

Eberhard Freitag

# Riemann Surfaces

Sheaf Theory, Riemann Surfaces, Automorphic Forms

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## Introduction

In many text books about Riemann surfaces, the theorem of Riemann–Roch for divisors or line bundles has been treated. But usually they do not contain the theorem for vector bundles. Here the reader usually is referred to the Hirzebruch–Riemann–Roch theorem. The Riemann–Roch theorem for vector bundles has very elementary applications. For example, one can use it to compute dimensions of vector spaces of vector valued elliptic modular forms as they are used for example in the theory of Borchers products.

One purpose of this book is to give a quick proof of the Riemann–Roch theorem for vector bundles on a compact Riemann surface, and, as an application, we will deduce these dimension formulae in a rather general context.

The approach to Riemann surfaces is sheaf theoretic, quite different than in our textbook [Fr1]. There are similarities with the important book of Forster [Fo]. Forster also uses the sheaf theoretic approach. To keep this as elementary as possible, he restricts to introduce the first Čech cohomology group of a sheaf. Instead of this we introduce a full cohomology theory of sheaves in a form which allows to use this approach also in other mathematical topics as for example in algebraic geometry. Actually, we use Godement’s canonical flabby resolution to introduce the cohomology groups of a sheaf.

In the one variable case, the Riemann–Roch theorem is just a consequence of the finiteness theorem which states that the cohomology groups  $H^i(X, \mathcal{M})$  of a vector bundle  $\mathcal{M}$  on a compact Riemann surface are finite dimensional and vanish for  $i > 1$ .

The reduction of the Riemann–Roch theorem to the finiteness theorem gets particularly transparent if one formulates and proves it not only for vector bundles but for coherent sheaves. It is not necessary to introduce here the general machinery of coherent sheaves. Coherent sheaves can be introduced as extensions of vector bundles by skyscraper sheaves.

This version of the Riemann–Roch theorem does not contain the duality theorem. As usual in the modern approaches, its proof is separated from the proof of the Riemann–Roch theorem and needs some extra effort as a cohomological version of the residue.

After the proof of the Riemann–Roch theorem we treat the Jacobi inversion problem in an extra Chapter VII. We start this chapter with the Hodge theory of Riemann surfaces which shows that the first cohomology group with coefficients in  $\mathbb{C}$  can be described by means of harmonic differentials. Readers who are mainly interested in the mentioned dimension formulae can skip this chapter and switch to the last chapter where these formulae are derived.

In the last Chapt. VIII we derive the mentioned dimension formulae. First

we give a very short introduction to discrete subgroups of  $SL(2, \mathbb{R})$  and the associated Riemann surfaces and their extension by cusps. We are interested in the case where these surfaces are compact. Then we introduce vector valued automorphic forms which belong to real weight and matrix valued multiplier systems. We use the Riemann-Roch theorem to derive basic dimension formulae for spaces of automorphic forms. We describe two different methods for the computation of the degree of the occurring vector bundles. The first one works for rational weight and multiplier systems which get trivial on a subgroup of finite index. This method is very simple and needs no further ingredients. For practical purposes, this special case is sufficient. But we don't want to forget about the general case completely and prove the dimension formulae also in the general case in a compendious appendix. Here we have to introduce the notion of the Chern form of a line bundle and the formula of Gauss-Bonnet comes into the game.

# Chapter I. Topological spaces

## 1. The notion of a topological space

A topology  $\mathcal{T}$  on a set  $X$  is a system of subsets – called open subsets – with the following properties:

- 1)  $\emptyset, X$  are open.
- 2) The intersection of finitely many open subsets is open.
- 3) The union of arbitrarily many open sets is open.

A *topological space* is a pair  $(X, \mathcal{T})$ , consisting of a set  $X$  and a topology  $\mathcal{T}$  on  $X$ . Usually we will write  $X$  instead of  $(X, \mathcal{T})$  when it is clear which topology is considered. We give some constructions for topological spaces:

### Metric spaces

Let  $(X, d)$  be a metric space. We denote by  $U_r(a)$  the ball around  $a$  of radius  $r$ . A subset  $U$  of  $X$  is called open if for every  $a \in X$  there exists  $\varepsilon > 0$  with the property  $U_\varepsilon(a) \subset U$ .

### The induced topology

Let  $Y$  be a subset of a topological space  $X = (X, \mathcal{T})$ . Then  $Y$  can be equipped with the *induced topology*  $\mathcal{T}|Y$ .

A subset  $V \subset Y$  belongs to  $\mathcal{T}|Y$  if and only if there exists a subset  $U \subset X$ ,  $U \in \mathcal{T}$ , such that

$$V = U \cap Y.$$

(When  $Y$  is an open subset of  $X$  then this means  $V \in \mathcal{T}$ .) Since a subset  $V \subset Y$  is also a subset of  $X$ , one has to make clear whether “open” means to be in  $\mathcal{T}$  or in  $\mathcal{T}|Y$ . If we say “open in  $X$ ” we mean that  $V$  is contained in  $\mathcal{T}$ . Similarly “open in  $Y$ ” means that it is contained in  $\mathcal{T}|Y$ .

### The quotient topology

Let  $X$  be a topological space and  $f : X \rightarrow Y$  a surjective map onto a set  $Y$ . Then  $Y$  can be equipped with the *quotient topology*. A subset  $V \subset Y$  is called open if and only if the inverse image  $U := f^{-1}(V)$  is open in  $X$ . There is an important special case: Let „ $\sim$ “ be an equivalence relation on  $X$  and let be  $Y = X/\sim$  the set of equivalence classes and  $f : X \rightarrow Y$  the canonical projection. Then  $Y$  is called the quotient space of  $X$ . Examples are a torus  $X = \mathbb{C}/L$  for a lattice  $L$  or the modular space  $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ .

### The product topology

Let  $X_1, \dots, X_n$  be a finite system of topological spaces. Then a natural topology on the cartesian product

$$X = X_1 \times \cdots \times X_n$$

can be defined.

A subset  $U \subset X$  is called open if and only if for every point  $a \in U$  there exist open subsets  $U_1 \subset X_1, \dots, U_n \subset X_n$ , such that

$$a \in U_1 \times \cdots \times U_n.$$

## 2. Continuous maps

A subset of a topological space  $A \subset X$  is called *closed* if the complement  $X - A$  is open.

A subset  $M \subset X$  is called a *neighborhood* of a point  $a \in X$  if there exists an open subset  $U \subset X$  with  $a \in U \subset M$ .

A point  $a \in X$  is called a *boundary point* of a subset  $M \subset X$  if every neighborhood of  $a$  contains points of  $M$  and of its complement  $X - M$ .

*Notation.*

$$\partial M := \text{set of boundary points,}$$

$$\bar{M} := M \cup \partial M.$$

One shows easily that  $\bar{M}$  is the smallest closed subset of  $X$  which contains  $M$ . In particular,

$$M \text{ closed} \iff M = \bar{M}.$$

We call  $\bar{M}$  the *closure* of  $M$ . Similarly there exists a biggest open subset  $M^\circ$  of a set  $M$ . We call it the *interior* of  $M$ .

A map  $f : X \rightarrow Y$  between topological spaces is called *continuous* at a point  $a \in X$  if the inverse image  $f^{-1}(V(b))$  of an arbitrary neighborhood of  $b := f(a)$  is a neighborhood of  $a$ .

The following conditions are equivalent.

- 1) The map  $f$  is continuous (i.e. continuous at every point).
- 2) The inverse image of an arbitrary open subset of  $Y$  is open in  $X$ .
- 3) The inverse image of an arbitrary closed subset of  $Y$  is closed in  $X$ .

The composition of two continuous maps is continuous. (This is true already in the pointwise sense.)



### Some universal properties

Let  $Y$  be a subset of a topological space equipped with the induced topology. A map  $f : Z \rightarrow Y$  of a third topological space into  $Y$  is continuous if and only if the composition with the natural inclusion  $Y \hookrightarrow X$  is a continuous map  $Z \rightarrow X$ .

Let  $f : X \rightarrow Y$  be a surjective map of topological spaces, where  $Y$  carries the quotient topology. A map  $Y \rightarrow Z$  into a third topological space  $Z$  is continuous if and only if the composition with  $f$  is a continuous map  $X \rightarrow Z$ .

Let  $X_1, \dots, X_n$  be topological spaces and

$$f : Y \longrightarrow X_1 \times \cdots \times X_n$$

a map of another topological space  $Y$  into the cartesian product (equipped with the product topology). The map  $f$  is continuous if and only if each component

$$f_j : Y \longrightarrow X_j \quad (f = (f_1, \dots, f_n))$$

is continuous.

### Topological maps

A map  $f : X \rightarrow Y$  between topological spaces is called *topological* if it is bijective and if  $f$  and  $f^{-1}$  both are continuous. Two topological spaces  $X, Y$  are called *topologically equivalent* or *homeomorphic* if there exists a topological map between them.

For example the 2-sphere

$$S^2 = \{ x \in \mathbb{R}^3; \quad \xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \}$$

and the *Riemann sphere* are homeomorphic. Here the Riemann sphere  $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is equipped with the following topology. A subset  $U \subset \bar{\mathbb{C}}$  is open if the following two conditions are satisfied:

- a)  $U \cap \mathbb{C}$  is open.
- b) If  $\infty \in U$  then there exists  $C > 0$  such that  $\{z \in \mathbb{C}; |z| > C\} \subset U$ .

The topological map  $S^2 \rightarrow \bar{\mathbb{C}}$  is constructed by means of the stereographic projection.

If  $L \subset \mathbb{C}$  is a lattice, then  $\mathbb{C}/L$  is homeomorphic to the cartesian product of two circles:

$$\mathbb{C}/L = \text{Torus} \simeq S^1 \times S^1.$$

### 3. Special classes of topological spaces

A topological space  $X$  is called *Hausdorff* if for two different points  $a, b \in X$  there exist disjoint neighborhoods  $U(a)$  and  $U(b)$ .

A topological space  $X$  is called *compact* if it is Hausdorff and if every open covering admits a finite sub-covering. A subset  $Y$  of a topological space  $X$  is called compact if it is – equipped with the induced topology – a compact topological space.

#### Properties of compact subsets

- a) Compact subsets are closed.
- b) A closed subset of a compact space is compact.
- c) Let  $f : X \rightarrow Y$  be a continuous map of Hausdorff spaces, then the image of a compact subset is compact.
- d) Let  $X$  be a compact and  $Y$  a Hausdorff space and  $f : X \rightarrow Y$  bijective and continuous. Then  $f$  is topological.
- e) The product  $X_1 \times \cdots \times X_n$  of compact spaces is compact.

A topological space is called *locally compact* if it is Hausdorff and if every point admits a compact neighborhood. The product  $X_1 \times \cdots \times X_n$  of locally compact spaces is locally compact.

#### Proper maps

A continuous map  $f : X \rightarrow Y$  of Hausdorff spaces is called *proper* if the inverse image of an arbitrary compact subset is compact. If  $Y$  consists only of one point then this means that  $X$  is a compact space. Hence “proper” should be considered as a relative version of compact. A simple example of a proper map is

$$\mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto z^n,$$

for natural numbers  $n$ .

**3.1 Lemma.** *Let  $f : X \rightarrow Y$  be a proper map. Let  $B \subset Y$  be a subset and  $A = f^{-1}(B)$  the full inverse image of  $f$ . Then the restriction  $A \rightarrow B$  is also proper.*

*Proof.* Let  $K \subset B$  be compact. Then the inverse image of  $K$  considered in  $X$  and in  $A$  is the same, since  $A$  is the full inverse image.  $\square$

## 4. Sequences

A sequence  $(a_n)$  in a topological space  $X$  converges to  $a \in X$  if for every neighborhood  $U(a)$  there exists  $N \in \mathbb{N}$  with  $a_n \in U$  for  $n \geq N$ . We simply write  $a_n \rightarrow a$  for this. If  $X$  is Hausdorff, the limit  $a$  is unique. If  $f : X \rightarrow Y$  is continuous in  $a$  then  $a_n \rightarrow a$  implies  $f(a_n) \rightarrow f(a)$ .

It is easy to show that in a compact space every sequence admits a convergent sub-sequence. The converse is only true under additional conditions.

One says that a topological space has a *countable* basis of the topology if there exists a countable family  $(U_i)$  of open subsets such that every open subset can be written as union of members of this system. If this is the case, also every subset equipped with the induced topology has countable basis of the topology. The space  $\mathbb{R}^n$  with the usual topology has countable basis. One can take open balls where the radius and the coordinates of the center are rational numbers.

**4.1 Proposition.** *Assume that  $X$  is a Hausdorff space such that every sequence admits a convergent sub-sequence. Then  $X$  is compact if one of the following two conditions is satisfied.*

- a) *The topology comes from a metric.*
- b) *There exists a countable basis of the topology.*

Let  $X$  be a locally compact space with countable basis  $(U_i)_{i \in I}$  of the topology. If we take the sub-system  $(U_i)_{i \in J}$ ,  $J \subset I$  of all  $U_i$  with compact closure, then this is still a basis of the topology in the sense that every open subset is a union of members of this system. Indeed, the open subsets  $U$  with compact closures can be covered by members of the restricted system and, since  $X$  is locally compact, every open subset can be covered by open subsets with compact closure. Hence we can write  $X = U_1 \cup U_2 \cup \dots$  as union of a sequence of open subsets with compact closure. Replacing inductively  $U_n$  by the union of the  $U_i$  with  $i < n$ , we can get  $U_1 \subset U_2 \subset \dots$ . For given  $n$ , the closure of  $U_n$  can be covered by finitely many of the  $U_i$ . Hence, taking a suitable sub-sequence we can obtain that the closure of  $U_n$  is contained in  $U_{n+1}$ . Hence we have obtained the following result.

**4.2 Lemma.** *Let  $X$  be a locally compact space with countable basis of the topology. Then  $X$  can be written as union of countable many compact subsets,*

$$X = K_1 \cup K_2 \cup \dots$$

*We can assume that  $K_n$  is contained in the interior of  $K_{n+1}$ .*

## 5. Connectedness

A topological space is called *arcwise connected* if every two points are contained in (the image of) an arc. An arc is a continuous map of a real interval into  $X$ .

A topological space is called *connected* if one of the following two equivalent conditions is satisfied.

- 1) Every locally constant map  $f : X \rightarrow M$  into an arbitrary set  $M$  is constant. It is sufficient to verify this for one set  $M$  which contains at least two elements.
- 2) If  $X = U \cup V$  is written as union of two disjoint open subsets  $U, V$ , then one of them is empty.

The mean value theorem of calculus shows that every real interval is connected. As a consequence, every arcwise connected space is connected. The converse is only true under additional assumptions (s. below).

### Arc components

Two points of a topological space are called equivalent if there exists an arc which contains both. The equivalence classes of this equivalence relation are called *arc components*.

A *topological manifold* is a Hausdorff space such that every point admits an open neighborhood which is homeomorphic to some open subset of  $\mathbb{R}^n$ . If  $n$  can be taken to be two, then  $X$  is called a (*topological*) *surface*.

**5.1 Remark.** *Let  $X$  be a topological manifold. Then the arc components are open. The manifold is connected if and only if it is arcwise connected.*

Since an open subset of a manifold is a manifold, the arc components of a manifold are also manifolds. Of course they are connected. We call the arc components of a manifold also the *connected components*.

## 6. Paracompactness

We recall the definition and basic properties of *paracompact* spaces. The proofs can be found in standard text books of topology, as for example [Qu].

A covering  $\mathfrak{U} = (U_i)_{i \in I}$  of a topological space is called *locally finite*, if for every point  $a \in X$  there exists a neighborhood  $W$ , such that the set of indices  $i \in I$  with  $U_i \cap W \neq \emptyset$  is finite.

A covering  $\mathfrak{V} = (V_j)_{j \in J}$  is called a *refinement* of the covering  $\mathfrak{U}$  if for every index  $j \in J$  there exists an index  $i \in I$  with  $V_j \subset U_i$ . If one chooses for each

$j$  such an  $i$  one obtains a so-called *refinement map*  $J \rightarrow I$ , which needs not to be unique.

**6.1 Definition.** A Hausdorff space is called **paracompact** if every open covering admits a locally finite open refinement.

Every compact space is paracompact. We need the following result.

**6.2 Lemma.** Every locally compact space with countable basis of the topology is paracompact.

We sketch the simple proof. We use Lemma 4.2 and write  $X = K_1 \cup K_2 \cup \dots$  as union of a sequence of compact sets such that  $K_n$  is contained in the interior  $K_{n+1}^\circ$  of  $K_{n+1}$ . We set  $K_0 = \emptyset$ . Now we consider an open covering  $(U_i)$  of  $X$ . For each natural number  $n$  we cover  $K_n - K_{n-1}^\circ$  by the open sets  $U_i \cap (K_{n+1}^\circ - K_{n-1})$ . There exists a finite subsystem which covers  $K_n$ . If we collect these sets for all  $n$  we get a refinement of  $(U_i)$  which is locally finite.  $\square$

Just for sake of completeness we mention that each metric space is paracompact and that every locally compact space with countable basis of the topology can be metricised.

Next we formulate the basic result about paracompactness. Let  $\mathfrak{U} = (U_i)$  be a locally finite open covering. A *partition of unity* with respect to  $\mathfrak{U}$  is family  $\varphi_i$  of continuous real valued functions on  $X$  with the following property.

- a) The support of  $\varphi_i$  is contained in  $U_i$ .
- b)  $0 \leq \varphi_i \leq 1$ .
- c)  $\sum_{i \in I} \varphi_i(x) = 1$  for all  $x \in X$ .

(This sum is finite for each  $x$ . Recall that the support is the closure of the set of all points where the function is different from zero.)

**6.3 Proposition.** Let  $X$  be a paracompact space. For every locally finite open covering there exists a partition of unity.

A proof can be found in many textbooks of topology, for example in [Qu]. We mention that the proof gets very simple if  $X$  is a Hausdorff topological manifold with a countable basis of topology. This is enough for our purposes.

We mention two related results.

**6.4 Proposition.** Let  $X$  be a paracompact space and  $\mathfrak{U} = (U_i)$  a locally finite open covering. There exist open subsets  $V_i \subset U_i$  whose closure  $\bar{V}_i$  (taken in  $X$ ) is contained in  $U_i$  and such that  $\mathfrak{V} = (V_i)$  is still a covering.

Let  $V \subset U$  are two open subsets of a paracompact space such the closure  $\bar{V}$  of  $V$  (taken in  $X$ ) is contained  $U$ . Then we can consider the open covering  $X = U \cup (X - \bar{V})$ . By means of a partition of unity with respect to this covering we obtain the following result.

**6.5 Proposition.** *Let  $X$  be a locally compact paracompact space,  $U$  an open subset and  $V \subset U$  a whose closure (taken in  $X$ ) is contained in  $U$ . Then there exists a continuous function on  $X$  which is one on  $V$  and whose support is contained in  $U$ .*

## 7. Frèchet spaces

A topological vector space is a complex vector space  $V$  equipped with a topology such that the maps

$$V \times V \longrightarrow V, (a, b) \longmapsto a + b, \quad \mathbb{C} \times V \longrightarrow V, (C, a) \longmapsto Ca,$$

are continuous. Here  $V \times V$  and  $\mathbb{C} \times V$  carry of course the product topology. For example  $\mathbb{C}^n$  equipped with the usual topology is a topological vector space. More generally, every finite dimensional vector space  $V$  can be equipped in a unique way with a structure as topological space such that each isomorphism  $V \rightarrow \mathbb{C}^n$  is a topological map.

In a topological vector space one can talk about *Cauchy sequences*. A sequence  $(a_n)$  is called a Cauchy sequence if for every neighbourhood  $U$  of 0 there exists  $N$  such that  $a_m - a_n \in U$  for all  $m, n \geq N$ . Every convergent sequence is a Cauchy sequence. A topological vector space is called *complete* if every Cauchy sequence converges.

A semi-norm  $p$  on a complex vector space  $V$  is a map  $p : V \rightarrow \mathbb{R}$  with the properties

- a)  $p(a) \geq 0$  for all  $a \in V$ ,
- b)  $p(ta) = |t|p(a)$  for all  $t \in \mathbb{C}$ ,  $a \in V$ ,
- c)  $p(a + b) \leq p(a) + p(b)$ .

The *ball* of radius  $r > 0$  is defined as

$$U_r(a, p) := \{ x \in V; p(a - x) < r \}.$$

Let  $\mathcal{M}$  be a set of semi-norms. A subset  $B \subset V$  is called a *semi-ball* around  $a$  with respect to  $\mathcal{M}$  if there exists a finite subset  $\mathcal{N} \subset \mathcal{M}$  such that  $B$  is the intersection of balls with respect to the elements of  $\mathcal{N}$

$$B = \bigcap_{p \in \mathcal{N}} U_{r(p)}(a, p).$$

A subset  $U$  of  $V$  is called open (with respect to  $\mathcal{M}$ ) if for every  $a \in U$  there exists a semi-ball  $B$  around  $a$  with  $B \subset U$ .

It is clear that this defines a topology on  $V$  and that  $V$  gets a topological vector space. All maps  $p : V \rightarrow \mathbb{C}$  are continuous and the topology is actually the weakest topology with this property.

A sequence  $(a_n)$  in  $V$  converges to  $a \in V$  if and only if  $p(a_n - a) \rightarrow 0$  for all  $p \in \mathcal{M}$ .

A sequence  $(a_n)$  in  $V$  is a *Cauchy sequence* with respect to  $\mathcal{M}$  if for every  $\varepsilon > 0$  and every  $p \in \mathcal{M}$  there exists an  $N = N(p, \varepsilon)$  such that

$$p(a_n - a_m) < \varepsilon \quad \text{for } n, m \geq N.$$

The set  $\mathcal{M}$  is called *definit* if

$$p(a) = 0 \quad \text{for all } p \in \mathcal{M} \quad \implies \quad a = 0.$$

It is easy to prove that  $V$  is a Hausdorff space if and only if  $\mathcal{M}$  is definit.

**7.1 Definition.** *A Fréchet space  $V$  is a Hausdorff topological vector space with the following property. It is complete and the topology can be defined by means of a countable set of semi-norms on  $V$ .*

We formulate a simple remark.

**7.2 Remark.** *A closed subspace of a Fréchet space is a Fréchet space. The direct product of finitely many Fréchet spaces is a Fréchet space.*

The proof is very easy. To construct the semi-norms for a product one uses the construction  $\max(p_1(a_1), \dots, p_n(a_n))$ . The details can be left to the reader.  $\square$

We recall that a semi-norm  $p$  is called a *norm* if it is definit in the sense that  $p(a) = 0$  implies  $a = 0$ . A *normed space* is a vector space with a distinguished norm. It is called a *Banach space* if it is complete with respect to this norm. So Fréchet spaces can be considered as generalizations of Banach spaces.

**7.3 Lemma.** *Fréchet spaces are metrizable.*

*Proof.* We choose a countable set of defining semi-norms  $\mathcal{M} = \{p_1, p_2, \dots\}$ . Then one defines

$$d(a, b) = \sum_{n=1}^{\infty} 2^{-n} \frac{p_n(a - b)}{1 + p_n(a) + p_n(b)}.$$

It is easy to show that this is a metric which defines the original topology.  $\square$

As a consequence, a subset  $M$  of a Fréchet space is compact if and only if each sequence has a convergent sub-sequence with limit in  $M$ .

### Basic examples of Frèchet spaces

Let  $U \subset \mathbb{C}$  be an open subset and  $\mathcal{O}(U)$  the set of all analytic functions on  $U$ . This is a complex vector space. For an arbitrary compact subset  $K \subset U$  we define

$$p(f) = p_K(f) := \max_{z \in K} |f(z)|.$$

This is a semi norm. (Actually it is norm if  $U$  is connected and if  $K$  contains a non-empty open subset of  $U$ .) A sequence  $(f_n)$  converges with respect to  $p_K$  if and only if  $f_n$  converges uniformly on  $K$ .

**7.4 Remark.** *Let  $U \subset \mathbb{C}$  be an open subset. The vector space  $\mathcal{O}(U)$  equipped with the set of all norms of the form  $p_K$ ,  $K \subset U$  compact, is a Frèchet space.*

*Proof.* Recall that  $U$  can be exhausted by a sequence of compact subsets  $K_n$  such that  $K_n$  is contained in the interior of  $K_{n+1}$ . It is enough to take for  $K$  the members of this sequence. Hence the topology can be defined by means of a countable set of semi-norms. The convergence of Cauchy sequences follows from the theorem of Weierstrass, which states that analyticity is stable under uniform convergence on compact subsets.  $\square$

The same argument shows that the space of continuous functions  $\mathcal{C}(U)$  gets a Frèchet space. The basic result about the Frèchet space  $\mathcal{O}(U)$  is the following.

**7.5 Theorem of Montel.** *Let  $U$  be an open subset of  $\mathbb{C}$  and  $C > 0$  a positive constant. The set*

$$\mathcal{O}(U, C) := \{ f \in \mathcal{O}(U); \quad |f(z)| \leq C \text{ for } z \in U \}$$

*is compact in  $\mathcal{O}(U)$ .*

For the proof one has to use the fact that a metric space is compact if every sequence admits a convergent subsequence. Hence the statement follows from the usual theorem of Montel which states that every sequence in  $\mathcal{O}(U, C)$  admits a locally convergent sub-sequence. A proof can be found in [FB], Kap. IV, Theorem 4.9. We notice that the analogue for differentiable continuous functions is false. The proof uses heavily the Cauchy integral.

### Compact operators

A well-known fact is that in a Banach space of infinite dimension the closed ball  $\|a\| \leq 1$  is not compact. This result is also true for Frèchet spaces in the following form:

*Assume that the Frèchet space admits a non-empty open subset with compact closure. Then it is of finite dimension.*

We need a generalization of this result. A continuous linear map  $f : E \rightarrow F$  between Frèchet spaces is called a *compact operator* if there exists a non-empty open subset of  $E$  such that the closure of its image is compact.

A linear map  $f : V \rightarrow W$  is called *nearly surjective* if  $W/f(V)$  has finite dimension. This is automatically the case when  $W$  is finite dimensional.



**7.6 Theorem of Schwartz.** *Let  $f : E \rightarrow F$  be a surjective continuous linear map between Fréchet spaces and let  $g : E \rightarrow F$  be a compact operator. Then  $f + g$  is nearly surjective.*

A short proof can be found in [GR], Appendix B, Theorem 12.

If one applies Schwartz's theorem in the case  $E = F$ ,  $f = -\text{id}$  and  $g = \text{id}$ , one obtains:

**7.7 Corollary.** *When the identity operator  $\text{id} : E \rightarrow E$  of a Fréchet space is compact, then  $E$  is finite dimensional.*

From Montel's theorem we obtain the following important example of a compact operator.

**7.8 Proposition.** *Let  $U \subset \mathbb{C}$  be an open subset and  $V \subset U$  an open subset whose closure (taken in  $\mathbb{C}$ ) is compact and contained in  $U$ . The natural restriction map  $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$  is compact.*

*Proof.* Consider  $K = \bar{V}$  and the semi-norm  $p_K$  on  $\mathcal{O}(U)$ . The set of all functions  $f \in \mathcal{O}(U)$  with  $p_K(f) < 1$  is open. Its image in  $\mathcal{O}(V)$  is contained in a compact set by Montel's theorem.

## Chapter II. Some algebra

### 1. Abelian groups

We assume that the reader is familiar with the notion of an abelian group and homomorphism between abelian groups. If  $A$  is a subgroup of an abelian group  $B$ , then the factor group  $B/A$  is well defined. All what one needs usually is that there is a natural surjective homomorphism  $f : B \rightarrow B/A$  with kernel  $A$ . Let  $f : B \rightarrow X$  be a homomorphism into some abelian group. Then  $f$  factors through a homomorphism  $B/A \rightarrow X$  if and only if the kernel of  $f$  contains  $A$ . That  $f$  factors means that there is a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & X \\ \downarrow & & \nearrow \\ B/A & & \end{array}$$

Let  $f : A \rightarrow B$  be a homomorphism of abelian groups. Then the image  $f(A)$  is a subgroup of  $B$ . If there is no doubt which homomorphism  $f$  is considered, we allow the notation

$$B/A := B/f(A).$$

**1.1 Lemma.** *A commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

*induces homomorphisms*

$$B/A \longrightarrow D/C, \quad C/A \longrightarrow D/B.$$

A (finite or infinite) sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow \cdots$$

is called *exact* at  $B$  if

$$\text{Kernel}(B \longrightarrow C) = \text{Image}(A \longrightarrow B).$$

It is called exact if it is exact at every place. An exact sequence  $A \rightarrow B \rightarrow C$  induces an injective homomorphism

$$B/A \hookrightarrow C.$$

The sequence  $0 \rightarrow A \rightarrow B$  is exact if and only if  $A \rightarrow B$  is injective. The sequence  $A \rightarrow B \rightarrow 0$  is exact if and only if  $A \rightarrow B$  is surjective. The sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact if and only if  $A \rightarrow B$  is injective and if the induced homomorphism  $B/A \rightarrow C$  is an isomorphism. A sequence of this form is called a *short exact sequence*. Hence the typical short exact sequence is

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0 \quad (A \subset B).$$

**1.2 The five term lemma.** *Let*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

be a commutative diagram with exact lines and such that  $f_1, f_2$  and  $f_4, f_5$  are isomorphisms. Then  $f_3$  is an isomorphism too.

The proof is easy and left to the reader. □

## 2. Some homological algebra

A *complex*  $A^\bullet$  is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow A_{n-1} \xrightarrow{d_{n-1}} A_n \xrightarrow{d_n} A_{n+1} \longrightarrow \cdots$$

such that the composition of two consecutive is zero,  $d_n \circ d_{n-1} = 0$ . Usually one omits indices at the  $d$ -s and writes simply  $d = d_n$  and hence  $d \circ d = 0$ , which sometimes is written as  $d^2 = 0$ . The cohomology groups of  $A^\bullet$  are defined as

$$H^n(A^\bullet) := \frac{\text{Kernel}(A^n \rightarrow A^{n+1})}{\text{Image}(A^{n-1} \rightarrow A^n)} \quad (n \in \mathbb{Z}).$$

They vanish if and only if the complex is exact. Hence the cohomology groups measure the absence of exactness of a complex.

A homomorphism  $f^\bullet : A^\bullet \rightarrow B^\bullet$  of complexes is a commutative diagram

$$\begin{array}{ccccccc} \dots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \dots \\ & & \downarrow f^{n-1} & & \downarrow f^n & & \downarrow f^{n+1} & & \\ \dots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \dots \end{array}$$

It is clear how to compose two complex homomorphisms  $f^\bullet : A^\bullet \rightarrow B^\bullet, g^\bullet : B^\bullet \rightarrow C^\bullet$  to a complex homomorphism  $g^\bullet \circ f^\bullet : A^\bullet \rightarrow C^\bullet$ . A sequence of complex homomorphisms

$$\dots \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow \dots$$

is called exact if all the induced sequences

$$\dots \longrightarrow A^n \longrightarrow B^n \longrightarrow C^n \longrightarrow \dots$$

are exact. There is also the notion of a short exact sequence of complexes

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0.$$

Here 0 stands for the zero complex ( $0^n = 0, d^n = 0$  for all  $n$ ).

A homomorphism of complexes  $A^\bullet \rightarrow B^\bullet$  induces natural homomorphisms

$$H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$$

of the cohomology groups (use Lemma 1.1). These homomorphisms are compatible with the composition of complex-homomorphisms. A less obvious construction is as follows: Let

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes. We construct a homomorphism

$$\delta : H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet).$$

Let  $[c] \in H^n(C^\bullet)$  be represented by an element  $c \in C^n$ . Take a pre-image  $b \in B^n$  and consider  $\beta = db \in B^{n+1}$ . Since  $\beta$  goes to  $d(c) = 0$  in  $C^{n+1}$  there exists a pre-image  $a \in A^{n+1}$ . This goes to 0 in  $A^{n+2}$  (because  $A^{n+2}$  is imbedded in  $B^{n+2}$  and  $b$  goes to  $d^2(b) = 0$  there). Hence  $a$  defines a cohomology class  $[a] \in H^{n+1}(A^\bullet)$ . It is easy to check that this class doesn't depend on the above choices.

**2.1 Fundamental lemma of homological algebra.** *Let*

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

*be a short exact sequence of complexes. Then the long sequence*

$$\dots \rightarrow H^{n-1}(C^\bullet) \xrightarrow{\delta} H^n(A^\bullet) \rightarrow H^n(B^\bullet) \rightarrow H^n(C^\bullet) \xrightarrow{\delta} H^{n+1}(C^\bullet) \rightarrow \dots$$

*is exact.*

We leave the details to the reader. □

There is a second lemma of homological algebra which we will need.

**2.2 Lemma.** *Let*

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} & \longrightarrow & A^{03} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} & \longrightarrow & A^{13} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} & \longrightarrow & A^{23} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A^{30} & \longrightarrow & A^{31} & \longrightarrow & A^{32} & \longrightarrow & A^{33} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & 
 \end{array}$$

*be a commutative diagram where all lines and columns are exact besides the first column and the first row (those containing  $A^{00}$ ). Then there is a natural isomorphism between the cohomology groups of the first row and the first column,*

$$H^n(A^{\bullet,0}) \cong H^n(A^{0,\bullet})$$

*For  $n = 0$  this is understood as*

$$\text{Kernel}(A^{00} \longrightarrow A^{01}) = \text{Kernel}(A^{00} \longrightarrow A^{10}).$$

The proof is given by “diagram chasing”. We only give a hint how it works. Assume  $n = 1$ . Let  $[a] \in H^1(A^{0,\bullet})$  be a cohomology class represented by an element  $a \in A^{0,1}$ . This element goes to 0 in  $A^{0,2}$ . As a consequence the image of  $a$  in  $A^{1,1}$  goes to 0 in  $A^{1,2}$ . Hence this image comes from an element  $\alpha \in A^{1,0}$ . Clearly this element goes to zero in  $A^{2,0}$  (since it goes to 0 in  $A^{2,1}$ .) Now  $\alpha$  defines a cohomology class  $[\alpha] \in H^1(A^{\bullet,0})$ . There is some extra work to show that this map is well-defined. □

### 3. The tensor product

All rings which we consider are assumed to be commutative and with unit element. Ring homomorphisms are assumed to map the unit element into the unit element. A module  $M$  over a ring  $A$  is an abelian group together with a map  $A \times M \rightarrow M$ ,  $(a, m) \mapsto am$ , such that the usual axioms of a vector space are satisfied including  $1_A m = m$  for all  $m \in M$ . The notion of linear maps, kernel, image of a linear map are as in the case of vector spaces. But in contrast to the case of vector spaces, a module has usually no basis. A module which admits a basis is called free. A finitely generated free module is isomorphic to  $R^n$ .

