Periodic Bounce Orbits Obtained by Various Approximation Schemes

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Abstract. In this thesis, we investigate which approximation schemes are suitable to achieve periodic bounce orbits of prescribed energy on a bounded domain in $\mathbb{R}^N$.

Zusammenfassung. In dieser Arbeit untersuchen wir welche Approximationsschemata geeignet sind, um „periodic bounce orbits“ auf einer beschränkten Domäne in $\mathbb{R}^N$ unter vorgegebener Energie zu erhalten. Periodic bounce orbits sind periodische Trajektorien, die am Rand der Domäne mit Reflexionsgesetz abprallen.

Contents

1. Introduction

Throughout this thesis, let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, satisfying the following: $\bar{\Omega}$ is a smooth manifold with manifold boundary that coincides with $\partial\Omega$, the topological boundary of Ω,[1] and let $V \in C^\infty(\bar{\Omega})$. We interpret V as a potential energy, and consider the Lagrangian system for a particle of normalized mass given by

$$\begin{aligned} L : T\bar{\Omega} = \bar{\Omega} \times \mathbb{R}^N &\to \mathbb{R} \\ (q, v) &\mapsto \frac{1}{2}|v|^2 - V(q). \end{aligned} \tag{1.1}$$

It is a fact that on $\bar{\Omega}$ there exists a non-constant periodic bounce orbit with at most $\dim \Omega + 1$ bounce points. Indeed, Benci and Giannoni show that there exists a nonconstant periodic bounce orbit of prescribed period, for small enough periods [BG89, Theorem 1.7]. Albers and Mazzucchelli expand on this result by showing that there exists a nonconstant periodic bounce orbit of prescribed energy, for any energy $E > \max_{\bar{\Omega}} V$ and they give a bound for the period [AM11, Theorem 1.2].

Both papers define a periodic bounce orbit the following way.

Definition 1.1. (cf. [AM11, Section 1]) *Let $\tau > 0$. A continuous, piecewise smooth*[2] *map $\gamma : \mathbb{R}/\tau\mathbb{Z} \to \bar{\Omega}$, is called a* **periodic bounce orbit** *if there exits a (possibly empty) finite subset $\mathcal{B} \subset [0, \tau)$ such that*

(i) *γ solves the Euler-Lagrange equation*

$$\gamma''(t) + \nabla V(\gamma(t)) = 0 \text{ for all } t \notin \mathcal{B} \tag{1.2}$$

(ii) *for each $t \in \mathcal{B}$ we have $\gamma(t) \in \partial\Omega$, the left, respectively right derivatives*

$$\gamma'(t^\pm) := \lim_{s \to t^\pm} \gamma'(s)$$

exist and γ satisfies the law of reflection

$$\begin{gathered} \left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle = -\left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle \neq 0, \\ \gamma'(t^+) - \left\langle \gamma'(t^+), \nu(\gamma(t)) \right\rangle \cdot \nu(\gamma(t)) = \gamma'(t^-) - \left\langle \gamma'(t^-), \nu(\gamma(t)) \right\rangle \cdot \nu(\gamma(t)), \end{gathered} \tag{1.3}$$

where ν is the outer normal of $\partial\Omega$.

Remark 1.2. (cf. [AM11, Remark 1.1])

- The times $t \in \mathcal{B}$ are called bounce times and $\gamma(t)$ bounce points. In case V is a constant funciton bounce orbits are billiard trajectories.
- A periodic bounce orbit with $\mathcal{B} = \emptyset$ is a smooth periodic solution of (1.2).

[1] This guarantees that $\partial\Omega$ has a outer normal, excluding cases like $\Omega = B_2(0) \setminus S^1 \subset \mathbb{R}^2$.
[2] In particular the differentiability in $0 \equiv \tau$ can be assessed by considering $\gamma(\cdot \mod \tau) : \mathbb{R} \to \mathbb{R}$

- For a periodic bounce orbit γ the energy

$$E(\gamma) := \frac{1}{2}|\gamma'(t)|^2 + V(\gamma(t))$$

is an integral of motion, namely it is independent of $t \notin \mathcal{B}$.

A physical interpretation of the result from Benci and Giannoni, respectively Albers and Mazzucchelli about periodic bounce orbits is the following: When neglecting friction, one can throw a bouncy ball inside a closed hollow of smooth shape, such that it is on a periodic trajectory with at most 4 bounce points. Similarly, one can put a ball on any minigolf course on a periodic trajectory with at most 3 bounce points. The minigolf course can be arbitrarily hilly, however the ball would have to be constrained not to lift off the ground.

In order to get smooth approximations of a periodic bounce orbit, the reflection property is imitated by adding a potential U, which is only non-constant near the boundary and may be interpreted as a repelling force. In both papers [BG89] and [AM11] the authors fix a potential U which is constant if not near the boundary and which fulfills $U = (\mathrm{dist}_{\partial\Omega})^{-2}$ near $\partial\Omega$. Then they consider the series of approximating models, given by the modified Lagrangians $L_\varepsilon := L - \varepsilon U$ and show that given a series of smooth solutions of the Euler-Lagrange equations in the respective approximating models, taking the limit of these yields a periodic bounce orbit.

In this thesis this result is generalised. We show that U may be replaced in such a way that $U = u(\mathrm{dist}_{\partial\Omega})$ near $\partial\Omega$, where $u \in C^\infty((0,1])$ with $\lim_{x\to 0} u(x) = \infty$ and the derivative u' is monotone. An interpretation of such a function u is that since the potential near the boundary is unbounded,[3] no trajectory can escape the domain Ω, and the repelling force gets monotonically stronger, if the particle moves closer to its boundary.

The new result here is that there is a whole class of approximation schemes suitable to achieve periodic bounce orbits. The papers above attest the existence of a nonconstant periodic bounce orbit. In addition to that, it would be interesting to find explicit solutions as well. The Euler-Lagrange equation provides ordinary differential equations for orbits in the respective approximating models, and using different approximation schemes, we get a larger array of ordinary differential equations, whose solutions converge to periodic bounce orbits (up to a subsequence). Hence, this thesis contributes to the search for explicit periodic bounce orbits.

Additionally, we will consider a special choice of approximating models, using the modified Lagrangians $L_\varepsilon := L - U_\varepsilon$, where U_ε is a specific function we will define, which among other things fulfills $U_\varepsilon = 1$ on $\partial\Omega$ and $U_\varepsilon = 0$ on $\{\mathrm{dist}_{\partial\Omega} \geq \varepsilon\}$. In this case we cap the energy of solutions to the Euler-Lagrange equations by the value 1.

[3] In fact, since we prescribe u' to be monotone, the property $\lim_{x\to 0} u(x) = \infty$ is equivalent to u being unbounded. Indeed, u' monotone implies that u is monotone on $(0, d)$ for some $d > 0$. We choose to continue using the former, to avoid repeating or repeatedly referencing this argument.

2. On the condition: u' monotone

The objective of this thesis is to find out what criteria towards the potential, used for approximating the bounce orbits, are necessary so that the result of the approximation is indeed a bounce orbit. Going through the proof of [AM11, Proposition 2.1], we find that instead of using $u(x) = \frac{1}{x^2}$, we could also use $u \in C^\infty((0,1])$ with $\lim_{x\to 0} |\frac{u'(x)}{u(x)}| = \infty$. In this section we will show that if we have $\lim_{x\to 0} u(x) = \infty$ and u' monotone, the property $\lim_{x\to 0} |\frac{u'(x)}{u(x)}| = \infty$ holds. We will use this to state the analogue to [AM11, Proposition 2.1] with $u \in C^\infty((0,1])$ with $\lim_{x\to 0} u(x) = \infty$ and u' monotone, instead of stating it with the (by Lemma 2.4 more general) property $\lim_{x\to 0} |\frac{u'(x)}{u(x)}| = \infty$. We prefer this because for u' monotone we have the interpretation that in the approximating models, the repelling force near the boundary is antitone to the (/falls monotonically with) distance to the boundary. The property $\lim_{x\to 0} u(x) = \infty$ means that if we start a trajectory in Ω with arbitrarily high kinetic energy, it can not shoot out of Ω.

To justify the upcoming rather complicated proof, we give the Example 2.1 of a quotient of two monotone functions of definite divergence, which does not converge nor diverges to infinity, to show that the result is not trivial. See also Example 2.5 for a function u with $u(x) \xrightarrow{x\to 0} \infty$ and u' monotone which by Lemma 2.4 does satisfy $|\frac{u'(x)}{u(x)}| \xrightarrow{x\to 0} \infty$, but for which $\frac{u'}{u}$ is not monotone on any interval $(0, x_0]$.

Example 2.1. Let $u, v : (0,1) \to \mathbb{R}$ with

$$u\big|_{[\frac{1}{n+\frac{3}{2}}, \frac{1}{n+\frac{1}{2}})} = n! \text{ and } v\big|_{[\frac{1}{n+1}, \frac{1}{n})} = -n!.$$

Then we have that $u(x) \xrightarrow{x\to 0} \infty$ and u is monotonically decreasing. Further $v(x) \xrightarrow{x\to 0} -\infty$, and v is monotonically increasing. However, we evaluate

$$\frac{v}{u}\Big|_{[\frac{1}{n+\frac{1}{2}}, \frac{1}{n})} = \frac{-n!}{(n-1)!} = -n \text{ and } \frac{v}{u}\Big|_{[\frac{1}{n+1}, \frac{1}{n+\frac{1}{2}})} = \frac{-n!}{n!} = -1.$$

Hence, $\frac{v}{u}$ is unbounded, but $\frac{v}{u}(x)$ does not diverge to $-\infty$ as $x \to 0$. In particular, the quotient of two monotone functions is not necessarily monotone itself.

Lemma 2.2. *Let $u \in C^\infty((0,1])$ such that $\lim_{x\to 0} u(x) = +\infty$ and u' monotonic. Then $\lim_{x\to 0} u'(x) = -\infty$.*

Proof. We assume that u' is bounded. By the fundamental theorem of calculus we have

$$\limsup_{x\to 0} u(x) = \limsup_{x\to 0} \left(u(1) - \int_x^1 u'(t)\,\mathrm{d}t \right) \le |u(1)| + \limsup_{x\to 0} \|u'\|_\infty |1-x| \le |u(1)| + \|u'\|_\infty.$$

This contradicts $\lim_{x\to 0} u(x) = +\infty$. Now since u' unbounded towards 0 and monotonically increasing, we have definite divergence towards $-\infty$. □

Seeking a simpler proof of Lemma 2.4, one might try to argue that $\frac{u'}{u} = (\ln(u))'$, $\ln(u(x)) \xrightarrow{x\to 0} \infty$ if $u(x) \xrightarrow{x\to 0} \infty$, and $(\ln(u))'$ unbounded by the same argument as in the proof of Lemma 2.2. The problem is that in general, $(\ln(u))' = \frac{u'}{u}$ is not monotone! See Example 2.5. So we cannot immediately apply Lemma 2.2 to $\ln(u)$ in order to conclude that $\lim\limits_{x\to 0} |\frac{u'(x)}{u(x)}| = \lim\limits_{x\to 0} |(\ln(u))'(x)| = \infty$.

Lemma 2.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a monotonic sequence in $\mathbb{R}_{>0}$ of definite divergence, i.e. $\lim\limits_{n\to\infty} x_n = \infty$. Then*

$$A_n := \frac{\sum_{k=1}^{n} \frac{x_k}{k^2}}{x_n} \to 0 \text{ as } n \to \infty$$

holds true.

Proof. Since $\lim\limits_{n\to\infty} x_n = \infty$, we find a subsequence x_{n_l} satisfying

$$x_{n_l} \geq l x_{n_{l-1}} \text{ for all } l \geq 2. \tag{2.1}$$

By monotony of $(x_n)_{n\in\mathbb{N}}$ we have

$$A_{n_l} = \frac{\sum_{k=1}^{n_l} \frac{x_k}{k^2}}{x_{n_l}} = \frac{\sum_{k=1}^{n_{l-1}} \frac{x_k}{k^2}}{x_{n_l}} + \frac{\sum_{k=n_{l-1}+1}^{n_l} \frac{x_k}{k^2}}{x_{n_l}} \leq \frac{x_{n_{l-1}}}{x_{n_l}} \sum_{k=1}^{n_{l-1}} \frac{1}{k^2} + \frac{x_{n_l}}{x_{n_l}} \sum_{k=n_{l-1}+1}^{n_l} \frac{1}{k^2}.$$

With (2.1) and $C := \frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ furthermore

$$A_{n_l} \leq \frac{C}{l} + \sum_{k=n_{l-1}}^{\infty} \frac{1}{k^2} \to 0 \text{ as } l \to \infty.$$

Moreover, by monotony we have

$$A_n = \frac{\sum_{k=1}^{n} \frac{x_k}{k^2}}{x_n} = \frac{\sum_{k=1}^{n-1} \frac{x_k}{k^2}}{x_n} + \frac{1}{n^2} \leq \frac{\sum_{k=1}^{n-1} \frac{x_k}{k^2}}{x_{n-1}} + \frac{1}{n^2} = A_{n-1} + \frac{1}{n^2}.$$

We set $\phi(n) := \max\{l \in \mathbb{N} | n_l \leq n\}$. In particular holds $n_{\phi(n)} \leq n$ and $\lim\limits_{n\to\infty} \phi(n) = \infty$. Recursively, we get

$$A_n \leq A_{n-1} + \frac{1}{n^2} \leq A_{n-2} + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq \cdots \leq A_{n_{\phi(n)}} + \sum_{k=n_{\phi(n)}+1}^{n} \frac{1}{k^2} \to 0 \text{ as } n \to \infty$$

as both terms converge to 0 and since A_n is positive.

□

Lemma 2.4. *Let $u \in C^\infty((0,1])$ such that $\lim\limits_{x\to 0} u(x) = \infty$ and u' monotonic. Then*

$$\lim_{x\to 0} \left| \frac{u'(x)}{u(x)} \right| = \infty$$

holds true.

