instead of T_1, we deduce that $\operatorname{tr}(TS_1T^{-1}S_1^{-1}) - 2$ must be a positive multiple of $4 - \operatorname{tr}^2(T)$.

By (2.8) we know that $4 - \operatorname{tr}^2(T)$ is a positive real number. Because of (2.6) it is less than 1. Hence, T is elliptic of order at least 7.

Furthermore, we see that $\operatorname{tr}(TST^{-1}S^{-1}) - 2$ is a positive real number. Thus (2.6) may be written as

$$\operatorname{tr}(TST^{-1}S^{-1}) - \operatorname{tr}^2(T) + 2 = 1$$

and since

$$\operatorname{tr}(TST^{-1}S^{-1}) + \operatorname{tr}(TSTS^{-1}) = \operatorname{tr}^2(T) \tag{2.10}$$

also the last assertion in Lemma 14 is proved. □

Theorem 9. *Suppose that $T, S \in \mathrm{PSL}(2, \mathbb{R})$ and $< T, S >$ is a discrete non-elementary group. Then*

$$|\operatorname{tr}^2(T) - 4| + |\operatorname{tr}(TST^{-1}S^{-1}) - 2| = 1$$

if and only if $< T, S >$ is a triangle group of signature $(2, 3, q)$ where $q \in \{7, 8, 9, ..., \infty\}$.

Proof. Consider the case in which T is parabolic, then we get from (2.5)

$$\operatorname{tr}(TST^{-1}S^{-1}) - 2 = (\operatorname{tr}(S) \pm \operatorname{tr}(TS))^2. \tag{2.11}$$

Together (2.6) and (2.11) give $\operatorname{tr}(TST^{-1}S^{-1}) = 3$. Using (2.10), we have $\operatorname{tr}(TSTS^{-1}) = 1$. So by Lemma 14, we have $\operatorname{tr}(TSTS^{-1}) = 1$ in any case, so $TSTS^{-1}$ is elliptic of order 3. As seen from the general identities (2.5) and (2.10), it means that

$$1 = \operatorname{tr}^2(S) + \operatorname{tr}^2(TS) - \operatorname{tr}(T)\operatorname{tr}(S)\operatorname{tr}(TS) \tag{2.12}$$

or, what is easily seen to be the same

$$1 = \operatorname{tr}^2(S) - \operatorname{tr}(TS)\operatorname{tr}(S^{-1}T). \tag{2.13}$$

Consider the subgroup $< T, S_1 >$ of $< T, S >$ where $S_1 = STS^{-1}$. By Lemma 13, we know, that

$$|\operatorname{tr}(T) - 2| + |\operatorname{tr}(TS_1T^{-1}S_1^{-1}| = 1.$$

We may as well take T and $S^* := TS_1$ (elliptic of order 3) as generators. Substituting S^* for S in (2.13), we obtain

$$0 = \operatorname{tr}(TS^* \operatorname{tr}(S_1)$$

and since $\operatorname{tr}(S_1) = \operatorname{tr}(T) \neq 1$, we see that TS^* is elliptic of order 2.

Example. Consider the group generated by $T(z) = z+1$ and $S(z) = -\frac{1}{z}$ as in the proof of the Jørgensen Inequality. We already know $< T, S >= \mathrm{PSL}(2, \mathbb{Z})$ is an extreme Fuchsian group. One can prove that $\mathrm{PSL}(2, \mathbb{Z})$ is a triangle group of signature $(2, 3, \infty)$.

Since T is either elliptic of order at least 7 or parabolic, we have proved that $< T, S_1 >$ is one of the triangle groups spoken of in Theorems 9. This is because being elliptic of order at least 7 means rotation of angle at least $\frac{2\pi}{7}$. But such groups are maximal, that is, such groups cannot be subgroups of strictly larger Fuchsian groups (see Greenberg [1]). Thus $< T, S_1 >=< T, S >$. □

Finally the question arises whether the condition as stated in the Jørgensen Lemma is a consequence of stronger inequalities such as

$$|\operatorname{tr}^2(T) + \operatorname{tr}(TST^{-1}S^{-1}) - 6| \geq 1$$

or

$$|\operatorname{tr}^2(T) - \operatorname{tr}(TST^{-1}S^{-1}) - 2| \geq 1$$

by means of "unnecessary" use of the triangle-inequality. The answer to each of these questions is in the negative. For that look at

$$T = \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{1}{4\sqrt{2}} \\ 2\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad S = \begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\ -2\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

One can prove that $< T, S >$ is indeed a Fuchsian group (see Jørgensen [2]). Then, we have $\mathrm{tr}^2(T) = (2\sqrt{2})^2 = 8$, $\mathrm{tr}(TST^{-1}S^{-1}) = -2$ and thus

$$\mathrm{tr}^2(T) + \mathrm{tr}(TST^{-1}S^{-1}) - 6 = 0.$$

References

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