If  $M \subset N$  is a submodule, then the factor group  $N/M$  carries a structure of an  $A$ -module. All what we have said about exact sequences of abelian groups is literally true for  $A$ -modules.

#### Tensor product

Recall that for two modules  $M, N$  over a ring  $R$ , there exists a module  $M \otimes_R N$  together with an  $R$ -bilinear map

$$M \times N \longrightarrow M \otimes_R N, \quad (a, b) \mapsto a \otimes b,$$

such that for each bilinear map  $M \times N \rightarrow P$  into an arbitrary third module  $P$  there exists a unique commutative diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad} & M \otimes_R N \\ & \searrow & \swarrow \\ & & P \end{array}$$

with an  $R$ -linear map  $M \otimes_R N \rightarrow P$ . The tensor product  $M \otimes_R N$  is generated by the special elements  $m \otimes n$ .

If  $f : M \rightarrow M'$  and  $g : N \rightarrow N'$  are  $R$ -linear maps, then one gets a natural  $R$ -linear map

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N', \quad (a, b) \mapsto f(a) \otimes g(b).$$

It is clear that this map is uniquely determined by this formula. The existence follows from the universal property applied to the map  $(a, b) \mapsto f(a) \otimes g(b)$ .

**Basic properties of the tensor product**

There is a natural isomorphism

$$R \otimes_R M \xrightarrow{\sim} M, \quad (r \otimes m) \mapsto rm,$$

and more generally

$$R^n \otimes M \xrightarrow{\sim} M^n.$$

As a special case we get

$$R^n \otimes_R R^m \cong R^{n \times m}.$$

This is related also to the formula

$$(M \times N) \otimes_R P \cong (M \otimes_R P) \times (N \otimes_R P) \quad (\text{canonically}).$$

The tensor product is associative: for usual  $R$ -modules  $M, N, P$  one has an isomorphism

$$(M \otimes_R N) \otimes_R P \xrightarrow{\sim} M \otimes_R (N \otimes_R P), \quad (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p).$$

The existence of this map follows from the universal property of the tensor product.

The tensor product is also commutative in the following sense:

$$M \otimes_R N \xrightarrow{\sim} N \otimes_R M, \quad m \otimes n \mapsto n \otimes m.$$

**Ring extension**

Let  $A \rightarrow B$  be a ring homomorphism and  $M$  an  $A$ -module. Then  $M \otimes_A B$  carries a natural structure as  $B$ -module. It is given by  $b(m \otimes b') = m \otimes (bb')$ . The existence follows from the universal property of the tensor product. A special case is

$$A^n \otimes_A B = B^n.$$

**Existence of the tensor product**

For an arbitrary set  $I$  we define  $R^I$  to be the set of all maps  $I \rightarrow R$ ,  $i \mapsto r_i$  such that  $r_i$  is 0 for almost all  $i$ . So  $R^I = R^n$  for  $I = \{1, \dots, n\}$ . By definition a module is free if and only if it is isomorphic to an  $R^I$  for suitable  $I$ . An arbitrary  $R$ -module  $M$  can be represented by an exact sequence

$$R^J \longrightarrow R^I \longrightarrow M \longrightarrow 0.$$

For another  $N$  module we define now the tensor product by the exact sequence

$$N^J \longrightarrow N^I \longrightarrow M \otimes_R N \longrightarrow 0.$$

The bilinear map  $M \times N \rightarrow M \otimes_R N$  and the universal property are obvious.

### Exactness properties

Let  $M \rightarrow N$  be an injective homomorphism of  $R$ -modules. For an  $R$ -module  $P$  the induced homomorphism  $M \otimes_R P \rightarrow N \otimes_R P$  needs not to be injective. But when  $P \cong R^n$  is free, injectivity is preserved. A slight and trivial extension of this observation is:

**3.1 Remark.** *Let  $M_1 \rightarrow M_2 \rightarrow M_3$  be an exact exact sequence of  $R$ -modules. Then for every **free** module  $P$  the sequence  $M_1 \otimes_R P \rightarrow M_2 \otimes_R P \rightarrow M_3 \otimes_R P$  remains exact.*

In this context we mention some other exactness properties. For two  $R$ -modules  $M, N$  we denote by  $\text{Hom}_R(M, N)$  the set of all  $R$ -linear maps  $M \rightarrow N$ . This is an  $R$ -module. Let  $M \rightarrow N$  be an  $R$ -linear map. Then for an arbitrary  $R$ -module  $P$  one has obvious  $R$ -linear maps

$$\text{Hom}_R(P, M) \longrightarrow \text{Hom}_R(P, N), \quad \text{Hom}_R(N, P) \longrightarrow \text{Hom}_R(M, P).$$

Since  $\text{Hom}(R^n, M) \cong M^n$ , one has:

**3.2 Remark.** *Let  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  be an exact sequence of  $R$ -modules and  $P$  also an  $R$ -module, then:*

a) *If  $P$  is free then*

$$0 \longrightarrow \text{Hom}_R(P, M_1) \longrightarrow \text{Hom}_R(P, M_2) \longrightarrow \text{Hom}_R(P, M_3) \longrightarrow 0$$

*remains exact.*

b) *If  $M_3$  is free than*

$$0 \longrightarrow \text{Hom}_R(M_3, P) \longrightarrow \text{Hom}_R(M_2, P) \longrightarrow \text{Hom}_R(M_1, P) \longrightarrow 0$$

*remains exact.*

We comment shortly b). When  $M_3$  is free one can chose a system of elements in  $M_2$  whose images in  $M_3$  define a basis. This system generates a submodule  $M'_3 \subset M_2$  which maps isomorphically to  $M_3$ . Now it is easy to see that  $M_2$  is isomorphic to  $M_1 \times M_3$  and the map  $M_1 \rightarrow M_2$  corresponds to  $m \mapsto (m, 0)$  and the map  $M_2 \rightarrow M_3$  corresponds to  $(m_1, m_3) \mapsto m_3$ . Now the exactness should be clear.



# Chapter III. Sheaves

## 1. Presheaves

We introduce the language presheaves of abelian groups. This consists mainly of definitions and simple remarks whose proofs are very simple. In most cases they can be left to the reader.

**1.1 Definition.** A presheaf  $F$  (of abelian groups) on a topological space  $X$  is a map which assigns to every open subset  $U \subset X$  an abelian group  $F(U)$  and to every pair  $U, V$  of open subsets with the property  $V \subset U$  a homomorphism

$$r_V^U : F(U) \longrightarrow F(V)$$

such that  $r_U^U = \text{id}$  and that for three open subsets  $U, V, W$  with the property  $W \subset V \subset U$

$$r_W^U = r_W^V \circ r_V^U$$

holds:

Example:  $F(U)$  is the set of all continuous functions  $f : U \rightarrow \mathbb{C}$  and  $r_V^U(f) := f|_V$  (restriction).

Many presheaves generalize this example. Hence the maps  $r_V^U$  are called “restrictions” in general and one uses the notation

$$s|_V := r_V^U(s) \quad \text{for } s \in F(U).$$

The elements of  $F(U)$  sometimes are called “sections” of  $F$  over  $U$ . In the special case  $U = X$  they are called “global” sections.

**1.2 Definition.** Let  $X$  be a topological space. A homomorphism of presheaves

$$f : F \longrightarrow G$$

is a family of group homomorphisms

$$f_U : F(U) \longrightarrow G(U),$$

such that the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & G(V) \end{array}$$

commutes for every pair  $V \subset U$  of open subsets, i.e.  $f_U(s)|_{G(V)} = f_V(s|_F(V))$ .

It is clear how to define the identity map  $\text{id}_F : F \rightarrow F$  of a presheaf and the composition  $g \circ f$  of two homomorphisms  $f : F \rightarrow G$ ,  $g : G \rightarrow H$  of presheaves.

There is also a natural notion of a sub-presheaf  $F \subset G$ . Besides  $F(U) \subset G(U)$  for all  $U$  one has to demand, that the restrictions are compatible. This means:

*The canonical inclusions  $i_U : F(U) \rightarrow G(U)$  define a homomorphism  $i : F \rightarrow G$  of presheaves.*

When  $f : F \rightarrow G$  is a homomorphism of presheaves, the images  $f_U(F(U))$  define a sub-presheaf of  $G$ . We call it the *presheaf-image* and denote it by

$$f_{\text{pre}}(F).$$

It is also clear that the kernels of the maps  $f_U$  define a sub-presheaf of  $F$ . We denote it by  $\text{Kernel}(f : F \rightarrow G)$ . When  $F$  is a sub-presheaf of  $G$  then one can consider the factor groups  $G(U)/H(U)$ . Using II.1.1 it is clear how to define restriction maps to get a presheaf  $G/\text{pre}F$ . We call this presheaf the *factor presheaf*.

Since we have defined Kernel and Image we can also introduce the notion of a *presheaf-exact sequence*. A sequence  $F \rightarrow G \rightarrow H$  is presheaf-exact if and only if  $F(U) \rightarrow G(U) \rightarrow H(U)$  is exact for all  $U$ . What we have said about exact sequences of abelian groups carries literally over to presheaf-exact sequences of presheaves of abelian groups.

## 2. Germs and Stalks

let  $F$  be a presheaf on a topological space  $X$  and let  $a \in X$  be a point. We consider pairs  $(U, s)$ , where  $U$  is an open neighbourhood of  $a$  and  $s \in F(U)$  a section over  $U$ . Two pairs  $(U, s)$ ,  $(V, t)$  are called equivalent if there exists an open neighborhood  $a \in W \subset U \cap V$ , such that  $s|_W = t|_W$ . This is an equivalence relation. The equivalence classes

$$[U, s]_a := \{ (V, t); \quad (V, t) \sim (U, s) \}$$

are called *germs* of  $F$  at the point  $a$ . The set of all germs

$$F_a := \{ [U, s]_a, \quad a \in U \subset X, \quad s \in F(U) \}$$

is the so-called *stalk* of  $F$  at  $a$ . The stalk carries a natural structure as abelian group. One defines

$$[U, s]_a + [V, t]_a := [U \cap V, s|_{U \cap V} + t|_{U \cap V}]_a.$$

We use frequently the simplified notation

$$s_a = [U, s]_a.$$

For every open neighborhood  $a \in U \subset X$  there is an obvious homomorphism

$$F(U) \longrightarrow F_a, \quad s \longmapsto s_a.$$

A homomorphism of presheaves  $f : F \rightarrow G$  induces natural mappings

$$f_a : F_a \longrightarrow G_a \quad (a \in X).$$

The image of a germ  $[U, s]_a$  is simply  $[U, f_U(s)]_a$ . It is easy to see that this is well-defined.

**2.1 Remark.** *Let  $F \rightarrow G$  and  $G \rightarrow H$  be homomorphism of presheaves and let  $a \in X$  be a point. Assume that every point  $a$  contains a small open neighborhood  $U$  such that  $F(U) \rightarrow G(U) \rightarrow H(U)$  is exact. Then  $F_a \rightarrow G_a \rightarrow H_a$  is exact.*

**Corollary.** *if  $F \rightarrow G \rightarrow H$  is presheaf-exact then  $f_a \rightarrow G_a \rightarrow H_a$  is exact for all  $a$ .*

The proof is easy and can be omitted. □

If  $F$  is a presheaf on  $X$ , one can consider for each open subset  $U \subset X$

$$F^{(0)}(U) := \prod_{a \in U} F_a.$$

The elements are families  $(s_a)_{a \in U}$  with  $s_a \in F_a$ . There is now coupling between the different  $s_a$ . Hence  $F^{(0)}(U)$  usually is very monstrous.

For open sets  $V \subset U$ , one has an obvious homomorphism  $F^{(0)}(U) \rightarrow F^{(0)}(V)$ . Hence we obtain a presheaf  $F^{(0)}$  together with a natural homomorphism  $F \rightarrow F^{(0)}$ . Each homomorphism  $F \rightarrow G$  of presheaves induces a homomorphism  $F^{(0)} \rightarrow G^{(0)}$  such that the diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F^{(0)} & \longrightarrow & G^{(0)} \end{array}$$

commutes.

### 3. Sheaves

**3.1 Definition.** A presheaf  $F$  is called a **sheaf** if the following conditions are satisfied:

- (G1) When  $U = \bigcup U_i$  is an open covering of an open subset  $U \subset X$  and if  $s, t \in F(U)$  are sections with the property  $s|_{U_i} = t|_{U_i}$  for all  $i$ , then  $s = t$ .
- (G2) When  $U = \bigcup U_i$  is an open covering of an open subset  $U \subset X$  and if  $s_i \in F(U_i)$  is a family of sections with the property

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j,$$

then there exists a section  $s \in F(U)$  with the property  $s|_{U_i} = s_i$  for all  $i$ .

- (G3)  $F(\emptyset)$  is the zero group.

Clearly the presheaf of continuous functions is a sheaf, since continuity is a local property. An example of a presheaf  $F$ , which usually is not a sheaf is the presheaf of constant functions with values in  $\mathbb{Z}$  ( $F(U) = \{f : U \rightarrow \mathbb{Z}, f \text{ constant}\}$ ). But the set of *locally constant* functions with values in  $\mathbb{Z}$  is a sheaf.

By a subsheaf of a sheaf  $F$  we understand a sub-presheaf  $G \subset F$  which is already a sheaf. If  $F, G$  are sheaves, then a homomorphism  $f : F \rightarrow G$  of presheaves is called also a homomorphism of sheaves.

**3.2 Remark.** Let  $F \subset G$  be a sub-presheaf. We assume that  $G$  (but not necessarily  $F$ ) is a sheaf. Then there is a smallest subsheaf  $\tilde{F} \subset G$  which contains  $F$ . For an arbitrary point  $a \in X$  the induced map  $f_a : F_a \rightarrow \tilde{F}_a$  is an isomorphism.

It is clear that  $\tilde{F}(U)$  has to be defined as set of all  $s \in G(U)$ , such that:

There exists an open covering  $U = \bigcup U_i$ , such that  $s|_{U_i}$  is in  $F(U_i)$  for all  $i$ . This is equivalent with: the germ  $s_a$  is in the image of  $F_a \rightarrow G_a$  for all  $a \in U$ .

**3.3 Definition.** Let  $F \rightarrow G$  be a homomorphism of sheaves. The sheaf-image  $f_{\text{sheaf}}(F)$  is the smallest subsheaf of  $G$ , which contains the presheaf-image  $f_{\text{pre}}(F)$ .

We have to differ between two natural notions of surjectivity.

#### 3.4 Definition.

- 1) A homomorphism of presheaves  $f : F \rightarrow G$  is called **presheaf-surjective** if  $f_{\text{pre}}(F) = G$ .
- 2) A homomorphism of sheaves  $f : F \rightarrow G$  is called **sheaf-surjective** if  $f_{\text{sheaf}}(F) = G$ .

When  $F$  and  $G$  both are sheaves then sheaf-surjectivity and presheaf-surjectivity are different things. We give an example which will be basic:

Let  $\mathcal{O}$  be the sheaf of holomorphic functions on  $\mathbb{C}$ , hence  $\mathcal{O}(U)$  is the set of all holomorphic functions on an open subset  $U$ . This is a sheaf of abelian groups (under addition). Similarly we consider the sheaf  $\mathcal{O}^*$  of holomorphic functions without zeros. This is also a sheaf of abelian groups (under multiplication). The map  $f \rightarrow e^f$  defines a sheaf homomorphism

$$\exp : \mathcal{O} \longrightarrow \mathcal{O}^*.$$

The map  $\mathcal{O}(U) \rightarrow \mathcal{O}^*(U)$  is not always surjective. For example for  $U = \mathbb{C}^*$  the function  $1/z$  is not in the image. Hence  $\exp$  is not presheaf-surjective. But it is known from complex calculus that  $\exp : \mathcal{O}(U) \rightarrow \mathcal{O}^*(U)$  is surjective if  $U$  is simply connected, for example for a disk  $U$ . Since a point admits arbitrarily small neighborhoods which are disks, it follows that  $\exp$  is sheaf-surjective.

**3.5 Remark.** *A homomorphism of sheaves  $f : F \rightarrow G$  is sheaf-surjective if and only if the maps  $f_a : F_a \rightarrow G_a$  are surjective for all  $a \in X$ .*

We omit the simple proof. □

Fortunately the notion “injective” does not contain this difficulty. For trivial reason the following remark is true.

**3.6 Remark.** *Let  $f : F \rightarrow G$  be a homomorphism of sheaves. The kernel in the sense of presheaves is already a sheaf.*

Hence we don't have to distinguish between presheaf-injective and sheaf-injective and also not between presheaf-kernel and sheaf-kernel.

**3.7 Remark.** *A homomorphism of sheaves  $f : F \rightarrow G$  is injective if and only if the maps  $f_a : F_a \rightarrow G_a$  are injective for all  $a \in X$ .*

A homomorphism of presheaves  $f : F \rightarrow G$  (sheaves) is called an isomorphism if all  $F(U) \rightarrow G(U)$  are isomorphisms. Their inverses then define a homomorphism  $f^{-1} : G \rightarrow F$ .

**3.8 Remark.** *A homomorphism of sheaves  $F \rightarrow G$  is an isomorphism if and only if  $F_a \rightarrow G_a$  is an isomorphism for all  $a$ .*

For presheaves this is false. As counter example one can take for  $F$  the presheaf of constant functions and for  $G$  the sheaf of locally constant functions.

It is natural to introduce the notion of sheaf-exactness as follows:

**3.9 Definition.** *A sequence  $F \rightarrow G \rightarrow H$  of sheaf homomorphisms is **sheaf-exact** at  $G$  if the kernel of  $G \rightarrow H$  and the sheaf-image of  $F \rightarrow G$  agree.*

Generalizing 3.5 and 3.7 one can easily show the following proposition.

**3.10 Proposition.** *A sequence  $F \rightarrow G \rightarrow H$  is exact if and only if  $F_a \rightarrow G_a \rightarrow H_a$  is exact for all  $a$ .*

Our discussion so far has obviously one gap: Let  $F \subset G$  be subsheaf of a sheaf  $G$ . We would like to have an exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The sheaf  $H$  should be the factor sheaf of  $G$  by  $F$ . But up to now we only defined the factor presheaf  $G/\text{pre}F$  which usually is no sheaf. In the next section we will give the correct definition for a factor sheaf  $G/\text{sheaf}F$ .

## 4. The generated sheaf

For a presheaf  $F$  we introduced the monstrous presheaf

$$F^{(0)}(U) = \prod_{a \in U} F_a.$$

Obviously  $F^{(0)}$  is a sheaf. Sometimes it is called the ‘‘Godement sheaf’’ or the ‘‘associated flabby sheaf’’. There is a natural homomorphism

$$F \rightarrow F^{(0)}.$$

We can consider its presheaf-image and then the smallest subsheaf which contains it. We denote this sheaf by  $\hat{F}$  and call it the ‘‘generated sheaf’’ by  $F$ . There is a natural homomorphism

$$F \rightarrow \hat{F}.$$

From the construction follows immediately

**4.1 Remark.** *Let  $F$  be a presheaf. The natural maps*

$$F_a \xrightarrow{\sim} \hat{F}_a$$

*are isomorphisms.*

A homomorphism  $F \rightarrow G$  of presheaves induces a homomorphism  $F^{(0)} \rightarrow G^{(0)}$ . Clearly  $\hat{F}$  is mapped into  $\hat{G}$ . This gives us the following result.

**4.2 Remark.** *Let  $f : F \rightarrow G$  be a homomorphism of presheaves. There is a natural homomorphism  $\hat{F} \rightarrow \hat{G}$ , such that the diagram*

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \hat{F} & \longrightarrow & \hat{G} \end{array}$$

*commutes.*

When  $F$  is already a sheaf then  $F \rightarrow F^{(0)}$  is injective. Then the map of  $F$  into the presheaf image is an isomorphism. This implies that the presheaf image is already a sheaf.

**4.3 Remark.** *Let  $F$  be a sheaf. Then  $F \rightarrow \hat{F}$  is an isomorphism.*

*If  $F$  is a sub-presheaf of a sheaf  $G$ , then the induced map  $\hat{F} \rightarrow \hat{G} \cong G$  is an isomorphism  $\hat{F} \rightarrow \tilde{F}$  between  $\hat{F}$  and the smallest subsheaf  $\tilde{F}$  of  $G$ , which contains  $F$ .*

Hence we can identify  $\tilde{F}$  and  $\hat{F}$ .

**Factor sheaves and exact sequences of sheaves**

Let  $F \rightarrow G$  be a homomorphism of presheaves. We introduced already the factor presheaf  $G/\text{pre}F$ , which associates to an open  $U$  the factor group  $G(U)/F(U)$ . Even if both  $F$  and  $G$  are sheaves this will usually not be a sheaf. Hence we define the factor sheaf as the sheaf generated by the factor presheaf.

$$G/\text{sheaf}F := \widehat{G/\text{pre}F}.$$

Since we are interested mainly in sheaves, we will write usually for a homomorphism for sheaves  $f : F \rightarrow G$ :

$$\begin{aligned} G/F &:= G/\text{sheaf}F && \text{(factor sheaf)} \\ f(F) &:= f_{\text{sheaf}}(F) && \text{(sheaf image)} \end{aligned}$$

Notice that there is no need to differ between sheaf- and presheaf-kernel. When we talk about an exact sequence of sheaves

$$F \longrightarrow G \longrightarrow H$$

we usually mean “sheaf exactness”. All what we have said about exactness properties of sequences of abelian groups is literally true for sequences of sheaves. For example: A sequence of sheaves  $0 \rightarrow F \rightarrow G$  ( $0$  denotes the zero sheaf) is exact if and only if  $F \rightarrow G$  is injective. A sequence of sheaves  $F \rightarrow G \rightarrow 0$  is exact if and only if  $F \rightarrow G$  is surjective (in the sense of sheaves of course). A sequence of sheaves  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  is exact if and only if there is an isomorphism  $H \cong G/F$  which identifies this sequence with

$$0 \longrightarrow F \longrightarrow G \longrightarrow G/F \longrightarrow 0.$$

**4.4 Remark.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be an exact sequence of sheaves. Then for open  $U$  the sequence*

$$0 \rightarrow F(U) \rightarrow G(U) \rightarrow H(U)$$

*is exact.*

**Corollary.** *The sequence*

$$0 \rightarrow F(X) \rightarrow G(X) \rightarrow H(X)$$

*is exact.*

The simple proof can be left to the reader.  $\square$

Usually  $G(X) \rightarrow H(X)$  is not surjective as the example

$$0 \rightarrow \mathbb{Z}_X \rightarrow \mathcal{O} \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{O}^* \rightarrow 0$$

shows. Cohomology theory will measure the absence the right exactness. The above sequence will be part of a long exact sequence

$$0 \rightarrow F(X) \rightarrow G(X) \rightarrow H(X) \rightarrow H^1(X, F) \rightarrow \dots$$



## Chapter IV. Cohomology of sheaves

### 1. The canonical flabby resolution

A sheaf  $F$  is called *flabby* if  $F(X) \rightarrow F(U)$  is surjective for all  $U$ . Then  $F(U) \rightarrow F(V)$  is surjective for all  $V \subset U$ . An example for a flabby sheaf is the Godement sheaf  $F^{(0)}$ . Recall that we have the exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)}.$$

We want to extend this sequence. For this we consider the sheaf  $F^{(0)}/F$  and embed it into its Godement sheaf,

$$F^{(1)} := (F^{(0)}/F)^{(0)}.$$

In this way we get a long exact sequence

$$0 \longrightarrow F \longrightarrow F^{(0)} \longrightarrow F^{(1)} \longrightarrow F^{(2)} \longrightarrow \dots$$

If  $F^{(n)}$  has been already constructed, then we define

$$F^{(n+1)} := (F^{(n)}/F^{(n-1)})^{(0)}.$$

The sheaves  $F^{(n)}$  are all flabby. We call this sequence the *canonical flabby resolution* or the *Godement resolution*. Sometimes it is useful to write the resolution in the form

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & F^{(0)} & \longrightarrow & F^{(1)} & \longrightarrow & F^{(2)} & \longrightarrow & F^{(3)} & \longrightarrow & \dots \end{array}$$

Both lines are complexes. The vertical arrows can be considered as a complex homomorphism. The induced homomorphism of the cohomology groups are isomorphisms. Notice that only the 0-cohomology group of both complexes is different from 0. This zero cohomology group is naturally isomorphic  $F$ .

Now we apply the global section functor  $\Gamma$  to the resolution. This is

$$\Gamma F := F(X).$$

We obtain a long sequence

$$0 \longrightarrow \Gamma F \longrightarrow \Gamma F^{(0)} \longrightarrow \Gamma F^{(1)} \longrightarrow \Gamma F^{(2)} \longrightarrow \dots$$

The essential point is that this sequence is no longer exact. We only can say that it is a complex. We prefer to write it in the form

$$\begin{array}{cccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \Gamma F & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \Gamma F^{(0)} & \longrightarrow & \Gamma F^{(1)} & \longrightarrow & \Gamma F^{(2)} & \longrightarrow & \Gamma F^{(3)} & \longrightarrow & \dots \end{array}$$

The second line is

$$\begin{array}{cccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \Gamma F^{(0)} & \longrightarrow & \Gamma F^{(1)} & \longrightarrow & \Gamma F^{(2)} & \longrightarrow & \dots \\ & & & & \uparrow & & & & & & \\ & & & & \text{zero position} & & & & & & \end{array}$$

Now we define the cohomology groups  $H^\bullet(X, F)$  to be the cohomology groups of this complex:

$$H^n(X, F) := \frac{\text{Kern}(\Gamma F^{(n)} \longrightarrow \Gamma F^{(n+1)})}{\text{Kern}(\Gamma F^{(n-1)} \longrightarrow \Gamma F^{(n)})}.$$

(We define  $\Gamma F^{(n)} = 0$  for  $n < 0$ .) Clearly

$$H^n(X, F) = 0 \quad \text{for } n < 0.$$

Next we treat the special case  $n = 0$ ,

$$H^0(X, F) = \text{Kernel}(\Gamma F^{(0)} \longrightarrow \Gamma F^{(1)}).$$

Since the kernel can be taken in the presheaf sense, we can write

$$H^0(X, F) = \Gamma \text{Kernel}(F^{(0)} \longrightarrow F^{(1)}).$$

Recall that  $F^{(1)}$  is a sheaf, which contains  $F^{(0)}/F$  as subsheaf. We obtain

$$H^0(X, F) = \Gamma \text{Kernel}(F^{(0)} \longrightarrow F^{(0)}/F)$$

This is the image of  $F$  in  $F^{(0)}$  and hence a sheaf which is canonically isomorphic to  $F$ .

**1.1 Remark.** *There is a natural isomorphism*

$$H^0(X, F) \cong \Gamma F = F(X).$$

If  $F \rightarrow G$  is a homomorphism of sheaves, then the homomorphism  $F_a \rightarrow G_a$  induces a homomorphism  $F^{(0)} \rightarrow G^{(0)}$ . If  $F \rightarrow G \rightarrow H$  is an exact sequence. Then  $F^{(0)} \rightarrow G^{(0)} \rightarrow H^{(0)}$  is also exact (already as sequence of presheaves). More generally, the following lemma holds.

**1.2 Lemma.** *Let  $F \rightarrow G \rightarrow H$  be an exact sequence of sheaves. Then the induced sequence  $F^{(n)} \rightarrow G^{(n)} \rightarrow H^{(n)}$  is exact for every  $n$ .*

*Proof.* By a general principle it is sufficient to prove the  $F \mapsto F^{(n)}$  maps short exact sequences  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  into short exact sequences  $0 \rightarrow F^{(n)} \rightarrow G^{(n)} \rightarrow H^{(n)} \rightarrow 0$ . The reason is that an arbitrary exact sequence  $F \xrightarrow{f} G \xrightarrow{g} H$  can be splitted into short exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{kernel}(f) \longrightarrow F \longrightarrow f(F) \longrightarrow 0, \\ 0 &\longrightarrow f(F) \longrightarrow G \longrightarrow g(G) \longrightarrow 0, \\ 0 &\longrightarrow g(G) \longrightarrow H \longrightarrow H/g(G) \longrightarrow 0. \end{aligned}$$

So we start with a short exact sequence  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ . The proof can now be given by induction. One needs the following lemma about abelian groups:

*Let*

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{00} & \longrightarrow & A^{01} & \longrightarrow & A^{02} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{10} & \longrightarrow & A^{11} & \longrightarrow & A^{12} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A^{20} & \longrightarrow & A^{21} & \longrightarrow & A^{22} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

*be a commutative diagram such that the three columns and the first two lines are exact. Then the third line is also exact.*

The proof is easy and can be omitted. □

Before we continue we need a basic lemma:

**1.3 Lemma.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be a short exact sequence of sheaves. Assume that  $F$  is flabby. Then*

$$0 \rightarrow \Gamma F \rightarrow \Gamma G \rightarrow \Gamma H \rightarrow 0$$

*is exact.*

*Proof.* Let  $h \in H(X)$ . We have to show that  $h$  is the image of an  $g \in G(X)$ . For the proof one considers the set of all pairs  $(U, g)$ , where  $U$  is an open subset and  $g \in G(U)$  and such that  $g$  maps to  $h|_U$ . This set is ordered by

$$(U, g) \geq (U', g') \iff U' \subset U \text{ and } g|_{U'} = g'.$$

From the sheaf axioms follows that every inductive subset has an upper bound. By Zorns's lemma there exists a maximal  $(U, g)$ . We have to show  $U = X$ . If this is not the case, we can find a pair  $(U', g')$  in the above set such that  $U'$  is not contained in  $U$ . The difference  $g - g'$  defines a section in  $F(U \cap U')$ . Since  $F$  is flabby, this extends to a global section. This allows us to modify  $g'$  such that it glues with  $g$  to a section on  $U \cup U'$ .  $\square$

An immediate corollary of Lemma 1.3 states:

**1.4 Lemma.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  an exact sequence of sheaves. If  $F$  and  $G$  are flabby then  $H$  is flabby too.*

Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be an exact sequence of sheafs. We obtain a commutative diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{(n-1)} & \longrightarrow & G^{(n-1)} & \longrightarrow & H^{(n-1)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{(n)} & \longrightarrow & G^{(n)} & \longrightarrow & H^{(n)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{(n+1)} & \longrightarrow & G^{(n+1)} & \longrightarrow & H^{(n+1)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

From Lemma 1.2 we know that all lines of this diagram are exact. From Lemma 1.3 follows that they remain exact after applying  $\Gamma$ . Hence the diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma F^{(n-1)} & \longrightarrow & \Gamma G^{(n-1)} & \longrightarrow & \Gamma H^{(n-1)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma F^{(n)} & \longrightarrow & \Gamma G^{(n)} & \longrightarrow & \Gamma H^{(n)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma F^{(n+1)} & \longrightarrow & \Gamma G^{(n+1)} & \longrightarrow & \Gamma H^{(n+1)} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

can be considered as a short exact sequence of complexes. We can apply Lemma II.2.1 to obtain the long exact cohomology sequence:

**1.5 Theorem.** *Every short exact sequence  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  induces a natural long exact cohomology sequence*

$$\begin{aligned} 0 \rightarrow \Gamma F \rightarrow \Gamma G \rightarrow \Gamma H \xrightarrow{\delta} H^1(X, F) \rightarrow H^1(X, G) \rightarrow H^1(X, H) \\ \xrightarrow{\delta} H^2(X, F) \rightarrow \dots \end{aligned}$$

The next Lemma shows that the cohomology of flabby sheaves is trivial.

**1.6 Lemma.** *Let*

$$0 \rightarrow F \rightarrow F_0 \rightarrow F_1 \rightarrow \dots$$

*be an exact sequence of flabby sheaves (finite or infinite). Then*

$$0 \rightarrow \Gamma F \rightarrow \Gamma F_0 \rightarrow \Gamma F_1 \rightarrow \dots$$

*is exact.*

**Corollary.** *For flabby  $F$  one has:*

$$H^i(X, F) = 0 \quad \text{for } i > 0.$$

*Proof.* We use the so-called splitting principle. The long exact sequence can be splitted into short exact sequences

$$0 \rightarrow F \rightarrow F_0 \rightarrow F_0/F \rightarrow 0, \quad 0 \rightarrow F_0/F \rightarrow F_1 \rightarrow F_1/F_0 \rightarrow 0, \dots$$

From 1.4 we get that the  $F_0/F, F_1/F_0, \dots$  are flabby. The claim now follows from Lemma 1.3.  $\square$

A sheaf  $F$  is called *acyclic* if  $H^n(X, F) = 0$  for  $n > 0$ . Hence flabby sheaves are acyclic. By an *acyclic* resolution of a sheaf we understand an exact sequence

$$0 \rightarrow F \rightarrow F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \dots$$

with acyclic  $F_i$ .