Proof. From Lemma 2.2 we get $\lim_{x\to 0} u'(x) = -\infty$. In particular, we get $d \in (0,1]$ such that $u'(x) < 0$ for all $x \le d$, allowing the following equivalence:

$$\lim_{x\to 0}\Big|\frac{u'(x)}{u(x)}\Big| = \infty \iff \lim_{x\to\infty}\Big|\frac{u'(\frac{1}{x})}{u(\frac{1}{x})}\Big| = \infty \iff \lim_{x\to\infty}\Big|\frac{u(\frac{1}{x})}{u'(\frac{1}{x})}\Big| = 0.$$

W.l.o.g. assume $d = 1$. Set $g(x) := \frac{1}{x}$. By the fundamental theorem of calculus we have

$$u(\tfrac{1}{x}) = u(g(x)) = u(1) - \int_{g(x)}^{1=g(1)} u'(t)\,\mathrm{d}t$$

and integration by substitution gives

$$u(\tfrac{1}{x}) = u(1) - \int_x^1 u'(g(s))g'(s)\,\mathrm{d}s = u(1) + \int_x^1 u'(\tfrac{1}{s})\tfrac{1}{s^2}\,\mathrm{d}s = u(1) - \int_1^x u'(\tfrac{1}{s})\tfrac{1}{s^2}\,\mathrm{d}s.$$

We establish an upper bound for the second term:

$$\Big|\int_1^x u'(\tfrac{1}{s})\tfrac{1}{s^2}\,\mathrm{d}s\Big| \le \sum_{k=1}^{\lfloor x\rfloor - 1}\int_k^{k+1} |u'(\tfrac{1}{s})|\tfrac{1}{s^2}\,\mathrm{d}s + \int_{\lfloor x\rfloor}^x |u'(\tfrac{1}{s})|\tfrac{1}{s^2}\,\mathrm{d}s.$$

Since $u' < 0$ is monotonic we have $|u'(\frac{1}{s})| \le |u'(\frac{1}{b})|$ and $\frac{1}{s^2} \le \frac{1}{a^2}$ on $[a,b] \subset [1,\infty)$. Hence

$$\dots \le \sum_{k=1}^{\lfloor x\rfloor - 1} |u'\big(\tfrac{1}{k+1}\big)|\tfrac{1}{k^2} + \frac{|u'(\frac{1}{x})|(x - \lfloor x\rfloor)}{\lfloor x\rfloor^2} \le \sum_{k=1}^{\lfloor x\rfloor - 1} |u'\big(\tfrac{1}{k+1}\big)|\tfrac{1}{k^2} + \frac{|u'(\frac{1}{x})|}{\lfloor x\rfloor^2}.$$

By monotony we have $|u'(\frac{1}{x})| \ge |u'(\frac{1}{\lfloor x\rfloor})|$, i.e. $\frac{1}{|u'(\frac{1}{x})|} \le \frac{1}{|u'(\frac{1}{\lfloor x\rfloor})|}$, therefore altogether we get the majorant

$$\Big|\frac{u(\frac{1}{x})}{u'(\frac{1}{x})}\Big| \le \Big|\frac{u(1)}{u'(\frac{1}{x})}\Big| + \frac{\sum_{k=1}^{\lfloor x\rfloor - 1} |u'\big(\frac{1}{k+1}\big)|\frac{1}{k^2}}{|u'\big(\frac{1}{\lfloor x\rfloor}\big)|} + \frac{1}{\lfloor x\rfloor^2} \to 0 \text{ as } x \to \infty,$$

where the convergence of the second term follows from Lemma 2.3 with the sequence $\big(|u'\big(\frac{1}{k+1}\big)|\big)_{k\in\mathbb{N}}$. With the equivalence at the top we conclude $\lim_{x\to 0}|\frac{u'(x)}{u(x)}| = \infty$. $\square$

Example 2.5. We want to give an example of a function $u \in C^\infty((0,1])$ with $u(x) \xrightarrow{x\to 0} \infty$, u monotone and u' monotone where $\frac{u'}{u}$ is not monotone on any $(0, x_0]$. Similar to the proof of Lemma 2.4, we prefer to work in the interval $[1,\infty)$ instead of $(0,1]$. To simplify notation we may sometimes mean "f" when we write "$f(x)$".

Fix

$$w \in C^\infty([1,\infty)) \text{ monotonic with } w = 2^n \text{ on } [n, n+\tfrac{3}{4}).$$

In particular, $w(x) = 2^n$ for all $x \in [n, n+\frac{3}{4})$ and on $[n+\frac{3}{4}, n+1)$, w is a smooth monotone connection from 2^n to 2^{n+1}, so $2^n \le w(x) \le 2^{n+1}$ for all $x \in [n+\frac{3}{4}, n+1)$.

Now set $v \in C^\infty([1,\infty))$ with

$$v(x) := \int_1^x w(s)\,\mathrm{d}s.$$

Finally set $u \in C^\infty((0,1])$ with
$$u(x) := v\left(\tfrac{1}{x}\right).$$
We will show below that $\eta(x) := \frac{x^2 w(x)}{v(x)}$ is not monotone on any $[x_0,\infty)$. Then
$$-\eta(x) = \frac{-x^2 w(x)}{v(x)} = \frac{-x^2 v'(x)}{v(x)} = \frac{-x^2 (u(\frac{1}{x}))'}{u(\frac{1}{x})} = \frac{u'(\frac{1}{x})}{u(\frac{1}{x})}$$
is not monotone on any $[x_0,\infty)$ and hence $\frac{u'(x)}{u(x)}$ is not monotone on any $(0,x_0]$. For $n \in \mathbb{N}$ we estimate
$$\begin{aligned} v(n) &= \int_1^n w(s)\,\mathrm{d}s = \sum_{k=1}^{n-1}\left(\int_k^{k+\frac{3}{4}} 2^k\,\mathrm{d}s + \int_{k+\frac{3}{4}}^{k+1} w(s)\,\mathrm{d}s\right) \\ &\le \sum_{k=1}^{n-1}\left(\frac{3}{4}2^k + \frac{1}{4}2^{k+1}\right) = \sum_{k=1}^{n-1}\frac{5}{4}2^k = \frac{5}{4}(2^n-2) \le \frac{5}{4}2^n. \end{aligned} \tag{2.2}$$
We get
$$\eta(n) = \frac{n^2 w(n)}{v(n)} = \frac{n^2 2^n}{v(n)} \ge \frac{4}{5}n^2. \tag{2.3}$$
Further,
$$\begin{aligned} v(n+\tfrac{1}{2}) &= \int_1^{n+\frac{1}{2}} w(s)\,\mathrm{d}s = \sum_{k=1}^{n-1}\left(\int_k^{k+\frac{3}{4}} 2^k\,\mathrm{d}s + \int_{k+\frac{3}{4}}^{k+1} w(s)\,\mathrm{d}s\right) + \int_n^{n+\frac{1}{2}} 2^n\,\mathrm{d}s \\ &\ge \sum_{k=1}^{n-1}\int_k^{k+1} 2^k\,\mathrm{d}s + \int_n^{n+\frac{1}{2}} 2^n\,\mathrm{d}s = \sum_{k=1}^{n-1} 2^k + \frac{1}{2}2^n = 2^n - 2 + \frac{1}{2}2^n = \frac{3}{2}2^n - 2, \end{aligned}$$
so
$$\begin{aligned} \eta(n+\frac{1}{2}) &= \frac{(n+\frac{1}{2})^2 w(n+\frac{1}{2})}{v(n+\frac{1}{2})} = \frac{(n^2+n+\frac{1}{4})2^n}{v(n+\frac{1}{2})} \le \frac{(n^2+n+\frac{1}{4})2^n}{\frac{3}{2}2^n - 2} \\ &= \frac{2}{3}\left(n^2+n+\frac{1}{4}\right)\frac{2^n}{2^n-\frac{4}{3}} = \frac{2}{3}\left(n^2+n+\frac{1}{4}\right)\frac{1}{1-\frac{4}{3}2^{-n}} \\ &= \frac{2}{3}\left(n^2+n+\frac{1}{4}\right)\left(1+\frac{1}{\frac{3}{4}2^n-1}\right). \end{aligned} \tag{2.4}$$
Now by (2.3) and (2.4), for any big enough $n \in \mathbb{N}$ we have
$$\eta(n) > \eta(n+\frac{1}{2}).$$
We further estimate
$$\begin{aligned} v(x) &= v(\lfloor x\rfloor) + \int_{\lfloor x\rfloor}^x w(s)\,\mathrm{d}s \le v(\lfloor x\rfloor) + \int_{\lfloor x\rfloor}^{\lfloor x\rfloor+1} 2^{\lfloor x\rfloor+1}\,\mathrm{d}s \\ &= v(\lfloor x\rfloor) + 2^{\lfloor x\rfloor+1} \le \frac{5}{4}2^{\lfloor x\rfloor} + 2^{\lfloor x\rfloor+1} = \frac{13}{4}2^{\lfloor x\rfloor}, \end{aligned}$$

where the last inequality follows from (2.2).

So we have

$$\eta(x) = \frac{x^2 w(x)}{v(x)} \geq \frac{x^2 2^{\lfloor x \rfloor}}{v(x)} \geq \frac{x^2 2^{\lfloor x \rfloor}}{\frac{13}{4} 2^{\lfloor x \rfloor}} = \frac{4}{13} x^2.$$

To summarize, we have

$$\lim_{x \to \infty} \eta(x) = \infty \text{ and } \eta(n) > \eta(n + \frac{1}{2}) \text{ for all } n \in \mathbb{N}.$$

We conclude that $\eta(x) = \frac{x^2 w(x)}{v(x)}$ is not monotone on any $[x_0, \infty)$ and so $\frac{u'}{u}$ is not monotone on any $(0, x_0]$, as argued above.

However, $\lim_{x \to 0} u(x) = \infty$ because $\lim_{x \to \infty} v(x) = \infty$. Lastly, we show u' is monotone. Indeed: $x^2 w(x)$ is monotone as $(x^2 w(x))' = 2xw(x) + x^2 w'(x)$ is positive because w is monotonically increasing, so $u'(\frac{1}{x}) = -x^2 v'(x) = -x^2 w(x)$ is monotone and hence u' is monotone as well. The last equality we get from calculating $v'(x) = (u(\frac{1}{x}))' = -\frac{1}{x^2} u'(\frac{1}{x})$. Additionally, note that we can verify the statement of Lemma 2.4 for our choice of u: The fact that $\lim\limits_{x \to \infty} \eta(x) = \infty$ implies $\lim\limits_{x \to 0} |\frac{u'(x)}{u(x)}| = \infty$.

3. Relevant function spaces

In this section, we introduce the function spaces that we will use. The properties of these we derive from function spaces that are treated in undergraduate studies. These involve spaces of real-valued functions on domains that are suitable subsets of $\mathbb{R}^N$, such as continuous or smooth functions, and spaces of equivalence classes of real-valued functions on domains that are suitable subsets of $\mathbb{R}^N$, such as Lebesgue- and Sobolev-spaces.

Since we are interested in periodic orbits, we define analogue spaces to some of those listed above, with $\mathbb{R}/_{\tau\mathbb{Z}}$ as a domain, where $\tau > 0$ is the period. We may always identify a function $f : \mathbb{R}/_{\tau\mathbb{Z}} \to \mathbb{R}$ with $\tilde{f} : [0, \tau) \to \mathbb{R}$, and if we pluck any time $t \in \mathbb{R}$ into f, we mean $\tilde{f}(t) = f(t \mod \tau)$. We will drop this distinction, for instance we might write $f(t \mod 1)$ when we really mean $\tilde{f}(t - \lfloor t \rfloor)$. To ensure that properties like continuity, differentiability or weak differentiability are met across $0 \equiv \tau$, for a function $f : \mathbb{R}/_{\tau\mathbb{Z}} \to \mathbb{R}$, we consider $\tilde{f} : \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f} = f(\cdot \mod \tau)$ and check if this function meets the condition at the time τ.

We define

$$C^0(\mathbb{R}/_{\tau\mathbb{Z}}) := \{f : [0, \tau) \to \mathbb{R} \mid \tilde{f} = f(\cdot \mod \tau) \in C^0(\mathbb{R})\}$$

and

$$C^\infty(\mathbb{R}/_{\tau\mathbb{Z}}) := \{f : [0, \tau) \to \mathbb{R} \mid \tilde{f} = f(\cdot \mod \tau) \in C^\infty(\mathbb{R})\}.$$

For Sobolev-functions we need to be more careful. Since periodic functions in general are not integrable on $\mathbb{R}$, for instance $f \equiv 1$, we define spaces of locally integrable Sobolev-functions:

$$W^{m,p}_{\mathrm{loc}}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \in W^{m,p}((-K, K)) \text{ for all } K \in \mathbb{N}\}.$$

Now we define spaces of τ-periodic Sobolev-funcitons:

$$W^{m,p}(\mathbb{R}/_{\tau\mathbb{Z}}) := W^{m,p}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}) := \{f \in W^{m,p}((0,\tau)) |\ \tilde{f} = f(\ \cdot \mod \tau) \in W^{m,p}_{\mathrm{loc}}(\mathbb{R})\}.$$

The spaces $C^0(\mathbb{R}/_{\tau\mathbb{Z}})$ and $W^{m,p}(\mathbb{R}/_{\tau\mathbb{Z}})$ inherit the respective norms, i.e.

$$\|\cdot\|_{C^0(\mathbb{R}/\tau\mathbb{Z})} = \|\cdot\|_\infty \text{ and } \|\cdot\|_{W^{m,p}(\mathbb{R}/\tau\mathbb{Z})} = \|\cdot\|_{W^{1,2}((0,\tau))}.$$

Now since our particle moves in $\mathbb{R}^N$, we define spaces of $\mathbb{R}^N$ valued functions. For a $\mathbb{R}^N$ valued function it comes down to whether all of the component functions are in the respective function spaces. For instance,

$$\Gamma \in C^0(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N) :\Leftrightarrow \Gamma_i \in C^0(\mathbb{R}/_{\tau\mathbb{Z}}) \text{ for all } i = 1, ..., N$$

or

$$\Gamma \in L^p((0,1), \mathbb{R}^N) :\Leftrightarrow \Gamma_i \in L^p((0,1)) \text{ for all } i = 1, ..., N.$$

In the cases of normed vector spaces, we equip these new function spaces with the norm that is the sum of the norms of the component functions, for instance

$$\|\Gamma\|_{m,p} = \|\Gamma\|_{W^{m,p}(\mathbb{R}/\tau\mathbb{Z},\mathbb{R}^N)} := \sum_{i=1}^{N} \|\Gamma_i\|_{W^{m,p}(\mathbb{R}/\tau\mathbb{Z})},$$

using such abbreviations when it is clear which norm is meant or which is the underlying space.

Actually, we want our particle to move only in $\Omega \subset \mathbb{R}^N$ or in $\bar{\Omega} \subset \mathbb{R}^N$. By Lemma 3.4 our Sobolev functions have concrete values, namely those of the continuous representatives. Hence, we can define

$$W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \Omega) := \{f \in W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N) | f(\mathbb{R}/_{\tau\mathbb{Z}}) \subset \Omega\},$$

respectively

$$W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \bar{\Omega}) := \{f \in W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N) | f(\mathbb{R}/_{\tau\mathbb{Z}}) \subset \bar{\Omega}\}.$$

Throughout we will write $S^1 := \mathbb{R}/_{\mathbb{Z}}$ interchangeably.

Lemma 3.1. *$W^{m,p}(\mathbb{R}/_{\tau\mathbb{Z}})$ is a Banach space, in particular it is a closed subspace of $W^{m,p}((0,\tau))$.*

Proof. We show this for only for $\tau = 1$ as it conveys the main point and avoids unnecessary trickiness. Let $(f_n)_{n\in\mathbb{N}} \subset W^{m,p}(S^1)$ be a Cauchy sequence. Since $W^{m,p}((0,1))$ is a Banach space, we have $f \in W^{m,p}((0,1))$ such that $\|\partial^k f - \partial^k f_n\|_p \to 0$ for all $k \leq m$. We need to establish the weak derivative of $\tilde{f} = f(\ \cdot \mod 1)$ on $(-K, K)$ for $K \in \mathbb{N}$. Let $\psi \in C_c^\infty((-K, K))$, we want to show

$$\int_{(-K,K)} \partial^k \psi \tilde{f} \,\mathrm{d}\lambda = (-1)^k \int_{(-K,K)} \psi \widetilde{\partial^k f} \,\mathrm{d}\lambda.$$

Indeed, since for the elements of the sequence holds

$$\int_{(-K,K)} \partial^k \psi \tilde{f}_n \,\mathrm{d}\lambda - (-1)^k \int_{(-K,K)} \psi \widetilde{\partial^k f_n} \,\mathrm{d}\lambda = 0 \text{ for all } n \in \mathbb{N},$$

we have

$$
\begin{aligned}
&|\int_{(-K,K)} \partial^k\psi \tilde{f} \ \mathrm{d}\lambda - (-1)^k \int_{(-K,K)} \psi \widetilde{\partial^k f} \ \mathrm{d}\lambda| \\
\leq &|\int_{(-K,K)} \partial^k\psi(\tilde{f} - \tilde{f}_n)\,\mathrm{d}\lambda| + |(-1)^k \int_{(-K,K)} \psi(\widetilde{\partial^k f_n} - \widetilde{\partial^k f})\,\mathrm{d}\lambda| \\
\leq &\|\partial^k\psi\|_q \|\tilde{f} - \tilde{f}_n\|_{p,(-K,K)} + \|\psi\|_q \|\widetilde{\partial^k f_n} - \widetilde{\partial^k f}\|_{p,(-K,K)} \\
\leq &(2K)^{\frac{1}{p}} \|\partial^k\psi\|_q \|f - f_n\|_{p,(0,1)} + (2K)^{\frac{1}{p}} \|\psi\|_q \|\partial^k f_n - \partial^k f\|_{p,(0,1)} \xrightarrow{n\to\infty} 0
\end{aligned}
$$

using the Hölder inequality with conjugated Hölder exponent q and then splitting the integral into intervals of length 1. Hence, we have the weak derivatives $\partial^k \tilde{f} = \widetilde{\partial^k f}$, which are in $L^p((-K,K)$, so $\tilde{f} \in W^{m,p}_{\mathrm{loc}}(\mathbb{R})$ and $f \in W^{m,p}(S^1)$. $\square$

Lemma 3.2. *$C^0(\mathbb{R}/\tau\mathbb{Z})$ is a Banach space.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset C^0(\mathbb{R}/\tau\mathbb{Z})$ be a Cauchy sequence. In particular with $f_n(\tau) = f_n(0)$ we have $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,\tau])$. Since this is a Banach space we have $f \in C^0([0,\tau])$ such that $f_n \xrightarrow{C^0([0,\tau])} f$. Further $f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} f_n(\tau) = f(\tau)$, so $f \in C^0(\mathbb{R}/\tau\mathbb{Z})$. $\square$

We adopt the language of Adams and Fournier:

Remark 3.3. (cf. [AF03, 1.25]) We say the normed vector space X is **imbedded** in the normed space Y, and write $X \hookrightarrow Y$ to designate this imbedding, provided that

(i) X is a vector subspace of Y, and
(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

Since I is linear, (ii) is equivalent to the existence of a constant C such that

$$\|Ix\|_Y \leq C\|x\|_X,\ x \in X.$$

Sometimes the requirement that X be a subspace of Y and I be the identity map is weakened to allow as imbeddings certain canonical transformations of X into Y, such as imbeddings of Sobolev spaces into spaces of continuous functions.

Let X and Y be Banach spaces. We say that X is **compactly imbedded** in Y, and write $X \subset\subset Y$, if the imbedding operator I is compact.

Lemma 3.4. *$W^{1,2}(\mathbb{R}/\tau\mathbb{Z})$ is compactly imbedded into $C^0(\mathbb{R}/\tau\mathbb{Z})$.*

Proof. By the Rellich-Kondrachov Theorem [AF03, 6.3 PART III] we have the compact imbeddings $W^{1,2}((a,b)) \subset\subset C^0([a,b])$ for $a < b \in \mathbb{R}$. Let $f \in W^{1,2}(\mathbb{R}/\tau\mathbb{Z})$. Then $\tilde{f} \in W^{1,2}((-K,K))$ and we find a continuous representative i.e. $\tilde{f} \in C^0([-K,K])$ for any $K \in \mathbb{N}$, hence $\tilde{f} \in C^0(\mathbb{R})$. Now by definition, $f \in C^0(\mathbb{R}/\tau\mathbb{Z})$. Furthermore we have $\|f\|_\infty \leq C\|f\|_{1,2,(0,\tau)}$ for some $C > 0$ by the imbedding above. This proves the imbedding $W^{1,2}(\mathbb{R}/\tau\mathbb{Z}) \hookrightarrow C^0(\mathbb{R}/\tau\mathbb{Z})$.

Now denote by B the unit ball in $W^{1,2}(\mathbb{R}/\tau\mathbb{Z})$ and by $\tilde{B}$ the unit ball in $W^{1,2}((0,\tau))$. $\tilde{B}$ is a precompact subset of $C^0([0,\tau])$ by the above imbedding, so the subset B is, too. From a finite covering with $\frac{\varepsilon}{4}$-balls of B in $C^0([0,\tau])$ we get a finite covering with ε-balls of B in $C^0(\mathbb{R}/_{\tau\mathbb{Z}})$. Hence, B is a precompact subset of $C^0(\mathbb{R}/_{\tau\mathbb{Z}})$ which proves that the imbedding $W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}) \hookrightarrow C^0(\mathbb{R}/_{\tau\mathbb{Z}})$ is compact.[4] □

Lemma 3.5. *$W^{2,1}(\mathbb{R}/_{\tau\mathbb{Z}})$ is compactly imbedded into $W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}})$.*

Proof. By the Rellich-Kondrachov Theorem [AF03, 6.3 PART I] we have the compact imbedding $W^{2,1}((a,b)) \subset\subset W^{1,2}((a,b))$ for $a < b \in \mathbb{R}$.

Let $f \in W^{2,1}(\mathbb{R}/_{\tau\mathbb{Z}})$. Then $\tilde{f} \in W^{2,1}((-K,K)) \subset W^{1,2}((-K,K))$ for any $K \in \mathbb{N}$, so $\tilde{f} \in W^{1,2}_{\text{loc}}(\mathbb{R})$ and $f \in W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}})$. Furthermore we have $\|f\|_{1,2,(0,\tau)} \leq C\|f\|_{2,1,(0,\tau)}$ for some $C > 0$ by the imbedding above. The compactness of the imbedding follows analogously to the proof of Lemma 3.4

□

Remark 3.6. Certainly we have

(1) $W^{m,p}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$ is a Banach space.
(2) $W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$ is compactly imbedded in $C^0(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$.
(3) $W^{2,1}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$ is compactly imbedded into $W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$.

Lemma 3.7. *Let $\Gamma \in L^p((0,1), \mathbb{R}^N)$ and $\Psi \in L^q((0,1), \mathbb{R}^N)$, where p and q are conjugated Hölder exponents. Then*

$$\int_{(0,1)} |\langle \Gamma, \Psi \rangle| \,\mathrm{d}\lambda \leq \|\Gamma\|_p \|\Psi\|_q$$

holds true.

Proof. By applying the Hölder inequality to the component functions we get

$$\int_{(0,1)} |\langle \Gamma, \Psi \rangle| \,\mathrm{d}\lambda \leq \sum_{i=1}^{N} \int_{(0,1)} |\Gamma_i \Psi_i| \,\mathrm{d}\lambda \leq \sum_{i=1}^{N} \|\Gamma_i\|_p \|\Psi_i\|_q \leq \sum_{i=1}^{N} \sum_{j=1}^{N} \|\Gamma_i\|_p \|\Psi_j\|_q = \|\Gamma\|_p \|\Psi\|_q.$$

□

We may later simply refer to this result with "by Hölder inequality" or in case $p = 2$ "by Cauchy-Schwartz inequality".

4. Differentiation on Banach spaces

In this section, we list the definitions and properties of differentiation on Banach spaces that we will use when working with the action-functionals that correspond to our respective dynamical systems.

In the following, let E be a (real) banach space and U an open subset. Let $f : U \to \mathbb{R}$ be a map.

[4] Note that by the lemmata above, both are Banach spaces.

Definition 4.1. (cf. [Lan93, XIII §2]) *Let $x \in U$. We shall say that f is Fréchet differentiable at x if there exists a continuous linear functional $T_x : E \to \mathbb{R}$ and a map ψ defined for all sufficiently small h in E, with values in $\mathbb{R}$, such that*

$$\lim_{h \to 0} \psi(0) = 0,$$

and such that

$$f(x+h) = f(x) + T_x h + \|h\|_E \psi(h).$$

If f is Fréchet differentiable at every point x of U, we say that f is ***Fréchet differentiable*** *on U. In that case, the Fréchet derivative $\mathrm{D}f$ defined by $\mathrm{D}f(x) = T_x$ is a map*

$$\mathrm{D}f : U \to E^*$$

from U into the space of continuous linear functionals E^*. If $\mathrm{D}f$ is continuous, we say that f is of class C^1.

Definition 4.2. (cf. [AH10, Appendix A]) *Let $x \in U$. We shall say that f is Gâteaux differentiable at x if there exists a continuous linear functional $T_x : E \to \mathbb{R}$, such that*

$$\forall v \in E, \ \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} = T_x v.$$

If f is Gâteaux differentiable at every point x of U, then we say that f is ***Gâteaux differentiable*** *on U. In that case, the Gâteaux derivative $\mathrm{D}_G f$ defined by $\mathrm{D}_G f(x) = T_x$ is a map*

$$\mathrm{D}_G f : U \to E^*$$

from U into the space of continuous linear functionals E^*.

Both derivatives are unique if they exist, as shown in [Lan93, XIII §2], respectively [AH10, Appendix A].

Lemma 4.3. (cf. [AH10, Proposition A.3.]) *If f is Gâteaux differentiable on U, and $\mathrm{D}_G f : U \to E^*$ is continuous, then f is also Fréchet differentiable on U and $\mathrm{D}_G f = \mathrm{D}f$. In particular holds f is of class C^1.*

Proof. Reviewing the proof of [AH10, Proposition A.3.], for a point $x \in U$, the linear operator used to prove the Fréchet differentiability at x, is the Gâteax derivative at x. □

(cf. [Lan93, XIII §7]) Now consider a product $E = E_1 \times E_2$ of two banach spaces. Let U_i be open in E_i and let $f : U_1 \times U_2 \to \mathbb{R}$ be a map. We write an element $x \in U_1 \times U_2$ in terms of "coordinates", namely $x = (x_1, x_2)$ with $x_i \in U_i$. For x_1 fixed, (respectively analogously for x_2 fixed,) we consider the partial map $x_2 \mapsto f(x_1, x_2)$ of U_2 into $\mathbb{R}$. If this map is Fréchet differentiable, we call its derivative the **partial derivative** of f and denote it by $\mathrm{D}_i f(x)$ at the point x.

Lemma 4.4. (cf. [Lan93, XIII Theorem 7.1.]) *The map $f : U_1 \times U_2 \to \mathbb{R}$ is of class C^1 if and only if each partial derivative $\mathrm{D}_i f : U_1 \times U_2 \to E_i^*$ is continuous. If this is the case, and $v = (v_1, v_2) \in E_1 \times E_2$, then*

$$\mathrm{D}f(x)v = \mathrm{D}_1 f(x) v_1 + \mathrm{D}_2 f(x) v_2.$$

Proof. See [Lan93, XIII Theorem 7.1]. $\square$

In order to differentiate under the integral sign, we use the notion of integrating a Banach space valued curve, which is defined on the space of regulated maps. This is the closure with respect to the sup norm of the space of step maps. For this we refer to [Lan93, XIII §1]. In particular, for a Banach space F and a continuous map $h : [0, 1] \to F$, the integral $\int_0^1 h(t)\,\mathrm{d}t$ is defined.

Lemma 4.5. (cf. [Lan93, XIII Theorem 8.1.]) *Let U be open in E and let $f : [0, 1]\times U \to \mathbb{R}$ be a continuous map such that the partial derivative $\mathrm{D}_2 f$ exists and is continuous. Let*

$$g(x) = \int_0^1 f(t, x)\,\mathrm{d}t.$$

Then g is Fréchet differentiable on U and

$$\mathrm{D}g(x) = \int_0^1 \mathrm{D}_2 f(t, x)\,\mathrm{d}t.$$

Proof. See [Lan93, XIII Theorem 8.1]. $\square$

5. The approximating models and the free-time action functional

For introducing the approximating models, we distinguish between the cases (I) and (II). First, we introduce the case (I) approximating models following [AM11, Section 2] and generalizing the definition of the potential U. Recall that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, such that $\bar{\Omega}$ is a smooth manifold whose manifold boundary coincides with the topological boundary $\partial\Omega$ of Ω, and $V \in C^\infty(\bar{\Omega})$. We fix $d_0 \in (0, \frac{1}{2})$ sufficiently small, such that the distance function $\mathrm{dist}_{\partial\Omega}(q)$ is smooth at all points $q \in \Omega$ with $\mathrm{dist}_{\partial\Omega}(q) \leq 2d_0$. The existence of such a d_0 is guaranteed by [GT83, Lemma 14.16].

Let $k : [0, \infty) \to [0, 2d_0]$ be a smooth function such that $0 \leq k' \leq 1$, $k(x) = x$ on $[0, d_0]$, and k constant on $[2d_0, \infty)$. We define $h \in C^\infty(\bar{\Omega})$ through

$$h(q) := k(\mathrm{dist}_{\partial\Omega}(q)).$$

Note that h satisfies the following properties.

- $h(q) = \mathrm{dist}_{\partial\Omega}(q)$ for all $q \in \bar{\Omega}$ with $\mathrm{dist}_{\partial\Omega} \leq d_0$,
- $h(q) > d_0$ for all $q \in \bar{\Omega}$ with $\mathrm{dist}_{\partial\Omega}(q) > d_0$,
- $0 \leq h \leq 1$ and h constant on $\{\mathrm{dist}_{\partial\Omega} \geq 2d_0\} \subset \Omega$,
- $|\nabla h| \leq 1$.

In case (I), we consider any choice of $u \in C^\infty((0, 1])$ with $\lim\limits_{x\to 0} u(x) = \infty$ and u' monotonic. For the pending choice of u, we fix

$$U := u \circ h \in C^\infty(\Omega).$$

Thus, $U = u(\mathrm{dist}_{\partial\Omega})$ near $\partial\Omega$ and U is constant on $\{\mathrm{dist}_{\partial\Omega} \geq 2d_0\}$. For a visualisation, see Figure 1, which has been copied from [AM11].

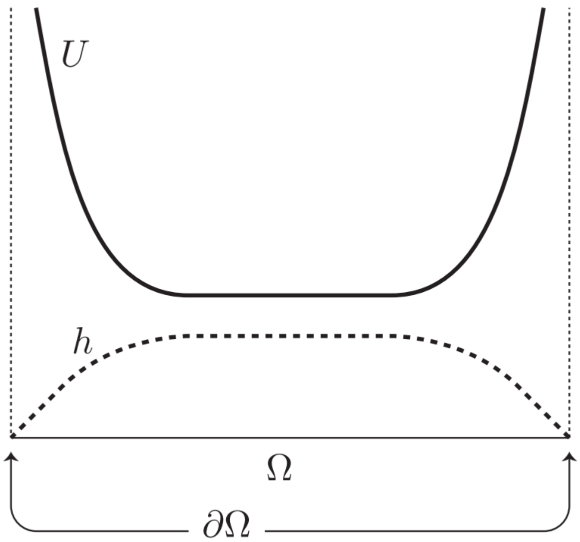

FIGURE 1. The functions U and h, from [AM11].

For $\varepsilon > 0$, the Lagrangian of our approximating model has the form

$$\begin{aligned} L_\varepsilon : T\Omega = \Omega \times \mathbb{R}^N &\to \mathbb{R} \\ (q, v) &\mapsto \tfrac{1}{2}|v|^2 - V(q) - \varepsilon U(q). \end{aligned}$$

We now define the free-time action functional, whose purpose becomes clear in Lemma 5.2 – the critical points are precisely the solutions of the corresponding Euler-Lagrange equation. It is called the free-time action functional, because opposed to the approach of Benci and Giannoni [BG89, Section 2], this approach by Albers and Mazzucchelli [AM11, Section 2] allows to prescribe the energy of the particle and leave the period flexible.