**1.7 Proposition.** *Let  $0 \rightarrow F \rightarrow F_0 \rightarrow F_1 \rightarrow \dots$  be an acyclic resolution of  $F$ . Then there is a natural isomorphism between the  $n$ -th cohomology group  $H^n(X, F)$  and the  $n$ -th cohomology group of the complex*

$$\begin{aligned} \dots \rightarrow 0 \rightarrow \Gamma F_0 \rightarrow \Gamma F_1 \rightarrow \Gamma F_2 \rightarrow \dots \\ \uparrow \\ \text{zero position} \end{aligned}$$

*Proof.* Taking the canonical flabby resolutions of  $F$  and of all  $F_n$  on gets a diagram

$$\begin{array}{cccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F & \longrightarrow & F_0 & \longrightarrow & F_2 & \longrightarrow & F_3 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F^{(0)} & \longrightarrow & F_0^{(0)} & \longrightarrow & F_1^{(0)} & \longrightarrow & F_2^{(0)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F^{(1)} & \longrightarrow & F_0^{(1)} & \longrightarrow & F_1^{(1)} & \longrightarrow & F_2^{(1)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F^{(2)} & \longrightarrow & F_0^{(2)} & \longrightarrow & F_1^{(2)} & \longrightarrow & F_2^{(2)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & 
 \end{array}$$

All lines and columns are exact. We apply  $\Gamma$  to this complex. Then all lines and columns besides the first ones remain exact. We can apply Lemma II.2.2.  $\square$

One may ask what “natural” means in Proposition 1.7 means. It means that certain diagrams in which this isomorphism appears are commutative. Since it is the best to check this when it is used we give just one example: Consider the above commutative diagram in the following new meaning: All occurring sheaves besides  $F$  are acyclic. Then II.2.2 gives an isomorphism between the  $n$ -th cohomology groups of the complexes  $0 \rightarrow \Gamma F_0 \rightarrow \Gamma F_1 \rightarrow \cdots$  and  $0 \rightarrow \Gamma F^{(0)} \rightarrow \Gamma F^{(1)} \rightarrow \cdots$ . Both are isomorphic to  $H^n(X, F)$ . This gives a commutative triangle.

## 2. Sheaves of rings and modules

A sheaf of  $A$ -modules is a sheaf  $F$  of abelian groups such that every  $F(U)$  carries a structure as  $A$ -module and such that the restriction maps  $F(U) \rightarrow F(V)$  for  $V \subset U$  are  $A$ -linear. A homomorphism  $F \rightarrow G$  is called  $A$ -linear if all  $F(U) \rightarrow G(U)$  are so. Then kernel and image carry natural structures of sheafs of  $A$ -modules. Also the stalks carry such a structure naturally. Hence the whole canonical flabby resolution is a sequence of sheafs of  $A$ -modules. This implies that the cohomology groups also are  $A$ -modules.

There is a refinement of this construction: By a sheaf of rings  $\mathcal{O}$  we understand a sheaf of abelian groups such that every  $\mathcal{O}(U)$  is not only an abelian group but a ring and such that all restriction maps  $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$  are ring homomorphisms. Then the stalks  $\mathcal{O}_a$  carry a natural ring structure such that the homomorphisms  $\mathcal{O}(U) \rightarrow \mathcal{O}_a$  ( $U$  is an open neighborhood of  $a$ ) are ring homomorphisms.

By an  $\mathcal{O}$ -module we understand a sheaf  $\mathcal{M}$  of abelian groups such every  $\mathcal{M}(U)$  carries a structure as  $\mathcal{O}(U)$ -module and such that the restriction maps are compatible with the module structure. To make this precise we give a short comment. Let  $M$  be an  $A$ -module and  $N$  be a module over a different ring  $B$ . Assume that a homomorphism  $r : A \rightarrow B$  is given. A homomorphism  $f : M \rightarrow N$  of abelian groups is called compatible with the module structures if the formula

$$f(am) = r(a)f(m) \quad (a \in A, m \in M)$$

holds. An elegant way to express this is as follows. We can consider  $N$  also as an module over  $A$  by means of the definition  $an := r(a)n$ . Sometimes this  $A$ -module is written as  $N_{[r]}$ . Then the compatibility of the map  $f$  simply means that it is an  $A$ -linear map

$$f : M \rightarrow N_{[r]}.$$

Usually we will omit the subscript  $[r]$  and simply say that  $f : M \rightarrow N$  is  $A$ -linear.

If  $\mathcal{M}$  is an  $\mathcal{O}$ -module then the stalk  $\mathcal{M}_a$  is naturally an  $\mathcal{O}_a$ -module. An  $\mathcal{O}$ -linear map  $f : \mathcal{M} \rightarrow \mathcal{N}$  between two  $\mathcal{O}$ -modules is a homomorphism of sheaves of abelian groups such the maps  $\mathcal{M}(U) \rightarrow \mathcal{N}(U)$  are  $\mathcal{O}(U)$  linear. Then the Kernel and image also carry natural structures of  $\mathcal{O}$ -modules. Clearly the canonical flabby resolution of an  $\mathcal{O}$ -module is naturally a sequence of  $\mathcal{O}$ -modules.

Since for every open subset  $U \subset X$  we have a ring homomorphism  $\mathcal{O}(X) \rightarrow \mathcal{O}(U)$  all  $\mathcal{M}(U)$  can be considered as  $\mathcal{O}(X)$ -modules. Hence a  $\mathcal{O}$ -module can be considered as sheaf of  $\mathcal{O}(X)$ -modules. Especially  $H^n(X, \mathcal{M})$  carries a natural structure as  $\mathcal{O}(X)$ -module.

We consider a very special case. We take for  $\mathcal{O}$  the sheaf  $\mathcal{C}$  of continuous functions. There are two possibilities:  $\mathcal{C}_{\mathbb{R}}$  is the sheaf of continuous real-valued and  $\mathcal{C}_{\mathbb{C}}$  the sheaf of continuous complex-valued functions. If we write  $\mathcal{C}$  we mean one of both. The sheaf  $\mathcal{C}$  or more generally a module over this sheaf have over paracompact spaces a property which can be considered as a weakend form of flabbiness.

**2.1 Remark.** *Let  $X$  be paracompact space and  $\mathcal{M}$  a  $\mathcal{C}$ -module on  $X$ . Assume that  $U$  is an open subset and  $V \subset U$  an open subset whose closure is contained in  $U$ . Assume that  $s \in \mathcal{M}(U)$  is a section over  $U$ . Then there is a global section  $S \in \mathcal{M}(X)$  such that  $S|V = s|V$ .*

*Proof.* We choose a continuous real valued function  $\varphi$  on  $X$ , which is one on  $V$  and whose support is compact and contained in  $U$ . Then we consider the open covering  $X = U \cup U'$ , where  $U'$  denotes the complement of the support of  $\varphi$ . On  $U$  we consider the section  $\varphi s$  and on  $U'$  the zero section. Since both are zero on  $U \cap U'$  they glue to a section  $S$  on  $X$ .  $\square$

**2.2 Lemma.** *Let  $X$  be a paracompact space and  $\mathcal{M} \rightarrow \mathcal{N}$  a surjective  $\mathcal{C}$ -linear map of  $\mathcal{C}$ -modules. Then  $\mathcal{M}(X) \rightarrow \mathcal{N}(X)$  is surjective.*

*Proof.* Let  $s \in \mathcal{N}(X)$ . There exists an open covering  $(U_i)_{i \in I}$  of  $X$  such that  $s|_{U_i}$  is the image of an section  $t_i \in \mathcal{M}(U_i)$ . We can assume that the covering is locally finite. We take open subsets  $V_i \subset U_i$  whose closure is contained in  $U_i$  and such that  $(V_i)$  is still a covering. Then we choose a partition of unity  $(\varphi_i)$  with respect to  $(V_i)$ . By 2.2 there exists global sections  $T_i \in \mathcal{M}(X)$  with  $T_i|_{V_i} = t_i|_{V_i}$ . We now consider

$$T := \sum_{i \in I} \varphi_i T_i.$$

Since  $I$  can be infinite, we have to explain what this means. Let  $a \in X$  a point. There exists an open neighborhood  $U(a)$  such  $V_i \cap U(a) \neq \emptyset$  only for a finite subset  $J \subset I$ . We can define the section

$$T(a) := \sum_{i \in J} \varphi_i T_i|_{U(a)}.$$

The sets  $U(a)$  cover  $X$  and the sections  $T(a)$  glue to a section  $T$ . Clearly  $T$  maps to  $s$ .  $\square$

**2.3 Lemma.** *Let  $X$  be a paracompact space and  $\mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{P}$  an exact sequence of  $\mathcal{C}$ -modules. Then  $\mathcal{M}(X) \rightarrow \mathcal{N}(X) \rightarrow \mathcal{P}(X)$  is exact too.*

*Proof.* The exactness of the sequence implies the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M} \rightarrow \mathcal{N}) \longrightarrow \mathcal{N} \longrightarrow \text{Kernel}(\mathcal{N} \rightarrow \mathcal{P}) \longrightarrow 0.$$

From 2.2 we get

$$0 \longrightarrow \text{Image}(\mathcal{M} \rightarrow \mathcal{N})(X) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N} \rightarrow \mathcal{P})(X) \longrightarrow 0.$$

Applying 2.2 to  $\mathcal{M} \rightarrow \text{Image}(\mathcal{M} \rightarrow \mathcal{N})$  we obtain

$$\text{Image}(\mathcal{M} \rightarrow \mathcal{N})(X) = \text{Image}(\mathcal{M}(X) \rightarrow \mathcal{N}(X)).$$

Since also

$$\text{Kernel}(\mathcal{N} \rightarrow \mathcal{P})(X) = \text{Kernel}(\mathcal{N}(X) \rightarrow \mathcal{P}(X))$$

we get the exactness of

$$0 \longrightarrow \text{Image}(\mathcal{M}(X) \rightarrow \mathcal{N}(X)) \longrightarrow \mathcal{N}(X) \longrightarrow \text{Kernel}(\mathcal{N}(X) \rightarrow \mathcal{P}(X)) \longrightarrow 0.$$

This proves Lemma 2.3.  $\square$

Let  $\mathcal{M}$  be an  $\mathcal{C}$ -module over a paracompact space. Then the canonical flabby resolution is also a sequence of  $\mathcal{C}$ -modules. From 2.3 follows that the resolution remains exact after the application of  $\Gamma$ . We obtain.



**2.4 Proposition.** *Let  $X$  be paracompact. Every  $\mathcal{C}$ -module is acyclic, i.e.  $H^n(X, \mathcal{M}) = 0$  for  $n > 0$ .*

### 3. Čech Cohomology

Here we will consider only the first Čech cohomology group of a sheaf. We have to work with open coverings  $\mathfrak{U} = (U_i)_{i \in I}$  of the given topological space  $X$ . Let  $F$  be a sheaf on  $X$ . A one-cocycle of  $F$  with respect to the covering  $\mathfrak{U}$  is family of sections

$$s_{ij} \in F(U_i \cap U_j), \quad (i, j) \in I \times I,$$

with the following property: For each triple  $i, j, k$  of indices one has

$$s_{ik} = s_{ij} + s_{jk} \quad \text{on} \quad U_i \cap U_j \cap U_k.$$

In more precise writing this means

$$s_{ik}|(U_i \cap U_j \cap U_k) = s_{ij}|(U_i \cap U_j \cap U_k) + s_{jk}|(U_i \cap U_j \cap U_k).$$

We denote by  $C^1(\mathfrak{U}, F)$  the group of all one-cocycles. Assume that a family of sections  $s_i \in F(U_i)$  is given. Then

$$s_{ij} = s_i|(U_i \cap U_j) - s_j|(U_i \cap U_j)$$

obviously is a cocycle. We denote it by

$$\delta(s_i)_{i \in I}.$$

A cocycle of this form is called a coboundary. The set of all coboundaries is a subgroup

$$B^1(\mathfrak{U}, F) \subset C^1(\mathfrak{U}, F).$$

The (first) Čech cohomology of  $F$  with respect to the covering  $\mathfrak{U}$  is defined as

$$\check{H}^1(\mathfrak{U}, F) := C^1(\mathfrak{U}, F)/B^1(\mathfrak{U}, F).$$

A homomorphism of sheaves  $F \rightarrow G$  induces a homomorphism

$$\check{F}^1(\mathfrak{U}, F) \longrightarrow \check{G}^1(\mathfrak{U}, F).$$

Let  $f : G \rightarrow H$  be a surjective homomorphism of sheaves and  $\mathfrak{U} = (U_i)$  an open covering of  $X$ . We denote by  $H_{\mathfrak{U}}(X)$  the set of all global sections of  $H$  with the following property:

For every index  $i$  there is a section  $t_i \in G(U_i)$  with  $f(t_i) = s|_{U_i}$ . By definition of (sheaf-)surjectivity for every global section  $s \in H(X)$  there exists an open covering  $\mathfrak{U}$  with  $s \in H_{\mathfrak{U}}(X)$ . It follows

$$H(X) = \bigcup_{\mathfrak{U}} H_{\mathfrak{U}}(X).$$

Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be an exact sequence and  $\mathfrak{U}$  an open covering. There exists a natural homomorphism

$$\delta : H_{\mathfrak{U}}(X) \longrightarrow \check{H}^1(\mathfrak{U}, F),$$

which is constructed as follows: Let be  $s \in H_{\mathfrak{U}}(X)$ . We choose elements  $t_i \in G(U_i)$  which are mapped to  $s|_{U_i}$ . The differences  $t_i - t_j$  come from sections  $t_{ij} \in F(U_i \cap U_j)$ . They define a 1-cocycle  $\delta(s)$ . It is easy to check that this corresponding element of  $\check{H}^1(\mathfrak{U}, F)$  doesn't depend on the choice of the  $t_i$ .

**3.1 Lemma.** *Let  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  be an exact sequence of sheaves and  $\mathfrak{U}$  an open covering. The sequence*

$$0 \longrightarrow F(X) \longrightarrow G(X) \longrightarrow H_{\mathfrak{U}}(X) \xrightarrow{\delta} \check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{U}, G) \longrightarrow \check{H}^1(\mathfrak{U}, H)$$

*is exact.*

**Remark.** *This sequence doesn't extend naturally to a long sequence.*

This Lemma indicates that Čech cohomology must be related to usual cohomology. Another result in this direction is:

**3.2 Remark.** *Let  $F$  be a flabby sheaf. Then for every open covering*

$$\check{H}^1(\mathfrak{U}, F) = 0.$$

*Proof.* We start with a little remark. Assume that the whole space  $X = U_{i_0}$  is a member of the covering. Then the Čech cohomology vanishes (for every sheaf): If  $(s_{ij})$  is a cocycle one defines  $s_i = s_{i, i_0}$ . Then  $\delta((s_i)) = (s_{ij})$ . For the proof of 3.2 we now consider the sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & F(X) & \longrightarrow & \prod_i F(U_i) & \longrightarrow & \prod_{ij} F(U_i \cap U_j) & \longrightarrow & \prod_{ijk} F(U_i \cap U_j \cap U_k) \\ & s & \longmapsto & (s|_{U_i}) & & (s_i) & \longmapsto & (s_i - s_j) \\ & & & & & & & (s_{ij}) & \longmapsto & (s_{ij} + s_{jk} - s_{ik}) \end{array}$$

We will proof that this sequence is exact. (Then Remark 3.2 follows.) The idea is to sheafify this sequence: For an open subset  $U \subset X$  one considers  $F|_U$

and also the restricted covering  $U \cap U_i$ . Repeating the above construction for  $U$  instead of  $X$  on obtains a sequence of sheaves

$$0 \longrightarrow F \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C}.$$

Since  $F$  is flabby, also  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  are flabby. The remark at the beginning of the proof shows that  $0 \longrightarrow F(U) \longrightarrow \mathcal{A}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{C}(U)$  is exact, when  $U$  is contained in some  $U_i$ . Hence the sequence of sheaves is exact. From Lemma 1.6 follows that the exactness is also true for  $U = X$ .  $\square$

Let now  $F$  be an arbitrary sheaf,  $F^{(0)}$  the associated flabby sheaf. We get an exact sequence  $0 \rightarrow F \rightarrow F^{(0)} \rightarrow H \rightarrow 0$ . let  $\mathfrak{U}$  be an open covering. We know that  $\check{H}^1(\mathfrak{U}, F^{(0)})$  vanishes (Remark 3.2). From Lemma 3.1 we obtain an isomophy

$$\check{H}^1(\mathfrak{U}, F) \cong H_{\mathfrak{U}}(X)/G(X).$$

From the long exact cohomology sequence we get for the usual cohomology

$$H^1(X, F) \cong H(X)/G(X).$$

This gives an *injective* homomorphism

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow H^1(X, F).$$

This gives us:

**3.3 Proposition.** *Let  $F$  be a sheaf. Then*

$$H^1(X, F) = \bigcup_U \check{H}^1(\mathfrak{U}, F).$$

The following commutative diagram shows that the Čech combining  $\delta$  from Lemma 3.1 and that of general sheaf theory Theorem 1.5 coincide:

**3.4 Remark.** *For a short exact sequence  $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$  the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H_{\mathfrak{U}}(X) & \xrightarrow{\delta} & \check{H}^1(\mathfrak{U}, F) \\ & & \parallel & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F(X) & \longrightarrow & G(X) & \longrightarrow & H(X) & \xrightarrow{\delta} & H^1(X, F) \end{array}$$

*is commutative.*

The proof is left to the reader.  $\square$

Let  $\mathfrak{V} = (V_j)_{j \in J}$  be a refinement of  $\mathfrak{U} = (U_i)_{i \in I}$  and  $\varphi : J \rightarrow I$  a refinement map ( $V_\varphi \subset U_i$ ). Using this refinement map one obtains a natural map

$$\check{H}^1(\mathfrak{U}, F) \longrightarrow \check{H}^1(\mathfrak{V}, F).$$

This shows the following result.

**3.5 Remark.** Let  $\mathfrak{V}$  be a refinement of  $\mathfrak{U}$  and  $\varphi : J \rightarrow I$  a refinement map. The diagram

$$\begin{array}{ccc} \check{H}^1(\mathfrak{U}, F) & \xrightarrow{\quad} & \check{H}^1(\mathfrak{V}, F) \\ & \searrow & \swarrow \\ & H^1(X, F) & \end{array}$$

commutes. Especially it doesn't depend on the choice of the refinement map.

Usually it is of course very difficult to control all open coverings of a topological space. But sometimes a single covering is sufficient:

**3.6 Theorem of Leray.** Let  $F$  be a sheaf on  $X$  and  $\mathfrak{U} = (U_i)$  an open covering of  $X$ . Assume that  $H^1(U_i, F|_{U_i}) = 0$  for all  $i$ . Then

$$H^1(X, F) = \check{H}^1(\mathfrak{U}, F).$$

*Proof.* Since two coverings admit a joint refinement it is sufficient to prove that  $\check{H}^1(\mathfrak{U}, F) \rightarrow \check{H}^1(\mathfrak{V}, F)$  is an isomorphism for each refinement  $\mathfrak{V}$  of  $\mathfrak{U}$ . Since the map is injective, it remains to prove surjectivity. We choose a refinement map  $\varphi : J \rightarrow I$ . We denote the indices from  $I$  by  $i, j, \dots$  and those of  $J$  by  $\alpha, \beta, \dots$ . Let be  $(s_{\alpha, \beta}) \in C^1(\mathfrak{V}, F)$ . We consider the covering  $U_i \cap \mathfrak{V} := (U_i \cap V_\alpha)_\alpha$  of  $U_i$ . From the assumption  $\check{H}^1(U_i \cap \mathfrak{V}, F|_{U_i}) = 0$  we get the existence of  $t_{i\alpha} \in F(U_i \cap V_\alpha)$  such that

$$s_{\alpha\beta} = t_{i\alpha} - t_{i\beta} \quad \text{on} \quad U_i \cap V_\alpha \cap V_\beta.$$

From this equation follows that

$$t_{j\alpha} - t_{j\alpha} = t_{j\beta} - t_{j\beta} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha \cap V_\beta.$$

Hence these differences glue to section  $T_{ij} \in F(U_i \cap U_j)$ ,

$$T_{ij} = t_{i\alpha} - t_{j\alpha} \quad \text{on} \quad U_i \cap U_j \cap V_\alpha.$$

Clearly  $(T_{ij})$  is a cocycle in  $C^1(\mathfrak{U}, F)$ . We consider its image  $(T_{(\varphi\alpha, \varphi\beta)})$  in  $C^1(\mathfrak{V}, F)$ . It is easy to check that this cocycle and the one we started with  $(s_{\alpha\beta})$  define the same cohomology class: They differ by the coboundary  $(h_\beta - h_\alpha)$  with  $h_\alpha = t_{\varphi\alpha, \alpha} \in F(V_\alpha)$ .  $\square$

## 4. Some vanishing results

Let  $X$  be a topological space and  $A$  an abelian group. We denote by  $A_X$  the sheaf of locally constant functions with values in  $A$ . This sheaf can be identified with the sheaf which is generated by the presheaf of constant functions. We will write

$$H^n(X, A) := H^n(X, A_X).$$

We mention that these groups under reasonable assumptions (for example for paracompact manifolds) agree with the singular cohomology in the sense of algebraic topology.

**4.1 Proposition.** *Let  $U$  be an open and convex subset of  $\mathbb{R}^n$ . Then for every abelian group  $A$*

$$H^1(U, A) = 0.$$

Actually this is true for all  $H^n$ ,  $n > 0$ . The best way to prove this to use the comparison theorem with singular cohomology as defined in algebraic topology. But we do not want to use this. Therefore we restrict to  $H^1$ , where we can give a simple argument.

*Proof of 4.1.* Every convex open subset of  $\mathbb{R}^n$  is topologically equivalent to  $\mathbb{R}^n$ . Hence it is sufficient to restrict to  $U = \mathbb{R}^n$ . Just for simplicity we assume  $n = 1$ . (The general case should then be clear.) We use Čech cohomology. Because of Proposition 3.3 and Remark 3.5 it is sufficient to show that every open covering admits a refinement  $\mathfrak{U}$  such that  $H^1(\mathfrak{U}, A_X) = 0$ . To show this we take a refinement of a very simple nature. It is easy to show that there exists a refinement of the following form. The index set is  $\mathbb{Z}$ . There exists a sequence of real numbers  $(a_n)$  with the following properties:

- a)  $a_n \leq a_{n+1}$
- b)  $a_n \rightarrow +\infty$  for  $n \rightarrow \infty$  and  $a_n \rightarrow -\infty$  for  $n \rightarrow -\infty$
- c)  $U_n = (a_n, a_{n+2})$ .

Assume that  $s_{n,m}$  is a cocycle with respect to this covering. Notice that  $U_n$  has non empty intersection only with  $U_{n-1}$  and  $U_{n+1}$ . Hence only  $s_{n-1,n}$  is of relevance. This a locally constant function on  $U_{n-1} \cap U_n = (a_n, a_{n+1})$ . Since this is connected, the function  $s_{n-1,n}$  is constant. We want to show that it is coboundary, i.e. we want to construct constant functions  $s_n$  on  $U_n$  such that  $s_{n-1,n} = s_n - s_{n-1}$  on  $(a_n, a_{n+1})$ . This is easy. One starts with  $s_0 = 0$  and then constructs inductively  $s_1, s_2, \dots$  and in the same way for negative  $n$ .  $\square$

Consider on the real line  $\mathbb{R}$  the sheaf of complex valued differentiable functions  $\mathcal{C}^\infty$ . Taking derivatives one gets a sheaf homomorphism  $\mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$ ,

$f \mapsto f'$ . The kernel is the sheaf of all locally constant functions, which we denote simply by  $\mathbb{C}$ . Hence we get an sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty \longrightarrow \mathcal{C}^\infty \longrightarrow 0.$$

This sequence is exact since every differentiable function has an integral. Hence this sequence can be considered as acyclic resolution of  $\mathbb{C}$ . We obtain  $H^n(\mathbb{R}, \mathbb{C}) = 0$  for all  $n > 0$ . For  $n = 1$  this follows already from 4.1. There is a generalization to higher dimensions. For example a standard result of vector analysis states in the case  $n = 2$ .

**4.2 Lemma.** *Let  $E \subset \mathbb{R}^n$  be an open and convex subset,  $f, g \in \mathcal{C}^\infty$  a pair of differentiable functions with the property*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

*Then there is a differentiable function  $h$  with the property*

$$f = \frac{\partial h}{\partial x}, \quad g = \frac{\partial h}{\partial y}.$$

In the language of exact sequences this means:

The sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E) \times \mathcal{C}^\infty(E) \longrightarrow \mathcal{C}^\infty(E) \longrightarrow 0$$

$$\begin{aligned} f &\longmapsto \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \\ (f, g) &\longmapsto \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} \end{aligned}$$

is exact. When  $E$  is not convex, this sequence needs not to be exact. But since every point in  $\mathbb{R}^2$  has an open convex neighborhood, the sequence of sheaves

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \times \mathcal{C}_X^\infty \longrightarrow \mathcal{C}_X^\infty \longrightarrow 0$$

is exact for every open subset  $X \subset \mathbb{R}^2$ . This is an acyclic resolution and we obtain:

**4.3 Proposition.** *For convex open  $E \subset \mathbb{R}^2$  we have*

$$H^i(E, \mathbb{C}) = 0 \quad \text{for } i > 0.$$

One can of course consider real valued differentiable functions and the same proof shows  $H^i(E, \mathbb{R}) = 0$  for  $i > 0$ .

As an application we prove the following proposition.

**4.4 Proposition.** *For convex open  $E \subset \mathbb{R}^n$  one has*

$$H^2(E, \mathbb{Z}) = 0.$$

*Proof.* We consider the homomorphism

$$\mathbb{C} \longrightarrow \mathbb{C}^\bullet, \quad z \longmapsto e^{2\pi iz}.$$

The kernel is  $\mathbb{Z}$ . This can be considered as an exact sequence of sheaves for example on an open convex  $E \subset \mathbb{R}^n$ . A small part of the long exact cohomology sequence is

$$H^1(E, \mathbb{C}^\bullet) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(E, \mathbb{C}).$$

Since the first and the third member of this sequence vanish (Propositions 4.1 and 4.3), we get the proof of Proposition 4.4.  $\square$

Next we treat an example of complex analysis.

**4.5 Lemma of Dolbeault.** *Let  $E \subset \mathbb{C}$  be an open disk. For every function  $f \in C^\infty(E)$  there exists  $g \in C^\infty(E)$  with*

$$f = \frac{\partial g}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial g}{\partial x} + i \frac{\partial g}{\partial y} \right).$$

A proof can be found in [Fo], Satz 13.2. We give a short sketch here. In a first step one restricts to the case where  $f$  has compact support. In this case the function  $g$  can be constructed as an integral:

$$g(z) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^1 f(z + re^{i\varphi}) e^{-i\varphi} dr d\varphi.$$

One can show that  $\partial g / \partial \bar{z} = f$ . But this is not trivial. One has to make use of the Theorem of Stokes V.7.6. We do not give the details.

If  $f$  has not compact support, one needs an approximation argument. One writes  $f$  as limit of a sequence of functions  $f_n$  with compact support such that  $f$  and  $f_n$  coincide for  $|z| < 1 - 1/n$ . Then one writes  $\partial g_n / \partial \bar{z} = f_n$  using the first part of the proof. We have the freedom to add to  $g_n$  a polynomial  $P_n(z)$ . In this way one can get the convergence of the sequence  $g_n$  to a solution  $g$ .  $\square$

The lemma of Dolbeault is related to an exact sequence of sheaves. Recall that a (real-) differentiable function is analytic (=holomorphic) if and only if  $\partial f / \partial \bar{z} = 0$ . We denote by  $\mathcal{O}$  the sheaf of holomorphic functions. The Lemma of Dolbeault shows that the sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}_X^\infty \xrightarrow{\bar{\partial}} \mathcal{C}_X^\infty \longrightarrow 0$$

is exact for an open subset  $X \subset \mathbb{C}$ . This is an acyclic resolution of  $\mathcal{O}_X$ . Applying the Lemma of Dolbeault once more, we get the following result.

**4.6 Proposition.** *Let  $E \subset \mathbb{C}$  be an open disk. Then*

$$H^i(E, \mathcal{O}_E) = 0 \quad \text{for } i > 0.$$

We denote by  $\bar{\mathbb{C}}$  the Riemann sphere.

**4.7 Theorem.** *One has*

$$H^1(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}}) = 0.$$

For the proof we consider a covering of  $\bar{\mathbb{C}}$  by two disks  $E_1, E_2$  such that the intersection is a circular ring  $1 < |z| < 2$ . By Leray's theorem 3.6 and the vanishing result Proposition 4.6 it is sufficient to show that the Čech cohomology with respect to this covering vanishes. A Čech cocycle simply is given by a holomorphic function on the circular ring. We have to show that it can be written as difference  $f_1 - f_2$  where  $f_i$  is holomorphic on the disk  $E_i$ . This is possible by the theory of the *Laurent decomposition*.  $\square$



# Chapter V. Basic facts about Riemann surfaces

## 1. Geometric spaces

In the following we denote by  $\mathcal{C}(X)$  the set of complex valued continuous functions on the topological space  $X$ .

**1.1 Definition.** *A geometric structure  $\mathcal{O}$  on a topological space is a collection of subrings  $\mathcal{O}(U) \subset \mathcal{C}(U)$ , where  $U$  runs through all open subsets, such that the following conditions are satisfied:*

- 1) *The constant functions are in  $\mathcal{O}(U)$ .*
- 2) *If  $V \subset U$  are open sets then*

$$f \in \mathcal{O}(U) \implies f|_V \in \mathcal{O}(V).$$

- 3) *Let  $(U_i)_{i \in I}$  be a system of open subsets and  $f_i \in \mathcal{O}(U_i)$  such that*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \quad \text{for all } (i, j),$$

*then there exist a  $f \in \mathcal{O}(U)$  where  $U = \bigcup_{i \in I} U_i$  with the property*

$$f|_{U_i} = f_i \quad \text{for all } i.$$

We see that  $\mathcal{O}$  is a sheaf of rings. We call the functions of  $\mathcal{O}(U)$  the distinguished functions. Conditions two and three mean that to be distinguished is a local property. Our main example at the moment is  $X = \mathbb{C}$ , where the distinguished functions are the holomorphic functions.

A *geometric space* is a pair  $(X, \mathcal{O})$  consisting of a topological space and a geometric structure.

**1.2 Definition.** *A morphism  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  of geometric spaces is a continuous map  $f : X \rightarrow Y$  with the following additional property. If  $V \subset Y$  is open and  $g \in \mathcal{O}_Y(V)$  then  $g \circ f$  is contained in  $\mathcal{O}_X(f^{-1}(V))$ .*

Quite trivial facts are:

*The composition of two morphisms is a morphism.*

*The identical map  $(X, \mathcal{O}) \rightarrow (X, \mathcal{O})$  is a morphism.*

A morphism  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  of geometric spaces is called an isomorphism if  $f$  is topological and if  $f^{-1} : (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$  is also a morphism. This means that the rings  $\mathcal{O}_X(U)$  and  $\mathcal{O}_Y(f(U))$  are naturally isomorphic.

Let  $U \subset X$  be an open subset of a geometric space  $(X, \mathcal{O})$ . We can define the restricted geometric structure  $\mathcal{O}|U$  by

$$\mathcal{O}|U(V) := \mathcal{O}(V) \quad (V \subset U \text{ open}).$$

It is clear that the natural embedding  $i : (U, \mathcal{O}_X|U) \hookrightarrow (X, \mathcal{O}_X)$  is a morphism and moreover that a map  $f : Y \rightarrow U$  from a geometric space  $(Y, \mathcal{O}_Y)$  into  $U$  is a morphism if and only if  $i \circ f$  is a morphism.

A morphism  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  is called an *open embedding* if it is the composition of an isomorphism  $(X, \mathcal{O}_X) \rightarrow (U, \mathcal{O}_Y|U)$ ,  $U \subset Y$  open, and the natural injection.

## 2. The notion of a Riemann surface

We equip  $\mathbb{C}$  and more generally an open subset  $V$  with the sheaf of holomorphic functions. The geometric space obtained in this way is denoted by  $(V, \mathcal{O}_V)$ . A Riemann surface is a geometric space which is locally isomorphic to such a space:

**2.1 Definition.** *A Riemann surface is a geometric space  $(X, \mathcal{O}_X)$ , such that for every point there exists an open neighborhood  $U$  and an open subset  $V \subset \mathbb{C}$  such that the geometric spaces  $(U, \mathcal{O}_X|U)$  and  $(V, \mathcal{O}_V)$  are isomorphic geometric spaces. We always assume that  $X$  is a Hausdorff space with countable basis of the topology.*

Of course  $(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$  is a Riemann surface. An open subspace of a Riemann surface (equipped with the induced geometric structure) is a Riemann surface. In particular, every open subset  $U \subset \mathbb{C}$  carries a natural structure of a Riemann surface. If  $U, V$  are two open subsets of  $\mathbb{C}$  then for a map  $f : U \rightarrow V$  the following two conditions are equivalent:

- a) The map  $f$  is analytic in the sense of complex analysis.
- b) The map  $f$  defines a morphism of geometric spaces  $f : (U, \mathcal{O}_U) \rightarrow (V, \mathcal{O}_V)$ .

This allows us to define:

**2.2 Definition.** *A map  $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$  between Riemann surfaces is called holomorphic if it is a morphism of geometric spaces.*

A biholomorphic map between Riemann surfaces of course is a bijective map which is analytic in both directions.

A (topological) chart on a Riemann surface  $X$  is a topological map from an open subset  $U \subset X$  onto an open subset  $V \subset \mathbb{C}$ . The chart is called analytic if it is moreover biholomorphic, i.e. an isomorphism of geometric spaces  $(U, \mathcal{O}_X|_U) \rightarrow (V, \mathcal{O}_V)$ .

Let  $\varphi : U \rightarrow V$  and  $\psi : U' \rightarrow V'$  be two charts on  $X$ . Then we can consider the topological map

$$\gamma := \psi \circ \varphi^{-1} : \varphi(U \cap U') \rightarrow \psi(U \cap U').$$

The notation “ $\psi \circ \varphi^{-1}$ ” is not in accordance with the strong rules of set theory, but we allow this and related notations when it is clear from the context what is meant.

The sets  $\varphi(U \cap U')$  and  $\psi(U \cap U')$  are open subsets of  $\mathbb{C}$ . If the charts  $\varphi, \psi$  are analytic then the chart change map  $\gamma$  is biholomorphic.

A set of charts  $\varphi : U \rightarrow V$  is called an *atlas* of  $X$  if the domains of definition  $U$  cover  $X$ . The set of all analytic charts is an atlas.

### Riemann surfaces via charts

Assume that a topological space  $X$  and a set  $\mathcal{A}$  of two dimensional charts (topological maps from open subsets of  $X$  to open subsets of  $\mathbb{C}$ ) is given. We assume that  $\mathcal{A}$  is an atlas. We also assume that all chart changes  $\gamma = \psi \circ \varphi^{-1}$  ( $\varphi, \psi \in \mathcal{A}$ ) are biholomorphic. We call then  $\mathcal{A}$  an analytic atlas.

**2.3 Remark.** *Let  $\mathcal{A}$  be an analytic atlas on  $X$ . Then there exists a unique structure as Riemann surface  $(X, \mathcal{O}_X)$  such that all elements of  $\mathcal{A}$  are analytic charts with respect to this structure*

It should be clear how  $\mathcal{O}_X$  has to be defined: A function  $f : U \rightarrow \mathbb{C}$  for open  $U \subset X$  belongs to  $\mathcal{O}_X(U)$  if and only if for every chart  $\varphi : U_\varphi \rightarrow V_\varphi$  the function

$$f_\varphi = f \circ \varphi^{-1} : \varphi(U \cap U_\varphi) \rightarrow \mathbb{C}$$

is analytic in the usual sense.

The atlas  $\mathcal{A}$  then is part of the atlas of all analytic charts of  $(X, \mathcal{O}_X)$ , but it can be smaller. The atlas of all analytic charts is the unique maximal analytic atlas which contains  $\mathcal{A}$ .

### 3. The notion of a differentiable manifold

An  $n$ -dimensional topological manifold  $X$  is a topological space such that each point contains an open neighbourhood which is homeomorphic to some open subset of  $\mathbb{R}^n$ . In the case  $n = 2$  we call this a topological surfaces. Every Riemann surface has an underlying topological surface. Hence Riemann surfaces should be considered as surfaces with an additional structure. We introduced this additional structure as a sheaf  $\mathcal{O}_X$ . This approach is very flexible and can be used to consider other geometric structures. For example, one can consider for an open subset  $U \subset \mathbb{R}^n$  the set  $\mathcal{C}^\infty(U)$  of all differentiable functions in the sense of real analysis. One can take them real- or complex valued. A complex valued function is differentiable if and only if real and imaginary part are differentiable. For our purposes it is better to take them complex valued. Hence from now on we will use the notation  $\mathcal{C}^\infty(U)$  for the set of *complex valued differentiable functions*. We then can consider the geometric space  $(U, \mathcal{C}_U^\infty)$  where  $\mathcal{C}_U^\infty(V) := \mathcal{C}^\infty(V)$ . Similar to the definition of a Riemann surface we define an  $n$ -dimensional differentiable manifold  $(X, \mathcal{C}_X^\infty)$  to be a geometric space that is locally isomorphic to an  $(U, \mathcal{C}_U^\infty)$ ,  $U \subset \mathbb{R}^n$ , open. A morphism between differentiable manifolds is just called a differentiable map. Isomorphisms between differentiable manifolds are called diffeomorphisms.

Let  $X$  be a Riemann surface and  $U$  be an open subset. One can define what it means that a function  $f : U \rightarrow \mathbb{C}$  is differentiable ( $\mathcal{C}^\infty$ ). For every point  $a \in U$  there exists an analytic chart  $\varphi : U_\varphi \rightarrow V_\varphi$  such that the transported function  $f_\varphi(z) = f(\varphi^{-1}(z))$  is differentiable in the usual sense. It is clear that this defines a structure  $\mathcal{C}_X^\infty$  as differentiable surface on  $X$ . We call this the underlying differentiable surface of  $X$ . Holomorphic maps between Riemann surfaces are differentiable maps for the underlying differentiable surfaces.

One reason that it is useful to consider the underlying differentiable surface of a Riemann surface is that in the differentiable world partitions of unity exist. Here a partition of unity  $(\varphi_i)$  is called differentiable if all functions  $\varphi_i$  are differentiable.

**3.1 Proposition.** *Let  $X$  be a paracompact differentiable manifold. For every locally finite open covering there exists a differentiable partition of unity.*

The same proof as in the case of continuous functions works. □

Hence there is the following variant of Proposition IV.2.4.

**3.2 Proposition.** *Let  $X$  be a paracompact differentiable manifold, then every  $\mathcal{C}_X^\infty$ -module is acyclic.*

## 4. Meromorphic functions

We recall the topology of the Riemann sphere. A subset  $U \subset \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  is open if  $U \cap \mathbb{C}$  is open in  $\mathbb{C}$  and if in the case  $\infty \in U$  there exists  $C > 0$  with the property

$$z \in \mathbb{C}, \quad |z| > C \implies z \in U.$$

Obviously  $\bar{\mathbb{C}}$  is a compact space. The subset  $\mathbb{C}$  is open and the induced topology is the usual one.

Let  $U \subset \bar{\mathbb{C}}$  an open set and  $f : U \rightarrow \mathbb{C}$  a function. We assume that  $\infty \in U$ . Then  $f(1/z)$  is defined in an open neighborhood of 0. We call  $f$  analytic at  $\infty$  if  $f(1/z)$  is analytic in an open neighborhood of zero. For any open set we define  $\mathcal{O}_{\bar{\mathbb{C}}}(U)$  to be the set of all functions with the following properties:

- a) The restriction of  $f$  to  $U \cap \mathbb{C}$  is analytic in the usual sense.
- b) When  $\infty \in U$  then  $f$  is analytic at  $\infty$ .

It is easy to see that  $(\bar{\mathbb{C}}, \mathcal{O}_{\bar{\mathbb{C}}})$  is a Riemann surface. We describe two analytic charts which cover  $\bar{\mathbb{C}}$ . They can be used to introduce  $\bar{\mathbb{C}}$  as Riemann surface via charts.

- 1)  $\bar{\mathbb{C}} - \{\infty\} = \mathbb{C} \xrightarrow{\text{id}_{\mathbb{C}}} \mathbb{C}$ ,
- 2)  $\bar{\mathbb{C}} - \{0\} \rightarrow \mathbb{C}, \quad z \mapsto 1/z \quad (1/\infty = 0).$

The chart change map is

$$\mathbb{C} - \{0\} \rightarrow \mathbb{C} - \{0\}, \quad z \mapsto 1/z.$$

We consider now holomorphic maps  $f : X \rightarrow \bar{\mathbb{C}}$  of an arbitrary Riemann surface into the Riemann sphere.

**4.1 Definition.** *A meromorphic function  $f$  on a Riemann surface  $X$  is an analytic map*

$$f : X \rightarrow \bar{\mathbb{C}},$$

*such that  $f^{-1}(\infty)$  is a discrete subset of  $X$ .*

The constant function  $f(z) = \infty$  is an analytic map but not meromorphic. Let  $S \subset X$  be a discrete subset and  $f : X \rightarrow \mathbb{C}$  a holomorphic function. It may happen that  $f$  extends to a meromorphic function on  $X$ . Then we call the points in  $S$  inessential singularities. Since the extension of  $f$  to  $X$  is unique we will use the same letter for it.

We denote by  $\mathcal{M}(X)$  the set of all meromorphic functions on  $X$ . Let  $f, g \in \mathcal{M}(X)$  be two meromorphic functions on  $X$  and denote by  $S$  the union of the points where  $f$  or  $g$  has the value  $\infty$ . Then we can define on  $X - S$  the analytic

functions  $f + g$  and  $f \cdot g$ . The points of  $S$  are inessential singularities. Hence we have defined

$$f + g, \quad fg \in \mathcal{M}(X).$$

It follows that  $\mathcal{M}(X)$  is a ring.

If  $f$  is a meromorphic function with discrete zero set then  $1/f$  is an analytic function on the complement of this set. Since the singularities are inessential it extends as meromorphic function on  $X$ .

When  $f \in \mathcal{O}_X(X)$  is an analytic function the composition with the natural inclusion  $\mathbb{C} \hookrightarrow \bar{\mathbb{C}}$  is a meromorphic function. We identify  $f$  with this meromorphic function. This means that we can consider  $\mathcal{O}_X(X)$  as a subring of  $\mathcal{M}(X)$ .

We want to show that  $\mathcal{M}(X)$  is a field, wenn  $X$  is connected. For this we need a a variant of the principle of analytic continuation.

**4.2 Lemma.** *Let  $f, g : X \rightarrow Y$  be two analytic maps of a connected Riemann surface  $X$  into a Riemann surface  $Y$ . Assume that there exists a subset  $A \subset X$  which is not discrete and such that  $f$  and  $g$  agree on  $A$ . Then  $f = g$ .*

**Corollary.** *Let  $f : X \rightarrow Y$ ,  $X$  connected, be a non-constant analytic map than  $f^{-1}(y)$  is discrete for every  $y \in Y$ .  $b \in Y$  discrete in  $X$ .*

**Corollary.** *The set  $\mathcal{M}(X)$  of all meromorphic functions on a connected Riemann surface is a field.*

*Proof.* Consider the set of all cumulation points of the set  $\{x \in X; f(x) = g(x)\}$ . It is sufficient to show that this set is open and closed. Since this is a statement of local nature, one can take analytic charts and reduce the statement to the case where  $X$  and  $Y$  are open subsets of  $\mathbb{C}$ . Now one can use the standard principle of analytic continuation.  $\square$

## 5. Ramification points

An analytic map  $f : X \rightarrow Y$  of Riemann surfaces is called *locally biholomorphic* at a point  $a \in X$  if  $f$  maps some open neighborhood of  $a$  biholomorphic to an open neighborhood of  $f(a)$ . If this is not the case,  $a$  is called a *ramification point*.

**5.1 Remark.** *Let  $f : X \rightarrow Y$  be a non-constant holomorphic map of connected Riemann surfaces. The set of ramification points is discrete in  $X$ .*

*Proof.* Taking charts one can assume that  $X$  and  $Y$  are open subsets of  $\mathbb{C}$ . The ramification points then are the zeros of the derivative of  $f$ .  $\square$

We recall a result of complex calculus. Let  $f : U \rightarrow \mathbb{C}$  be a non-constant analytic function on an open connected neighborhood of 0 with the property

$f(0) = 0$ . There exists a small open neighborhood  $0 \in V \subset V$  and an analytic function  $h : V \rightarrow \mathbb{C}$  with the properties

$$f(z) = h(z)^n, \quad h'(0) \neq 0.$$

If  $V$  is taken small enough then  $h$  maps  $V$  biholomorphic onto an open neighborhood of 0. (The number  $n$  is the zero order of  $f$ .) We refer to [FB] for a proof. There it has been used for the proof of the Open Mapping Theorem III.3.3.

We want to reformulate this result for Riemann surfaces. For sake of convenience we will use the following notation:

A **disk** around  $a \in X$  on a Riemann surface  $X$  is a biholomorphic map (=analytic chart)

$$\varphi : U \xrightarrow{\sim} \mathbb{E}, \quad \varphi(0) = 0.$$

Here

$$\mathbb{E} := \{ q \in \mathbb{C}; |q| < 1 \}$$

denotes the unit disk.

If  $b$  is a point of  $U$ , we also say that the disk contains  $a$ . If  $U$  is a subset of a subset  $A \subset X$ , then we say that the disk is contained in  $A$ . (This is not quite correct since a disk is a map and not only the set  $U$ .)

If  $a$  is a point of a Riemann surface, then of course there exists a disk around  $a$ . One simply takes an arbitrary analytic chart  $\varphi : U \rightarrow V, a \in U$ . Then one replaces  $V$  by a small disk around  $f(a)$  and  $U$  by the inverse image. For trivial reasons there exists a biholomorphic map of an arbitrary disk to the unit disk, such that the center goes to the center. One can say a little more:

*Around an arbitrary point  $a$  of a Riemann surface there exist arbitrary small disks.*

This means of course that for a given neighborhood one can find a disk which is contained in it.

**5.2 Remark.** *Let  $f : X \rightarrow Y$  be a non-constant map of connected Riemann surfaces and  $a \in X$  a given point. There exist disks  $\psi : V \rightarrow E$  around  $f(a) \in Y$  and  $\varphi : U \rightarrow E$  around  $a$  such that the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & E \\ f \downarrow & & \downarrow \\ V & \xrightarrow{\psi} & E \end{array} \quad \begin{array}{c} q \\ \downarrow \\ q^n \end{array}$$

*commutes for some natural number  $n$  which is uniquely determined.*

This is just a reformulation of the discussion above. □

A simple way to express Remark 5.2 is:

*Analytic maps of Riemann surfaces look locally like “ $q \mapsto q^n$ ”.*

The point is a ramification point if and only if  $n > 1$ .

### Proper maps

For *proper* analytic maps  $f : X \rightarrow Y$  there are better results. We will use in the following frequently the following trivial fact: Let  $V \subset Y$  an open subset and  $U = f^{-1}(V)$  its (full!) inverse image. Then  $f : U \rightarrow V$  is proper too (Lemma I.3.1).

The basic fact which we have to use is the following purely topological result.

**5.3 Proposition.** *Let  $f : X \rightarrow \mathbb{E}^\bullet$  be a locally topological proper map of a connected Hausdorff space into the punctured unit disk  $\mathbb{E}^\bullet = \mathbb{E} - \{0\}$ . Then there exists a topological map  $\sigma : X \rightarrow \mathbb{E}^\bullet$  such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \mathbb{E}^\bullet \\ & \searrow f & \swarrow \\ & \mathbb{E}^\bullet & \\ & & \swarrow q \\ & & q^n \end{array}$$

*commutes.*

The proof uses topological covering theory. A quick proof can be found in [Fr1], Chapt. I, Sect 3, Appendix A. We will not give the details of the proof here. We just mention the topological background. Locally topological and proper maps are special cases of (unramified) topological coverings. These are classified by the fundamental group of the space. We will introduce the fundamental group in Sect. 8 of this chapter. The fundamental group of  $\mathbb{E}^\bullet$  is isomorphic to  $\mathbb{Z}$ . This implies that for each natural number  $n$  there exists only one covering of  $\mathbb{E}^\bullet$  of degree  $n$ .  $\square$

We point out as a consequence.

**5.4 Corollary of 5.3.** *Let in addition  $X$  be a Riemann surface and  $f$  analytic. Then the map  $\sigma$  is biholomorphic.*

A simple consequence of Proposition 5.3 states:

**5.5 Lemma.** *Let  $f : X \rightarrow \mathbb{E}$ ,  $a \mapsto 0$ , be a holomorphic and proper map of a connected Riemann surface, which is locally biholomorphic outside  $a$ . Then there exists a biholomorphic map  $\sigma : X \rightarrow \mathbb{E}$  such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & \mathbb{E} \\ & \searrow f & \swarrow \\ & \mathbb{E} & \\ & & \swarrow q \\ & & q^n \end{array}$$

*commutes.*



*Proof.* From Proposition 5.3 follows the existence of a biholomorphic map  $f : X - \{a\} \rightarrow \mathbb{E}^\bullet$ . We extend by  $f(a) = 0$ . The Riemann extension theorem shows that this map is still holomorphic.  $\square$

We treat a topological application of Proposition 5.3. We take (in the notation of Proposition 5.3) a symbol  $a$  which is not contained in  $X$ . We define  $\bar{X} = X \cup \{a\}$ . We extend the map  $\sigma$  to the map

$$\bar{\sigma} : \bar{X} \longrightarrow \mathbb{E}, \quad \bar{\sigma}(a) = 0.$$

This is bijective map. Then we equip  $\bar{X}$  with the unique topology such that this map is topological. Then  $X$  is an open subspace of  $\bar{X}$ . We also extend  $f$  to a map

$$\bar{f} : \bar{X} \longrightarrow \mathbb{E}, \quad \bar{f}(a) = 0.$$

This map still is continuous, even proper but not locally topological.

This trivial construction admits an important extension:

**5.6 Proposition.** *Let  $\bar{Y}$  be a surface and  $Y \subset \bar{Y}$  an open subset such that  $T := \bar{Y} - Y$  is a finite set. Assume that  $f : X \rightarrow Y$  is a locally topological proper map of a Hausdorff space  $X$  into  $Y$ . Then there exists a surface  $\bar{X}$  with the following property:*

- a)  $X$  is an open subset of  $\bar{X}$  and the topology of  $X$  is the induced topology.
- b)  $T := \bar{X} - X$  is a finite set. The map  $f$  extends to a continuous and proper map

$$\bar{f} : \bar{X} \longrightarrow \bar{Y}.$$

- c) For every point  $a \in T$  there exist neighborhoods  $a \in U \subset \bar{X}$  and  $b \in V \subset \bar{Y}$  and topological maps  $U \rightarrow \mathbb{E}$  and  $V \rightarrow \mathbb{E}$  such that the diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{E} \\ f \downarrow & & \downarrow \\ V & \longrightarrow & \mathbb{E} \end{array} \quad \begin{array}{c} q \\ \downarrow \\ q^n \end{array}$$

*commutes.*

**Additional Remark.** *When  $\bar{Y}$  is compact then so is  $\bar{X}$ .*

*Proof.* For each  $b \in T$  we choose an open neighborhood  $V(b)$  and a topological map  $V(b) \rightarrow \mathbb{E}$ ,  $b \mapsto 0$ . We can assume that the  $V(b)$  for different  $b$  are disjoint. Next we take the inverse image  $f^{-1}(V(b))$  in  $X$ . This needs not to be connected. Hence we consider the connected components. From the properness of  $f$  follows that there are only finitely many connected components. Let  $U$  be one of them. The map

$$U \rightarrow V(b) - \{b\} \cong \mathbb{E}^\bullet$$

still is proper. To this situation we apply the above construction adding one extra point to  $U$ . To be precise we take for each point  $b \in T$  and for each connected component of  $f^{-1}(V(b))$  an extra symbol  $a_U$ . Then we add to  $U$  this symbol to get  $\bar{U} = U \cup \{a_U\}$ . Then we consider

$$\bar{X} = X \cup \bigcup_{b \in T, U} \{a_U\}.$$

Then  $\bar{U}$  is a subset of  $\bar{X}$ . It is clear how to extend  $f$  to a map  $\bar{f} : \bar{X} \rightarrow \bar{Y}$ . It is also clear how to define the topology on  $\bar{X}$ : a subset is called open if the intersection with  $X$  and all  $U$  is open.  $\square$

If  $f : U \rightarrow \mathbb{C}$  is a non-constant function on a connected open subset  $U \subset \mathbb{C}$ . Then for  $a \in U$  the zero order of  $f(z) - f(a)$  at  $a$  is called the multiplicity of  $f$  at  $a$ . If  $f : X \rightarrow Y$  is a non-constant holomorphic map of Riemann surfaces, then the multiplicity of  $f$  at a point  $a \in X$  can be defined in an obvious way via charts. It also can be defined by means of Remark 5.2.

**5.7 Definition.** *Let  $f : X \rightarrow Y$  be a non-constant holomorphic maps of Riemann surfaces. The multiplicity of  $f$  at a point  $a$  is the number  $n$  such that  $f$  looks locally at  $a$  like  $q \mapsto q^n$  (see Remark 5.2).*

From Lemma 5.5 and the above discussion we deduce:

**5.8 Proposition.** *Let  $f : X \rightarrow Y$  be a proper non-constant holomorphic map of Riemann surfaces. Assume that  $Y$  is connected. Denote for  $b \in Y$  by  $d$  the number of all  $a \in X$  with  $f(a) = b$  counted with multiplicity. The number  $d$  is independent of  $b$ . In particular,  $f$  is surjective.*

We denote the number  $d$  the *covering degree* of  $f$ .

*Proof of Proposition 5.8.* The statement is trivial for the map  $\mathbb{E} \rightarrow \mathbb{E}$ ,  $q \mapsto q^n$ . The covering degree in this case is  $n$ . We claim that the function that associates to  $b \in Y$  the number  $d$  is locally constant. It is sufficient to prove this in the case  $Y = \mathbb{E}$ . In this case we can apply Lemma 5.5 to the connected components of  $X$ . So we have proved that the function  $b \mapsto d$  is locally constant. Since  $Y$  is connected, it is constant.  $\square$

## 6. Examples of Riemann surfaces

The only Riemann surfaces which we introduced so far is the Riemann sphere and its open subsets. We give some more interesting examples but keep short since they are not needed for the development of the general theory. We will treat now three examples, complex tori, the Riemann surface of an analytic

function and the compact Riemann surfaces that can be associated to algebraic functions. In the last Chapter VIII we will introduce another important example, namely the Riemann surface that can be associated to a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$ .

### Tori

Let  $L \subset \mathbb{C}$  be a lattice, i.e.  $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ , where  $\omega_1, \omega_2$  is an  $\mathbb{R}$ -basis of  $\mathbb{C}$ . The quotient  $X_L = \mathbb{C}/L$  carries a natural structure as Riemann surface. The topology is the quotient topology. A function  $f : U \rightarrow \mathbb{C}$  on an open subset  $U \subset X_L$  is called holomorphic if and only if the composition with the natural projection  $p : \mathbb{C} \rightarrow X_L$  is a holomorphic function in the usual sense on  $p^{-1}(U)$ . The natural projection  $p$  is holomorphic and even more locally biholomorphic. It follows the meromorphic functions  $f : X_L \rightarrow \bar{\mathbb{C}}$  are in 1-1-correspondence to the elliptic functions  $F = f \circ p$  (meromorphic functions on  $\mathbb{C}$  which are periodic with respect to  $L$ ).

It can be shown that two tori  $X_{L_1}$  and  $X_{L_2}$  are biholomorphic equivalent if and only if there exists a complex number  $a \in \mathbb{C}$  such that  $L_2 = aL_1$ . Since this is usually not the case and since two tori are always topologically equivalent, we see that topologically equivalent Riemann surfaces are usually not biholomorphic equivalent.

### Concrete Riemann surfaces

A function element  $(a, P)$  is point  $a \in \mathbb{C}$  together with an element  $P \in \mathcal{O}_a$ . We can think of  $P$  as power series with center  $a$  and positive convergence radius. Let  $\alpha : [0, 1] \rightarrow \mathbb{C}$  be a curve with starting point  $a = \alpha(0)$ . Assume that a function element  $(a, P)$  is given. We say that  $(a, P)$  admits analytic continuation along  $\alpha$  if there exist for every  $t \in [0, 1]$  a function element  $(\alpha(t), P_t)$  such that the following condition hold:

- a)  $(a, P) = (\alpha(0), P_0)$ .
- b) Let  $t \in [0, 1]$ . If  $t' \in [0, 1]$  is close enough to  $t$  than the (open) convergence disks of  $P_t$  and  $P_{t'}$  have non empty intersection and both functions agree in the intersection.

It is easy to see that such an analytic continuation is unique. Two function elements  $(a, P)$  and  $(b, Q)$  are called equivalent if there exists a path  $\alpha$  with  $\alpha(0) = a$  and  $\alpha(1) = b$  such that  $(a, P)$  can be continued analytically to  $b$  with the end-element  $(b, Q)$ .

The basic truth is that  $(b, Q)$  depends on the choice of the curve  $\alpha$ . As an example one considers the function element  $(1, \sqrt{z})$ , where  $\sqrt{z}$  is defined by the principal part of the logarithm. This function element can be continued to  $-1$ . But if one takes the continuation along the half circle in the upper half plane one gets a different result as if one takes the half circle in the lower half plane. The two continuations differ by a sign.

The idea of the Riemann surface is to count the point  $-1$  twice. This means one takes a two fold covering of  $\mathbb{C}^*$ . On this covering  $\sqrt{z}$  can be defined as an unambiguous.

To define this in mathematical rigorous way, one starts with an analytic function  $f : D \subset \mathbb{C}$  on some connected open subset of  $\mathbb{C}$ . The elements  $(a, f_a)$  of course all are equivalent. Now we introduce the set  $\mathcal{R} = \mathcal{R}(f)$  of all functions element which are equivalent to the elements  $(a, f_a)$ . So  $\mathcal{R}$  contains all possible analytic continuations of  $f$ .

We introduce a topology on  $\mathcal{R}$ . Let  $(a, P) \in \mathcal{R}$ . Consider a positive number  $\varepsilon$  which is smaller than the convergence radius of  $r$ . Then we define

$$U_\varepsilon(a, P) := \{ (b, Q); \quad b \in U_\varepsilon(a), \quad Q \text{ is the germ of } P \text{ in } b \}.$$

A subset  $U \subset \mathcal{R}$  is called open if for every point  $a \in U$  there exists a small  $\varepsilon > 0$  such that

$$U_\varepsilon(a, P) \subset U.$$

It is clear that this is a (Hausdorff) topology on  $\mathcal{R}$ . The natural projection

$$p : \mathcal{R} \longrightarrow \mathbb{C}, \quad (a, P) \longmapsto a,$$

is continuous and moreover the restriction

$$U_\varepsilon(a, P) \longrightarrow U_\varepsilon(a) \quad (\varepsilon \text{ small enough})$$

is topological. Hence we see that the map  $\mathcal{R} \rightarrow \mathbb{C}$  is locally topological. This is enough information to equip  $\mathcal{R}$  with a structure as Riemann surface.

**6.1 Remark.** *Let  $Y$  be a Riemann surface and  $f : X \rightarrow Y$  a locally topological map of a Hausdorff space  $X$  into  $Y$ . Then  $X$  carries a unique structure as Riemann surface such that  $f$  is holomorphic.*

*Proof.* One defines  $\mathcal{O}_X(U)$  to be the set of all function  $f : U \rightarrow \mathbb{C}$  such that for all  $U_\varepsilon(a, P) \subset U$  the composition

$$U_\varepsilon(a) \xrightarrow{\sim} U_\varepsilon(a, P) \xrightarrow{f} \mathbb{C}$$

is analytic in the usual sense. It is easy to verify the demanded properties.  $\square$

Besides the projection  $\mathcal{R} \rightarrow \mathbb{C}$  one can consider the map

$$F : \mathcal{R} \longrightarrow \mathbb{C}, \quad F(a, P) = P(a).$$

This function is of course analytic. There is a natural map

$$D \longrightarrow \mathcal{R}, \quad a \longmapsto (a, f_a),$$

which obviously is an open imbedding. The basic fact is the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad} & \mathcal{R} \\ & \searrow f & \swarrow F \\ & \mathbb{C} & \end{array}$$

shows  $F$  to be an extension of the original function  $f$ . This extension includes all possible analytic continuations of  $f$ . In a naive sense they are a multi-valued function. The idea of the Riemann surface  $\mathcal{R}$  is to obtain a single-valued function on a covering of  $\mathbb{C}$ .

### The Riemann surface of an algebraic function

We show how a compact Riemann surface can be associated to an algebraic function. We will keep rather short. More details can be found in [Fr1], Chapt. I, Sect. 3.

Let  $P(z, w)$  by a polynomial of two variables. We assume in the following that  $P$  is irreducible. This means that  $P$  cannot be written as product of two non-constant polynomials. We also assume that  $P$  depends on both variables properly.

The “algebraic function”  $f$  related to  $P$  is the solution of  $P(z, f(z)) = 0$ . This is not a function in the usual sense, since it is multi-valued. The origin of the theory of Riemann surface can be seen in the wish to get a precise definition of  $f$  as a single valued function on a suitable surface. One could use the technique of concrete Riemann surfaces to construct such a surface. We prefer to use the algebraic curve related to  $P$ . In both approaches the main problem is to get a *compact* Riemann surface. One can show that every compact Riemann surface is biholomorphic equivalent to one which comes from an algebraic function. Hence compact Riemann surfaces and algebraic functions describe the same objects.

The affine curve associated to  $P$  is defined as

$$\mathcal{N} := \{ (z, w) \in \mathbb{C} \times \mathbb{C}; \quad P(w, z) = 0 \}.$$

**6.2 Lemma.** *The fibres of the map  $p : \mathcal{N} \rightarrow \mathbb{C}$ ,  $p(z, w) = z$ , are finite.*

This is an easy application of the fact, that a non-zero polynomial in one variable has only finitely many zeros.  $\square$

**6.3 Proposition.** *There are finite subsets  $S \subset \mathbb{C}$  and  $T = p^{-1}(S)$  such that*

$$p : \mathcal{N} - T \longrightarrow \mathbb{C} - S$$

*is proper and locally topological.*

We just describe the set  $S$ : A point  $a \in \mathbb{C}$  is contained in  $S$  if one of the following conditions is satisfied:

- a) There is a  $b$  such that  $P$  and  $\partial P/\partial w$  both vanish at  $(a, b)$ .
- b) Write  $P$  in the form

$$P(z, w) = a_0(z) + \cdots + a_n(z)w^n, \quad a_n \neq 0.$$

Then  $a_n(s) = 0$ .

We don't give more details and mention just that b) implies that  $p : \mathcal{N} - T \rightarrow \mathbb{C} - T$  is locally topological. (One has to use the theorem of implicit functions.) That  $p$  is proper uses a).  $\square$

Now we imbed  $\mathbb{C} - S$  into the Riemann sphere  $\bar{\mathbb{C}}$  and apply Proposition 5.6.

**6.4 Theorem.** *There exists a compact Riemann surface  $X$  which contains  $\mathcal{N} - T$  as open sub-surface (with the induced topology and the restricted geometric structure). The complement is a finite set. The map  $p : \mathcal{N} - T \rightarrow \mathbb{C} - T$  extends to a meromorphic function*

$$p : X \rightarrow \bar{\mathbb{C}}.$$

We also mention a result, which often is clear in concrete situations but needs a proof in general. We refer to [Fr1] for a proof.

**6.5 Theorem.** *The compact Riemann surface  $X$  attached to an algebraic function is connected. The second projection  $q(z, w) = w$  extends also to a meromorphic function*

$$q : X \rightarrow \bar{\mathbb{C}}.$$

We end this section with a special example in which the above results can be proved less more easily. Let  $Q(z)$  be a non-constant polynomial in 1 variable without multiple zeros and take  $P(z, w) = w^2 - Q(z)$ . It is easy to show that  $P$  and  $\partial P/\partial w$  have no common zeros. It follows that  $\mathcal{N}$  is already a surface (an imbedded manifold in the sense of analysis). It carries a structure as Riemann surface such that  $p : \mathcal{N} \rightarrow \mathbb{C}$  is proper and holomorphic. The covering degree is two. The compactification  $\mathcal{N} \subset X$  needs one or two additional points. If it is one, the map  $p$  is locally of the form  $z \mapsto z^2$ . If there are two, the map is locally biholomorphic at both. Later we will see that the compactification by one (resp. two) points depends on whether the degree of  $Q$  is odd (resp. even).

Recall that we have two projections  $p, q : X \rightarrow \bar{\mathbb{C}}$  which are induced by  $p(z, w) = z$  and  $q(z, w) = w$ . The first projection plays the role of the natural "coordinate" on  $X$ . Hence we write simply  $z$  instead of  $q$ . The second projection  $q$  describes the solution of the equation  $w^2 = Q(z)$ . Hence we write simply  $\sqrt{Q(z)}$  for  $q$ . We see that the a priori "double valued function  $\sqrt{Q(z)}$ " appears as usual single-valued function  $q$  on a two fold covering of  $\bar{\mathbb{C}}$ .

## 7. Differential forms

Let  $U \subset \mathbb{C}$  be an open subset. We consider the space  $A^p(U)$  ( $p = 0, 1, 2$ ) of complex valued  $C^\infty$ -differential forms,

$$A^0(U) = C^\infty(U), \quad A^1(U) = C^\infty(U) \times C^\infty(U), \quad A^2(U) = C^\infty(U).$$

As usually we write the elements of  $A^1(U)$  in the form  $fdx + gdy$  and the elements of  $A^2(U)$  in the form  $fdx \wedge dy$ . We recall the operators

$$\begin{aligned} A^1(X) \times A^1(X) &\longrightarrow A^2(X), \\ (f_1dx + g_1dy) \wedge (f_2dx + g_2dy) &= (f_1g_2 - f_2g_1)dx \wedge dy, \end{aligned}$$

(exterior product) and the exterior derivatives

$$d : A^0(U) \longrightarrow A^1(U), \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

and

$$d : A^1(U) \longrightarrow A^2(U), \quad d(fdg + gdy) = \left( \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) dx \wedge dy.$$

One has  $d \circ d = 0$ . From Lemma IV.4.2 we obtain the following result.