For an energy value $E \in \mathbb{R}$ we define the free-time action functional $\mathcal{L}_\varepsilon^E : W^{1,2}(S^1, \Omega) \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\mathcal{L}_\varepsilon^E(\Gamma, \tau) := \tau \int_0^1 [L_\varepsilon\Big(\Gamma(t), \frac{1}{\tau}\Gamma'(t)\Big) + E]\,\mathrm{d}t = \int_0^\tau [L_\varepsilon(\gamma(t), \gamma'(t)) + E]\,\mathrm{d}t$$

where $\gamma(t) := \Gamma(t/\tau)$.

As explained by Abbondandolo [Abb13, Section 3], the idea here is that we want to study the Lagrangian action on a space of periodic curves, with arbitrary period. In order to do so, we re-parameterise a τ-periodic curve γ as above and identify it with the pair (Γ, τ), keeping track of the actual period.

This functional is well-defined. Indeed:

$$
\begin{aligned}
&\Big|\int_0^1 L_\varepsilon\Big(\Gamma(t), \frac{1}{\tau}\Gamma'(t)\Big)\,\mathrm{d}t\Big| \\
\leq &\int_0^1 \Big|L_\varepsilon\Big(\Gamma(t), \frac{1}{\tau}\Gamma'(t)\Big)\Big|\,\mathrm{d}t \\
\leq &\int_0^1 \frac{1}{2\tau^2}|\Gamma'(t)|^2\,\mathrm{d}t + \|V\|_\infty + \varepsilon\int_0^1 |U(\Gamma(t))|\,\mathrm{d}t \\
\leq &\frac{1}{2\tau^2}\|\Gamma\|_{1,2}^2 + \|V\|_\infty + \varepsilon C,
\end{aligned}
$$

where C is some constant, since $\inf_{t\in S^1}\mathrm{dist}_{\partial\Omega}(\Gamma(t)) > 0$ as Γ is continuous by Lemma 3.4, and $\|V\|_\infty$ is finite because $\bar{\Omega}$ is compact by Heine-Borel.

Proposition 5.1. *$\mathcal{L}_\varepsilon^E$ is of class C^1 and for any $(\Gamma,\tau) \in W^{1,2}(S^1,\Omega)\times\mathbb{R}_{>0}$ and $(\Psi,\sigma)\in W^{1,2}(S^1,\mathbb{R}^N)\times\mathbb{R}$ holds*

$$
\begin{aligned}
\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma,\tau)(\Psi,\sigma) = &\frac{1}{\tau}\int_0^1 \big\langle\Gamma'(t),\Psi'(t)\big\rangle\,\mathrm{d}t - \tau\int_0^1 \big\langle\nabla V(\Gamma(t)),\Psi(t)\big\rangle\,\mathrm{d}t \\
&- \tau\int_0^1 \big\langle\varepsilon\nabla U(\Gamma(t)),\Psi(t)\big\rangle\,\mathrm{d}t + \sigma\int_0^1 \Big[-\frac{1}{2\tau^2}|\Gamma'(t)|^2 - V(\Gamma(t)) - \varepsilon U(\Gamma(t)) + E\Big]\,\mathrm{d}t.
\end{aligned}
\tag{5.1}
$$

Proof. First of all we have that $W^{1,2}(S^1,\Omega)$ is an open subset of $W^{1,2}(S^1,\mathbb{R}^N)$ with the C^0-norm, so by Lemma 3.4 it is an open subset with respect to $\|\cdot\|_{W^{1,2}(S^1,\mathbb{R}^N)}$ as well, and $\mathbb{R}_{>0}$ is an open subset of $\mathbb{R}$. The Cartesian product is open with respect to the product norm ($\|(\cdot,\cdot)\|_{X\times Y} = \|\cdot\|_X + \|\cdot\|_Y$). We want to use Lemma 4.4 and show that

(i) the partial derivative $\mathrm{D}_1\mathcal{L}_\varepsilon^E$ exists and is of class C^0, and
(ii) the partial derivative $\mathrm{D}_2\mathcal{L}_\varepsilon^E$ exists and is of class C^0.

Fix $\mathcal{L} := \mathcal{L}_\varepsilon^E$ and $\mathcal{V} := V + \varepsilon U$.

$\boxed{\text{On (i):}}$ We proceed similarly to [AS09]. Let $\Gamma\in W^{1,2}(S^1,\Omega)$, $\Psi\in W^{1,2}(S^1,\mathbb{R}^N)$, $\tau>0$ and $s\neq 0$ small enough that $\Gamma + s\Psi\in W^{1,2}(S^1,\Omega)$. Then

$$
\begin{aligned}
\frac{\mathcal{L}(\Gamma+s\Psi,\tau)-\mathcal{L}(\Gamma,\tau)}{s} = &\ \tau\int_0^1 \frac{1}{\tau^2}\big\langle\Gamma'(t),\Psi'(t)\big\rangle\,\mathrm{d}t + s\tau\int_0^1 \frac{1}{2\tau^2}|\Psi'(t)|^2\,\mathrm{d}t \\
&- \frac{\tau}{s}\int_0^1 [\mathcal{V}(\Gamma(t)+s\Psi(t)) - \mathcal{V}(\Gamma(t))]\,\mathrm{d}t.
\end{aligned}
\tag{5.2}
$$

In order to use Lemma 4.5 on $g(\Gamma) := \int_0^1 f(t,\Gamma)\,\mathrm{d}t$ with $f(t,\Gamma) = \mathcal{V}(\Gamma(t))$ we show that f is continuous, and that the partial derivative $\mathrm{D}_2 f$ exists and is continuous as well. Let $(t_n,\Gamma_n)\to(t,\Gamma)$. Then $|\Gamma_n(t_n)-\Gamma(t)| \leq |\Gamma_n(t_n)-\Gamma(t_n)| + |\Gamma(t_n)-\Gamma(t)| \xrightarrow{n\to\infty} 0$ since the first term converges to 0, by the uniform convergence of Γ_n to Γ by Lemma 3.4, and the second term converges to 0 by the continuity of Γ. By the continuity of V and U we get that f is continuous. Since $V, U \in C^\infty(\Omega)$, we get the partial derivative $\mathrm{D}_2 f$ with

$$\mathrm{D}_2 f(t,\Gamma)\Psi = \lim_{s\to 0} \frac{1}{s}[\mathcal{V}(\Gamma(t)+s\Psi(t)) - \mathcal{V}(\Gamma(t))] = \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle. \tag{5.3}$$

Now we show that $\mathrm{D}_2 f$ is continuous. Let $(t_n, \Gamma_n) \to (t, \Gamma)$.

$$\begin{aligned}
&\| \mathrm{D}_2 f(t_n,\Gamma_n) - \mathrm{D}_2 f(t,\Gamma)\|_{W^{1,2}(S^1,\mathbb{R}^N)^*} \\
&= \sup_{\|\Psi\|\le 1} |\big\langle \nabla\mathcal{V}(\Gamma_n(t_n)), \Psi(t_n)\big\rangle - \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle| \\
&\le \sup_{\|\Psi\|\le 1} |\big\langle \nabla\mathcal{V}(\Gamma_n(t_n)), \Psi(t_n)\big\rangle - \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t_n)\big\rangle| \\
&\quad + \sup_{\|\Psi\|\le 1} |\big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t_n)\big\rangle - \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle| \\
&\le \sup_{\|\Psi\|\le 1} |\big\langle \nabla\mathcal{V}(\Gamma_n(t_n)) - \nabla\mathcal{V}(\Gamma(t)), \Psi(t_n)\big\rangle| + \sup_{\|\Psi\|\le 1} |\big\langle \nabla\mathcal{V}(\Gamma(t), \Psi(t_n) - \Psi(t)\big\rangle| \\
&\le |\nabla\mathcal{V}(\Gamma_n(t_n) - \nabla\mathcal{V}(\Gamma(t))| \sup_{\|\Psi\|\le 1} |\Psi(t_n)| + |\nabla\mathcal{V}(\Gamma(t))| \sup_{\|\Psi\|\le 1} |\Psi(t_n) - \Psi(t)| \\
&\quad \to 0 \text{ as } n \to \infty,
\end{aligned}$$

where the first term converges to 0 analogously to the proof of f being continuous and because $|\Psi(t_n)| \le \|\Psi\|_\infty \le C\|\Psi\|_{W^{1,2}} \le C$ for some $C > 0$ by Lemma 3.4. For the convergence of the second term we note that by Lemma 3.4 we have that the closed unit ball of $W^{1,2}(S^1)$ is a compact subset of $C^0(S^1)$. By the Arzelà-Ascoli Theorem [Alt16, 4.12] it is equicontinuous, implying the desired convergence.

By Lemma 4.5 we get that g is Fréchet differentiable with $\mathrm{D}g(\Gamma) = \int_0^1 \mathrm{D}_2 f(t,\Gamma)\,\mathrm{d}t$. Let $\Psi \in W^{1,2}(S^1,\mathbb{R}^N)$. Set $\lambda \in W^{1,2}(S^1,\mathbb{R}^N)^{**}$ with $T \mapsto T\Psi$. Then by [Lan93, XIII Proposition 1.1] we get

$$\mathrm{D}g(\Gamma)\Psi = \lambda(\mathrm{D}g(\Gamma)) = \int_0^1 \lambda(\mathrm{D}_2 f(\Gamma(t))\,\mathrm{d}t = \int_0^1 \mathrm{D}_2 f(\Gamma(t))\Psi\,\mathrm{d}t = \int_0^1 \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle\,\mathrm{d}t,$$

using (5.3). So by taking the limit in equation (5.2) we get

$$\begin{aligned}
\mathrm{D}_{1,G}\mathcal{L}(\Gamma,\tau)\Psi &:= \lim_{s\to 0} \frac{\mathcal{L}(\Gamma+s\Psi,\tau) - \mathcal{L}(\Gamma,\tau)}{s} \\
&= \frac{1}{\tau}\int_0^1 \big\langle \Gamma'(t), \Psi'(t)\big\rangle\,\mathrm{d}t - \tau\int_0^1 \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle\,\mathrm{d}t.
\end{aligned}$$

Now $\mathrm{D}_{1,G}\mathcal{L}(\Gamma,\tau) : W^{1,2}(S^1,\mathbb{R}^N) \to \mathbb{R}$ is the Gâteaux-differential as it is a bounded linear functional: $|\,\mathrm{D}_{1,G}(\Gamma,\tau)\Psi| \le C\|\Psi\|_{W^{1,2}(S^1,\mathbb{R}^N)}$ with some $C > 0$ by using the Cauchy-Schwartz inequality.[5]

In order to use Lemma 4.3 we show that $\mathrm{D}_{1,G}\mathcal{L} : W^{1,2}(S^1,\Omega) \times \mathbb{R}_{>0} \to W^{1,2}(S^1,\mathbb{R}^N)^*$ is continuous at (Γ,τ). Let $(\Gamma_n,\tau_n) \to (\Gamma,\tau)$ in $W^{1,2}(S^1,\Omega) \times \mathbb{R}_{>0}$. Again using the

[5]Note that $U(\Gamma) \in L^\infty((0,1),\mathbb{R}^N) \subset L^2((0,1),\mathbb{R}^N)$ as Γ is continuous and stays away from the boundary.

Cauchy-Schwartz inequality we get

$$\begin{aligned}
&\| \mathrm{D}_{1,G}\mathcal{L}(\Gamma_n,\tau_n) - \mathrm{D}_{1,G}\mathcal{L}(\Gamma,\tau)\|_{W^{1,2}(S^1,\mathbb{R}^N)^*}\\
&= \sup_{\|\Psi\|_{1,2}\le 1} \Big| \frac{1}{\tau_n}\int_0^1 \big\langle \Gamma_n'(t),\Psi'(t)\big\rangle\,\mathrm{d}t - \tau_n \int_0^1 \big\langle \nabla\mathcal{V}(\Gamma_n(t)),\Psi(t)\big\rangle\,\mathrm{d}t\\
&\quad - \Big(\frac{1}{\tau}\int_0^1 \big\langle \Gamma'(t),\Psi'(t)\big\rangle\,\mathrm{d}t - \tau\int_0^1 \big\langle \nabla\mathcal{V}(\Gamma(t)),\Psi(t)\big\rangle\,\mathrm{d}t\Big)\Big|\\
&\le \sup_{\|\Psi\|_{1,2}\le 1}\Bigg(\Big|\Big(\frac{1}{\tau_n}-\frac{1}{\tau}\Big)\int_0^1 \big\langle \Gamma_n'(t),\Psi'(t)\big\rangle\,\mathrm{d}t\Big| + \Big|\frac{1}{\tau}\int_0^1 \big\langle \Gamma_n'(t)-\Gamma'(t),\Psi'(t)\big\rangle\,\mathrm{d}t\Big|\\
&\quad + \Big|(\tau_n-\tau)\int_0^1 \big\langle \nabla\mathcal{V}(\Gamma_n(t)),\Psi(t)\big\rangle\,\mathrm{d}t\Big| + \Big|\tau\int_0^1 \big\langle \nabla\mathcal{V}(\Gamma_n(t))-\nabla\mathcal{V}(\Gamma(t)),\Psi(t)\big\rangle\,\mathrm{d}t\Big|\Bigg),
\end{aligned}$$

and using the Hölder inequality, we further estimate

$$\begin{aligned}
\ldots &\le \sup_{\|\Psi\|_{1,2}\le 1}\Bigg(\Big|\frac{1}{\tau_n}-\frac{1}{\tau}\Big|\|\Gamma_n'\|_2\|\Psi'\|_2 + \frac{1}{\tau}\|\Gamma_n'-\Gamma'\|_2\|\Psi'\|_2\\
&\quad + |\tau_n-\tau|\|\nabla\mathcal{V}(\Gamma_n)\|_2\|\Psi\|_2 + \tau\|\nabla\mathcal{V}(\Gamma_n)-\nabla\mathcal{V}(\Gamma)\|_2\|\Psi\|_2\Bigg)\\
&\le \Big|\frac{1}{\tau_n}-\frac{1}{\tau}\Big|\|\Gamma_n'\|_2 + \frac{1}{\tau}\|\Gamma_n'-\Gamma'\|_2\\
&\quad + |\tau_n-\tau|\|\nabla\mathcal{V}(\Gamma_n)\|_2 + \tau\|\nabla\mathcal{V}(\Gamma_n)-\nabla\mathcal{V}(\Gamma)\|_2 \to 0 \text{ as } n\to\infty,
\end{aligned}$$

where the convergence of the fourth term follows from dominated convergence.[6] The first term converges because the convergence of $\Gamma_n \to \Gamma$ in $W^{1,2}(S^1,\mathbb{R}^N)$ implies that $(\|\Gamma_n'\|_2)_{n\in\mathbb{N}}$ is uniformly bounded. The third term converges because $(\nabla\mathcal{V}(\Gamma_n(\cdot)))_{n\in\mathbb{N}}$ is uniformly bounded[7] and the second term converges simply because of $\Gamma_n\to\Gamma$ in $W^{1,2}(S^1,\mathbb{R}^N)$.

Finally, by Lemma 4.3 we get the partial derivative $\mathrm{D}_1\mathcal{L} = \mathrm{D}_{1,G}\mathcal{L}\in C^0$.

$\boxed{\text{On (ii):}}$ Let $\Gamma\in W^{1,2}(S^1,\Omega)$. We set $f:[0,1]\times\mathbb{R}_{>0}\to\mathbb{R}$ with

$$f(t,\tau) = \tau(\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \mathcal{V}(\Gamma(t)) + E = \frac{1}{2\tau}|\Gamma'(t)|^2 - \tau\mathcal{V}(\Gamma(t)) + \tau E.$$

Note that f is continuous and continuously differentiable in τ:

$$\partial_2 f(t,\tau) = -\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \mathcal{V}(\Gamma(t)) + E,$$

i.e. $\mathrm{D}_2 f(t,\tau)\sigma = -\frac{\sigma}{2\tau^2}|\Gamma'(t)|^2 - \sigma\mathcal{V}(\Gamma(t)) + \sigma E$ for $\sigma\in\mathbb{R}$. We may again differentiate under the integral sign and get $\mathrm{D}_2\mathcal{L}$ with

$$\mathrm{D}_2\mathcal{L}(\Gamma,\tau)\sigma = \sigma\int_0^1 \Big[-\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \mathcal{V}(\Gamma(t)) + E\Big]\,\mathrm{d}t.$$

[6] $\big(\nabla U(\Gamma_n) - \nabla U(\Gamma)\big)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty((0,1),\mathbb{R}^N)$ as $\inf_{t\in(0,1),n\in\mathbb{N}}\mathrm{dist}_{\partial\Omega}(\Gamma_n(t)) > 0$ because $\Gamma_n\to\Gamma$ in C^0 by Lemma 3.4.

[7] As in the last footnote.

It remains to be shown that $\mathrm{D}_2\mathcal{L}$ is continuous. Let $(\Gamma_n, \tau_n) \to (\Gamma, \tau)$ in $W^{1,2}(S^1, \mathbb{R}^N) \times \mathbb{R}_{>0}$. We have

$$\begin{aligned}
&\| \mathrm{D}_2\mathcal{L}(\Gamma_n, \tau_n) - \mathrm{D}_2\mathcal{L}(\Gamma, \tau)\|_{\mathbb{R}^*} \\
&= \sup_{|\sigma|\leq 1} \Big|\sigma \int_0^1 \Big[-\frac{1}{2\tau_n^2}|\Gamma_n'(t)|^2 - \mathcal{V}(\Gamma_n(t)) + E\Big] \,\mathrm{d}t - \sigma \int_0^1 \Big[-\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \mathcal{V}(\Gamma(t)) + E\Big] \,\mathrm{d}t\Big| \\
&\leq \int_0^1 \Big|\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \frac{1}{2\tau_n^2}|\Gamma_n'(t)|^2 + \mathcal{V}(\Gamma(t)) - \mathcal{V}(\Gamma_n(t)) + E\Big| \,\mathrm{d}t \to 0 \text{ as } n \to \infty,
\end{aligned}$$

by dominated convergence because $(\Gamma_n')_{n\in\mathbb{N}}$ is uniformly bounded in L^2 and more of the same arguments from above.

With Lemma 4.4 we conclude that $\mathcal{L}_\varepsilon^E$ is of class C^1 with

$$\begin{aligned}
\mathrm{D}\mathcal{L}(\Gamma, \tau)(\Psi, \sigma) &= \mathrm{D}_1\mathcal{L}(\Gamma, \tau)\Psi + \mathrm{D}_2\mathcal{L}(\Gamma, \tau)\sigma = \frac{1}{\tau}\int_0^1 \big\langle \Gamma'(t), \Psi'(t)\big\rangle \,\mathrm{d}t \\
&- \tau \int_0^1 \big\langle \nabla\mathcal{V}(\Gamma(t)), \Psi(t)\big\rangle \,\mathrm{d}t + \sigma \int_0^1 \Big[-\frac{1}{2\tau^2}|\Gamma'(t)|^2 - \mathcal{V}(\Gamma(t)) + E\Big] \,\mathrm{d}t.
\end{aligned}$$

□

Now we will show that we can identify physically possible trajectories, i.e. solutions of the Euler Lagrange equation, with critical points of $\mathcal{L}_\varepsilon^E$. The upcoming proof was compiled using the lecture notes of Hans Joachim Oberle on calculus of variations and optimal control [Obe].

Lemma 5.2. $(\Gamma, \tau) \in W^{1,2}(S^1, \Omega) \times \mathbb{R}_{>0}$ *is a critical point of* $\mathcal{L}_\varepsilon^E$, *i.e.* $\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma, \tau) = 0$, *if and only if for the corresponding curve* $\gamma : \mathbb{R}/\tau\mathbb{Z} \to \Omega$ *with* $\gamma(t) = \Gamma(t/\tau)$ *holds that* $\gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z}, \mathbb{R}^N)$ *and that* γ *is a solution of the Euler-Lagrange system*

$$\gamma'' + \nabla V(\gamma) + \varepsilon\nabla U(\gamma) = 0,$$

with energy

$$E \equiv \tfrac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) + \varepsilon U(\gamma(t)).$$

Proof. "⇒" Let $\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma, \tau)(\Psi, \sigma) = 0$ for all $(\Psi, \sigma) \in W^{1,2}(S^1, \mathbb{R}^N) \times \mathbb{R}$. Especially for e_i a canonical basis vector of $\mathbb{R}^N$, $\psi \in C_0^1([0, 1]) := \{f \in C^1([0, 1]) | 0 = f(0) = f(1) = f'(0) = f'(1)\}$, $\Psi = \psi e_i$[8] and $\sigma = 0$, by (5.1) we have

$$\frac{1}{\tau^2}\int_0^1 \Gamma_i'(t)\psi' \,\mathrm{d}t - \int_0^1 \big[\partial_i V(\Gamma(t)) + \varepsilon\partial_i U(\Gamma(t))\big]\psi(t) \,\mathrm{d}t = 0.$$

We use partial integration[9] on the second term and with $F(t) := \int_0^t \partial_i(V + \varepsilon U)(\Gamma(s)) \,\mathrm{d}s$ we get

$$\int_0^1 \Big(\frac{1}{\tau^2}\Gamma_i'(t) + F(t)\Big)\psi'(t) \,\mathrm{d}t = 0.$$

[8] Indeed $\psi \in W^{1,2}(S^1)$ because $\psi(\cdot \mod 1) \in C^1(\mathbb{R})$

[9] Note that the term $\big[F(t)\psi(t)\big]_0^1$ vanishes.

Applying the Du Bois-Reymond Lemma [Obe, Theorem 4.1] to this we get that the first factor is a constant polynomial, i.e.

$$\frac{1}{\tau^2}\Gamma_i'(t) = -F(t) + C \tag{5.4}$$

for some $C \in \mathbb{R}$. Since F is continuously differentiable, we get $\Gamma_i' \in C^1([0,1])$ with

$$\frac{1}{\tau^2}\Gamma_i''(t) = -F'(t) = -\partial_i(V + \varepsilon U)(\Gamma(t)).$$

In particular, with $\gamma(t) = \Gamma(t/\tau)$ for $t \in [0,\tau]$ we get

$$\gamma''(t) = \frac{1}{\tau^2}\Gamma''(t/\tau) = -\nabla(V + \varepsilon U)(\Gamma(t/\tau)) = -\nabla V(\gamma(t)) - \varepsilon\nabla U(\gamma(t)),$$

i.e.

$$\gamma'' + \nabla V(\gamma) + \varepsilon\nabla U(\gamma) = 0. \tag{5.5}$$

This also implies that the antiderivative $\frac{1}{2}|\gamma'|^2 + V(\gamma) + \varepsilon U(\gamma)$ is constant. With (5.1), from $\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma,\tau)(\Psi,\sigma) = 0$ with $\Psi = 0$ and $\sigma = 1$ we get that this constant must be E, i.e.

$$\tfrac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) + \varepsilon U(\gamma(t)) \equiv E.$$

The C^∞ property remains to be shown. We have already shown $\gamma \in C^2([0,\tau],\mathbb{R}^N)$. Applying this and $U, V \in C^\infty(\Omega)$ to (5.5), we get that γ'' as well is twice continuously differentiable, i.e. $\gamma \in C^4([0,\tau],\mathbb{R}^N)$. This however, by the same argument as above, implies that γ'' is four times continuously differentiable so $\gamma \in C^6([0,\tau],\mathbb{R}^N)$ and so on. By repeating this bootstrap argument, we get $\gamma \in C^\infty([0,\tau],\mathbb{R}^N)$.

We revisit (5.4) and get $\Gamma_i'(0^+) = \Gamma_i'(1^-)$, denoting the left and right derivative, because $F(0) = 0 = \partial_i(V + \varepsilon U)(\Gamma(1)) - \partial(\Gamma(0)) = F(1)$. So $\gamma(\,\cdot\mod\tau) \in C^1(\mathbb{R},\mathbb{R}^N)$. Now since $\gamma(0) = \gamma(\tau)$, using (5.5) we may also conclude that the right second derivative of γ in 0 and the left second derivative of γ in τ coincide. Differentiating both sides of the differential equation (5.5) yields that also the higher derivatives coincide, since by chain rule we get that the left and right derivatives always depend on the lower derivatives that already coincide. We conclude $\gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z},\mathbb{R}^N)$.

"$\Leftarrow$" Let $\tau > 0$ and let $\gamma : \mathbb{R}/\tau\mathbb{Z} \to \Omega$ satisfy $\gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z},\mathbb{R}^N)$ and be a solution to the Euler-Lagrange system with energy E. Certainly we have $\Gamma \in W^{1,2}(S^1,\Omega)$. The energy property immediately implies that the third term of $\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma,\tau)(\Psi,\sigma)$ is 0 for all $(\Psi,\sigma) \in W^{1,2}(S^1,\mathbb{R}^N) \times \mathbb{R}$, i.e.

$$\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma,\tau)(\Psi,\sigma) = \frac{1}{\tau}\int_0^1 \big\langle \Gamma'(t), \Psi'(t)\big\rangle\,\mathrm{d}t - \tau\int_0^1 \big\langle \nabla V(\Gamma(t)) + \varepsilon\nabla U(\Gamma(t)), \Psi(t)\big\rangle\,\mathrm{d}t.$$

This time we use partial integration on the first term[10] and get

$$\mathrm{D}\mathcal{L}_\varepsilon^E(\Gamma,\tau)(\Psi,\sigma) = \tau \int_0^1 \left\langle \tfrac{1}{\tau^2}\Gamma''(t) - \nabla(V+\varepsilon U)(\Gamma(t)), \Psi(t)\right\rangle \mathrm{d}t.$$

This is equal to 0 by the Euler-Lagrange equation, i.e. (Γ,τ) is a critical point of $\mathrm{D}\mathcal{L}_\varepsilon^E$. $\square$

We now introduce the approximating models of case (II). For $\varepsilon > 0$, consider the function

$$u_\varepsilon(x) := \begin{cases} \exp(\frac{1}{\varepsilon})\exp(\frac{1}{x-\varepsilon}) & x < \varepsilon \\ 0 & x \geq \varepsilon. \end{cases} \tag{5.6}$$

It is well-known that $\mathbb{1}_{(0,\infty)}(x)\exp(-\frac{1}{x}) \in C^\infty(\mathbb{R})$, so $u_\varepsilon \in C^\infty(\mathbb{R})$ as well. Note that $u_\varepsilon(0) = 1$.

In the case (II) approximating models, instead of adding the potential $\varepsilon U = \varepsilon u \circ h$, we add $U_\varepsilon = u_\varepsilon \circ h$. We only consider trajectories of energy $E = 1$, and we also restrict the choice of V to be compactly supported, to ensure that trajectories which solve the Euler-Lagrange equation may not escape $\bar{\Omega}$.

We specify this below, but first we thicken up our set Ω, to define an action functional which we can differentiate in trajectories that hit the boundary $\partial\Omega$. Consider the signed distance function

$$S(q) := \begin{cases} \mathrm{dist}_{\partial\Omega}(q) & q \in \bar{\Omega} \\ -\mathrm{dist}_{\partial\Omega}(q) & q \in \mathbb{R}^N \setminus \bar{\Omega}. \end{cases}$$

Similar to the choice of d_0 above, now choose $d_0 \in (0, \frac{1}{2})$, such that the signed distance function S is smooth at all points $q \in \mathbb{R}^N$ with $\mathrm{dist}_{\partial\Omega}(q) \leq 2d_0$. Set

$$\Xi := \{q \in \mathbb{R}^N |\, \operatorname*{dist}_{\partial\Omega}(q) < d_0\} \cup \Omega.$$

Recall k from above and extend it onto $[-d_0,\infty)$, i.e. $k : [-d_0,\infty) \to [-d_0, 2d_0]$ be a smooth function such that $0 \leq k' \leq 1$, $k(x) = x$ on $[-d_0, d_0]$, and k constant on $[2d_0,\infty)$. We extend h to $h \in C^\infty(\Xi)$ through

$$h(q) := k(S(q)).$$

Note that h still satisfies all properties that where stated in case (I).

Now fix $V \in C_c^\infty(\Omega)$ and

$$U_\varepsilon := u_\varepsilon \circ h \in C^\infty(\Xi).$$

Thus, $U_\varepsilon = 1$ on $\partial\Omega$ and $U_\varepsilon = 0$ on $\{q \in \Omega |\, \mathrm{dist}_{\partial\Omega}(q) \geq \varepsilon\}$. For $\varepsilon > 0$, the Lagrangian of our approximating model is

$$\begin{aligned} L_\varepsilon^{(\mathrm{II})} &: T\Xi = \Xi \times \mathbb{R}^N \to \mathbb{R} \\ &(q,v) \mapsto \tfrac{1}{2}|v|^2 - V(q) - U_\varepsilon(q). \end{aligned}$$

[10] Note that now with Γ twice continuously differentiable this is possible, opposed to above where we needed to use partial integration on the second term because a priori we did not know if Γ had a second (weak) derivative. Also note that the terms $\left[\Gamma_i'(t)\Psi_i(t)\right]_0^1$ vanish because $\Psi_i(0) = \Psi_i(1)$ and $\Gamma'(0) = \Gamma'(1)$ as $\gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z}, \mathbb{R}^N)$.

Fixing the energy value to be 1, we define the free-time action functional $\mathcal{L}_\varepsilon^{(\mathrm{II})} : W^{1,2}(S^1, \Xi) \times \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\mathcal{L}_\varepsilon^{(\mathrm{II})}(\Gamma, \tau) := \tau \int_0^1 [L_\varepsilon^{(\mathrm{II})}\Big(\Gamma(t), \frac{1}{\tau}\Gamma'(t)\Big) + 1]\,\mathrm{d}t = \int_0^\tau [L_\varepsilon^{(\mathrm{II})}(\gamma(t), \gamma'(t)) + 1]\,\mathrm{d}t,$$

where $\gamma(t) = \Gamma(t/\tau)$.

For case (II), we get the analogous statements to those above in case (I):

Corollary 5.3. *$\mathcal{L}_\varepsilon^{(\mathrm{II})}$ is of class C^1 and*

$$\begin{aligned}\mathrm{D}\mathcal{L}_\varepsilon^{(\mathrm{II})}(\Gamma, \tau)(\Psi, \sigma) = \frac{1}{\tau}\int_0^1 \big\langle \Gamma'(t), \Psi'(t)\big\rangle\,\mathrm{d}t - \tau\int_0^1 \big\langle \nabla V(\Gamma(t)) + \nabla U_\varepsilon(\Gamma(t)), \Psi(t)\big\rangle\,\mathrm{d}t \\ + \sigma\int_0^1 \Big[-\frac{1}{2\tau^2}|\Gamma'(t)|^2 - V(\Gamma(t)) - U_\varepsilon(\Gamma(t)) + 1\Big]\,\mathrm{d}t,\end{aligned}$$

for all $(\Gamma, \tau) \in W^{1,2}(S^1, \Xi) \times \mathbb{R}_{>0}$ and $(\Psi, \sigma) \in W^{1,2}(S^1, \mathbb{R}^N) \times \mathbb{R}$.