### 7.1 Lemma of Poincaré. *The sequence*

$$0 \longrightarrow \mathbb{C} \longrightarrow A^0(U) \longrightarrow A^1(U) \longrightarrow A^2(U) \longrightarrow 0$$

*is exact for convex  $U$ .*

We use the notations

$$dz = dx + idy, \quad d\bar{z} = dx - idy$$

and

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We can write a one-form also as  $fdz + gd\bar{z}$ . The operator  $d$  can be rewritten as

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \quad \text{and} \quad d(fdz + gd\bar{z}) = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial g}{\partial z} dz \wedge d\bar{z}$$

Notice

$$dz \wedge d\bar{z} = -d\bar{z} \wedge dz = 2idx \wedge dy.$$

We obtain a splitting

$$A^1(U) = A^{1,0}(U) \oplus A^{0,1}(U)$$

with  $A^{1,0}(U) = \mathcal{C}^\infty(U)dz$  and  $A^{0,1}(U) = \mathcal{C}^\infty(U)d\bar{z}$ . A one-differential is called holomorphic if it is of the form  $f dz$  with holomorphic  $f$ . From the Cauchy–Riemann equations follows

$$d(f(z)dz) = f'(z)dz \quad \text{for holomorphic } f.$$

The set of all holomorphic one-forms is denoted by  $\Omega(U)$ . It is a subspace of  $A^{1,0}(U)$ . We also introduce the operators

$$\bar{\partial} : A^0(U) \longrightarrow A^{0,1}(U), \quad \bar{\partial}(f) := \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

and

$$\bar{\partial} : A^{1,0}(U) \longrightarrow A^{1,1}(U), \quad \bar{\partial}(f dz) := \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$$

The latter coincides with  $d$ . We can reformulate the Lemma of Dolbeault IV.4.5 as follows

**7.2 Lemma of Dolbeault.** *For a disk  $U$  the sequence*

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow A^0(U) \xrightarrow{\bar{\partial}} A^{0,1}(U) \longrightarrow 0$$

*is exact or equivalently*

$$0 \longrightarrow \Omega(U) \longrightarrow A^{1,0}(U) \xrightarrow{\bar{\partial}} A^{1,1}(U) \longrightarrow 0$$

*is exact.*

### Transformation of differential forms

Let  $\gamma = \gamma_1 + i\gamma_2 : U \rightarrow V$  be a  $\mathcal{C}^\infty$ -map of open subsets  $U, V \subset \mathbb{C}$ . One defines the pull-back

$$\gamma^* : A^p(V) \longrightarrow A^p(U)$$

as follows:

- a) for 0-forms:  $\gamma^*(f) = f \circ \gamma$ ,
- b) for 1 forms:  $\gamma^*(f dx + g dy) = (f \circ \gamma) d\gamma_1 + (g \circ \gamma) d\gamma_2$ ,
- c) for 2-forms:  $\gamma^*(f dx \wedge dy) = (f \circ \gamma) \left( \frac{\partial \gamma_1}{\partial x} \frac{\partial \gamma_2}{\partial y} - \frac{\partial \gamma_1}{\partial y} \frac{\partial \gamma_2}{\partial x} \right) dx \wedge dy$ .



The chain rule says:  $\gamma_1^* \circ \gamma_2^* = (\gamma_2 \circ \gamma_1)^*$ . Another consequence of the chain rule is that the derivative  $d$  is compatible with transformation:

$$\gamma^*(d\omega) = d(\gamma^*\omega).$$

When  $w = \gamma(z)$  is holomorphic and  $f(w)dw$  a holomorphic 1-form, then the transformation formula simplifies because of the Cauchy–Riemann equations as follows:

$$\gamma^*(f(w)dw) = f(\gamma z)\gamma'(z)dz \quad (f \text{ holomorphic}).$$

Finally we mention

$$\gamma^*(\omega \wedge \omega') = (\gamma^*\omega) \wedge (\gamma^*\omega')$$

which is easy to verify.

### Differential forms on Riemann surfaces

By a  $p$ -form on a Riemann surface one understands a family of  $p$ -forms  $\omega_\varphi \in A^p(V_\varphi)$ , where  $\varphi : U_\varphi \rightarrow V_\varphi$  runs through all analytic charts, such that the following condition is satisfied:

If  $\varphi, \psi$  are two analytic charts and  $\gamma = \psi \circ \varphi^{-1}$  is the coordinate change map then

$$\omega_\psi = \gamma^*\omega_\varphi$$

holds on the open subsets where  $\gamma$  is defined. If one has a family, which is only defined for all  $\varphi$  form an atlas of analytic charts then this family extends uniquely to the atlas of all analytic charts. All what one has to use to us is the chain rule  $\gamma_1^* \circ \gamma_2^* = (\gamma_2 \circ \gamma_1)^*$  and the fact that transformation and derivatives are compatible with restriction to smaller open subsets.

A zero form especially is a family of functions  $f_\varphi : V_\varphi \rightarrow \mathbb{C}$  such that the transported functions in  $U_\varphi$  coincide in the intersections. Hence they glue to a function  $f : X \rightarrow \mathbb{C}$ . We will identify  $f$  with the family  $(f_\varphi)$ . We denote by  $A^p(X)$  the space of  $p$ -forms. We have  $A^0(X) = \mathcal{C}^\infty(X)$ .

A 1-form is called holomorphic if all its components  $\omega_\varphi$  are holomorphic. It is enough to demand this for an atlas of analytic charts, since the transform of a holomorphic 1-form by a holomorphic transformation is holomorphic again. (Here we make use of the fact that we have a Riemann surface and not only a differentiable surface in the sense of real analysis). We denote by  $\Omega(X)$  the space of holomorphic 1-forms.

Another obvious property of *holomorphic* transformations is that it preserves the type  $A^{0,1}$  and  $A^{0,1}$ . This also follows from the Cauchy Riemann differential equations. Hence we can define  $A^{0,1}(X)$  and  $A^{1,0}(X)$  for a Riemann

surface componentwise. The operations  $d$  and  $\partial$  commute with holomorphic transformations. Hence we can define operators

$$\begin{aligned} d : \mathcal{C}^\infty(X) &\longrightarrow A^1(X), & d : A^1(X) &\longrightarrow A^2(X), \\ \bar{\partial} : \mathcal{C}^\infty(X) &\longrightarrow A^{0,1}(X), & \bar{\partial} : A^{1,0}(X) &\longrightarrow A^2(X) \end{aligned}$$

also componentwise (for example  $d(\omega)_\varphi := d(\omega_\varphi)$ .)

Also the exterior product generalizes to Riemann surfaces via charts.

### Sheaves of differential forms

If one attaches to each open subset  $U$  of a Riemann surface the various spaces of differential forms one obtains sheaves which we denote by  $\mathcal{A}_X^p$ ,  $\mathcal{A}_X^{0,1}$ ,  $\mathcal{A}_X^{1,0}$ ,  $\Omega_X$ . We also obtain sequences of sheaves:

$$\begin{aligned} 0 &\longrightarrow \mathbb{C}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{A}_X^1 \longrightarrow \mathcal{A}_X^2 \longrightarrow 0, \\ 0 &\longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}_X^\infty \longrightarrow \mathcal{A}_X^{0,1} \longrightarrow 0, \\ 0 &\longrightarrow \Omega_X \xrightarrow{\text{inclusion}} \mathcal{A}_X^{1,0} \longrightarrow \mathcal{A}_X^2 \longrightarrow 0. \end{aligned}$$

The Lemma of Poincaré 7.1 and the Lemma of Dolbeault 7.2 imply that these sequences of sheaves are exact. Hence we obtain the following two theorems:

**7.3 Theorem of de Rham.** *Let  $X$  be a connected Riemann surface. Then*

$$\begin{aligned} H^0(X, \mathbb{C}) &= \mathbb{C}, & H^1(X, \mathbb{C}) &= \frac{\text{Kernel}(A^1(X) \longrightarrow A^2(X))}{\text{Image}(\mathcal{C}^\infty(X) \longrightarrow A^1(X))} \\ H^2(X, \mathbb{C}) &= \frac{A^2(X)}{\text{Image}(A^1(X) \longrightarrow A^2(X))}, & H^q(X, \mathbb{C}) &= 0 \text{ for } q > 2 \end{aligned}$$

**7.4 Theorem of Dolbeault.** *Let  $X$  be a connected Riemann surface. Then*

$$H^1(X, \mathcal{O}_X) = \frac{A^{0,1}(X)}{\text{Image}(\mathcal{C}^\infty(X) \longrightarrow A^{0,1}(X))}, \quad H^q(X, \mathcal{O}_X) = 0 \text{ for } q > 1.$$

and

$$H^1(X, \Omega_X) = \frac{A^2(X)}{\text{Image}(A^{0,1}(X) \longrightarrow A^2(X))}, \quad H^q(X, \Omega_X) = 0 \text{ for } q > 1.$$

Who is familiar with alternating differential forms in arbitrary dimensions will know the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow A_X^0 \longrightarrow A_X^1 \longrightarrow \cdots \longrightarrow A_X^n \longrightarrow 0.$$

Here  $X$  is a differentiable manifold of dimension  $n$  and  $A_X^i$  denotes the sheaf of alternating differential forms of degree  $i$ . The general Lemma of Poincaré states that one gets an exact sequence if one takes global sections on an open subset which is diffeomorphic to an open convex domain in  $\mathbb{R}^n$ . Hence this sequence is an acyclic resolution of  $\mathbb{R}_X$  for arbitrary  $X$ . We obtain

**Theorem of de Rham.** *For a differentiable manifold  $X$  one has*

$$\dim H^i(X, \mathbb{C}_X) \cong \frac{\text{Kern}(A^i(X) \longrightarrow A^{i+1}(X))}{\text{Image}((A^{i-1}(X) \longrightarrow A^i(X)))}.$$

*TdR*

### The Stokes formula

We explain shortly the integration of differential forms. If  $\omega = f_1 dx + f_2 dy$  is a 1-form on an open set  $V \subset \mathbb{C}$  and  $\alpha : [0, 1] \rightarrow U$  a smooth (=infinitely differentiable) curve,  $\alpha = \alpha_1 + i\alpha_2$ , then one defines

$$\int_{\alpha} \omega := \int_0^1 (f_1(\alpha(t))\alpha_1'(t) + f_2(\alpha(t))\alpha_2'(t)) dt.$$

When  $\gamma : U \rightarrow V$  is a diffeomorphism then

$$\int_{\alpha} \omega = \int_{\alpha \circ \gamma} \gamma^* \omega.$$

This allows us to generalize this to Riemann surfaces: Let  $\omega \in A^1(X)$  and  $\alpha : [0, 1] \rightarrow X$  a smooth curve. (It is clear how to define smoothness for  $\alpha$ . One uses charts.) Then one can define  $\int_{\alpha} \omega$ . One divides  $\alpha$  in pieces which lie in analytic charts. Then one uses the local formula. Because of the transformation invariance this is independent of the choice of the charts.

Similarly one defines the integral of a two-form  $\omega$ . In the local case of an open subset  $V \subset \mathbb{C}$  one takes for

$$\int_V f(z) dx \wedge dy$$

the usual 2-dimensional integral of  $f$  (if it exists). Again we need transformation invariance. If  $\gamma : U \rightarrow V$  is a diffeomorphism then

$$\int_{\alpha} \omega = \int_{\alpha \circ \gamma} \gamma^* \omega$$

only holds if the (real) Jacobi determinant of  $\gamma$  is positive. When  $\gamma$  is biholomorphic, this is automatically the case, since the real Jacobi determinant of  $\gamma$  is  $|\gamma'|^2$ . This is an important fact because otherwise we could not integrate 2-forms on Riemann surfaces. We want to keep it in mind:

**7.5 Proposition.** *Riemann surfaces are **oriented** in the following sense: The chart change map of two analytic charts has positive real Jacobi determinant.*

If  $\omega$  is a 2-form on  $X$  which vanishes outside a chart  $\varphi$  then we can define

$$\int_X \omega := \int_{V_\varphi} \omega_\varphi.$$

The general case is reduced to this one by means of a partition of unity. Now we can formulate the formula of Stokes:

**7.6 Theorem of Stokes.** *Let  $U \subset X$  an open subset of a Riemann surface with compact closure. Assume that the boundary of  $U$  is the union of a finite number of (the images of) closed double point free regular curves  $\alpha_i : [0, 1] \rightarrow X$ ,  $1 \leq i \leq n$ . Assume furthermore that  $U$  is on the left of these curves. Then for every one-form  $\omega$  on  $X$*

$$\int_U d\omega = \sum_{i=1}^n \int_{\alpha_i} \omega.$$

We recall that a closed curve  $\alpha$  is called double point free if  $\alpha(t) = \alpha(t')$  if only for  $t, t' \in \{0, 1\}$ . Regular means  $\alpha'(t) \neq 0$  for all  $t$  in case of a plain curve  $\alpha : [0, 1] \rightarrow \mathbb{C}$ . The general case is reduced to that one by means of charts. An open subset  $U \subset \mathbb{C}$  is called to the left of a regular curve  $\alpha$  if for every  $t$  the following condition is satisfied. Consider the two vectors of norm 1 which are orthogonal to the tangent vector  $\alpha'(t)$ . Call  $\mathbf{n}^+$  the one which goes to the right (this makes sense in the plane) and  $\mathbf{n}^-$  the other one. Then there exists  $\varepsilon > 0$  such that for all  $0 \leq s \leq \varepsilon$

$$\alpha(t) + s\mathbf{n}^- \in U, \quad \alpha(t) + s\mathbf{n}^+ \notin U.$$

This can be generalized to open subsets  $U$  in a Riemann surface by means of analytic charts. One has to use 7.5.

### The residue theorem on compact Riemann surfaces

Recall that a holomorphic differential (=1-form) on a Riemann surface is a collection  $f_\varphi(z)dz$ , where  $f_\varphi$  is a holomorphic function  $f_\varphi : V_\varphi \rightarrow \mathbb{C}$  for every analytic chart (an atlas of analytic charts is enough) such that for two charts  $\varphi, \psi$  with chart change map  $\gamma$  the transformation formula

$$f_\psi(\gamma z) = \gamma'(z)f_\varphi(z)$$

holds. Instead of holomorphic functions one can take meromorphic functions. This leads to the notion of a *meromorphic differential* on a Riemann surface.

We recall the following residue formula from complex calculus: Let  $\gamma : U \rightarrow V$  be a biholomorphic map of open subsets  $U, V \subset \mathbb{C}$ . Let  $f : V \rightarrow \mathbb{C}$  be a holomorphic function and  $a \in U$ . Then

$$\operatorname{Res}(f(\omega), \gamma(a)) = \operatorname{Res}(\gamma'(z)f(\gamma z), a).$$

It is an important fact that the factor  $\gamma'(z)$  occurs. This means that it makes no sense to talk about residues of *meromorphic functions* on Riemann surfaces. But for a *meromorphic differential*  $\omega$  the definition

$$\operatorname{Res}(\omega, a) := \operatorname{Res}(\omega_\varphi, \varphi(a)) \quad (a \in U_\varphi)$$

makes sense since it is independent of the choice of a chart.

**7.7 Residue theorem.** *Let  $\omega$  be a meromorphic differential on a compact Riemann surface. Then*

$$\sum_{a \in X} \operatorname{Res}(\omega; a) = 0.$$

The sum is of course a finite sum. For the proof we take for each pole  $a$  a small neighborhood  $U(a)$  such that the closures of two different are disjoint and such that the boundaries are nice (take disks with respect to charts). Then we apply the Stokes formula to the set

$$U = X - \bigcup_a \overline{U(a)}.$$

Since  $\omega$  is holomorphic we have  $d\omega = 0$ . It follows

$$\sum_a \int_{\partial U(a)} \omega = 0.$$

This proves 7.7. □

If  $f$  is a non-vanishing meromorphic function on a connected Riemann surface, then we can consider the meromorphic differential  $df/f$ . From the formula

$$\operatorname{Res}\left(\frac{df}{f}; a\right) = \operatorname{Ord}(f, a)$$

follows:

**7.8 Remark.** *Let  $f$  be a non-constant meromorphic function on a connected compact Riemann surface. The number of poles and zeros – counted with multiplicity – is the same.*

This remark can also be proved without residue theorem. In Sect. 5 (Proposition 5.5.8) we have seen that this result has a merely topological background.

**7.9 Remark.** *Every holomorphic function on a connected compact Riemann surface is constant.*

This follows from Remark 7.8, but it follows also already from the maximum principle of complex calculus.

# Chapter VI. The Riemann–Roch theorem

## 1. Generalities about vector bundles

In the following we assume that the considered Riemann surfaces are connected.

**1.1 Definition.** A *vector bundle* on a Riemann surface  $X$  is a locally free  $\mathcal{O}_X$ -module, i.e. a  $\mathcal{O}_X$  module such that every point admits an open neighborhood such that  $\mathcal{M}|_U$  is isomorphic to  $(\mathcal{O}_X|_U)^n$  for some integer  $n \geq 0$ . If  $n$  can be taken to be 1, then  $\mathcal{M}$  is called a *line bundle*.

When  $n$  is the same for all  $U$ , for example when  $X$  is connected, we call  $n$  then the rank of  $\mathcal{M}$ . Hence line bundles are vector bundles of rank 1.

We give a basic example for a line bundle. A *divisor* on a Riemann surface is a map  $D : X \rightarrow \mathbb{Z}$  such that  $D(a) = 0$  outside a discrete set. We are mainly interested in compact surfaces. Then this means that  $D(a) = 0$  outside a finite set. We write  $D$  as formal linear combination

$$D = \sum_{a \in X} D(a)a.$$

The set of all divisors is an additive group  $\text{Div}(X)$  (componentwise addition). To every meromorphic function which is not zero on a component, we associate the so-called *principal divisor*  $(f)$ .

$$(f)(a) = \text{Ord}(f, a).$$

From the formula  $(fg) = (f) + (g)$  follows that the set of all principal divisors is a subgroup  $H(X) \subset \text{Div}(X)$ . The elements of the factor group  $\text{Div}(X)/H(X)$  are called *divisor classes*. Let  $f$  be a meromorphic function on some open subset  $U$ . The notation

$$(f)(a) \geq n \quad (a \in X, n \in \mathbb{Z})$$

means that  $\text{Ord}(f, a) \geq n$  if  $f$  doesn't vanish in a neighborhood of  $a$ . Hence  $(f)(a) \geq n$  is always true if  $f = 0$  in a neighborhood of  $a$ . Let  $D$  be a divisor on  $X$ . We write  $(f) \geq D$  if  $(f)(a) \geq D(a)$  for all  $a$ . Now we associate to a divisor  $D$  on  $X$  the following sheaf.

$$\mathcal{O}_D(U) := \{ f : U \rightarrow \bar{\mathbb{C}} \text{ meromorphic, } (f) \geq -D \}.$$

Clearly  $\mathcal{O}_D$  is an  $\mathcal{O}_X$ -module. We claim that it is a line bundle: let  $U$  be a sufficiently small neighborhood of a given point. Then there exists a meromorphic function  $g$  whose divisor on  $U$  coincides with  $-D$ . One simply has to take  $U$  so small that it is in the domain of definition of a chart and that at most one  $a \in U$  with  $D(a) \neq 0$  exists. The map

$$\mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{O}_D(U), \quad f \mapsto gf,$$

then is an isomorphism. Since  $U$  can be replaced by smaller open subsets, we obtain  $\mathcal{O}_D|U \cong \mathcal{O}_X|U$ .

Two line bundles are called isomorphic if they are isomorphic as  $\mathcal{O}_X$ -module.

**1.2 Proposition.** *Let  $X$  be a Riemann surface. Two divisors  $D$  and  $D'$  are in the same divisor class, if and only if the line bundles  $\mathcal{O}_D$  and  $\mathcal{O}_{D'}$  are isomorphic.*

*Proof.* We use the following trivial algebraic fact. Every homomorphism  $f : M \rightarrow N$  between  $R$ -modules which are isomorphic to  $R$  is given by multiplication with some element from  $R$ . From this follows easily: let  $\mathcal{O}_D \rightarrow \mathcal{O}_{D'}$  be an isomorphism of line bundles. Then each point admits an open neighbourhood  $U$  and a meromorphic function  $f_U : U \rightarrow \mathbb{C}$ , such that  $\mathcal{O}_D|U \rightarrow \mathcal{O}_{D'}|U$  is given by multiplication with  $f_U$ . The functions  $f_U$  glue to a meromorphic function  $f$  on  $X$ . We have  $D' = D + (f)$ .  $\square$

If  $D$  is a divisor on a compact Riemann surface, then the degree

$$\deg(D) = \sum_{a \in X} D(a)$$

is defined. As we know, the degree of a principal divisor is zero. Hence  $\deg D$  only depends on the divisor class. This enables us to define:

**1.3 Remark.** *Let  $\mathcal{L}$  be a line bundle on a compact Riemann surface. Assume that there exists a divisor  $D$  such that  $\mathcal{L} \cong \mathcal{O}_D$ . The **degree***

$$\deg \mathcal{L} := \deg D$$

*does not depend on the choice of  $D$ .*

Let  $\mathcal{M}, \mathcal{N}$  be two  $\mathcal{O}_X$ -modules. We denote by  $\text{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$  the set of all  $\mathcal{O}_X$ -linear maps  $\mathcal{M} \rightarrow \mathcal{N}$ . This is an  $\mathcal{O}_X(X)$ -module. More generally, we can consider for every open  $U \subset X$

$$U \mapsto \text{Hom}_{\mathcal{O}_X|U}(\mathcal{M}|U, \mathcal{N}|U).$$

It is clear that this is presheaf. It is easy to check that it is actually a sheaf and moreover an  $\mathcal{O}_X$ -module. We denote it by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}).$$

We denote by the  $\mathcal{O}_X(U)^{p \times q}$  the set of all  $p \times q$ -matrices with entries from  $\mathcal{O}_X(U)$ . This is a free  $\mathcal{O}_X$ -module. There is an obvious natural isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p, \mathcal{O}_X^q) \cong \mathcal{O}_X^{p \times q}.$$

Hence  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$  is a vector bundle if  $\mathcal{M}$  and  $\mathcal{N}$  is a vector bundle. It is a line bundle if both are line bundles. The dual bundle of a vector bundle  $\mathcal{M}$  is

$$\mathcal{M}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X).$$

It has the same rank as  $\mathcal{M}$ .

There is another construction which rests on the tensor-product of modules. Let  $\mathcal{M}, \mathcal{N}$  be two  $\mathcal{O}_X$ -modules. The assignment

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

defines a presheaf. This is usually not a sheaf. Hence we consider the generated sheaf and call it by  $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ . Clearly this is an  $\mathcal{O}_X$ -module. The notion of an  $\mathcal{O}_X$ -bilinear map  $\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$  for  $\mathcal{O}_X$ -modules  $\mathcal{M}, \mathcal{N}, \mathcal{P}$  and the following universal property should be clear. For an  $\mathcal{O}_X$ -bilinear map  $\mathcal{M} \times \mathcal{N} \rightarrow \mathcal{P}$  of  $\mathcal{O}_X$ -modules there exist a unique commutative diagram

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} & \longrightarrow & \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \\ & \searrow & \swarrow \\ & \mathcal{P} & \end{array}$$

with an  $\mathcal{O}_X$ -linear map  $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \rightarrow \mathcal{P}$ .

One also has  $\mathcal{O}_X^n \otimes_{\mathcal{O}_X} \mathcal{O}_X^m \cong \mathcal{O}_X^{n \times m}$ . Since the construction of the tensor product is compatible with the restriction to open subsets,

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})|_U \cong \mathcal{M}|_U \otimes_{\mathcal{O}_X|_U} \mathcal{N}|_U,$$

we obtain that the tensor product of two vector bundles is a vector bundle and moreover, the tensor product of two line bundles is a line bundle.

When  $M$  is an  $R$ -module and  $M^* = \text{Hom}_R(M, R)$  its dual, one has a natural bilinear map  $M \times M^* \rightarrow R$ ,  $(a, l) \mapsto l(a)$ . This induces a linear map  $M \otimes_R M^* \rightarrow R$ . Sheafifying we get an  $\mathcal{O}_X$ -bilinear map

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M}^* \rightarrow \mathcal{O}_X.$$

Clearly this is an isomorphism when  $\mathcal{M}$  is a line-bundle.

We denote by  $[\mathcal{L}]$  the class of all line bundles which are isomorphic to  $\mathcal{L}$ . We denote by  $\text{Pic}(X)$  (Picard group) the set of all isomorphy classes of line-bundles.



Since constructions as the “set of sets” are forbidden in set theory, there seems to be a logical difficulty. But this is not really there because one can easily prove that there exists a *set* of line-bundles such that every line bundle is isomorphic to one of this set.

We define the tensor product in  $\text{Pic}(X)$  by

$$[\mathcal{L}] \otimes [\mathcal{L}'] := [\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'].$$

This is clearly well-defined. We notice that this product is associative and commutative.

The element  $[\mathcal{O}_X]$  is neutral element in  $\text{Pic}(X)$ . Finally we notice that  $[\mathcal{L}] \otimes [\mathcal{L}^*] = [\mathcal{O}_X]$ . Hence every element of  $\text{Pic}(X)$  has an inverse.

**1.4 Remark.** *The set  $\text{Pic}(X)$  of isomorphism classes of line bundles is an abelian group under the tensor product.*

This group is called the Picard group.

## 2. The finiteness theorem

We introduce the notion of a coherent sheaf.

**2.1 Definition.** *An  $\mathcal{O}_X$ -module  $\mathcal{M}$  on a Riemann surface  $X$  is called a skyscraper sheaf if  $\mathcal{M}_a$  is zero for all  $a$  outside a discrete set and if  $\mathcal{M}_a$  is a finite dimensional complex vector space for all  $a$ .*

The skyscraper sheaf is determined by the stalks:

**2.2 Remark.** *For a skyscraper sheaf the natural map*

$$\mathcal{M}(X) \xrightarrow{\sim} \prod_{a \in X} \mathcal{M}_a$$

*is an isomorphism.*

*Proof.* The map is injective because  $\mathcal{M}$  is a sheaf. To prove the surjectivity, we consider the discrete set  $S$  of all  $a$  with  $\mathcal{M}_a \neq 0$ . Let  $(s_a) \in \prod \mathcal{M}_a$ . We choose for each  $a \in X$  an open neighborhood  $U(a)$  such that  $U(a) \cap S$  is  $\{a\}$  or empty and such that  $s_a$  can be represented by a section  $s_{U(a)} \in \mathcal{M}(U(a))$ . These sections and the zero section on  $X - S$  glue to a global section.  $\square$

The notion of a “coherent sheaf” includes skyscraper sheafs and vector bundles.

**2.3 Definition.** A *coherent sheaf* on a Riemann surface is an  $\mathcal{O}_X$ -module such that there exists an exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow 0$$

with a skyscraper sheaf  $\mathcal{W}$  and a vector bundle  $\mathcal{N}$ .

An element  $r$  of a ring  $R$  is called a *zero divisor* if multiplication by  $r$  is not an injective map  $R \rightarrow R$ . An element  $a$  of an  $R$ -module  $M$  is called a *torsion element* if there exists a non-zero divisor  $r \in R$  such that  $ra = 0$ . The set of all torsion elements is the torsion submodule  $M^{\text{tor}}$ .

**2.4 Lemma.** *Let*

$$0 \longrightarrow \mathcal{W}_a \longrightarrow \mathcal{M}_a \longrightarrow \mathcal{N}_a \longrightarrow 0$$

be an exact sequence as in Definition 2.3. Then  $\mathcal{W}_a$  is the torsion sub-module of  $\mathcal{M}_a$ .

*Proof.* It is helpful to make use of some very simple structure theory of algebra. Recall that an  $R$ -sub-module  $\mathfrak{a}$  of a ring  $R$  is called an *ideal* in  $R$ . If  $a$  is an element of  $R$  then  $Ra$  is an ideal and such an ideal is called a *principal ideal*. A ring is called *principal ideal ring* if every ideal is a principal ideal.

We are interested in the ring  $\mathbb{C}\{z\}$  of all convergent power series in one variable. If  $a$  is a point of a Riemann surface, then  $\mathcal{O}_{X,a}$  is isomorphic to  $\mathbb{C}\{z\}$ . From the point of view of algebra, the ring  $\mathbb{C}\{z\}$  is an extremely simple ring. Every ideal is a principal ideal which is generated by  $z^n$  for a suitable integer  $n \geq 0$ . A known result of algebra states that every finitely generated module  $M$  over a principal ideal ring  $R$  is isomorphic to the finite direct product of modules of the type  $R/\mathfrak{a}$ .

Since  $\mathcal{W}_a$  is a finite dimensional vector space, the structure theorem shows that all its elements are torsion elements. Hence it is contained in  $\mathcal{M}_a^{\text{tor}}$ . Conversely every torsion element of  $\mathcal{M}_a$  maps to 0, since  $\mathcal{N}_a$  is torsion free. This proves Lemma 2.4.  $\square$

Lemma 2.4 implies that “coherence” is a local property:

*Let  $(U_i)$  be an open covering. An  $\mathcal{O}_X$  module  $\mathcal{M}$  is coherent if and only if all  $\mathcal{M}|_{U_i}$  are coherent.*

**2.5 Theorem.** *Let  $\mathcal{M}$  be a coherent sheaf on a compact Riemann surface. Then:*

- a)  $H^n(X, \mathcal{M})$  is finite dimensional for all  $n$ .
- b)  $H^n(X, \mathcal{M}) = 0$  for  $n > 1$ .

*Proof.* Skyscraper sheaves are flabby. The statement is clear in this case. Hence we can restrict to vector bundles:

First we show the vanishing statement b). We will use the Dolbeault complex  $0 \rightarrow \mathcal{O}_X \rightarrow A_X^0 \rightarrow A_X^{0,1} \rightarrow 0$ . We know that this is an exact sequence (Lemma of Dolbeault V.7.2). The essential point is that  $\bar{\partial} : A_X^0 \rightarrow A_X^{0,1}$  is  $\mathcal{O}_X$ -linear. This comes from the fact that  $\bar{\partial}f = 0$  for holomorphic functions. Hence we can consider the Dolbeault complex as sequence of  $\mathcal{O}_X$ -modules and we can tensor it with  $\mathcal{M}$ . The sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow A_X^0 \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow A_X^{0,1} \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow 0$$

remains of course exact. Since  $A_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}$  carries a natural structure as  $\mathcal{C}_X^{\infty}$ -module we have an acyclic resolution of  $\mathcal{M}$  (Proposition V.3.2). The long exact cohomology sequence gives  $H^i(X, \mathcal{M}) = 0$  for  $i > 1$ .

Next we prove the finiteness of  $H^0(X, \mathcal{M})$ . We choose a finite open covering  $\mathfrak{U} = (U_i)$  such that each  $U_i$  has compact closure and that it is biholomorphic equivalent to a disk in the plane. This is possible since  $X$  is compact. We can take the  $U_i$  so small that  $\mathcal{M}|_{U_i}$  is free. We choose a trivialization  $\mathcal{M}|_{U_i} \cong \mathcal{O}_{U_i}^n$ . We choose open subsets  $V_i \subset U_i$  whose closure (taken in  $X$ ) is contained in  $U_i$  and such that  $\mathfrak{V} = (V_i)$  is still a covering of  $X$ . We can assume that all  $V_i$  are biholomorphic equivalent to disks. By means of the isomorphism  $\mathcal{M}(U_i) \cong \mathcal{O}(U_i)^n$  we can equip  $\mathcal{M}(U_i)$  with a structure as a Frèchet space. Here we use Remark I.7.4 and the trivial fact that a finite direct product of Frèchet spaces is Frèchet space (Remark I.7.2). In particular,  $\prod_i \mathcal{M}(U_i)$  also carries a structure as Frèchet space. We can consider  $\mathcal{M}(X)$  as a (closed) subspace of this space. A closed subspace of a Frèchet space is also a Frèchet space (Remark I.7.2). Hence  $\mathcal{M}(X)$  is a Frèchet space too. We can repeat this with the covering  $\mathfrak{V}$ . It is easy to see that both Frèchet structures on  $\mathcal{M}(X)$  are the same. From Montel's theorem (in the form of Proposition I.7.8) follows that the identity map  $\mathcal{M}(X) \rightarrow \mathcal{M}(X)$  is compact. From Corollary I.7.7 follows that  $\mathcal{M}(X)$  is finite dimensional.

Next we prove the finiteness of  $H^1(X, \mathcal{M})$ . From Leray's lemma 4.3.6 we know

$$H^1(X, \mathcal{M}) = \check{H}^1(\mathfrak{U}, \mathcal{M}) = \check{H}^1(\mathfrak{V}, \mathcal{M}).$$

We consider the space of Čech cocycles  $C^1(\mathfrak{U}, \mathcal{M})$ . This is a closed subspace of  $\prod_{i,j} \mathcal{M}(U_i \cap U_j)$  and hence a Frèchet space. The same is true for  $C^1(\mathfrak{V}, \mathcal{M})$ . The natural restriction operator

$$r : C^1(\mathfrak{U}, \mathcal{M}) \longrightarrow C^1(\mathfrak{V}, \mathcal{M})$$

is a compact operator by Montel's theorem (in the form of Proposition I.7.8). Now we consider the map

$$\begin{aligned} C^1(\mathfrak{U}, \mathcal{M}) \times \prod_i \mathcal{M}(V_i) &\longrightarrow C^1(\mathfrak{V}, \mathcal{M}), \\ (A, (s_i)) &\longmapsto r(A) + \delta((s_i)). \end{aligned}$$

This map is a continuous surjective linear map of Frèchet spaces. It differs from the map

$$C^1(\mathfrak{U}, \mathcal{M}) \times \prod_i \mathcal{M}(V_i) \longrightarrow C^1(\mathfrak{V}, \mathcal{M}),$$

$$(A, (s_i)) \longmapsto \delta((s_i)).$$

only by a compact operator (essentially  $r$ ). By the Schwartz theorem I.7.6 this map has a finite dimensional cokernel. But the cokernel is  $\check{H}^1(\mathfrak{V}, \mathcal{M})$ .  $\square$

### 3. The Picard group

In section 1 we introduced the Picard group  $\text{Pic}(X)$ , i.e. the group of isomorphism classes of line bundles. We want to compare it with the group  $\text{Div}(X)/H(X)$  of divisor classes. Recall that we associated to a divisor  $D$  on a Riemann surface  $X$  a line bundle  $\mathcal{O}_D$ . For two divisors  $D, D'$  there is a natural  $\mathcal{O}_X$ -bilinear multiplication map

$$\mathcal{O}_D \times \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{D+D'}.$$

It is clear that the induced homomorphism

$$\mathcal{O}_D \otimes_{\mathcal{O}_X} \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{D+D'}$$

is an isomorphism. Hence we get a *homomorphism*  $\text{Div}(X) \rightarrow \text{Pic}(X)$ . Since principal divisors are in the kernel (Proposition 1.2) we get a homomorphism of the group of divisor classes into  $\text{Pic}(X)$ . This homomorphism is injective (1.2). We claim more:

**3.1 Proposition.** *Let  $X$  be a compact Riemann surface. The natural homomorphism*

$$\text{Div}(X)/H(X) \xrightarrow{\sim} \text{Pic}(X)$$

*is an isomorphism.*

The proof is non-trivial and will use the finiteness theorem. We will use twists of sheaves. The twist of a coherent sheaf  $\mathcal{M}$  with a line bundle  $\mathcal{L}$  simply is  $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{L}$ . We will use a very special twist. For the rest of the section we fix a point  $a \in X$ . For an integer  $n$  we consider the divisor  $(a)$

$$(a)(x) = \begin{cases} 1 & \text{for } x = a, \\ 0 & \text{elsewhere.} \end{cases}$$

We denote by  $\mathcal{O}(n)$  the associated line bundle and by  $\mathcal{M}(n)$  the twist of  $\mathcal{M}$  with  $\mathcal{O}(n)$ . (This construction depends on the choice of  $a$ .)