Proof. This follows from Proposition 5.1, by choosing Ω as Ξ, V as $V + U_\varepsilon$, $E = 1$ and U as 0. ☐

Corollary 5.4. *$(\Gamma, \tau) \in W^{1,2}(S^1, \Xi) \times \mathbb{R}_{>0}$ is a critical point of $\mathcal{L}_\varepsilon^{(\mathrm{II})}$, i.e. $\mathrm{D}\mathcal{L}_\varepsilon^{(\mathrm{II})}(\Gamma, \tau) = 0$, if and only if for the corresponding curve $\gamma : \mathbb{R}/\tau\mathbb{Z} \to \Xi$ with $\gamma(t) = \Gamma(t/\tau)$ holds that $\gamma \in C^\infty(\mathbb{R}/\tau\mathbb{Z}, \mathbb{R}^N)$ and that γ is a solution of the Euler-Lagrange system*

$$\gamma'' + \nabla V(\gamma) + \nabla U_\varepsilon(\gamma) = 0$$

with energy

$$1 = E \equiv \tfrac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) + U_\varepsilon(\gamma(t)).$$

Proof. This follows from Lemma 5.2, by choosing Ω as Ξ, V as $V + U_\varepsilon$, $E = 1$ and U as 0. ☐

6. Main result

In this section we prove that periodic bounce orbits can be obtained as limits of periodic orbits in the approximating models that have been introduced in the previous section. In particular, in this section let $u \in C^\infty((0, 1])$ with $\lim_{x\to 0} u(x) = \infty$ and u' monotone for case (I) and in case (II) let u_ε as in (5.6).

We carry out the proofs for the cases (I) and (II) simultaneously, because they have the same structure. By default, we refer to case (I), and whenever the arguments or details for case (II) alter from those of case (I), we add them in the format $\boxed{\text{...special in case (II)...}}$.

To facilitate this, in case (I) we denote

(6.1) $$U_\varepsilon := \varepsilon U \text{ and } u_\varepsilon := \varepsilon u.$$

In this notation, in case (I) we have the free-time action functional

$$\mathcal{L}_\varepsilon^E(\Gamma, \tau) := \tau \int_0^1 [\frac{1}{2\tau^2}|\Gamma'(t)|^2 - V(\Gamma(t)) - U_\varepsilon(\Gamma(t)) + E]\,\mathrm{d}t,$$

and in case (II) we have the free-time action functional

$$\mathcal{L}_\varepsilon^{(II)}(\Gamma,\tau) := \tau \int_0^1 [\frac{1}{2\tau^2}|\Gamma'(t)|^2 - V(\Gamma(t)) - U_\varepsilon(\Gamma(t)) + 1]\,\mathrm{d}t.$$

We note with caution that U_ε differs for the cases (I) and (II).

To simplify the notation we may sometimes omit the integration variable and write "$\int f\,\mathrm{d}t$" instead of "$\int f(t)\,\mathrm{d}t$".

Set $E_{\min} = \min\{\inf_{q\in\Omega} U(q), 0\} + V(q)$.[11] For $E < E_{\min}$, there are no critical points of $\mathcal{L}_\varepsilon^E$, by Lemma 5.2.

Theorem 6.1. (cf. [AM11, Proposition 2.1]) *Let $K > 0$ and $T_2 > T_1 > 0$. For each $\varepsilon > 0$,[12] let $T_1 \le \tau_\varepsilon \le T_2$, $E_{\min} \le E_\varepsilon \le K$ $\boxed{\text{in case (II), } E_\varepsilon \equiv 1}$ and let $(\Gamma_\varepsilon, \tau_\varepsilon)$ be a critical point of $\mathcal{L}_\varepsilon^{E_\varepsilon}$, i.e. $(\Gamma_\varepsilon,\tau_\varepsilon) \in W^{1,2}(S^1,\Omega)\times\mathbb{R}_{>0}$ and $\mathrm{D}\mathcal{L}_\varepsilon^{E_\varepsilon}(\Gamma_\varepsilon,\tau_\varepsilon) = 0$.*

$\boxed{(\Gamma_\varepsilon,\tau_\varepsilon) \text{ be a critical point}^{13} \text{of } \mathcal{L}_\varepsilon^{(\mathrm{II})}, \text{ i.e. } (\Gamma_\varepsilon,\tau_\varepsilon) \in W^{1,2}(S^1,\bar\Omega)\times\mathbb{R}_{>0} \text{ and } \mathrm{D}\mathcal{L}_\varepsilon^{(\mathrm{II})}(\Gamma_\varepsilon,\tau_\varepsilon) = 0.}$

Then, up to a subsequence, $(\Gamma_\varepsilon,\tau_\varepsilon)$ converges to $(\Gamma,\tau) \in W^{1,2}(S^1,\bar\Omega)\times\mathbb{R}_{>0}$ as $\varepsilon\to 0$. Moreover, if we define the curve $\gamma(t) := \Gamma(t/\tau)$, there exists a finite Borel meausure μ on $\mathcal{C} = \{t \in \mathbb{R}/\tau\mathbb{Z} \,|\, \gamma(t) \in \partial\Omega\}$ such that

(i) $\int_0^\tau [\langle\gamma',\psi'\rangle - \langle\nabla V(\gamma),\psi\rangle]\,\mathrm{d}t = \int_{\mathcal{C}} \langle\nu(\gamma),\psi\rangle\,\mathrm{d}\mu$ *for all* $\psi \in W^{1,2}(\mathbb{R}/\tau\mathbb{Z},\mathbb{R}^N)$

(ii) *Outside of* $\mathrm{supp}(\mu)$, *γ is a smooth solution of the Euler-Lagrange system of L as in* (1.1), *with energy* $E(\gamma) = \frac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) = \lim_{\varepsilon\to 0} E_\varepsilon$.

(iii) *γ has left and right derivatives that are left and right continuous on $\mathbb{R}/\tau\mathbb{Z}$, respectively. Moreover, γ satisfies the law of reflection* (1.3) *at each time t which is an isolated point of* $\mathrm{supp}(\mu)$.

In particular, if $\mathrm{supp}(\mu)$ *is a finite set, then γ is a periodic bounce orbit of the Lagrangian system given by L and $\mathcal{B} :=$* $\mathrm{supp}(\mu)$ *is its set of bouncing times.*

Proof. We proceed as in [AM11, Proposition 2.1], with generalizations to account for the more general choice of u.

Since the families $(\tau_\varepsilon)_{\varepsilon>0}$ and $(E_\varepsilon)_{\varepsilon>0}$ are bounded, up to a subsequence for $\varepsilon\to 0$, we have $\tau_\varepsilon \to \tau$ and $E_\varepsilon \to E$ with $T_1 \le \tau \le T_2$ and $E_{\min} \le E \le K$.

We want to show that up to further passing to a subsequence, Γ_ε also converges in $W^{1,2}(S^1,\mathbb{R}^N)$. Planing to use the compact imbedding from Lemma 3.5, we now show that $(\Gamma_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $W^{2,1}(S^1,\mathbb{R}^N)$.

---

[11] The value of $E_{\min}$ is finite because U diverges to $+\infty$ on $\partial\Omega$, so it is bounded from below.

[12] Sometimes for uniform boundedness in ε we need ε bounded. We mean $\varepsilon \in (0, d_0]$ when we write $\varepsilon > 0$ to enhance readability. We use d_0 from the definition of h as a bound, because this makes one argument a little easier for case (II).

[13] The image of a critical point is contained in $\bar\Omega$ by Corollary 5.4, because $U_\varepsilon > 1$ outside of $\bar\Omega$, and the energy of a curve corresponding to a critical point is equal to 1.

Let $\gamma_\varepsilon(t) = \Gamma_\varepsilon(t/\tau_\varepsilon)$ be the periodic orbit corresponding to $(\Gamma_\varepsilon, \tau_\varepsilon)$. By Lemma 5.2 we get that the energy of γ_ε is equal to E_ε, written in terms of $(\Gamma_\varepsilon, \tau_\varepsilon)$:[14]

$$\frac{1}{2\tau_\varepsilon}|\Gamma_\varepsilon'|^2 + V(\Gamma_\varepsilon) + U_\varepsilon(\Gamma_\varepsilon) \equiv E_\varepsilon. \tag{6.2}$$

Since γ_ε is a solution of the Euler-Lagrange Equation associated with L_ε by Lemma 5.2, in terms of $(\Gamma_\varepsilon, \tau_\varepsilon)$ we have

$$\frac{1}{\tau_\varepsilon^2}\Gamma_\varepsilon'' + \nabla V(\Gamma_\varepsilon) + \nabla U_\varepsilon(\Gamma_\varepsilon) = 0. \tag{6.3}$$

Since $(\Gamma_\varepsilon, \tau_\varepsilon)$ is a critical point of $\mathcal{L}_\varepsilon^{E_\varepsilon}$, for each $(\Psi, \sigma) \in W^{1,2}(S^1, \mathbb{R}^N) \times \mathbb{R}$ we have $\mathrm{D}\mathcal{L}_\varepsilon^{E_\varepsilon}(\Gamma_\varepsilon, \tau_\varepsilon)(\Psi, \sigma) = 0$ and choosing $\sigma = 0$ in equation (5.1) since $\nabla U_\varepsilon(p) = u_\varepsilon'(h(p))\nabla h(p)$ for all $p \in \Omega$, we get

$$\int_0^1 [\tau_\varepsilon^{-2}\langle \Gamma_\varepsilon', \Psi'\rangle - \langle \nabla V(\Gamma_\varepsilon), \Psi\rangle]\,\mathrm{d}t = \int_0^1 \langle \nabla U_\varepsilon(\Gamma_\varepsilon), \Psi\rangle\,\mathrm{d}t = \int_0^1 \langle u_\varepsilon'(h(\Gamma_\varepsilon))\nabla h(\Gamma_\varepsilon), \Psi\rangle\,\mathrm{d}t. \tag{6.4}$$

We fix $\Psi = \Psi_\varepsilon = -\nabla h(\Gamma_\varepsilon)$ which is in $W^{1,2}(S^1, \mathbb{R}^N)$ (actually in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^N)$ by chain rule and Lemma 5.2).[15] The sequence $(\Gamma_\varepsilon')_{\varepsilon\in(0,1]}$ is uniformly bounded in L^∞. Indeed, by equation (6.2) we have

$$|\Gamma_\varepsilon'|^2 = 2\tau_\varepsilon(E_\varepsilon - V(\Gamma_\varepsilon) - U_\varepsilon(\Gamma_\varepsilon)) \le 2T_2(K + E_{\min}) < \infty.$$

By chain rule we have $\Psi_\varepsilon' = -D^2h(\Gamma_\varepsilon)\Gamma_\varepsilon'$, so as $h \in C^\infty(\bar{\Omega})$, $(\Psi_\varepsilon')_{\varepsilon>0}$ is uniformly bounded in L^∞ as well. Hence, with our choice of Ψ the first two summands on the left-hand side of (6.4) are uniformly bounded in ε. Indeed,

$$\begin{aligned}
&\left|\int_0^1 [\tau_\varepsilon^{-2}\langle \Gamma_\varepsilon', \Psi_\varepsilon'\rangle - \langle \nabla V(\Gamma_\varepsilon), \Psi_\varepsilon\rangle]\,\mathrm{d}t\right| \\
&\le T_1^{-2}\int_0^1 |\langle \Gamma_\varepsilon', \Psi_\varepsilon'\rangle|\,\mathrm{d}t + \int_0^1 |\langle \nabla V(\Gamma_\varepsilon), \Psi_\varepsilon\rangle|\,\mathrm{d}t \\
&\le T_1^{-2}\|\Gamma_\varepsilon'\|_\infty\|\Psi_\varepsilon'\|_\infty + \|V\|_\infty\|\nabla h\|_\infty \le C \text{ for all } \varepsilon > 0
\end{aligned}$$

for some $C \in \mathbb{R}$. Thus, we get

$$\int_0^1 \langle \varepsilon\nabla U(\Gamma_\varepsilon), \Psi_\varepsilon\rangle\,\mathrm{d}t = -\int_0^1 \varepsilon u'(h(\Gamma_\varepsilon)|\nabla h(\Gamma_\varepsilon)|^2\,\mathrm{d}t \le C \text{ for all } \varepsilon > 0. \tag{6.5}$$

Let $\Omega' \subset \Omega$ be the compact neighborhood of $\partial\Omega$ given by $\Omega' = \{q \in \Omega | h(q) \le d_0\}$, where d_0 is the positive constant that enters the definition of the function h.

In order to show that $(\nabla U_\varepsilon(\Gamma_\varepsilon))_{\varepsilon>0} = (u_\varepsilon'(h(\Gamma_\varepsilon))\nabla h(\Gamma_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in L^1, we calculate the following upper bound

$$\int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))|\,\mathrm{d}t = \int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))|\mathbb{1}_{\{\Gamma_\varepsilon\in\Omega'\}}\,\mathrm{d}t + \int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))|\mathbb{1}_{\{\Gamma_\varepsilon\in\Omega\setminus\Omega'\}}\,\mathrm{d}t.$$

[14] Recall that in case (I) we denote $U_\varepsilon := \varepsilon U$, see (6.1).

[15] Note that the integrabilities are satisfied because $\mathbb{R}/\mathbb{Z}$ or rather [0,1] is compact.

Now as $|\nabla h| \equiv 1$ on Ω' and $u \in C^\infty$ i.e. $u'|_{[d_0,1]} \in C^0([d_0,1])$ bounded, we have

$$\int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))| \,\mathrm{d}t \leq \int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))||\nabla h(\Gamma_\varepsilon)|^2 \,\mathrm{d}t + \varepsilon \|u'|_{[d_0,1]}\|_\infty.$$

The first term splits into

$$-\int_0^1 \varepsilon u'(h(\Gamma_\varepsilon))|\nabla h(\Gamma_\varepsilon)|^2 \,\mathrm{d}t + 2\int_0^1 \varepsilon u'(h(\Gamma_\varepsilon))|\nabla h(\Gamma_\varepsilon)|^2 \mathbb{1}_{\{u'(h(\Gamma_\varepsilon))\geq 0\}} \,\mathrm{d}t$$

where the first term is bounded by equation (6.5) and since $u' < 0$ on some $(0,d)$, the second term is bounded by $2\varepsilon\|u'|_{[d,1]}\|_\infty$. Hence

$$\int_0^1 |\varepsilon u'(h(\Gamma_\varepsilon))| \,\mathrm{d}t \leq C + 3\varepsilon\|u'|_{[min\{d_0,d\},1]}\|_\infty. \tag{6.6}$$

This proves that $(\nabla U_\varepsilon(\Gamma_\varepsilon))_{\varepsilon>0} = (u'_\varepsilon(h(\Gamma_\varepsilon))\nabla h(\Gamma_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in L^1, since $\nabla h \in L^\infty$.

We show that $\nabla(U_\varepsilon(\Gamma_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in L^1. Since the first two summands in (6.4) are uniformly bounded, so is the third, that is $-\int_0^1 u'_\varepsilon(h(\Gamma_\varepsilon))|\nabla h(\Gamma_\varepsilon)|^2 \,\mathrm{d}t = \int_0^1 \left\langle \nabla U_\varepsilon(\Gamma_\varepsilon), \Psi_\varepsilon \right\rangle \mathrm{d}t \leq C$. We now write $\Omega' = \{q \in \bar{\Omega} | h(q) \leq d_0\}$, and calculate

$$\begin{aligned}\int_0^1 |u'_\varepsilon(h(\Gamma_\varepsilon))| \,\mathrm{d}t &= \int_0^1 |u'_\varepsilon(h(\Gamma_\varepsilon))|\mathbb{1}_{\{\Gamma_\varepsilon\in\Omega'\}} \,\mathrm{d}t = \int_0^1 |u'_\varepsilon(h(\Gamma_\varepsilon))||\nabla h(\Gamma_\varepsilon)|^2\mathbb{1}_{\{\Gamma_\varepsilon\in\Omega'\}} \,\mathrm{d}t \\ &\leq \int_0^1 |u'_\varepsilon(h(\Gamma_\varepsilon))||\nabla h(\Gamma_\varepsilon)|^2 \,\mathrm{d}t = -\int_0^1 u'_\varepsilon(h(\Gamma_\varepsilon))|\nabla h(\Gamma_\varepsilon)|^2 \,\mathrm{d}t \leq C\end{aligned}$$

for any $0 < \varepsilon < d_0$ since $u'_\varepsilon = 0$ on $[\varepsilon,\infty)$ and because $u'_\varepsilon \leq 0$. This proves that $(\nabla U_\varepsilon(\Gamma_\varepsilon))_{\varepsilon>0} = (u'_\varepsilon(h(\Gamma_\varepsilon))\nabla h(\Gamma_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in L^1, since $|\nabla h| \leq 1$.

Together with the Euler-Lagrange equation (6.3) and $\tau_\varepsilon \leq T_2$ this implies that $(\Gamma''_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in L^1, since $\nabla V \in L^\infty$. Indeed, from (6.3) we get

$$\begin{aligned}\|\Gamma''_\varepsilon\|_{L^1} &= \tau_\varepsilon^2\|\nabla V(\Gamma_\varepsilon) + \nabla U_\varepsilon(\Gamma_\varepsilon)\|_{L^1} \\ &\leq T_2^2\|\nabla V(\Gamma_\varepsilon)\|_{L^1} + T_2^2\|\nabla U_\varepsilon(\Gamma_\varepsilon)\|_{L^1} \leq T_2^2\|\nabla V\|_\infty + T_2^2\|\nabla U_\varepsilon(\Gamma_\varepsilon)\|_{L^1},\end{aligned}$$

where the last term is uniformly bounded by the previous statement.

We conclude that $(\Gamma_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $W^{2,1}(S^1,\mathbb{R}^N)$. Hence, by compactness of the embedding $W^{2,1}(S^1,\mathbb{R}^N) \subset\subset W^{1,2}(S^1,\mathbb{R}^N)$, see Lemma 3.5, up to further passing to a subsequence for $\varepsilon \to 0$, we have Γ_ε converges to some $\Gamma \in W^{1,2}(S^1,\mathbb{R}^N)$. Actually, by Lemma 3.4 we have $\Gamma \in W^{1,2}(S^1,\bar{\Omega})$. Mind that the image of Γ may contain points in $\partial\Omega$.

Points (i) - (iii) remain to be shown. Set $\tilde{\mu}_\varepsilon := -u'_\varepsilon(h(\Gamma_\varepsilon))$. $(\tilde{\mu}_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in L^1 by (6.6). For $f \in C^0([0,1])$ and $\tilde{\mu}_\varepsilon(f) := \int_0^1 f\tilde{\mu}_\varepsilon \,\mathrm{d}t$ holds $|\tilde{\mu}_\varepsilon| \leq \|f\|_\infty\|\tilde{\mu}_\varepsilon\|_{L^1}$ so by linearity of the integral we have $\tilde{\mu}_\varepsilon \in (C^0([0,1]))^*$. Since the $\tilde{\mu}_\varepsilon$ are uniformly bounded in L^1, the corresponding functionals are uniformly bounded in $(C^0([0,1]))^*$. Since $C^0([0,1])$ is separable [Alt16, 4.18], by [Alt16, Theorem 8.5] the closed unit ball in $(C^0([0,1]))^*$ is weakly* sequentially compact. We conclude that up to a subsequence, $\tilde{\mu}_\varepsilon$ converges to $\tilde{\mu} \in (C^0([0,1]))^*$ weak*. Using the linear isometric isomorphism from the Riesz-Radon Theorem

[Alt16, 6.23] we identify $\tilde{\mu}$ with a regular, finite, signed Borel measure, i.e. $\tilde{\mu}(f) = \int_0^1 f \,\mathrm{d}\tilde{\mu}$ for all $f \in C^0([0,1])$. We set

$$\mathcal{C}' := \{t \in S^1 | \Gamma(t) \in \partial\Omega\}.$$

We show that for $t \notin \mathcal{C}'$, there is a neighborhood O_t of t such that $\tilde{\mu}_\varepsilon|_{O_t}$ converges uniformly to 0. Indeed, let $O_t = [t-\delta, t+\delta]$ for some $\delta > 0$ such that $O_t \cap \mathcal{C}' = \emptyset$. This exists since $\partial\Omega$ is closed and Γ is continuous by Lemma 3.4, which also implies that $\Gamma_\varepsilon \to \Gamma$ in C^0, so there is an $\varepsilon' > 0$ such that for all $\varepsilon < \varepsilon' : \inf_{s \in O_t} \mathrm{dist}_{\partial\Omega}(\Gamma_\varepsilon(s)) \geq \frac{1}{2} \min_{s \in O_t} \mathrm{dist}_{\partial\Omega}(\Gamma(s)) > 0$. This implies that $\tilde{\mu}_\varepsilon|_{O_t} = -u'_\varepsilon(h(\Gamma_\varepsilon))|_{O_t}$ converges uniformly to 0. We conclude that the support of $\tilde{\mu}$ is contained in $\mathcal{C}'$.

We want to obtain (i) by taking the limit in (6.4). Therefore we evaluate the limits of the individual terms. For any $\Psi \in W^{1,2}(S^1, \mathbb{R}^N)$ we have

$$\lim_{\varepsilon \to 0} \int_0^1 \tau_\varepsilon^{-2} \big\langle \Gamma'_\varepsilon, \Psi \big\rangle \,\mathrm{d}t = \int_0^1 \tau^{-2} \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t.$$

Indeed,

$$\begin{aligned}
&\Big| \tau_\varepsilon^{-2} \int_0^1 \big\langle \Gamma'_\varepsilon, \Psi' \big\rangle \,\mathrm{d}t - \tau^{-2} \int_0^1 \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| \\
&\leq \Big| \tau_\varepsilon^{-2} \int_0^1 \big\langle \Gamma'_\varepsilon, \Psi' \big\rangle \,\mathrm{d}t - \tau_\varepsilon^{-2} \int_0^1 \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| + \Big| \tau_\varepsilon^{-2} \int_0^1 \big\langle \Gamma', \Psi' \big\rangle - \tau^{-2} \int_0^1 \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| \\
&= \Big| \tau_\varepsilon^{-2} \int_0^1 \big\langle \Gamma'_\varepsilon - \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| + |\tau_\varepsilon^{-2} - \tau^{-2}| \Big| \int_0^1 \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| \\
&\leq T_1^{-2} \|\Gamma'_\varepsilon - \Gamma'\|_2 \|\Psi'\|_2 + |\tau_\varepsilon^{-2} - \tau^{-2}| \Big| \int_0^1 \big\langle \Gamma', \Psi' \big\rangle \,\mathrm{d}t \Big| \xrightarrow{\varepsilon \to 0} 0,
\end{aligned}$$

using the Cauchy-Schwartz inequality. Furthermore we have

$$\lim_{\varepsilon \to 0} \int_0^1 \big\langle \nabla V(\Gamma_\varepsilon), \Psi \big\rangle \,\mathrm{d}t = \int_0^1 \big\langle \nabla V(\Gamma), \Psi \big\rangle \,\mathrm{d}t,$$

again applying the Cauchy-Schwartz inequality:

$$\Big| \int_0^1 \big\langle \nabla V(\Gamma_\varepsilon) - \nabla V(\Gamma), \Psi \big\rangle \,\mathrm{d}t \Big| \leq \|\nabla V(\Gamma_\varepsilon) - \nabla V(\Gamma)\|_2 \|\Psi\|_2 \xrightarrow{\varepsilon \to 0} 0,$$

using dominated convergence on the first factor with the majorant $2\|\nabla V\|_\infty$. Concerning the third term in (6.4), we note that $\big\langle \nabla h(\Gamma_\varepsilon), \Psi \big\rangle \to \big\langle \nabla h(\Gamma), \Psi \big\rangle$ in C^0. Indeed, for each component i we have

$$\sup_{t \in [0,1]} |\nabla h(\Gamma_\varepsilon(t))_i \Psi_i(t) - \nabla h(\Gamma(t))_i \Psi_i(t)| \leq \sup_{t \in [0,1]} |\nabla h(\Gamma_\varepsilon(t))_i - \nabla h(\Gamma(t))_i| \|\Psi_i\|_\infty \to 0,$$

since $\Psi \in W^{1,2} \subset C^0$, $\Gamma_\varepsilon \to \Gamma$ in C^0 and ∇h equicontinuous. Now note that $\nabla h(\Gamma) = -\nu(\Gamma)$ on $\mathcal{C}'$. Using $\mathrm{supp}(\tilde{\mu}) \subseteq \mathcal{C}'$ and applying [Alt16, Remark 8.3 (6)], we conclude from the convergence $\langle \nabla h(\Gamma_\varepsilon), \Psi \rangle \to \langle \nabla h(\Gamma), \Psi \rangle$ in C^0 that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^1 u_\varepsilon'(h(\Gamma_\varepsilon))\langle \nabla h(\Gamma_\varepsilon), \Psi \rangle \,\mathrm{d}t &= -\lim_{\varepsilon\to 0}\int_0^1 \tilde{\mu}_\varepsilon \langle \nabla h(\Gamma_\varepsilon), \Psi \rangle \,\mathrm{d}t \\ = -\tilde{\mu}(\langle \nabla h(\Gamma), \Psi \rangle) = -\int_{\mathcal{C}'} \langle \nabla h(\Gamma), \Psi \rangle \,\mathrm{d}\tilde{\mu} &= \int_{\mathcal{C}'} \langle \nu(\Gamma), \Psi \rangle \,\mathrm{d}\tilde{\mu}.\end{aligned}$$

Hence, taking the limit in (6.4) yields

$$\int_0^1 \tau^{-2}\langle \Gamma', \Psi' \rangle \,\mathrm{d}t + \int_0^1 \langle \nabla V(\Gamma), \Psi \rangle \,\mathrm{d}t = \int_{\mathcal{C}'} \langle \nu(\Gamma), \Psi \rangle \,\mathrm{d}\tilde{\mu}.$$

By the reparameterisation $\mathbb{R}/_{\tau\mathbb{Z}} \to S^1$ given by $t \mapsto t/\tau$ the measure $\tilde{\mu}$ is pulled back to a measure μ on $\mathcal{C} := \{t \in \mathbb{R}/_{\tau\mathbb{Z}} \,|\, \gamma(t) \in \partial\Omega\}$ and the above equation can be rewritten as in point (i) of the statement.

Now, let $t \notin \operatorname{supp}(\mu)$ and $\delta > 0$ small enough that $[t-\delta, t+\delta] \cap \operatorname{supp}(\mu) = \emptyset$. This exists because the support is closed by definition. For a $\psi \in W^{1,2}(\mathbb{R}/_{\tau\mathbb{Z}}, \mathbb{R}^N)$ that is supported in $[t-\delta, t+\delta]$, from (i) we get

$$\int_{t-\delta}^{t+\delta} [\langle \gamma', \psi' \rangle - \langle \nabla V(\gamma), \psi \rangle] \,\mathrm{d}t = 0.$$

Analogously to the proof of Lemma 5.2, we get that γ is a smooth solution to the Euler-Lagrange equation of L on $[t-\delta, t+\delta]$.

We need to show that the energy is E. For that, first we will show that $U_\varepsilon(\gamma_\varepsilon) \to 0$ almost everywhere for the subsequence from above. Set

$$I = \{t \in \mathbb{R}/_{\tau\mathbb{Z}} \,|\, U_\varepsilon(\gamma_\varepsilon(t)) = u_\varepsilon(h(\gamma_\varepsilon(t))) \text{ does not converge to zero}\} \tag{6.7}$$

Note that for each $t \in I$ we have $h(\gamma_\varepsilon(t)) \to 0$ since $\gamma_\varepsilon \to \gamma$ in C^0 implies $\gamma_\varepsilon(t) \to \gamma(t)$, $u_\varepsilon = \varepsilon u$ and u is bounded on any $[d, 1]$ for $d > 0$. $\boxed{h(\gamma_\varepsilon(t)) \to 0 \text{ since } u_\varepsilon = 0 \text{ on } [\varepsilon, \infty).}$

Hence, for $t \in I$ we have

$$\varepsilon \nabla U(\gamma_\varepsilon(t)) = \varepsilon u'(h(\gamma_\varepsilon(t)))\nabla h(\gamma_\varepsilon(t)) = \varepsilon u(h(\gamma_\varepsilon(t))\frac{u'(h(\gamma_\varepsilon(t)))}{u(h(\gamma_\varepsilon(t)))}\nabla h(\gamma_\varepsilon(t))$$

for ε small enough that $U(\gamma_\varepsilon(t)) \neq 0$. Since $\lim_{x\to 0} u(x) = \infty$ and u' monotone, by Lemma 2.4 we have $|\frac{u'(h(\gamma_\varepsilon))}{u(h(\gamma_\varepsilon))}| \to \infty$ pointwise on I as $h(\gamma_\varepsilon) \to 0$ pointwise on I. We conclude that

$$|\varepsilon \nabla U(\gamma_\varepsilon)| = |\varepsilon u(h(\gamma_\varepsilon))| \left|\frac{u'(h(\gamma_\varepsilon))}{u(h(\gamma_\varepsilon))}\right| |\nabla h(\gamma_\varepsilon)| \to \infty \text{ pointwise on } I,$$

as the first factor does not converge to zero, nor does a subsequence converge to zero since $\gamma_\varepsilon(t)$ does converge, and the last factor converges to 1 pointwise on I.

For $t \in I$ we have

$$\nabla U_\varepsilon(\gamma_\varepsilon(t)) = u'_\varepsilon(h(\gamma_\varepsilon(t))\nabla h(\gamma_\varepsilon(t)) = u_\varepsilon(h(\gamma_\varepsilon(t)))\frac{u'_\varepsilon(h(\gamma_\varepsilon(t)))}{u_\varepsilon(h(\gamma_\varepsilon(t)))}\nabla h(\gamma_\varepsilon(t))$$

for ε small enough that $U_\varepsilon(\gamma_\varepsilon(t)) \neq 0$. Calculate

$$u'_\varepsilon(x) = \begin{cases} -\exp(\frac{1}{\varepsilon})\exp(\frac{1}{x-\varepsilon})\frac{1}{(x-\varepsilon)^2} & ,x < \varepsilon \\ 0 & ,x \geq \varepsilon. \end{cases}$$

Hence

$$\Big|\frac{u'_\varepsilon(h(\gamma_\varepsilon(t)))}{u_\varepsilon(h(\gamma_\varepsilon(t)))}\Big| = \frac{1}{(h(\gamma_\varepsilon(t)) - \varepsilon)^2} \to \infty \text{ as } \varepsilon \to 0,$$

remembering that since $t \in I$, $0 \leq h(\gamma_\varepsilon(t)) < \varepsilon$. We conclude that

$$|\nabla U_\varepsilon(\gamma_\varepsilon)| = |u_\varepsilon(h(\gamma_\varepsilon))|\Big|\frac{u'_\varepsilon(h(\gamma_\varepsilon))}{u_\varepsilon(h(\gamma_\varepsilon))}\Big||\nabla h(\gamma_\varepsilon)| \to \infty \text{ pointwise on } I.$$

considering the other factors the same way as above.