Next we use the natural inclusion  $\mathcal{O}(n) \hookrightarrow \mathcal{O}(n+1)$ . It induces  $\mathcal{M}(n) \hookrightarrow \mathcal{M}(n+1)$ . We assume now for simplicity that  $\mathcal{M}$  is a vector bundle. Then this map is injective and we have an exact sequence

$$0 \longrightarrow \mathcal{M}(n) \longrightarrow \mathcal{M}(n+1) \longrightarrow \mathcal{W}(n) \longrightarrow 0$$

with a skyscraper sheaf  $\mathcal{W}(n)$ . From the exact cohomology sequence we get that  $H^1(X, \mathcal{M}(n)) \rightarrow H^1(X, \mathcal{M}(n+1))$  is surjective. By the finiteness theorem these are finite dimensional vector spaces. The dimension  $d_n$  of  $H^1(X, \mathcal{M}(n))$  gives a decreasing sequence of non-negative integers. This must be constant for big  $n$ . Hence we get:

**3.2 Lemma.** *Let  $\mathcal{M}$  be a vector bundle on a compact Riemann surface  $X$ . For sufficiently large  $n$  the homomorphism*

$$H^1(X, \mathcal{M}(n)) \rightarrow H^1(X, \mathcal{M}(n+1))$$

*is an isomorphism*

Now we make use of the exact sequence

$$H^0(X, \mathcal{M}(n+1)) \longrightarrow H^0(X, \mathcal{W}(n)) \longrightarrow H^1(X, \mathcal{M}(n)) \longrightarrow H^1(X, \mathcal{M}(n+1)).$$

For large  $n$  the last arrow is an isomorphism, hence  $H^0(X, \mathcal{M}(n+1)) \longrightarrow H^0(X, \mathcal{W}(n))$  is surjective. Since  $H^0(X, \mathcal{W}(n))$  is not zero we get:

**3.3 Proposition.** *Let  $\mathcal{M}$  be a vector bundle of positive rank on a compact Riemann surface. Then  $\mathcal{M}(n)$  admits for large enough  $n$  a non-zero global section.*

Taking products, one gets a bilinear map  $\mathcal{O}(-n) \times \mathcal{O}(n) \rightarrow \mathcal{O}_X$ . More generally one gets for a vector bundle  $\mathcal{M}$  a natural map

$$\mathcal{M} \times \mathcal{O}(-n) \times \mathcal{M}^* \times \mathcal{O}(n) \longrightarrow \mathcal{O}_X.$$

This induces an  $\mathcal{O}_X$ -linear map

$$\mathcal{M}(-n)^* \xrightarrow{\sim} \mathcal{M}^*(n).$$

A local computation shows that this is an isomorphism. For suitable (large enough)  $n$  one has a non-zero global section  $S \in H^0(X, \mathcal{M}(n))$ . We can use this section to define a non-zero  $\mathcal{O}_X$ -linear map

$$\mathcal{M}(-n) \longrightarrow \mathcal{O}_X, \quad (a \mapsto S(a)).$$

Next we need a lemma.

**3.4 Lemma.** *Let  $\mathcal{M}$  be a vector bundle on a Riemann surface and  $\mathcal{M} \rightarrow \mathcal{O}_X$  a non-zero  $\mathcal{O}_X$ -linear map. Then the image is line bundle of the form  $\mathcal{O}_{-D}$  with  $D \geq 0$ .*

*Proof.* We can replace  $X$  by a small open neighborhood of a given point  $a$ . Hence we may assume that  $X = U$  is an open neighborhood of  $a = 0$  in  $\mathbb{C}$ . Taking  $U$  small enough, we can assume  $\mathcal{M} = \mathcal{O}_U^d$ . The map  $\mathcal{M} \rightarrow \mathcal{O}_U$  is given a system of holomorphic functions  $f_1, \dots, f_d \in \mathcal{O}(U)$ . We can assume that non of the  $f_i$  is zero and then that  $f_i = z^{n_i} g_i$  where  $g_i$  is without zero. Modifying the map  $\mathcal{M} \rightarrow \mathcal{O}_U$  we reduce to the case  $g_i = 1$ . Now we see that the image is  $z^n \mathcal{O}_U$ , where  $n$  is the minimum of the  $n_1, \dots, n_d$ . This proves 3.4.  $\square$

The same type of argument gives also information about the kernel. We can assume  $n_1 \leq \dots \leq n_d$ . But then the first component of a section of the kernel can be computed from the rest. Hence we get that the kernel again is free. This shows:

**3.5 Coherence lemma.** *Let  $\mathcal{M} \rightarrow \mathcal{L}$  be an  $\mathcal{O}_X$ -linear map of a vector bundle into a line bundle (on a Riemann surface). Then the kernel is a vector bundle too.*

We go back to the map  $\mathcal{M}(n) \rightarrow \mathcal{O}_X$ . We know that the image is a line bundle. Tensoring with  $\mathcal{O}(-n)$  we obtain:

**3.6 Lemma.** *Every vector bundle  $\mathcal{M}$  of rank  $n > 0$  sits in an exact sequence  $0 \rightarrow \mathcal{N} \rightarrow \mathcal{M} \rightarrow \mathcal{L} \rightarrow 0$ , where  $\mathcal{L}$  is a line bundle and  $\mathcal{N}$  a vector bundle of rank  $n - 1$ .*

The proof shows a little more, namely that this line bundle can be taken in the form  $\mathcal{O}_D$  for some divisor. Since a vector bundle of rank 0 is zero, we obtain the proof of Proposition 3.1.  $\square$

When we apply Proposition 3.3 to  $\mathcal{M} = \mathcal{O}_X$  we get:

**3.7 Theorem.** *Every compact Riemann surface admits a non-constant meromorphic function*

## 4. Riemann–Roch

For a vector bundle on a Riemann surface we have defined the rank  $n = \text{rank}(\mathcal{M})$ . This can be generalized to a coherent sheaf  $\mathcal{M}$ : There exists a discrete subset  $S \subset X$  such that  $\mathcal{M}|(X - S)$  is a vector bundle. We define  $\text{rank}(\mathcal{M}) := \text{rank}(\mathcal{M}|(X - S))$ . The rank of a skyscraper sheaf is 0.

**4.1 Remark.** *For a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

*one has*

$$\text{rank}(\mathcal{M}_2) = \text{rank}(\mathcal{M}_1) + \text{rank}(\mathcal{M}_3).$$

*Proof.* One has to use the following. Assume that  $R$  is an integral domain and

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow R^p \longrightarrow 0$$

an exact sequence of  $R$ -modules. Then  $n = m + p$ . This is well-known from linear algebra when  $R$  is a field and can be reduced to this case by imbedding  $R$  into a field.  $\square$

There are other quantities, which have the additive property as in 4.1. For a coherent sheaf we define

$$\chi(\mathcal{M}) = \sum_i (-1)^i \dim H^i(X, \mathcal{M}) = \dim H^0(X, \mathcal{M}) - \dim H^1(X, \mathcal{M}).$$

in analogy to 4.1 we have:

**4.2 Remark.** *For a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

*one has*

$$\chi(\mathcal{M}_2) = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_3).$$

*Proof.* One has to use the long exact cohomology sequence and the following simple result of linear algebra: Let

$$0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

an exact sequence of finite-dimensional vector spaces, then

$$\sum_{i=1}^n (-1)^i \dim V_i = 0. \quad \square$$

**4.3 Theorem.** *Let  $X$  be a compact Riemann surface. There exists a unique function which associates to an arbitrary coherent sheaf  $\mathcal{M}$  on  $X$  a non-negative integer  $\deg \mathcal{M}$  such that the following properties are satisfied:*

1)  $\deg \mathcal{M}$  depends only on the isomorphism class of  $\mathcal{M}$ .

2) For a skyscraper sheaf  $\mathcal{W}$

$$\deg(\mathcal{W}) = \sum_{a \in X} \dim \mathcal{W}_a.$$

3) If  $D$  is a divisor and  $\mathcal{O}_D$  the associated line-bundle then

$$\deg(\mathcal{O}_D) = \deg D.$$

4. For a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$$

one has

$$\deg(\mathcal{M}_2) = \deg(\mathcal{M}_1) + \deg(\mathcal{M}_3).$$

We will prove this together with the

**4.4 Riemann–Roch theorem.** *Let  $\mathcal{M}$  be a coherent sheaf on a compact Riemann surface  $X$ . Then*

$$\chi(\mathcal{M}) = \deg(\mathcal{M}) + \text{rank}(\mathcal{M})(1 - g).$$

**Corollary.**  $\dim H^0(X, \mathcal{M}) \geq \deg(\mathcal{M}) + \text{rank}(\mathcal{M})(1 - g).$

So far, the degree has been defined for skyscraper sheaves. The Riemann–Roch theorem is trivial in this case. Since the rank of a skyscraper sheaf is 0 it reduces to the equation  $\dim \mathcal{M}(X) = \deg \mathcal{M}$  which follows from 2.2.

The Riemann–Roch theorem is also trivial for  $\mathcal{O}_X$ , since

$$\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(\mathcal{O}_X) = 1 - g.$$

Next we prove the Riemann–Roch theorem for a divisor  $D$ . We add to the divisor  $D$  a point divisor  $(a)$ . (This divisor is 1 in  $a$  and 0 elsewhere). We have an exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D+(a)} \longrightarrow \mathcal{W} \longrightarrow 0$$

with a skyscraper sheaf  $\mathcal{W}$ . Since we have  $\deg(D + (a)) = \deg(D) + \deg(\mathcal{W})$ , the Riemann–Roch theorem holds for  $\mathcal{O}_D$  if and only if it holds for  $\mathcal{O}_{D+(a)}$ . Adding and subtracting point divisors one can reduce the case of an arbitrary  $D$  to the zero divisor in a finite number of steps. But for the zero divisor ( $\mathcal{L} = \mathcal{O}_X$ ) Riemann–Roch has been proved.

We now make a little trick. We *define*  $\deg(\mathcal{M})$  by the Riemann–Roch formula. For skyscraper sheaves and for line bundles this coincides with the definition we gave already. But also 4) in 4.3 is true since additivity holds for  $\chi$  and rank. The uniqueness of  $\deg$  with the properties 1)–4) follows from 3.1 and 3.6. This proves the existence of the degree and the Riemann–Roch theorem.  $\square$



**4.5 Lemma.** *Let  $\mathcal{M}, \mathcal{N}$  be a two vector bundles, then*

$$\deg(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) = \deg \mathcal{M} \cdot \text{rank} \mathcal{N} + \deg \mathcal{N} \cdot \text{rank} \mathcal{M}.$$

*Let  $\mathcal{M}, \mathcal{N}$  be two vector bundles than*

$$\deg(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})) = \deg \mathcal{N} - \deg \mathcal{M}.$$

*Proof.* For line bundles the statements are clear. The general case can be reduced to this one by induction on the ranks using exact sequences of the type as  $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{L} \rightarrow 0$  where  $\mathcal{L}$  is a line bundle (Lemma 3.6).  $\square$

There is another useful formula. Since we will not use it, we keep short:

Let  $\mathcal{M}$  be a vector bundle of rank  $n$ . One can define  $\bigwedge^p \mathcal{M}$  as the sheaf of  $\mathcal{O}_X$ -multilinear alternating forms  $\mathcal{M}^* \times \cdots \times \mathcal{M}^* \rightarrow \mathcal{O}_X$  ( $n$ -copies). It should be clear what this means. In the case  $p = n$  one obtains a line bundle which sometimes is called the *determinant* of  $\mathcal{M}$ .

**4.6 Remark.** *A vector bundle and its determinant have the same degree.*

We indicate the proof. If  $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M}_3 \rightarrow 0$  is a short exact sequence, then the determinant of  $\mathcal{M}_2$  and  $\mathcal{M}_1 \times \mathcal{M}_3$  are isomorphic. This allows to reduce the statement to the case where  $\mathcal{M} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$  is product of line bundles. But then the determinant equals  $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$ .  $\square$

## 5. A residue map

In this section  $X$  is a compact (and connected) Riemann surface. We now have to make use of the sheaf  $\Omega_X$  of holomorphic differentials. For sake of simplicity we will write  $\mathcal{O} = \mathcal{O}_X$ ,  $\Omega = \Omega_X$ . Basic will be the „residue map“

$$\text{Res} : H^1(X, \Omega) \rightarrow \mathbb{C}.$$

**5.1 Remark and first definition of the residue map.** *Consider the exact sequence*

$$0 \rightarrow \Omega \rightarrow A^{1,0} \xrightarrow{\partial} A^2 \rightarrow 0$$

*and the resulting isomorphism*

$$H^1(X, \Omega) \cong A^2(X) / \partial A^{1,0}.$$

*For an  $\xi \in H^1(X, \Omega)$  consider a representant  $\omega \in A^2(X)$ . The integral*

$$\text{Res} \xi := \frac{1}{2\pi i} \int_X \omega$$

*is independent of the choice of the representant  $\omega$ .*

The independence follows from Stokes' theorem  $\int_X d\omega = 0$ . Notice that on the level of  $A^{1,0}$  we have  $\partial = d$ .  $\square$

We come to another property which justifies the notation “residue”. For this we consider the sheaf  $\mathfrak{M}$  of meromorphic 1-forms and the exact sequence

$$0 \longrightarrow \Omega \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}/\Omega \longrightarrow 0.$$

This induces the map

$$H^0(X, \mathfrak{M}/\Omega) \xrightarrow{\delta} H^1(X, \Omega).$$

We define now a residue map

$$\text{Res} : H^0(X, \mathfrak{M}/\Omega) \longrightarrow \mathbb{C}.$$

Recall that the elements of  $(\mathfrak{M}/\Omega)(X)$  are families  $(\omega_a)_{a \in X}$ ,  $\omega \in \mathcal{M}_a/\mathcal{O}_a$ , which locally fit together. This means that there exists an open covering  $\mathfrak{U} = (U_i)$  and meromorphic differentials  $\omega_i \in \mathfrak{M}(U_i)$ , which represent the  $\omega_a$  for all  $a \in U_i$ . The differences  $\omega_i - \omega_j$  are holomorphic in  $U_i \cap U_j$ . Hence the residue  $\text{Res}(\omega_a, a)$  is well defined.

**5.2 Remark and second definition of the residue.** *There is a natural linear map*

$$\text{Res} : H^0(X, \mathfrak{M}/\Omega) \longrightarrow \mathbb{C},$$

*which can be defined as*

$$\text{Res}((\omega_a)) = \sum_{a \in X} \text{Res}(\omega_a, a).$$

Both definitions of the residue are related:

**5.3 Proposition.** *The diagram*

$$\begin{array}{ccc} H^0(X, \mathfrak{M}/\Omega) & \xrightarrow{\delta} & H^1(X, \Omega) \\ & \searrow \text{Res} & \swarrow \text{Res} \\ & \mathbb{C} & \end{array}$$

*commutes.*

*Proof.* We start with an element from  $H^0(X, \mathfrak{M}/\Omega)$ . As we have seen there exists an open covering  $(U_i)$  and meromorphic differentials  $\omega_i$  on  $U_i$ , such that  $\omega_{ij} := \omega_i - \omega_j$  is holomorphic on  $U_i \cap U_j$ . The family  $\omega_{ij}$  is a Čech cocycle with coefficients in  $\Omega$  and defines an element of  $H^1(X, \Omega)$ . This is the image of  $(\omega_i) \in H^0(X, \mathfrak{M}/\Omega)$  under the combining map  $\delta : H^0(X, \mathfrak{M}/\Omega) \rightarrow H^1(X, \Omega)$ . We can consider the forms  $(\omega_{ij})$  also as a Čech-cocycle of  $A^{1,0}$ . Since the first cohomology of this sheaf vanishes there exist elements  $\eta_i \in A^{1,0}(U_i)$  such that

$$\eta_i - \eta_j = \omega_{ij} \quad (= \omega_i - \omega_j)$$

The advantage of the  $\eta_i$  compared to the  $\omega_i$  is that they have no poles in  $U_i$ . The prize which we have to pay is that they are not holomorphic. Since  $\omega_i - \omega_j$  is holomorphic we have  $d\omega_i = d\omega_j$  and hence  $d\eta_i = d\eta_j$  on  $U_i \cap U_j$ . Hence the  $d\eta_i$  glue to a differential  $\omega \in A^2(X)$ . This defines a cohomology class in  $H^1(X, \Omega) = A^2(X)/\partial A^{1,0}$ . One can check (we leave this to the reader) that this cohomology class corresponds to the cocycle  $(\omega_{ij})$ .

We have to compute  $\int_X \omega$  and to relate it to the residues of  $(\omega_i)$ . Hence we consider the finite set of poles  $S$ . The differences  $\eta_i - \omega_i$  are smooth on  $X' = X - S$  and glue to a differential  $\eta \in A^{1,0}(X')$ . We have  $d\eta = -\omega$  on  $X'$ . Hence  $d\eta$  (but not  $\eta$ ) has a smooth extension to  $X$ .

We now consider the integral  $\int_X \omega$ . We cut out the disk of radius  $\varepsilon$  around each  $a \in S$  (with respect to some chart) and denote the complement of the disks by  $X(\varepsilon)$ . We have

$$\int_X \omega = \lim_{\varepsilon \rightarrow 0} \int_{X(\varepsilon)} \omega.$$

Now Stokes' formula applies and gives

$$\int_{X(\varepsilon)} \omega = \sum_{a \in S} \oint_{\alpha} \eta,$$

where  $\alpha$  denotes a small circle around  $a$  (mathematically negative orientation).

The proof of Proposition 5.3 will be complete if we show

$$\lim_{\varepsilon \rightarrow 0} \oint_{\alpha} \eta = 2\pi i \operatorname{Res} \omega_i \quad (a \in U_i).$$

Here we have chosen  $i$  such that  $a \in U_i$ . On a small punctured neighborhood of  $a$  we have  $\eta = \eta_i - \omega_i$ . Since the integral of  $-\omega_i$  produces the residue, it remains to show

$$\lim_{\varepsilon \rightarrow 0} \oint_{\alpha} \eta_i = 0.$$

But this is clear since  $\eta_i$  is smooth on  $U_i$  (including  $a$ ). □

**5.4 Proposition.** *The residue map  $H^1(X, \Omega) \rightarrow \mathbb{C}$  is different from zero.*

*Proof.* Because of 5.3 it is sufficient to show that the map  $H^0(X, \mathfrak{M}/\Omega) \rightarrow \mathbb{C}$  is not zero. For this one takes a point  $a \in X$  and a meromorphic differential  $\omega_1$  on a small open neighborhood  $U_1$  which has a simple pole at  $a$  and no other pole. Then we consider the zero form  $\omega_2$  on  $U_2 = X - \{a\}$ . Both together glue to a global section of  $\mathfrak{M}/\Omega$ . Its residue is different from 0.  $\square$

## 6. Serre duality

Let  $\mathcal{M}$  be a vector bundle on the compact Riemann surface  $X$ . We consider the vector bundle  $\mathcal{H}om(\mathcal{M}, \Omega)$ . There is a natural  $\mathcal{O}_X$ -bilinear map

$$\mathcal{M} \times \mathcal{H}om(\mathcal{M}, \Omega) \longrightarrow \Omega.$$

Locally it is given by  $(m, l) \mapsto l(m)$ . For a fixed global section from  $\mathcal{H}om(\mathcal{M}, \Omega)$  this induces a map

$$\mathcal{M} \longrightarrow \Omega.$$

Taking cohomology we get a map

$$H^1(X, \mathcal{M}) \longrightarrow H^1(X, \Omega).$$

Varying the global section we get a bilinear map

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \times H^1(X, \mathcal{M}) \longrightarrow H^1(X, \Omega).$$

Combining it with the residue map we get

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \times H^1(X, \mathcal{M}) \longrightarrow \mathbb{C}.$$

This can be considered as a linear map

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \longrightarrow H^1(X, \mathcal{M})^*$$

where  $V^*$  denotes the dual of a vector space ( $V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ ). We call this map the *duality map*.

**6.1 Serre duality.** *For a vector bundle  $\mathcal{M}$  on a compact Riemann surface the duality map*

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \longrightarrow H^1(X, \mathcal{M})^*$$

*is an isomorphism.*

**Corollary.**  $\dim H^1(X, \mathcal{M}) = \dim H^0(X, \mathcal{H}om(\mathcal{M}, \Omega))$ .

*Proof.* In a first step we prove the injectivity of the duality map. Let  $f : \mathcal{M} \rightarrow \Omega$  be a non-zero  $\mathcal{O}$ -linear map. We have to show that its image in  $H^1(X, \mathcal{M})^*$  is not zero. For this, we use the sequence

$$0 \rightarrow \text{Kernel}(f) \rightarrow \mathcal{M} \rightarrow f(\mathcal{M}) \rightarrow 0.$$

We know that  $f(\mathcal{M})$  and  $\text{Kernel}(f)$  are vector bundles (see Lemma 3.4 and the Coherence Lemma 3.5). Since  $f$  is non-trivial,  $f(\mathcal{M})$  is a line bundle. Since  $H^2(X, \text{Kernel}(f)) = 0$  we get, that  $H^1(X, \mathcal{M}) \rightarrow H^1(X, f(\mathcal{M}))$  is surjective.

Now we use the sequence

$$0 \rightarrow f(\mathcal{M}) \rightarrow \Omega \rightarrow f(\mathcal{M})/\Omega \rightarrow 0.$$

Since  $f(\mathcal{M})/\Omega$  is a skyscraper sheaf, its first cohomology group vanishes and we get the surjectivity of  $H^1(X, f(\mathcal{M})) \rightarrow H^1(X, \Omega)$ . As a consequence, the composition  $H^1(\mathcal{M}) \rightarrow H^1(\Omega)$  is surjective. Because of Proposition 5.4 the composition with the residue map  $H^1(\mathcal{M}) \rightarrow \mathbb{C}$  is non-zero. It is easy to check that this is the image of  $f$  under the duality map. So we have shown that the image of  $f$  under the duality map is not 0. This proves the injectivity of the duality map.  $\square$

Next we will proof the surjectivity of the duality map. This is more involved, since it needs some control that  $H^1(X, \mathcal{M})$  is not too big. The idea is to consider twists  $\mathcal{M}(-n)$ . Recall that to define  $\mathcal{M}(-n)$ , one has to choose a point  $a$  and then to consider the divisor  $-n(a)$  which is concentrated on  $a$  with multiplicity  $-n$ . The associated line bundle  $\mathcal{O}(-n)$  is contained in  $\mathcal{O}$  when  $n \geq 0$ . Hence  $\mathcal{M}(-n)$  can be considered as subsheaf of  $\mathcal{M}$  for  $n \geq 0$ . Especially  $\dim H^0(X, \mathcal{M}(-n)) \leq \dim H^0(X, \mathcal{M})$ . We use the formula  $\deg(\mathcal{M}(-n)) = \deg(\mathcal{M}) - n \cdot \text{rank} \mathcal{M}$  (Lemma 4.5). Of course  $\text{rank}(\mathcal{M}) = \text{rank}(\mathcal{M}(n))$ . Hence Riemann–Roch gives

$$\begin{aligned} \dim H^1(X, \mathcal{M}(-n)) &= \dim H^0(X, \mathcal{M}(-n)) + n \cdot \text{rank} \mathcal{M} \\ &\quad - \deg(\mathcal{M}) + \text{rank}(\mathcal{M})(1 - g). \end{aligned}$$

As we have seen,  $\dim H^0(X, \mathcal{M}(-n))$  is bounded for  $n > 0$ .

**6.2 Lemma.** *For  $n \rightarrow \infty$  the asymptotic formula*

$$\dim H^1(X, \mathcal{M}(-n)) = n \cdot \text{rank} \mathcal{M} + O(1)$$

*holds.*

Similarly we get the asymptotic behavior of  $\dim H^0(X, \mathcal{H}om(\mathcal{M}(-n), \Omega))$ . By Lemma 4.5 one has  $\deg \mathcal{H}om(\mathcal{M}(-n), \Omega) = n \text{rank} \mathcal{M} - \deg \mathcal{M} + \deg(\Omega)$ . The inclusion  $\mathcal{M}(-n) \hookrightarrow \mathcal{M}$  gives a surjection  $\mathcal{H}om(\mathcal{M}, \Omega) \rightarrow \mathcal{H}om(\mathcal{M}(-n), \Omega)$  and this induces a surjection in the first cohomology (vanishing of  $H^2$ ). We get the boundedness of  $\dim H^1(X, \mathcal{H}om(\mathcal{M}(-n), \Omega))$ . Now Riemann–Roch shows the following asymptotic formula which, together with Lemma 6.2, can be considered as a weak form of the duality theorem.

**6.3 Lemma.** For  $n \rightarrow \infty$  the asymptotic formula

$$\dim H^0(X, \mathcal{H}om(\mathcal{M}(-n), \Omega)) = n \cdot \text{rank} \mathcal{M} + O(1)$$

holds.

The following asymptotic formula can be proved in the same way.

**6.4 Lemma.** For  $n \rightarrow \infty$  the asymptotic formula

$$\dim H^0(X, \mathcal{O}(n)) = n + O(1)$$

holds.

Next we have to investigate the behavior of the duality map under twisting. From the exact sequence  $0 \rightarrow \mathcal{M}(-1) \rightarrow \mathcal{M} \rightarrow \mathcal{M}/\mathcal{M}(-1) \rightarrow 0$  we obtain the exact sequence

$$H^0(X, \mathcal{M}/\mathcal{M}(-1)) \rightarrow H^1(X, \mathcal{M}) \rightarrow H^1(X, \mathcal{M}(-1)).$$

Dualizing gives the sequence

$$H^1(X, \mathcal{M}(-1))^* \rightarrow H^1(X, \mathcal{M})^* \rightarrow (\mathcal{M}_a/\mathcal{M}(-1)_a)^*$$

We have to construct a second sequence: the imbedding  $\mathcal{M}(-1) \subset \mathcal{M}$  gives a map

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \Omega) \rightarrow \mathcal{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega).$$

Next we use the fact that an  $\mathcal{O}$ -linear map of  $\mathcal{O}$ -modules  $\mathcal{M}_1 \rightarrow \mathcal{M}_2$  induces a natural map  $\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{L} \rightarrow \mathcal{M}_2 \otimes_{\mathcal{O}} \mathcal{L}$ . We can read this as an isomorphism  $\mathcal{H}om(\mathcal{M}_1, \mathcal{M}_2) \cong \mathcal{H}om(\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{L}, \mathcal{M}_2 \otimes_{\mathcal{O}} \mathcal{L})$ . In particular, we can identify

$$\mathcal{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega) = \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \Omega(1)).$$

Let  $\mathcal{M}'$  be the sheaf of meromorphic differentials which are holomorphic outside  $a$  and have possibly in  $a$  a pole of order at most one. There is an obvious isomorphism  $\Omega(1) \rightarrow \mathcal{M}'$  which sends  $\omega \otimes f$  to  $f\omega$ . The residue map  $\mathcal{M}'_a \rightarrow \mathbb{C}$  induces a map  $\Omega(1)_a \rightarrow \mathbb{C}$ .

Let  $\varphi : \mathcal{M}(-1) \rightarrow \Omega$  be an  $\mathcal{O}$ -linear map. As we mentioned, this corresponds to a map  $\varphi : \mathcal{M} \rightarrow \Omega(1)$ . It induces a map  $\mathcal{M}_a \rightarrow \Omega(1)_a$ . We can combine it with the residue map above to produce a map  $\varphi : \mathcal{M}_a \rightarrow \mathbb{C}$ . This map is zero on  $\mathcal{M}_a(-1)$ . Hence it induces a linear map  $\varphi : \mathcal{M}_a/\mathcal{M}(-1)_a \rightarrow \mathbb{C}$ . We can read this construction as a map

$$H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega)) \rightarrow (\mathcal{M}_a/\mathcal{M}(-1)_a)^*.$$

By means of the natural embedding  $\mathcal{M} \rightarrow \mathcal{M}(-1)$  we obtain the sequence

$$H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \Omega)) \rightarrow H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega)) \rightarrow (\mathcal{M}_a/\mathcal{M}(-1)_a)^*$$

The two framed sequences can be combined to a commutative diagram.

**6.5 Lemma.** *The diagram*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1(X, \mathcal{M})^* & \longrightarrow & H^1(X, \mathcal{M}(-1))^* & \longrightarrow & (\mathcal{M}_a/\mathcal{M}(-1)_a)^* \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) & \longrightarrow & H^0(X, \mathcal{H}om(\mathcal{M}(-1), \Omega)) & \longrightarrow & (\mathcal{M}_a/\mathcal{M}(-1)_a)^* \\
 & & \uparrow & & \uparrow & & \parallel \\
 & & 0 & & 0 & & 
 \end{array}$$

*is commutative. Its lines and columns are exact.*

*Proof.* It is easy to check that the rows are exact. The exactness of the columns follows from the injectivity of the duality map.  $\square$

Let  $\lambda \in H^1(X, \mathcal{M})^*$ . Our goal is to show that it is contained in the image of  $H^0(X, \mathcal{H}om(\mathcal{M}, \Omega))$ . We make the weaker assumption that the image of  $\lambda$  in  $H^1(X, \mathcal{M}(-1))^*$  comes from  $H^0(\mathcal{H}om_{\mathcal{O}}(\mathcal{M}(-1), \Omega))$ . Simple diagram chasing in Lemma 6.5 shows that then  $\lambda$  itself comes from  $H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \Omega))$ . We can apply this several times and obtain the following result.

**6.6 Lemma.** *Let  $\lambda \in H^1(X, \mathcal{M})^*$  and  $D \geq 0$  a divisor. We write  $\mathcal{M}_{-D} := \mathcal{M} \otimes \mathcal{O}_{-D}$ . (This is imbedded in  $\mathcal{M}$ .) Assume that the image of  $\lambda$  under  $H^1(X, \mathcal{M})^* \hookrightarrow H^1(X, \mathcal{M}_{-D})^*$  comes from  $H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}_{-D}, \Omega))$ . Then  $\lambda$  comes from  $H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \Omega))$ .*

We need a last preparation of the proof of the duality theorem. Assume that a function  $f \in H^0(X, \mathcal{O}(n))$  is given, i.e. a meromorphic function which is holomorphic outside  $a$  and has in  $a$  at most a pole of order  $n > 0$ . Multiplication by  $f$  defines a  $\mathcal{O}$ -linear map  $f : \mathcal{M}(-n) \rightarrow \mathcal{M}$ . (We considered up to now this map for  $f = 1$ .) This map induces a linear map

$$f : H^1(X, \mathcal{M})^* \longrightarrow H^1(X, \mathcal{M}(-n))^*.$$

Now we fix a non-zero element  $\lambda \in H^1(X, \mathcal{M})^*$ . (Recall that our goal is to prove that it is in the image of the duality map.) We use it to define the map

$$H^0(X, \mathcal{O}(n)) \longrightarrow H^1(X, \mathcal{M}(-n))^*, \quad f \longmapsto f(\lambda).$$

**6.7 Lemma.** *For non zero  $\lambda \in H^1(X, \mathcal{M}(-m))^*$ , the map*

$$H^0(X, \mathcal{O}(n)) \longrightarrow H^1(X, \mathcal{M}(-n))^*, \quad f \longmapsto f(\lambda),$$

*is injective.*

*Proof.* We have to show that  $f(\lambda) \neq 0$  for  $f \neq 0$ . This can also be interpreted as follows. For fixed  $f \neq 0$  the map  $f : H^1(X, \mathcal{M})^* \longrightarrow H^1(X, \mathcal{M}(-n))^*$  is

injective, or equivalently, the map  $f : H^1(X, \mathcal{M}(-n)) \rightarrow H^1(X, \mathcal{M})$  is surjective. This follows from the long exact cohomology sequence since  $\mathcal{M}/f(\mathcal{M}(-n))$  is a skyscraper sheaf.  $\square$

Now we consider the triangle

$$\begin{array}{ccc} H^0(X, \mathcal{O}(n)) & \xrightarrow{f \mapsto f(\lambda)} & H^1(X, \mathcal{M}(-n))^* \\ & & \uparrow \\ & & H^0(\mathcal{H}om(\mathcal{M}(-n)), \Omega) \end{array}$$

We know the both arrows are injective. We use the asymptotic formulas in the Lemmas 6.2, 6.3 and 6.4. They show that the images of  $H^0(X, \mathcal{O}(n))$  and  $H^0(\mathcal{H}om(\mathcal{M}(-n)), \Omega)$  for large  $n$  have a non-zero intersection in the space  $H^1(X, \mathcal{M}(-n))$ . Therefore, for large  $n$ , there must exist a non-zero  $f \in H^0(X, \mathcal{O}(n))$  such that  $f(\lambda)$  is in the image of the vertical arrow (=duality map for  $\mathcal{M}(-n)$ ). We consider the divisor  $D = n(a) + (f)$  with the property  $D \geq 0$ . Hence  $\mathcal{M}_{-D}$  is naturally imbedded in  $\mathcal{M}$ . Multiplication by  $f$  gives already a map  $f : \mathcal{M}_{-D} \rightarrow \mathcal{M}$ . Hence the map  $f : H^1(X, \mathcal{M})^* \rightarrow H^1(X, \mathcal{M}(-n))^*$  factors as

$$\begin{array}{ccccc} H^1(X, \mathcal{M})^* & \xrightarrow{\text{natural imbedding}} & H^1(X, \mathcal{M}_{-D})^* & \xrightarrow{f} & H^1(X, \mathcal{M}(-n))^* \\ \lambda & \longmapsto & \lambda & \longmapsto & f(\lambda) \end{array}$$

In the same way we get a factorization

$$H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) \xrightarrow{\text{nat. i.}} H^0(X, \mathcal{H}om(\mathcal{M}_{-D}, \Omega)) \xrightarrow{f} H^0(X, \mathcal{H}om(\mathcal{M}(-n), \Omega)).$$

This shows that  $\lambda \in \mathcal{H}om(\mathcal{M}_{-D}, \Omega)$  is contained in the image of the duality map. Now Lemma 6.6 applies and shows that already  $\lambda \in \mathcal{H}om(\mathcal{M}, \Omega)$  is contained in the duality map. This completes the proof of the duality theorem.  $\square$

## 7. Some comments on the Riemann–Roch theorem

The Riemann–Roch theorem can now be written in the cohomology free form

$$\dim H^0(X, \mathcal{M}) - \dim H^0(X, \mathcal{H}om(\mathcal{M}, \Omega)) = \deg(\mathcal{M}) + \text{rank}(\mathcal{M})(1 - g).$$

We formulate it in its classical form for divisors. A divisor  $K$  is called *canonical* if the associated line bundle is isomorphic to  $\Omega$ . One can get a canonical divisor



as follows: Let  $\omega$  be a meromorphic 1-form. It is clear how to associate to  $\omega$  a divisor  $D = (\omega)$  which describes the poles and zeros of  $\omega$ . This is obviously a canonical divisor, since the sections of  $\mathcal{O}_D$  are the meromorphic functions  $f$  with the property that  $f\omega$  is holomorphic. This gives an isomorphism  $\mathcal{O}_D \cong \Omega$ . By the way, a non vanishing meromorphic 1-form exists. One can take one of the form  $df$ , where  $f$  is a non-constant meromorphic function, which exists by Theorem 3.7.