Now, assume that I has a positive Lebesgue measure. By Fatou's Lemma we get

$$\liminf_{\varepsilon\to 0} \int_I |\nabla U_\varepsilon(\gamma_\varepsilon)| \,\mathrm{d}t \geq \int_I \liminf_{\varepsilon\to 0} |\nabla U_\varepsilon(\gamma_\varepsilon)| \,\mathrm{d}t = \infty,$$

which contradicts the fact that $(\nabla U_\varepsilon(\gamma_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in L^1 which has been established above. Our claim that $U_\varepsilon(\gamma_\varepsilon) \to 0$ almost everywhere, holds true.

We apply this to take the limit in (6.2) which we rewrite to

$$|\Gamma'_\varepsilon|^2 = 2\tau_\varepsilon(E_\varepsilon - V(\Gamma_\varepsilon) - U_\varepsilon(\Gamma_\varepsilon)).$$

The right-hand side of the equation converges almost everywhere to $2\tau(E - V(\Gamma))$. Hence, the left-hand side converges almost everywhere as well, and since $|\Gamma'_\varepsilon| \to |\Gamma'|$ in L^2, we get $|\Gamma'_\varepsilon|^2 \to |\Gamma'|^2$ almost everywhere. We conclude

$$\frac{1}{2}|\gamma'|^2 + V(\gamma) = E \text{ almost everywhere.} \tag{6.8}$$

This establishes point (ii).

From point (i) we deduce that the components of the weak derivative γ' are functions of bounded variation. This follows immediately from the the definition as in [AFP00, Definition 3.1.]. For these it is well known that the components are differences of two monotone functions by Jordan decomposition, and that therefore they possess left and right limits. We conclude that γ has left and right derivatives and that they are left and right continuous, respectively.

This allows to deduce from (6.8) that

$$\frac{1}{2}|\gamma'(t^\pm)|^2 + V(\gamma(t)) = E \text{ for all } t \in {}^{\mathbb{R}}\!/_{\tau\mathbb{Z}}\,. \tag{6.9}$$

Now let us consider a time t which is an isolated point in $\operatorname{supp}(\mu)$. In order to complete the proof, we only need to establish that the reflection rule is satisfied in t. Since t is an isolated point, we may choose $\delta > 0$ sufficiently small such that $[t-\delta, t+\delta] \cap \operatorname{supp}(\mu) = \{t\}$. In point (i) of the statement, let us choose ψ to be supported in the interval $[t-\delta, t+\delta]$. Then point (i) reduces to

$$\int_{t-\delta}^{t+\delta} \langle \gamma', \psi' \rangle \, \mathrm{d}s - \int_{t-\delta}^{t+\delta} \langle \nabla V(\gamma), \psi \rangle \, \mathrm{d}s = \langle \nu(\gamma(t)), \psi(t) \rangle \mu(\{t\}). \tag{6.10}$$

Note that outside of $\{t\}$, γ'' exists. We use partial integration to rearrange the first term: For each component i and a null sequence $(a_n)_{n\in\mathbb{N}} \subset (0, \delta)$ we have

$$\begin{aligned}
\int_{t-\delta}^{t+\delta} \gamma_i'\psi_i' \, \mathrm{d}s &= \int_{t-\delta}^{t-a_n} \gamma_i'\psi_i' \, \mathrm{d}s + \int_{t-a_n}^{t+a_n} \gamma_i'\psi_i' \, \mathrm{d}s + \int_{t+a_n}^{t+\delta} \gamma_i'\psi_i' \, \mathrm{d}s \\
&= [\gamma_i'\psi_i]_{t-\delta}^{t-a_n} - \int_{t-\delta}^{t-a_n} \gamma_i''\psi_i \, \mathrm{d}s + \int_{t-a_n}^{t+a_n} \gamma_i'\psi_i' \, \mathrm{d}s + [\gamma_i'\psi_i]_{t+a_n}^{t+\delta} - \int_{t+a_n}^{t+\delta} \gamma_i''\psi_i \, \mathrm{d}s \\
&\xrightarrow{n\to\infty} \gamma_i'(t^-)\psi(t) - \int_{(t-\delta,t)} \gamma_i''\psi \, \mathrm{d}s - \gamma_i'(t^+)\psi(t) - \int_{(t,t+\delta)} \gamma_i''\psi \, \mathrm{d}s.
\end{aligned}$$

Inserting this result into (6.10) and using that by (ii), outside of $\{t\}$ the Euler Lagrange equation corresponding to L is satisfied, we get

$$\langle \gamma'(t^-) - \gamma'(t^+), \psi(t) \rangle = \langle \nu(\gamma(t)), \psi(t) \rangle \mu(\{t\}),$$

and by varying the choice of ψ we get

$$\langle \gamma'(t^-) - \gamma'(t^+), v \rangle = \langle \nu(\gamma(t)), v \rangle \mu(\{t\}) \text{ for all } v \in \mathbb{R}^N. \tag{6.11}$$

We observe that $\langle \gamma'(t^-) - \gamma'(t^+), v \rangle = 0$ for all $v \in T_{\gamma(t)}\partial\Omega$, so $\gamma'(t^-) - \gamma'(t^+) \in \operatorname{span}(\nu(\gamma(t)))$. Hence, $\gamma'(t^-) - \gamma'(t^+)$ must coincide with its projection onto $\operatorname{span}(\nu(\gamma(t)))$, i.e.

$$\gamma'(t^-) - \gamma'(t^+) = \nu(\gamma(t)) \cdot \langle \nu(\gamma(t)), \gamma'(t^-) - \gamma'(t^+) \rangle,$$

which can be rearranged to

$$\gamma'(t^-) - \langle \gamma'(t^-), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t)) = \gamma'(t^+) - \langle \gamma'(t^+), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t)), \tag{6.12}$$

which is the second equation of (1.3). Using the pythagorean theorem we decompose

$$\begin{aligned}
|\gamma(t^-)|^2 &= |\gamma'(t^-) - \langle \gamma'(t^-), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t))|^2 + |\langle \gamma'(t^-), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t))|^2 \\
&= |\gamma'(t^-) - \langle \gamma'(t^-), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t))|^2 + |\langle \gamma'(t^-), \nu(\gamma(t)) \rangle|^2;
\end{aligned}$$

and

$$|\gamma'(t^+)|^2 = |\gamma'(t^+) - \langle \gamma'(t^+), \nu(\gamma(t)) \rangle \cdot \nu(\gamma(t))|^2 + |\langle \gamma'(t^+), \nu(\gamma(t)) \rangle|^2.$$

By the conservation of energy (6.9), we have $|\gamma'(t^-)|^2 = |\gamma'(t^+)|^2$, so using (6.12) we get

$$|\langle \gamma'(t^-), \nu(\gamma(t)) \rangle| = |\langle \gamma'(t^+), \nu(\gamma(t)) \rangle|.$$

If we now insert $v = \nu(\gamma(t))$ in (6.11), we get

$$\langle \gamma'(t^-), \nu(\gamma(t)) \rangle - \langle \gamma'(t^+), \nu(\gamma(t)) \rangle = \mu(\{t\}) \neq 0.$$

Since the absolute values of both terms on the left-hand side coincide, one must be positive and the other negative, i.e.

$$\big\langle \gamma'(t^-), \nu(\gamma(t))\big\rangle = -\big\langle \gamma'(t^+), \nu(\gamma(t))\big\rangle \neq 0.$$

This concludes the proof of point (iii). □

A limit of the approximation above does not necessarily have any bounce points. In fact, it is possible that the orbit of the limit is contained in $\partial\Omega$, but that there are no bounce times as in the following Example 6.2. This is also an example where the limit γ does not solve the Euler-Lagrange equation corresponding to L from (1.1) at any time t. The statement of Theorem 6.1 still holds true however.

Example 6.2. Consider the unit disc $\Omega = B_1(0) \subset \mathbb{R}^2$ and the approximation scheme of case (I) with $u(x) = \frac{1}{x}$. Set $V = 0$. We can choose $d_0 = \frac{1}{4}$. Periodic trajectories in the approximating models are

$$\gamma_\varepsilon(t) = r_\varepsilon \begin{pmatrix} \sin(t/\tau_\varepsilon) \\ \cos(t/\tau_\varepsilon) \end{pmatrix},$$

for $\varepsilon \leq \frac{1}{16} = d_0^2$, $r_\varepsilon = 1 - \sqrt{\varepsilon}$ and $\tau_\varepsilon = \sqrt{1-\sqrt{\varepsilon}}$. Indeed, the relevant derivatives are

$$\gamma_\varepsilon'(t) = r_\varepsilon/\tau_\varepsilon \begin{pmatrix} \cos(t/\tau_\varepsilon) \\ -\sin(t/\tau_\varepsilon) \end{pmatrix} \text{ and } \gamma_\varepsilon''(t) = r_\varepsilon/\tau_\varepsilon^2 \begin{pmatrix} -\sin(t/\tau_\varepsilon) \\ -\cos(t/\tau_\varepsilon) \end{pmatrix} = -\frac{1}{\tau_\varepsilon^2}\gamma_\varepsilon(t).$$

The Euler-Lagrange equation is satisfied:

$$\begin{aligned}\varepsilon\nabla U(\gamma_\varepsilon(t)) &= \varepsilon u'(\gamma_\varepsilon(t))\nabla h(\gamma_\varepsilon(t)) = -\frac{\varepsilon}{\operatorname{dist}_{\partial\Omega}(\gamma_\varepsilon(t))^2}\Big(-\frac{\gamma_\varepsilon(t)}{r_\varepsilon}\Big) = \frac{\varepsilon}{(1-r_\varepsilon)^2}\frac{\gamma_\varepsilon(t)}{r_\varepsilon} \\ &= \frac{1}{1-\sqrt{\varepsilon}}\gamma_\varepsilon(t) = \frac{1}{\tau_\varepsilon^2}\gamma_\varepsilon(t) = -\gamma_\varepsilon''(t).\end{aligned}$$

The energy of γ_ε is:

$$\begin{aligned}E(\gamma_\varepsilon) &= \frac{1}{2}|\gamma_\varepsilon'(t)|^2 + \varepsilon U(\gamma_\varepsilon)) = \frac{1}{2}|\gamma_\varepsilon'(t)|^2 + \varepsilon u(h(\gamma_\varepsilon)) = \frac{r_\varepsilon^2}{2\tau_\varepsilon^2} + \frac{\varepsilon}{\operatorname{dist}_{\partial\Omega}(\gamma_\varepsilon(t))} \\ &= \frac{r_\varepsilon^2}{2\tau_\varepsilon^2} + \frac{\varepsilon}{1-r_\varepsilon} = \frac{(1-\sqrt{\varepsilon})^2}{2(1-\sqrt{\varepsilon})} + \sqrt{\varepsilon} = \frac{1-\sqrt{\varepsilon}}{2} + \sqrt{\varepsilon} = E_\varepsilon.\end{aligned}$$

Both the energy and the period are uniformly bounded. We can verify the convergence to $\gamma(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$ of Theorem 6.1 by hand: We have

$$\Gamma_\varepsilon(t) = r_\varepsilon \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \text{ and } \Gamma_\varepsilon'(t) = r_\varepsilon \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.$$

Now for $\Gamma = \gamma$ we evaluate the L^2 convergence of the first component

$$\int_0^1 (r_\varepsilon sin(t) - sin(t))^2 \,\mathrm{d}t = \varepsilon \int_0^1 sin(t)^2 \,\mathrm{d}t \leq \varepsilon \xrightarrow{\varepsilon\to 0} 0,$$

and analogously for the other component and the components of the first derivative. Hence $(\Gamma_\varepsilon, \tau_\varepsilon) \to (\Gamma, 1)$ in $W^{1,2}(S^1, \mathbb{R}^2) \times \mathbb{R}$. Note that $\gamma(t) \in \partial\Omega$ for all t and that none of these are bounce times. Also note that γ does not solve the Euler-Lagrange equation $\gamma'' = 0$. This is no contradiction to point (ii) of Theorem 6.1, since the measure μ is supported at any time t. Indeed $\tilde{\mu}_\varepsilon = -u'(h(\Gamma_\varepsilon)) = \frac{1}{(\mathrm{dist}_\partial \Omega(\Gamma_\varepsilon))^2} = \frac{1}{(1-r_\varepsilon)^2} \xrightarrow{\varepsilon \to 0} 1$. I.e. point (i) of Theorem 6.1 holds with μ being the Lebesgue measure and point (ii) and (iii) become empty statements in this case, since there is no "outside $\mathrm{supp}(\mu)$" and there are no isolated points of $\mathrm{supp}(\mu)$.

Remark 6.3. In case (I) we prescribed that $\lim_{x\to 0} u(x) = \infty$. In fact, the statement of Theorem 6.1 still holds if we omit this condition. Indeed, the only time this condition appeared in the proof was to show that the set I from (6.7) had Lebesgue measure 0.[16] Now if $u(x)$ does not diverge to ∞, u must be bounded, since u' monotone implies that u is monotone on $(0, d)$ for some $d > 0$. Hence, in this case the set I is already empty.

If u' is bounded as well, then $\tilde{\mu}_\varepsilon = -\varepsilon u'(h(\Gamma_\varepsilon))$ converges uniformly to 0, so $\mu = 0$ and the limit γ solves the Euler-Lagrange equation corresponding to L from (1.1) at any time t. In particular there are no bounce times in this case.

If u is bounded but u' is not bounded, as for instance if $u(x) = 1 - \sqrt{x}$ and $u'(x) = -\frac{1}{2\sqrt{x}}$, there still might be bounce points, because in this case $\tilde{\mu}_\varepsilon = -\varepsilon u'(h(\Gamma_\varepsilon)) = \frac{\varepsilon}{2\sqrt{h(\Gamma_\varepsilon)}}$ might still diverge to ∞ at some times t. This has not been further investigated for this thesis.

The choice to investigate only the case $\lim_{x\to 0} u(x) = \infty$ was made, because then trajectories in the approximating models can not escape the domain Ω.

[16]Actually, if we omit the condition $\lim_{x\to 0} u(x) = \infty$, then u' is not necessarily negative near 0. However it is either non-negative or non-positive near 0. For the discussion leading up to (6.6), we might have to replace our choice of Ψ_ε by $-\Psi_\varepsilon$. However the arguments stay completely analogous.

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Erklärung zur Bachelorarbeit. Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst habe und dass keine anderen als die angegebenen Quellen und Hilfsmittel benutzt wurden.

Weiterhin erkläre ich, dass diese Arbeit bisher keinem anderen Prüfungsamt in gleicher oder vergleichbarer Form vorgelegt und bisher nicht veröffentlicht wurde.

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