Using the notation

$$l(D) := H^0(X, \mathcal{O}_D) = \dim\{ f \text{ meromorphic on } X; (f) \geq -D \}$$

we now get

**7.1 Classical Riemann–Roch.** *Let  $D$  be a divisor on a compact Riemann surface and let  $K$  be a canonical divisor. Then*

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

We consider some special cases: let  $D$  be the zero divisor. Then  $l(D) = 1$ ,  $\deg(D) = 0$  and we obtain a different definition for the genus.

**7.2 Theorem.** *One has  $\dim \Omega(X) = g$ .*

We now take  $D = K$  in the Riemann–Roch formula. We get  $g - 1 = \deg(D) + 1 - g$ :

**7.3 Theorem.** *The degree of a canonical divisor (or of  $\Omega$ ) is  $2g - 2$ .*

If  $f$  is a non zero meromorphic function with the property  $(f) + D \geq 0$ , then  $\deg(D) \geq 0$ . Hence  $l(D) = 0$  for  $\deg(D) < 0$ . Applying this to  $K - D$  we get:

**7.4 Theorem.** *Let  $D$  be a divisor with  $\deg(D) > 2g - 2$ . Then*

$$l(D) = \deg(D) + 1 - g.$$

Finally we derive a formula which allows to compute the genus. Assume that  $f : X \rightarrow Y$  is a non-constant holomorphic map of compact Riemann surfaces. Let  $a \in X$ . Recall that one can define the multiplicity  $v(f, a)$  of  $f$  in  $a$ . We call

$$b(f, a) := v(f, a) - 1$$

the ramification order. This is zero if and only if  $f$  is locally biholomorphic at  $a$ . We denote by

$$b(f) = \sum_{a \in X} b(f, a)$$

the *total ramification order*. The sum is of course finite.

**7.5 Riemann–Hurwitz ramification formula.** *Let  $f : X \rightarrow Y$  a non-constant holomorphic map of compact Riemann surfaces. Let  $n$  be the covering degree of  $f$  (see Proposition V.5.8). Then the following relation between the genus  $g_X$  of  $X$  and  $g_Y$  of  $Y$  holds:*

$$g_X = \frac{b(f)}{2} + n(g_Y - 1) + 1.$$

We just give a short hint for the proof. Consider a meromorphic function  $g$  on  $Y$  and compare the divisors of the differentials  $dg$  on  $Y$  and  $d(g \circ f)$  on  $X$ . Use Theorem 7.3.  $\square$

We treat some simple examples.

- 1) Since  $H^1(\bar{\mathbb{C}}, \mathcal{O}) = 0$  (Theorem IV.4.7) we have that the genus of the Riemann sphere is zero.
- 2) A meromorphic differential on  $\bar{\mathbb{C}}$  is given by  $dz$ . It has pole of order 2 in  $\infty$  (use the chart  $1/z$  and the fact that the derivative of  $1/z$  is  $-1/z^2$ ) and no further poles or zeros. Hence the degree of the canonical divisor is  $-2$  in concordance with the expression  $2g - 2$ .
- 3) Consider the map  $\bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}, z \mapsto z^2$ . There are two ramification points  $0, \infty$  of ramification order 1. The covering degree is 2, the total ramification number is also 2. The Riemann–Hurwitz formula gives  $0 = 1 + 2(0 - 1) + 1$  which is correct.
- 4) Consider a complex torus  $X = \mathbb{C}/L$ . The differential  $dz$  on  $\mathbb{C}$  is invariant under translations. Hence it gives a holomorphic differential on  $X$ . It has no poles and zeros, hence its degree is zero. From  $2g - 2 = 0$  we get  $g = 1$ . Hence the genus of a torus is one.
- 5) Another way to see it is to use the Weierstrass  $\wp$ -function. This is holomorphic map  $\mathbb{C}/L \rightarrow \bar{\mathbb{C}}$  with 4 ramification points (the zeros of  $\wp'$  and  $\infty$ ). The covering degree is 2, the total ramification number is 4. The Riemann–Hurwitz formula gives  $g = 2 + 2(-1) + 1 = 1$ .
- 5) At the end of chapter V and section 5, we explained the Riemann surface of the “function  $\sqrt{P(z)}$ ”, where  $P$  is polynomial without multiple zeros. Let the degree of  $P$  be  $n = 2g + 1$  or  $n = 2g + 2$ . We explained that this Riemann surface is a two fold covering  $f : X \rightarrow \bar{\mathbb{C}}$ . The number  $b$  of ramification points (all of order two) turned out to be  $n \leq b \leq n + 1$ . Since we know from the Riemann–Hurwitz formula 7.5 that the total ramification order is even, we can conclude that in both cases  $b = 2g$  is the correct number. From the Riemann–Hurwitz formula we get:

*The genus of the Riemann surface “ $\sqrt{P(z)}$ ” is  $g$ , where where the degree of  $P$  is  $2g + 1$  or  $2g + 2$ .*

We consider the projection  $z : X \rightarrow \bar{\mathbb{C}}$ . Then  $dz$  is a meromorphic differential on  $X$ . Recall that  $\sqrt{P(z)}$  has been defined as a meromorphic function on  $X$ . Hence we can consider the meromorphic differentials on  $X$

$$\frac{z^k dz}{\sqrt{P(z)}}.$$

If one goes carefully through the construction of  $X$  one can work out when this differential has no poles. The result is that this is the case if and only if

$$0 \leq k < g.$$

Since these differentials are linearly independent, by 7.2 they must give a basis of  $\Omega(X)$ .

The integration of these integrals is a major problem. In the case  $g = 0$  one has so-called circle integrals. The integration is elementary and leads to the functions  $\sin$  and  $\cos$ . The case  $g = 1$  is more involved. The integrals are called elliptic integrals. Their integration leads to the theory of elliptic functions. In the language of Riemann surfaces this means that one can show that  $X$  is biholomorphic equivalent to a complex torus  $\mathbb{C}/L$  and that every complex torus arises in this way. (The fact that both types have genus 1 may be considered as a weak hint that this is true.) In the case  $g > 1$  the integrals are called “hyperelliptic integral” and the corresponding Riemann surfaces are called *hyperelliptic Riemann surfaces*. The integration of hyperelliptic differentials leads deep into the theory of Riemann surfaces. With their help deep open problems as the Jacobi inversion problem could be solved.

# Chapter VII. The Jacobi inversion problem

## 1. Harmonic differentials

Recall that a function  $u$  on an open subset  $U \subset \mathbb{C}$  is called *harmonic* if it satisfies

$$\Delta u := \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0.$$

A function is harmonic if and only if its real and imaginary part are harmonic. It is known from complex calculus that a real function  $u$  is harmonic if and only if it is locally the real part of an analytic function. This enables a quick generalization to Riemann surfaces.

**1.1 Definition.** *A real function  $f$  on a Riemann surface  $X$  is called harmonic, if every point admits an open neighborhood  $U$  such that  $u$  can be written on  $U$  as real part of an analytic function  $f_U : U \rightarrow \mathbb{C}$ . An arbitrary (complex valued) function  $u$  is called harmonic if its real and imaginary part are harmonic.*

We also need the notion of a harmonic differential.

**1.2 Definition.** *A differential  $\omega \in A^1(X)$  on a Riemann surface is called harmonic, if every point admits an open neighborhood  $U$  such that  $\omega$  can be written on  $U$  as  $du$  with a harmonic function  $u : U \rightarrow \mathbb{C}$ .*

Analytic functions are also harmonic. Hence we obtain that holomorphic differentials are also harmonic.

We define the complex conjugate  $\bar{\omega}$  of a differential  $\omega$ . In coordinates this is defined as

$$\overline{f dx + g dy} := \bar{f} dx + \bar{g} dy.$$

One checks  $\gamma^* \bar{\omega} = \overline{\gamma^* \omega}$  for arbitrary differential maps. This allows to generalize complex conjugation on Riemann surfaces via charts. A differential  $\omega$  is called *real* if  $\omega = \bar{\omega}$ . The formula  $\overline{dz} = d\bar{z}$  shows that complex conjugation defines an isomorphism

$$A^{1,0}(X) \xrightarrow{\sim} A^{0,1}(X), \quad \omega \longmapsto \bar{\omega}.$$

The complex conjugate of a harmonic differential is harmonic as well. Hence we see that not only holomorphic differentials are harmonic but also antiholomorphic ones. (A function or a differential is called antiholomorphic if its complex conjugate is holomorphic.)

**1.3 Remark.** *A harmonic differential  $\omega$  can be written as the sum of a holomorphic and an antiholomorphic differential.*

*Proof.* First we notice that a harmonic function locally is the sum of a holomorphic and a antiholomorphic function. This shows that a harmonic differential locally can be written in the form  $\omega = \omega_1 + \omega_2$  with a holomorphic differential  $\omega_1$  and an antiholomorphic differential  $\omega_2$ . Since this decomposition is obviously unique, it extends to a global decomposition.  $\square$

We introduce the so-called star operator. Recall that we have the direct decomposition

$$A^1(X) = A^{1,0}(X) \oplus A^{0,1}(X).$$

**1.4 Definition.** *The star operator*

$$* : A^1(X) \longrightarrow A^1(X)$$

*is defined by*

$$*(\omega_1 + \omega_2) = i(\bar{\omega}_1 - \bar{\omega}_2), \quad \omega_1 \in A^{1,0}(X), \quad \omega_2 \in A^{0,1}(X).$$

Before we continue we formulate some rules which we will use in the following:

**1.5 Some rules.**

- a)  $d(f\omega) = (df) \wedge \omega + f\omega$  ( $f \in \mathcal{C}^\infty(X)$ ,  $\omega \in A^1(X)$ ).
- b) *The same as in a) but  $d$  replaced by  $\partial$  or  $\bar{\partial}$ .*
- c)  $**\omega = -\omega$ .
- d)  $\overline{*\omega} = *\bar{\omega}$ .
- e)  $d(*\omega) = i\partial\bar{\omega}$  for  $\omega \in A^{1,0}(X)$ ;  $d(*\omega) = -i\bar{\partial}\omega$  for  $\omega \in A^{0,1}(X)$ .
- f)  $*\partial f = i\bar{\partial}\bar{f}$ ;  $*\bar{\partial} f = -i\partial\bar{f}$ .
- g)  $d(*df) = 2i\partial\bar{\partial}f$ .

The proof can be given by direct calculations with coordinates.  $\square$

The announced new characterization of harmonic differentials uses this star operator:

**1.6 Proposition.** *A differential  $\omega$  on a Riemann surface is harmonic if and only if*

$$d\omega = d(*\omega) = 0.$$

*Proof.* Let  $\omega$  be harmonic. We show the two differential equations. Because of 1.3 we can assume that  $\omega$  is holomorphic or antiholomorphic. But then  $*\omega$  is antiholomorphic or holomorphic and the statement is clear.

Now we assume conversely the two differential equations. From  $d\omega = 0$  follows that  $\omega$  can be locally written as  $df$  with some function  $f$ . A calculation in coordinates shows that  $d*\omega = d(*df) = 0$  means that  $f$  is harmonic (use 1.5, g)).  $\square$

We denote by  $\text{Harm}^1(X)$  the space of all harmonic differentials on  $X$ . We reformulate 1.3 in the form

$$\text{Harm}^1(X) = \Omega(X) \oplus \bar{\Omega}(X).$$

Hence for compact  $X$  we get

$$\dim_{\mathbb{C}} \text{Harm}^1(X) = 2g.$$

## 2. Hodge theory of compact Riemann surfaces

For two differentials  $\alpha, \beta$  on a Riemann surface we define

$$[\alpha, \beta] = \alpha \wedge *\beta.$$

This is  $\mathbb{C}$ -linear in the first variable and satisfies

$$[\beta, \alpha] = \overline{[\alpha, \beta]}.$$

We express  $[\alpha, \alpha]$  in local coordinates and write  $\alpha$  for this purpose in the form

$$\alpha = f dz + g d\bar{z}.$$

Then

$$*\alpha = i(\bar{f} d\bar{z} - \bar{g} dz)$$

and hence

$$[\alpha, \alpha] = i(|f|^2 + |g|^2) dz \wedge d\bar{z} = 2(|f|^2 + |g|^2) dx \wedge dy.$$

The essential point here is that  $(|f|^2 + |g|^2)$  is non-negative and zero only if the (smooth) functions  $f, g$  are zero.

Now we will assume that the Riemann surface  $X$  is compact. Then we can define

$$\langle \alpha, \beta \rangle = \int_X [\alpha, \beta].$$

This is a hermitian form on  $A^1(X)$  ( $\mathbb{C}$ -linear in the first variable and with the property  $\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$ ). Moreover this form is positive definit. Hence  $A^1(X)$  has been equipped with a structure as unitary vector space.

**2.1 Proposition.** *On any compact Riemann surface one has an orthogonal (with respect to  $\langle \cdot, \cdot \rangle$ ) decomposition*

$$A^{0,1}(X) = \bar{\partial}C^\infty(X) \oplus \bar{\Omega}(X).$$

**Corollary.** *A form  $\sigma$  in  $A^{0,1}(X)$  can be written in the form  $\sigma = \bar{\partial}f$  if and only if*

$$\int_X \sigma \wedge \omega = 0$$

for all holomorphic differentials  $\omega$ .

*Proof.* Firstly we prove that  $\bar{\partial}C^\infty(X)$  and  $\bar{\Omega}(X)$  are orthogonal: We have to consider  $\omega \wedge * \bar{\partial}f$  for antiholomorphic  $\omega$ . Using the rules 1.5 one shows

$$\omega \wedge * \bar{\partial}f = -\text{id}(\bar{f}\omega).$$

From Stokes formula follows  $\langle \omega, df \rangle = 0$ . From the orthogonality we obtain that the natural map

$$\bar{\Omega}(X) \rightarrow A^{0,1}(X)/\bar{\partial}C^\infty(X)$$

is injective. Hence for the proof of 2.1 it remains to show that both sides have the same dimension. The dimension of  $\bar{\Omega}$  is  $g$  (by duality). By Dolbeault's theorem V.7.4 the right hand side is isomorphic to  $H^1(X, \mathcal{O}_X)$ , which by definition has also the dimension  $g$ . This completes the proof of 2.1.  $\square$

We mention that by trivial reason the spaces  $A^{1,0}(X)$  and  $A^{0,1}(X)$  are orthogonal (since  $dz \wedge dz = 0$ ). Hence not only  $\bar{\partial}C^\infty(X)$  but also  $\partial C^\infty(X)$  is orthogonal to  $\bar{\Omega}(X)$ . Hence also  $dC^\infty(X)$  is orthogonal to  $\bar{\Omega}(X)$ . Since  $dC^\infty(X)$  is invariant under complex conjugation it is also orthogonal to  $\Omega(X)$ . It follows that  $dC^\infty(X)$  and  $\text{Harm}^1(X)$  are orthogonal. Let now  $u$  be a harmonic function on  $X$ . Then  $du$  is a harmonic differential which is orthogonal to  $\text{Harm}^1(X)$  hence to itself. This means  $du = 0$  and we obtain:

**2.2 Proposition.** *Every harmonic function on a compact Riemann surface is constant.*

We give another application of 2.1. complex conjugation gives

$$A^{1,0}(X) = \partial C^\infty(X) \oplus \Omega(X)$$

or

$$A^1(X) = \partial C^\infty(X) \oplus \bar{\partial}C^\infty(X) \oplus \text{Harm}^1(X).$$

Now we use the rule 1.5, f). It shows that

$$dC^\infty(X) + *dC^\infty(X) \subset \partial C^\infty(X) + \bar{\partial}C^\infty(X).$$

From the same rule follows also  $df + *d(\bar{f}) = \partial f$  and hence the converse inclusion. The spaces  $dC^\infty(X)$  and  $*dC^\infty(X)$  are also orthogonal. This follows from  $df \wedge **dg = -df \wedge dg = -d(fdg)$ . This gives us

**2.3 Proposition.** *On a compact Riemann surface one has the orthogonal decomposition*

$$A^1(X) = dC^\infty(X) \oplus *dC^\infty(X) \oplus \text{Harm}^1(X).$$

We want to determine the position of  $K := \text{Kernel}(A^1(X) \xrightarrow{d} A^2(X))$  with respect to this position. We claim that it is orthogonal to  $*dC^\infty(X)$ . For  $d\omega = 0$  one gets

$$\omega \wedge *(df) = -\omega \wedge df = d(f\omega).$$

This shows  $\langle \omega, *df \rangle = 0$ . On the other side  $dC^\infty(X)$  and  $\oplus \text{Harm}^1(X)$  are contained in  $K$ . Hence  $K$  equals their sum:

**2.4 Proposition.** *On a compact Riemann surface one has the orthogonal decomposition*

$$\text{Kernel}(A^1(X) \xrightarrow{d} A^2(X)) = dC^\infty(X) \oplus \text{Harm}^1(X).$$

This proposition induces an isomorphism

$$\text{Harm}^1(X) \cong \frac{\text{Kernel}(A^1(X) \xrightarrow{d} A^2(X))}{dC^\infty(X)}.$$

By the theorem of de Rham the right hand side is isomorphic to  $H^1(X, \mathbb{C})$ . This gives us

**2.5 Theorem of de Rham–Hodge.** *For a compact Riemann surface one has*

$$H^1(X, \mathbb{C}) \cong \text{Harm}^1(X).$$

*Hence the dimension of  $H^1(X, \mathbb{C})$  is  $2g$ .*

Notice that  $H^1(X, \mathbb{C})$  depends only on the topological space  $X$  and not on the complex structure. Hence we see that the genus  $g$  for homeomorphic Riemann surfaces is the same. But homeomorphic Riemann surfaces need not to be biholomorphic equivalent as already the example of tori shows.



### 3. Integration of closed forms and homotopy

A differential  $\omega$  is called *closed* if  $d\omega = 0$  and *total* if it is of the form  $\omega = df$ . From the main theorem of calculus follows the formula

$$\int_{\alpha} df = \alpha(1) - \alpha(0)$$

for every smooth curve  $\alpha$ . In the following it will be useful to weaken the smoothness of a curve. First of all the curve integral  $\int_{\alpha} \omega$  can be generalized to *piecewise* smooth curves in an obvious way. But for *closed* differentials it is possible to extend the integral to arbitrary continuous curves as follows. What we have to use is that closed forms are locally total by the Poincaré lemma. Hence every point admits a neighborhood  $U$ , such that  $\int_{\alpha} \omega = 0$  for every closed curve in  $U$  when  $\omega$  is closed. Let now  $\alpha : [0, 1] \rightarrow X$  be an arbitrary (continuous) curve. By a compactness argument we can find a partition  $0 = a_0 < \dots < a_n = 1$  and open subsets  $U_1, \dots, U_n$  such that

$$\alpha([a_{i-1}, a_i]) \subset U_i \quad (1 \leq i \leq n)$$

and such that  $\omega$  is total in  $U_i$ . Then we combine  $\alpha(a_{i-1})$  and  $\alpha(a_i)$  by some smooth curve inside  $U_i$ . This defines a new piecewise smooth curve  $\beta$  with the same origin and end as  $\alpha$ . It is easy to see that  $\int_{\beta} \omega$  is independent of the choice of  $\beta$ . (Here we use that  $\omega$  is total in  $U_i$ .) Hence we can define

$$\int_{\alpha} \omega := \int_{\beta} \omega.$$

Let be  $Q \subset \mathbb{C}$  be a compact rectangle, parallel to the axes and let  $H : Q \rightarrow X$  be a continuous map. In an obvious way there can be defined a closed curve  $\alpha$  which runs through the image of the boundary of  $Q$ . We claim

$$\int_{\alpha} \omega = 0 \quad (\omega \text{ closed}).$$

The proof is very simple. One divides  $Q$  into small rectangles such that the image of each of them is contained in an open subset on which  $\omega$  is total. Then the corresponding integral for the small rectangles are zero. Summing up all the integrals over the small rectangles one gets obviously the integral over the original rectangle. We formulate a special case of this:

**3.1 Definition.** Two curves  $\alpha : [0, 1] \rightarrow X$  and  $\beta : [0, 1] \rightarrow X$  with the property  $\alpha(0) = \beta(0)$  and  $\alpha(1) = \beta(1)$  are called **homotopic** if there exists a continuous map  $H : [0, 1] \times [0, 1] \rightarrow X$ , such that

$$H(t, 0) = \alpha(t), \quad H(t, 1) = \beta(t)$$

and

$$H(0, s) = \alpha(0), \quad H(1, s) = \alpha(1).$$

The above observation shows:

**3.2 Proposition.** *Let  $\alpha, \beta$  be two homotopic curves. Then*

$$\int_{\alpha} \omega = \int_{\beta} \omega$$

*for all closed differentials  $\omega$ .*

In the special case, where  $\omega$  is holomorphic, this is called the homotopy version of the Cauchy integral theorem.

### The fundamental group

We fix a point  $a \in X$  and consider now closed curves, which start from  $a$  and end in  $a$ . Homotopy defines an equivalence relation on this set. The set of all homotopy classes is denoted by  $\pi_1(X, a)$ . It is easy to show that composition of curves defines a well-defined product in  $\pi_1(X, a)$ . It is not difficult to show that this product makes  $\pi_1(X, a)$  to a group. Details can be found in [Fr1], Chap. III, Appendix A. This group is called the fundamental group of  $(X, a)$ .

**3.3 Remark.** *Let  $\omega$  be a closed differential on  $X$ . Then integration defines a homomorphism*

$$\pi_1(X, a) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \int_{\alpha} \omega.$$

## 4. Periods

Let  $\omega$  be a (smooth) closed differential on a Riemann surface. A complex number  $C$  is called a *period* of  $\omega$ , if there exists a closed curve  $\alpha$  with the property

$$C = \int_{\alpha} \omega.$$

To explain the notion “period” we consider the case of a torus  $X = \mathbb{C}/L$  and “ $\omega = dz$ ”. Every closed curve  $\alpha$  in  $X$  lifts to a curve  $\beta : [0, 1] \rightarrow \mathbb{C}$ . Then

$$\int_{\alpha} dz = \beta(1) - \beta(0).$$

Since  $\alpha$  is closed we have  $\beta(0) \equiv \beta(1) \pmod{L}$ . Hence we have that the periods are precisely the elements of  $L$ .

It is sometimes necessary to choose a base point  $a \in X$  and to consider only curves which start from  $a$ . If  $\alpha$  is an arbitrary closed curve, one can

combine  $a$  with  $\alpha(0)$ , then run through  $\alpha$  and then go back the same way as one started. This shows that each period can be obtained by a curve starting and ending in  $a$ . Since the integral of a closed differential is homotopy invariant (Proposition 3.2) we get a map

$$\pi_1(X, a) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \int_{\alpha} \omega,$$

which is obviously a homomorphism.

The structure of the fundamental group of a compact surface can be determined by topological methods. A result, which can be obtained without any further theory is the following lemma.

**4.1 Lemma.** *Let  $S \subset X$  be a finite subset of a compact Riemann surface,  $a \in X - S$ . The fundamental group  $\pi_1(X - S, a)$  is countable.*

*Proof.* If  $X = \bar{\mathbb{C}}$  this can be seen as follows. Take the base point  $a$  to be rational. This means that real and imaginary part are rational numbers. It is easy to see that each closed curve with origin  $a$  is homotopic to piecewise linear curve with rational vertices. This is a countable set of curves.

The general case can be settled similarly choosing a non-constant meromorphic function  $f : X \rightarrow \bar{\mathbb{C}}$ . Then we consider curves in  $X$ , which map under  $f$  to piece wise linear curves with rational vertices as considered in the first case. This is a countable set of curves and the same argument as in the first case works.  $\square$

The importance of the periods shows the following

**4.2 Remark.** *A closed differential ( $d\omega = 0$ ) is total, i.e. of the form  $\omega = df$  if and only if all its periods vanish.*

*Proof.* The main theorem of calculus says

$$\int_{\alpha} df = f(\alpha(1)) - f(\alpha(0)).$$

This shows that the periods of  $df$  vanish. To prove the converse we *define*

$$f(x) := \int_a^x \omega.$$

Here  $a$  is a fixed chosen base point. The integral is understood as a curve integral along a curve which connects  $a$  with  $x$ . Since the periods vanish this integral is independent of the choice of this curve. A local computation shows that  $df = \omega$ .  $\square$

Harmonic differentials are closely tied to their periods:

**4.3 Proposition.** *A harmonic differential on a compact Riemann surface vanishes if all its periods are zero.*

*Proof.* Assume that  $\omega$  is a harmonic differential whose periods vanish. Then  $\omega = df$  by 4.2. Now 2.3 shows that  $\omega$  is orthogonal to  $\text{Harm}^1(X)$  hence to itself. This shows  $\omega = 0$ .  $\square$

A variant of Proposition 4.3 states:

**4.4 Proposition.** *A holomorphic differential on a compact Riemann surface vanishes if the real parts of all its periods are zero.*

*Proof.* From Proposition 4.3 follows that  $\text{Re } \omega = (\omega + \bar{\omega})/2$  is zero. Locally  $\omega$  can be written as  $df$  with a holomorphic function. It follows that  $f$  has constant real part. But then  $f$  is constant and  $\omega = 0$ .  $\square$

We now want to consider the periods of *all* holomorphic differentials together. This leads to the following

**4.5 Definition.** *A  $\mathbb{C}$ -linear form  $l : \Omega(X) \rightarrow \mathbb{C}$  is called a **period** of the compact Riemann surface  $X$ , if there exists a closed curve  $\alpha$  with the property*

$$l(\omega) = \int_{\alpha} \omega.$$

The set  $L$  of all periods is a subset of the dual space  $\Omega(X)^*$ . Actually it is an additive subgroup  $L \subset \Omega(X)^*$ . Hence we can consider the factor group

$$\text{Jac}(X) := \Omega(X)/L.$$

We will show later that  $L$  is a lattice and therefore  $\text{Jac}(X)$  a torus of real dimension  $2g$ . This will be explained in detail. Here we take it just as justification to call  $\text{Jac}(X)$  the *Jacobian variety*. We only mention that from Lemma 4.1 follows that  $L$  is a countable set.

**4.6 Definition of the Abel–Jacobi map.** *Let  $X$  be a Riemann surface with a base point  $a$ . Then the Abel–Jacobi map*

$$A : X \longrightarrow \text{Jac}(X)$$

*is defined as follows. For a point  $x \in X$  one chooses a curve  $\alpha$  which combines  $a$  with  $x$ . Then one considers the linear form*

$$\omega \longmapsto \int_a^x \omega := \int_{\alpha} \omega.$$

*The image of this linear form in  $\text{Jac}(X)$  is independent of the choice of  $\alpha$  and is defined to be  $A(x)$ .*

The Abel–Jacobi map admits certain important variants, which we will all denote by the same letter  $A$ : For example one can consider for any natural number  $d$  the  $d$ -fold cartesian product  $X^d := X \times \dots \times X$  and define

$$A : X^d \longrightarrow \text{Jac}(X), \quad A(\xi_1, \dots, \xi_d) = A(\xi_1) + \dots + A(\xi_d).$$

Here we used that  $\text{Jac}(X)$  carries a natural structure as abelian group.

A closely related extension is obtained as follows. Consider a divisor  $D$  on  $X$ . Then we define

$$A(D) := \sum_{a \in X} D(a)A(a).$$

This is obviously a homomorphism

$$A : \text{Div}(X) \longrightarrow \text{Jac}(X).$$

Of course this homomorphism depends on the choice of the base point  $a$ . But now we restrict  $A$  to the subgroup  $\text{Div}^0(X)$  of divisors of degree 0. Then we get:

**4.7 Remark.** *The Abel–Jacobi map  $A : \text{Div}^0(X) \longrightarrow \text{Jac}(X)$  restricted to the divisors of degree zero is independent of the choice of the base point.*

*Proof.* We take another base point and choose a fixed curve  $\beta$  which combines  $a$  and  $b$ . We use this curve to transform closed curves with origin  $a$  to curves with origin  $b$ . Then the two Abel–Jacobi maps differ by

$$\sum_{x \in X} D(x) \int_{\beta} \omega.$$

This is zero since  $D$  has degree zero.  $\square$

The Abel–Jacobi map on  $\text{Div}^0(X)$  can be defined in a slightly different way:

**4.8 Remark.** *Let  $D = (a_1) + \dots + (a_n) - (b_1) - \dots - (b_n)$  be a divisor (of degree zero) on the compact Riemann surface  $X$ . Let  $\gamma_i$  be curves which join  $a_i$  with  $b_i$ . Then  $A(D)$  is represented by the linear form*

$$\omega \longmapsto \sum_{i=1}^n \int_{\gamma_i} \omega.$$

*Proof.* One chooses a base point and joins  $a$  and  $a_i$  by curves  $\alpha_i$ . Then one defines curves from  $a$  to  $b_i$  by joining  $\alpha_i$  and  $\gamma_i$ .  $\square$

Special divisors of degree 0 are the principal divisors ( $f$ ). Recall that we define a subgroup  $\mathcal{H}(X) \subset \text{Div}^0(X)$ . The factor group

$$\mathcal{D}^0(X)/\mathcal{H}(X)$$

can be identified with a subgroup  $\text{Pic}^0(X)$  of the Picard group (VI.3.1). It can be considered as the group of all isomorphism classes of line bundles of degree 0.

**4.9 Proposition.** *The image of a principal divisor in  $\text{Jac}(X)$  under the Abel–Jacobi map is zero. Hence it induces a homomorphism*

$$A : \text{Pic}^0(X) \longrightarrow \text{Jac}(X).$$

*Proof.* Let  $f$  be a non-constant meromorphic function on the compact Riemann surface  $X$ . We consider it as a map  $f : X \rightarrow \bar{\mathbb{C}}$ . For each  $z \in \bar{\mathbb{C}}$  we consider the fibre  $D_z := f^{-1}(z)$ . Recall that we can talk about the multiplicity of a point with value  $a$ . Hence  $D_z$  can be considered naturally as a divisor. We will show that all  $D_z$  have the same image in  $\text{Jac}(X)$ . (This proves 4.9, since  $(f) = D_0 - D_\infty$ .) We have to show that the map

$$\bar{\mathbb{C}} \longrightarrow \text{Jac}(X), \quad z \longmapsto A(D_z),$$

is constant. What we will actually prove, that this map is *locally liftable*.

A map  $h : \bar{\mathbb{C}} \rightarrow \text{Jac}(X)$  is called *locally liftable*, if every point of  $\bar{\mathbb{C}}$  admits an open neighborhood  $U$ , such that  $h|_U$  lifts to a holomorphic map  $H : U \rightarrow \Omega(X)^*$ .

It should be clear what “holomorphic map” here means. For example one can say that  $H(z)(\omega)$  is holomorphic in the usual sense for every  $\omega \in \Omega(X)$ . Before we prove the lifting property we show that it will solve our problem:

**4.10 Lemma.** *Every locally liftable map  $h : \bar{\mathbb{C}} \rightarrow \text{Jac}(X)$  is constant.*

*Proof of the Lemma.* We fix a form  $\omega \in \Omega(X)$ . For a local lift  $H$  defined on some  $U$  we consider the holomorphic differential  $\omega_H = d(H(z)(\omega))$  on  $U$ . Let  $G$  be a local lift on some other  $V$ . Then we have on the intersection

$$H(z) = G(z) + \lambda(z), \quad \lambda(z) \in L.$$

The function  $\lambda(z)$  is a holomorphic function whose values lie in the countable set  $L$  (use Lemma 4.1). This means that  $\lambda$  is locally constant. So  $\omega_H$  and  $\omega_G$  coincide on the intersection. Hence they glue to a holomorphic differential on  $\bar{\mathbb{C}}$ . Since every holomorphic differential on the sphere is zero, we get that  $h$  is locally constant. Since  $\bar{\mathbb{C}}$  is connected we get that  $h$  is constant.  $\square$

*Proof of Proposition 4.9 continued.* It remains to show that the map  $h : \bar{\mathbb{C}} \rightarrow \text{Jac} X$ ,  $h(z) = D_z$ , is locally liftable. We fix a point  $b \in \bar{\mathbb{C}}$  and investigate the map  $h$  close to  $b$ . Let  $b_1, \dots, b_m$  be the points in  $X$  over  $a$  with multiplicities  $k_1, \dots, k_m$ . Then  $n = k_1 + \dots + k_m$  is the covering degree of  $f$ . For the Abel–Jacobi map we need a base point  $a \in X$  and curves  $\alpha_1, \dots, \alpha_n$  which combine  $a$  with the  $b_i$ . This is understood as follows. The first  $m_1$  curves go from  $a$  to  $b_1$ , the next  $m_2$  from  $a$  to  $b_2$  and so on. We consider small disk  $U$  around  $b$ . We know that then  $f^{-1}(z)$  is the union of disjoint disks  $U_1, \dots, U_m$  disks such

that  $b_i \in U_i$  for  $1 \leq i \leq m$ . Now take a point  $z \in U$  which is different from  $a$ . Since  $U$  can be taken small enough the point  $z$  will have  $n$  inverse images, where  $n$  is the covering degree of  $f$ . The  $n$  points are distributed over the  $m$  disks  $U_1, \dots, U_m$ . We can assume that the first  $m_1$  are in  $U_1$ , the next  $m_2$  in  $U_2$  and so on. Now consider curves  $\beta_i$ ,  $1 \leq i \leq n$  as follows. The first  $m_1$  ones lie in  $U_1$  and combine  $b_1$  with the  $z$ -s which lie in  $U_1$ . The next  $m_2$  ones lie in  $U_2$  and combine  $b_2$  with the  $z$ -s which lie in  $U_2$  and so on. Now we consider the sum of the integrals

$$\int_{\beta_i} \omega \quad (\omega \in \Omega(X)).$$

It is independent of the choice of the  $\beta_i$  and depends holomorphically on  $z$ . This sum represents the difference of  $A(D_b) - A(D_z)$ . We can take this sum to get a local lifting of  $h$ .  $\square$

## 5. Abel's theorem

Abel's theorem states that the map  $A : \text{Pic}^0(X) \rightarrow \text{Jac}(X)$  is injective. To prove this, we need some local preparation. We consider the unit disk  $E$ .

**5.1 Definition.** *A  $C^\infty$ -function  $f$  on  $E - \{0\}$  has order  $k$  in 0, if  $g(z) = f(z) \cdot z^{-k}$  extends to a  $C^\infty$ -function on  $E$ , which has no zero in 0.*

We are interested in the logarithmic derivative

$$\frac{df}{f} = k \frac{dz}{z} + \frac{dg}{g}.$$

Notice that  $dg/g$  is smooth on the whole  $E$ . Let  $\omega$  be a differential on  $E$  with compact support. We want to consider the integral

$$\int_E \frac{df}{f} \wedge \omega.$$

Since  $df/f$  is singular at 0, we have to check the existence of this integral. We write  $\omega = h_1 dz + h_2 d\bar{z}$ . Then

$$\frac{dz}{z} \wedge \omega = \frac{h_1}{z} dz \wedge d\bar{z} = 2i \frac{h_1}{z} dx \wedge dy.$$

To compute the integral we use polar coordinates  $z = re^{i\varphi}$ . Then

$$\int_E \frac{dz}{z} \wedge \omega = 2i \int_0^1 \int_0^{2\pi} \frac{h_1(re^{i\varphi})}{re^{i\varphi}} r dr d\varphi.$$

Since the factor  $r$  cancels, there is no problem with the existence of the integral.

Let  $g$  be a  $C^\infty$ -function with compact support. Then

$$\frac{df}{f} \wedge dg = d\left(g \cdot \frac{df}{f}\right).$$

We want to integrate over a circle  $|z| = r$ , where  $r < 1$  has been chosen so close to 1 such that  $g$  vanishes for  $|z| \geq r$ . Since  $gdf/f$  has a singularity in the origin, we have to be careful with the application of the Stokes formula. We have to choose a small  $\varepsilon > 0$  and then we can say

$$\int_E \frac{df}{f} \wedge dg = - \oint_{|z|=\varepsilon} g \cdot \frac{df}{f}.$$

(We take the integral in the mathematical positive sense, hence we need a minus sign.) We will take the limit  $\varepsilon \rightarrow 0$ . Then we can replace  $g$  by the constant  $g(0)$  and  $f$  by  $z^k$ . This gives:

**5.2 Lemma.** *Let  $f$  be a  $C^\infty$ -function on the punctured disk  $E - \{0\}$ , which is of order  $k$  at the origin. Let  $g$  be a  $C^\infty$ -function on  $E$  with compact support. Then*

$$\int_E \frac{df}{f} \wedge dg = kg(0).$$

The definition of order 5.1 can be generalized to Riemann surfaces in an obvious way: If  $f$  is a differentiable function in a punctured neighborhood of a point  $a$  of a Riemann surface  $X$ , one chooses a disk  $\varphi : U \rightarrow E$ ,  $\varphi(a) = 0$  around this point inside this neighborhood. Then  $f$  has order  $k$  in  $a$  if  $f_\varphi$  has order  $k$  in 0. It is clear that this definition doesn't depend on the choice of the disk.

**5.3 Definition.** *Let  $D$  be a divisor on a Riemann surface  $X$ . By a weak solution of  $D$  one understands a differentiable function in the complement of the support of  $D$ , such that  $f$  has order  $D(a)$  for each  $a$  of the support.*

Lemma 5.2 has then the obvious generalization:

**5.4 Lemma.** *Let  $f$  be a weak solution of a divisor  $D$  on a Riemann surface  $X$  and let  $g$  be a differentiable function on  $X$  with compact support. Then*

$$\frac{1}{2\pi i} \int_X \frac{df}{f} \wedge dg = \sum_{a \in X} D(a)g(a).$$

(The sum is finite.)

Next we have to construct weak solution. The essential part is a local construction:



**5.5 Lemma.** *Let  $a, b$  be points of the unit disk  $E$ . There is a weak solution  $f$  of the divisor  $(b) - (a)$  with the following additional property. There exists  $0 < r < 1$  such that  $f(z)$  is identical one for  $r \leq |z| < 1$ .*

*Proof.* In the case  $a = b$  we can take  $f \equiv 1$ , hence we can assume  $a \neq b$ . We need a simple but basic observation from complex calculus. The function  $(z-1)/(z+1)$  takes values on the negative real axis of and only if  $z$  is contained in the interval  $[-1, 1]$ . Hence the principal value of the logarithm defines a holomorphic function

$$\mathbb{C} - [-1, 1] \longrightarrow \mathbb{C}, \quad z \longmapsto \log \frac{z-1}{z+1}.$$

An easy consequence is:

*Let  $a, b$  be two different points in the disk  $|z| \leq r$ . Then there exists in the complement  $|z| > r$  a holomorphic branch of the logarithm  $\log \frac{z-b}{z-a}$ .*

In our situation we can take  $r < 1$ . Now we consider a differentiable function  $\psi$  on  $[r, 1]$  which is 1 close to  $r$  and 0 close to 1. The function

$$\exp\left(\psi \log \frac{z-b}{z-a}\right) \quad r \leq |z| < 1$$

is 1 if  $|z|$  is close to 1 and  $\frac{z-a}{z-b}$  if  $|z|$  is close to  $r$ . Hence we can glue it with the function

$$\frac{z-b}{z-a} \quad |z| \leq r \quad (z \neq a, b).$$

This gives the proof of 5.5. □

Again there is a generalization to Riemann surfaces:

**5.6 Lemma.** *Let  $\alpha : [0, 1] \rightarrow X$  be a curve on a Riemann surface,  $a = \alpha(0)$ ,  $b = \alpha(1)$ . There exists a weak solution of the divisor  $(b) - (a)$  with the following properties:*

- 1) *The function  $f$  is constant one outside some compact set.*
- 2) *For every closed differential  $\omega$  on  $X$  one has*

$$\int_{\alpha} \omega = \frac{1}{2\pi i} \int_X \frac{df}{f} \wedge \omega.$$

*Proof.* First we mention that  $df$  vanishes outside a compact set. Hence the existence of the integral on the right hand side is only a local problem around the singularities  $a$  and  $b$ , which we settled above.

Let  $c = \alpha(t_0)$ ,  $0 \leq t_0 < 1$  be another point on the curve. Assume that the problem has been solved for the curve  $\alpha|_{[0, t_0]}$ . Denote the solution by  $f_1$ . Let similarly  $f_2$  be a solution for the curve  $\alpha|_{[t_0, 1]}$ . Then  $f_1 f_2$  gives a solution

for the total curve  $\alpha$ . This observation allows us to restrict to the case, where the curve is contained in a disk  $U$ . In this disk we get a weak solution from 5.5. This solution can be extended by one to a solution on  $X$ . Now we have to prove the integral formula. In the disk  $U$  the differential  $\omega$  is total,  $\omega = dg$ . Now we take a smaller disk  $V \subset\subset U$  such that  $f$  is one outside  $V$ . We can modify  $g$  to get a differentiable function with compact support such that the equation  $\omega = dg$  still holds on  $V$ . Then

$$\int_{\alpha} \omega = g(b) - g(a)$$

and by 5.4 we also have

$$\frac{1}{2\pi i} \int_X \frac{df}{f} \wedge \omega = g(b) - g(a). \quad \square$$

Now we can see that every divisor of degree zero on a compact Riemann surface admits a weak solution. We write the divisor in the form

$$D = (a_1) + \cdots + (a_n) - (b_1) - \cdots - (b_n).$$

We join  $a_i$  and  $b_i$  by a curve  $\gamma_i$ . Using 5.6 we find a weak solution  $f$  of the divisor  $D$  which has the additional property

$$\sum_{i=1}^n \int_{\alpha_i} \omega - \sum_{i=1}^n \int_{\beta_i} \omega = \frac{1}{2\pi i} \int_X \frac{df}{f} \wedge \omega$$

for all closed differentials  $\omega$ . We will use this for *holomorphic*  $\omega$ . Then

$$\frac{df}{f} \wedge \omega = \frac{\bar{\partial}f}{f} \wedge \omega.$$

Recall that  $f$  locally is of the form  $z^k g(z)$  with a differentiable function  $g$  without zero. This implies (locally)

$$\frac{\bar{\partial}f}{f} = \frac{\bar{\partial}g}{g}.$$

Hence

$$\sigma := \frac{\bar{\partial}f}{f} \in A^{0,1}(X)$$

is differentiable everywhere.

Let's assume now that the divisor  $D$  is in the kernel of the Abel–Jacobi map. Then we get

$$\frac{1}{2\pi i} \int_X \sigma \wedge \omega = \sum_{i=1}^n \int_{\gamma_i} \omega = 0$$

for all holomorphic  $\omega$ . But this implies  $\sigma = \bar{\partial}h$  (corollary of 2.1). We use  $h$  to modify the weak solution  $f$ :

$$F = e^{-h} f.$$

We obtain

$$\bar{\partial}F = -e^{-h} f \bar{\partial}h + e^{-h} \bar{\partial}f = 0.$$

Hence  $F$  is a meromorphic solution. This gives

**5.7 Abel's theorem.** *The homomorphism*

$$A : \text{Pic}^0(X) \longrightarrow \text{Jac}(X)$$

*is injective.*

*In other words: A divisor of degree zero is principal if and only if its image in  $\text{Jac}(X)$  is zero.*

As an example we treat a torus  $X = \mathbb{C}/L$ . Recall that  $\Omega(X)$  is generated by “ $\omega = dz$ ”. We represent the divisor  $D$  of degree zero by a divisor  $(a_1) + \cdots + (a_n) - (b_1) - \cdots - (b_n)$  in the interior of some fundamental parallelogram. We join  $a_i$  and  $b_i$  by curves  $\gamma_i$  inside this interior. Since

$$\sum_{i=1}^n \int_{\gamma_i} dz = \sum_{i=1}^n b_i - \sum_{i=1}^n a_i.$$

This gives the well-known Abel theorem for elliptic functions:

*A divisor of degree zero  $(a_1) + \cdots + (a_n) - (b_1) - \cdots - (b_n)$  on a torus is principal if and only if  $a_1 + \cdots + a_n - b_1 - \cdots - b_n = 0$ .*

**6. The Jacobi inversion problem**

All what we have proved about the group  $L \subset \Omega(X)^*$  is that it is countable group. We want to prove more, namely that it is a lattice. By a lattice of a finite dimensional real vector space  $V$  we understand a subgroup  $L$  of the form

$$L = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$$

where  $e_1, \dots, e_n$  is a basis of  $V$ . A lattice is obviously a discrete subgroup which generates  $V$  as vector space. The converse is also true. We will use this without proof.

By a lattice of a finite dimensional complex vector space we understand a lattice of the underlying real vector space.

**6.1 Proposition.** *The set of periods  $L \subset \Omega(X)^*$  of a compact Riemann surfaces is a lattice. Hence  $\text{Jac}(X)$  is a torus of real dimension  $2g$ .*

*First part of the proof:  $L$  is discrete.*

Let  $V$  be a complex vector space of dimension  $g$ . If  $\mathfrak{M}$  is a set of sub-vector spaces whose intersection is 0, then there exist  $g$  vector spaces in  $\mathfrak{M}$  whose intersection is zero. For a point  $a \in X$  we can consider the subspace of all  $\omega \in \Omega(X)$ , which vanish in  $a$ . The above remark shows, that there exist  $g$

points  $a_1, \dots, a_g$  such that a form  $\omega \in \Omega(X)$  vanishes if it vanishes in these points. Let  $\omega_1, \dots, \omega_g$  be a basis of  $\Omega(X)$ . We choose disks  $z_i : U_i \rightarrow E$  around the points  $a_i$ . In these charts  $\omega_i = f_{ij} dz_j$ . Here we write  $f_{ij}$  as a function on  $U_i$  and not on  $E$ . Then

$$A = (f_{ij}(a_j))_{1 \leq i, j \leq g}$$

is an invertible matrix. In the disk  $U_i$  and we define the holomorphic function

$$F_i(x) = \int_{a_i}^x \omega_i,$$

where the integral is taken along a path from  $a_j$  to  $x$  inside  $U_i$ . then we consider the map

$$F : U_1 \times \cdots \times U_g \longrightarrow \mathbb{C}^g, \quad F(\xi_1, \dots, \xi_g) = F_1(\xi_1) + \cdots + F_g(\xi_g).$$

The (complex) Jacobian of  $F$  with respect to the charts is  $(f_{ij})$ . From the theorem of invertible functions follows that

$$W := F(U_1 \times \cdots \times U_g) \subset \mathbb{C}^g$$

is a neighborhood of  $F(a_1, \dots, a_g)$ .

Here we use a complex version of the theorem of invertible functions of several variables. But this is a consequence of the real version. The point is that  $F$  is differentiable in the real sense and the real functional determinant is the square of the absolute value of the complex functional determinant. To prove this one has to express the real derivatives by the complex ones and then use the Cauchy Riemann equations.

The basis  $\omega_1, \omega_g$  induces a dual basis  $\omega_1^*, \dots, \omega_g^*$  of  $\Omega(X)^*$ , namely

$$\omega_i^*(\omega_j) = \delta_{ij}.$$

We use this basis to identify the space  $\Omega(X)^*$  with  $\mathbb{C}^g$ . Then the map  $F$  can be read as a map

$$F : U_1 \times \cdots \times U_g \longrightarrow \Omega(X)^*.$$

The advantage is that it doesn't depend on the choice of the basis, since

$$F(\xi_1, \dots, \xi_g)(\omega) = \sum_{i=1}^g \int_{a_i}^{\xi_i} \omega.$$

(We used the coordinates just since the theorem of invertible functions often is not formulated in a coordinate invariant manner.)

We claim the the neighborhood  $0 \in W$  has empty intersection with  $L$ . (Since  $L$  is a group this implies the discreteness of  $L$ .) We argue by contradiction and assume  $F(\xi_1, \dots, \xi_g) = 0$  Since  $F(\xi_1, \dots, \xi_g)$  represents the image of the divisor

$D = (\xi_1) + \cdots + (\xi_g) - (a_1) - \cdots - (a_g)$  under the Abel–Jacobi map, we can apply Abels’s theorem 5.7. There exists a meromorphic function  $f$  with  $(f) = D$ . The function  $f$  has poles of order one in the  $a_i$ . We denote by  $C_i \neq 0$  the residue of  $f$  with respect to the chart  $z_i$ . Then

$$\text{Res}_{a_i}(f\omega_i) = C_i f_{ij}(a_j).$$

From the residue theorem applied to the meromorphic differentials  $f\omega_i$  follows

$$\sum_{j=1}^g C_j \varphi_{ij}(a_j) = 0.$$

But then the matrix  $A$  cannot be invertible, which gives a contradiction.

*Second part of the proof:  $L$  generates  $\Omega(X)^*$  as real vector space.*

We have to show that a real linear form  $L$  on  $\Omega(X)^*$ , which vanishes on  $L$  is zero. Every real linear form is the real part of a  $\mathbb{C}$ -linear complex linear form. Every complex linear form on  $\Omega(X)^*$  is of the form

$$l \longmapsto l(\omega)$$

for some  $\omega \in \Omega(X)$ . The real part of this linear form vanishes on  $L$  if and only if the real parts of the periods of  $\omega$  are zero. But then  $\omega$  is zero (4.4).  $\square$

The first part of the proof of 6.1 shows that a full neighbourhood of  $0 \in \text{Jac}(X)$  is contained in the image of the Abel–Jacobi map. The image is also a group. Obviously a torus is generated as group by any neighborhood of the origin. Hence we obtain that the Abel–Jacobi map is not only injective but also surjective.

**6.2 Theorem.** *The Abel–Jacobi map*

$$A : \text{Pic}^0(X) \xrightarrow{\sim} \text{Jac}(X)$$

*is an isomorphism.*

We com back to the Abel–Jacobi map in the form

$$A : X^d \longrightarrow \text{Jac}(X).$$

It depends on the choice of a base point  $a$ .

**6.3 Remark.** *Let  $X$  be compact Riemann surface of genus  $g > 1$ . Then*

$$A : X \longrightarrow \text{Jac}(X)$$

*is injective.*

*Proof.* We mention that a bijective holomorphic map  $f : X \rightarrow Y$  between Riemann surfaces is biholomorphic. This is known from complex calculus in the case of open domains in  $\mathbb{C}$ . The general case works in the same way. Let  $f$  be a meromorphic function on a compact Riemann surface with divisor  $(b) - (a)$ . Then it is of order one and hence defines a bijective map  $X \rightarrow \bar{\mathbb{C}}$ . Hence  $X$  is biholomorphic equivalent to  $\bar{\mathbb{C}}$ . This implies 6.3.  $\square$

We call a point  $a \in X^g$  *generic*, if every holomorphic differential  $\omega$ , which vanishes at all points occurring in  $a$ , vanishes identically. In the first part of the proof of 6.1 we have shown that there are generic points. The argument shows a little more, namely:

**6.4 Remark.** *The set of generic points is open and dense in  $X^g$ . The map  $A : X^g \rightarrow \text{Jac}(X)$  is locally topological on the set of generic points.*

Let now  $X$  be a compact Riemann surface of genus one. Then  $\text{Jac}(X) = \mathbb{C}/L$  is also a Riemann surface. The map  $A : X \rightarrow \text{Jac}(X)$  is injective. Since it is also injective we obtain:

**6.5 Theorem.** *Every compact Riemann surface of genus one is biholomorphic equivalent to a torus  $\mathbb{C}/L$ .*

Now we switch to  $A : X^d \rightarrow \text{Jac}(X)$ . One may ask, whether this map is surjective for suitable  $d$ . Since the real dimension of  $X^d$  is  $2d$  and of  $\text{Jac}(X)$  is  $2g$ , one can hope, that this is true for  $d = g$ .

**6.6 Theorem.** *The Abel–Jacobi map*

$$A : X^g \rightarrow \text{Jac}(X)$$

*is surjective.*

*Proof.* let  $a \in X$  be the base point. Since  $\text{Pic}^0(X) \rightarrow \text{Jac}(X)$  is surjective, we only must show the following:

*Every divisor  $D$  of degree 0 is equivalent to a divisor  $(b_1) + \cdots + (b_g) - g \cdot (a)$ .*

For the proof we consider the divisor  $D' = D + g(a)$ . It has degree  $g$ . Riemann–Roch implies that  $\dim \mathcal{O}_{D'}(X) \geq 1$ . Hence there exists a non-zero meromorphic function  $f$  such that  $(f) + D' \geq 0$ . Since the degree of this divisor is  $g$  we get  $(f) + D' = (b_1) + \cdots + (b_g)$ . This gives  $(f) + D = (b_1) + \cdots + (b_g) - g(a)$ .  $\square$

The symmetric group  $S_g$  acts on  $X^g$  by permutation of the components. The quotient

$$X^{(g)} = X^g / S_g$$

can be considered as the set of unordered  $n$ -tuples of  $X$ . Since the map  $X^g \rightarrow \text{Jac}(X)$  does not depend on the ordering of the points, it factors through the natural projection  $X^g \rightarrow X^{(g)}$ . We obtain the *Jacobi map*

$$J : X^{(g)} \rightarrow \text{Jac}(X).$$

The Jacobi inversion problem asks for the inversion of this map. Actually for  $g > 1$  this map is not bijective but it is close to a bijective map. The correct statement is:

**6.7 Jacobi inversion theorem, unprecise statement.** *The map*

$$J : X^{(g)} \longrightarrow \text{Jac}(X).$$

*is bimeromorphic.*

Up to now we didn't define what bimeromorphic means. This will be part of what follows.

## 7. The fibres of the Jacobi map

We can identify the elements of  $X^{(g)}$  with divisors  $D \geq 0$  of degree  $g$ . We have to determine the fibre  $J^{-1}(J(D))$ . It consists of all divisors  $D' \geq 0$  of degree  $g$  which are equivalent to  $D$ . Then there exist a meromorphic function with  $D' = D + (f)$ . Because of  $D' \geq 0$  we have  $f \in \mathcal{O}_D(X)$ . The function  $f$  is determined up to a constant factor. Hence it is better to identify two  $f$  if they are contained in the same one-dimensional complex sub-vector space. Recall that the set of all one-dimensional sub-vector spaces of a vector space  $V$  is called the projective space  $P(V)$ . There is a natural map  $V - \{0\} \rightarrow P(V)$ . We use this (for finite dimensional  $V$ ) to define a topology on  $P(V)$ , the quotient topology. It is easy to show that this is a compact space.

**7.1 Remark.** *Let  $D \in X^{(g)}$ . There is a natural bijective map*

$$P(\mathcal{O}_D(X)) \xrightarrow{\sim} f^{-1}(f(D)), \quad "f \longmapsto D + (f)".$$

The space  $X^g$  carries the product topology and  $X^{(g)} = X^g/S_g$  the quotient topology. It is easy to see that this is a Hausdorff space.

**7.2 Lemma.** *The map*

$$P(\mathcal{O}_D(X)) \xrightarrow{\sim} f^{-1}(f(D)), \quad f \longmapsto D + (f),$$

*is topological.*

**Corollary.** *The fibres of the Jacobi map are connected.*

From Remark 6.4 we know that a generic point in  $X^g$  is isolated in the fibre of  $X^g \rightarrow \text{Jac}(X)$ . As a consequence its image in  $X^{(g)}$  is isolated in its fibre as well. From Lemma 7.2 follows that is the only point in its fibre. Hence we obtain:

**7.3 Theorem.** *The Jacobi map  $J : X^{(g)} \rightarrow \text{Jac}(X)$  is surjective. There exists an open and dense subset of  $X^{(g)}$ , on which  $J$  is injective.*

This can be considered as a weak version of the Jacobi inversion theorem, which states that  $J$  is bimeromorphic.

## 8. The Jacobian inversion problems and abelian functions

We need some basic facts about holomorphic functions in several variables. A function  $f : D \rightarrow \mathbb{C}^m$ , where  $D \subset \mathbb{C}^n$  is called complex differentiable in a point  $a \in D$  if there exists a  $\mathbb{C}$ -linear map  $J(f, a) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ , such that

$$f(z) - f(a) = J(f, a)(z - a) + r(z), \quad \lim_{z \rightarrow a} \frac{r(z)}{z - a} = 0.$$

Hence complex differentiable implies real differentiable. The usual permanent properties including the theorem of invertible functions carry over to the complex case. The function  $f$  is called holomorphic if and only if it is complex differentiable in each point. This is the case if and only if every component of  $f$  is holomorphic. There is a basic

**8.1 Lemma.** *Let  $f : D \rightarrow \mathbb{C}$  be a continuous function on an open domain  $D \subset \mathbb{C}^n$  such that  $f$  is holomorphic in each variable separately. Then  $f$  is holomorphic. Moreover for every point  $a \in D$  there exists a power series, which converges absolutely and locally uniform in some ball  $U_r(a) \subset D$  and represents there  $f$ :*

$$f(z) = \sum_{\nu \in \mathbb{N}_0^n} (z - a)^\nu \quad (a \in U_r(a)).$$

Here we use the usual writing with multiindices. The proof uses a simple generalization of the Cauchy integral formula. Applying the usual one in each variable one obtains:

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{|\zeta_1 - a_1| = r} \cdots \oint_{|\zeta_n - a_n| = r} \frac{f(\zeta) d\zeta_1 \cdots d\zeta_n}{(z_1 - \zeta_1) \cdots (z_n - \zeta_n)}$$

Now the same argument as in the case  $n = 1$  works: One expands the integrand in to a (multivariable) geometric series and interchanges summation and integration.

The power series expansion shows a weak form of the principle of analytic continuation: If two analytic functions  $f, g : D \rightarrow \mathbb{C}^m$  on a connected open subset  $D \subset \mathbb{C}^n$  agree on some non empty open subset then they agree everywhere.



### Analytic manifolds

Analytic manifolds of complex dimension  $n \geq 1$  are the straight forward generalization of Riemann surfaces ( $n = 1$ ). Hence we can keep short:

*An analytic manifold is a geometric space, which is locally isomorphic to a space  $(U, \mathcal{O}_U)$ , where  $U \subset \mathbb{C}^n$  is an open subset and  $\mathcal{O}_U$  is the sheaf of holomorphic functions. An analytic map between analytic manifolds is just a morphism of geometric spaces.*

Basis facts of Riemann surfaces carry over to analytic manifolds. We mention just the following weak form of the principle of analytic continuation which easily follows by means of power series:

*Let  $f, g : X \rightarrow X$  be two analytic maps between to analytic manifolds. Assume that  $X$  is connected and that  $f$  and  $g$  agree on some non-empty open subset. Then  $f$  and  $g$  agree on the whole  $X$ .*

### Meromorphic functions

Zero sets of analytic functions are not discrete in general as for example the function  $z_1 \cdot z_2$  on  $\mathbb{C}^2$  shows. This makes the notion of a meromorphic function more delicate in the higher dimensional case. An example of a meromorphic function on  $\mathbb{C}^2$  should be  $z_1/z_2$ . There is now clean way to define a value for this function at the origin 0. Hence it would be false to define meromorphic functions simply as holomorphic maps into  $\bar{\mathbb{C}}$ . This works only in the case  $n = 1$ . We proceed as follows:

Let  $X$  be an analytic manifold. We consider pairs  $(U, f)$ , where  $U \subset X$  is an open an dense subset of  $X$  and  $f : U \rightarrow \mathbb{C}$  is an analytic function. We call this pair meromorphic on  $X$ , if for every point  $a \in X$  (only  $a \notin U$  is of interest), there exists a small open connected neighbourhood  $U$  and two analytic functions  $g, h : U \rightarrow \mathbb{C}$ , such that  $h$  is not identically zero and such that

$$f(x) = \frac{g(x)}{h(x)} \quad \text{for all } x \in U, h(x) \neq 0.$$

two meromorphic pairs  $(U, f)$  and  $(V, g)$  are called equivalent, if  $f$  and  $g$  agree on  $U \cap V$ . A *meromorphic function* on  $X$  is a full equivalence class of such pairs. It is easy to define the sum and product of two meromorphic functions. It is also not difficult to prove that for connected  $X$  the set of all meromorphic functions is a field. (We don't know that  $U$  is connected. But this doesn't matter, since the condition of meromorphicity concerns the whole domain  $D$ .) We denote this field by  $K(X)$ .

### Examples of analytic manifolds

The first example is a torus  $X_L := \mathbb{C}^n/L$ , where  $L$  is a lattice. As in the case  $n = 1$  the meromorphic functions on  $X_L$  correspond uniquely to the  $L$ -periodic

meromorphic functions on  $\mathbb{C}^n$ . Such functions are called “*abelian functions*”. They generalize the elliptic functions.

The direct product  $X \times Y$  of two analytic manifolds carries also a structure as analytic manifold. As a consequence the power  $X^n = X \times \dots \times X$  of a Riemann surface is an analytic manifold.

We go back to the Abel–Jacobi map

$$A : X^g \longrightarrow \text{Jac}(X).$$

Pulling back a function we get an imbedding of fields

$$K(\text{Jac}(X)) \longrightarrow K(X^g).$$

The image is contained in the subfield  $K(S^g)^{S_g}$  of all function which are invariant under arbitrary permutations.

It can be shown that The symmetric power  $X^{(g)}$  is an analytic manifold as well and that  $K(X^{(g)})$  can be naturally identified with  $K(X^g)^{S_g}$ . We will not prove this and take  $K(S^g)^{S_g}$  just as substitute for the correct field  $K(X^{(g)})$ .

## 8.2 Jacobi inversion theorem, precise statement. *The natural map*

$$K(\text{Jac}(X)) \longrightarrow K(X^g)^{S_g}$$

*is an isomorphism of fields.*

We will not prove this here.

We work out a special case of Theorem 8.2, which will lead us to a formulation which is close to what hat Jacobi in mind. Let  $f : X \rightarrow \bar{\mathbb{C}}$  be a non-constant meromorphic function. It induces an analytic map

$$f^g : X^g \rightarrow \bar{\mathbb{C}}^g.$$

The projections give  $g$  analytic maps

$$f_\nu : X^g \longrightarrow \bar{\mathbb{C}} \quad (1 \leq \nu \leq g).$$

They are just defined by

$$f_\nu(\xi_1, \dots, \xi_g) = f(\xi_\nu).$$

They can be considered as meromorphic functions on  $X^g$ . But they are not invariant under  $S_g$ . Hence we consider the elementary symmetric expressions

$$E_k = \sum_{1 \leq \nu_1 \leq \dots \leq \nu_k \leq g} f(\xi_{\nu_1}) \dots f(\xi_{\nu_k})$$

They are also meromorphic functions on  $X^g$  with the advantage to be symmetric. The Jacobi inversion theorem predicts the *existence of abelian functions*  $F_\nu$  on  $X_L$  whose pull-back are the  $E_k$ .

**Historical cases**

We consider the Riemann surface  $X$  of the function  $\sqrt{P(z)}$  where  $P$  is a polynomial of degree 3 or 4 without multiple zero. Recall that the corresponding Riemann surface has genus one. Up to a finite number of points it is the curve  $(z, w)$ ,  $w^2 = P(z)$ . The natural projection  $(z, w) \mapsto z$  is a meromorphic function on  $X$ , which we take for  $f$ . Recall that  $dz/\sqrt{P(z)}$  generates  $\Omega(X)$ . The Abel–Jacobi map is

$$X \longrightarrow \mathbb{C}/L, \quad x \longmapsto \int_a^x \frac{dz}{\sqrt{P(z)}}.$$

The inversion theorem says in this case, that the function  $z$  is the pull-back of an elliptic function  $\varphi$ . Pull-map means that the composition of  $\varphi$  with the Abel–Jacobi map gives  $x$ , i.e.

$$x = \varphi \left( \int_a^x \frac{dz}{\sqrt{P(z)}} \right).$$

This reflects the classical observation of Abel that the inversion of the elliptic integral gives an elliptic function.

The big question was how to generalize this to the hyperelliptic case. Let’s consider for the case  $\sqrt{P(z)}$ , where  $P$  now is of degree 5 or 6 without multiple zero. Recall that a basis of  $\Omega(X)$  in this case is given by

$$\frac{dz}{\sqrt{P(z)}}, \quad \frac{zdz}{\sqrt{P(z)}}.$$

The Abel–Jacobi map now is given by

$$(\xi_1, \xi_2) \longmapsto \left( \int_a^{\xi_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{dz}{\sqrt{P(z)}}, \int_a^{\xi_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{zdz}{\sqrt{P(z)}} \right)$$

Now we have to consider the two elementary symmetric functions  $\xi_1 + \xi_2, \xi_1\xi_2$ . The Jacobi inversion theorem states:

*The are two abelian functions  $\varphi_1, \varphi_2$  such that*

$$\begin{aligned} \xi_1 + \xi_2 &= \varphi_1 \left( \int_a^{\xi_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{dz}{\sqrt{P(z)}}, \int_a^{\xi_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{zdz}{\sqrt{P(z)}} \right) \\ \xi_1\xi_2 &= \varphi_2 \left( \int_a^{\xi_1} \frac{dz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{dz}{\sqrt{P(z)}}, \int_a^{\xi_1} \frac{zdz}{\sqrt{P(z)}} + \int_a^{\xi_2} \frac{zdz}{\sqrt{P(z)}} \right) \end{aligned}$$

Jacobi formulated this as a problem before the theorem of Riemann surfaces existed. The theory of Riemann surfaces enabled to proof this and moreover to reformulate it in a precise and very natural way. Moreover Jacobi’s inversion problem opened the door to the theory of complex analytic functions in several variables.

## Chapter VIII. Dimension formulae for automorphic forms

In his paper [Bo1], at the end of Sect. 4, Borchers mentions without proof a beautiful formula for dimensions of spaces of vector valued modular forms of weight  $\geq 2$  with respect to the full modular group (here reproduced as Theorem 8.6.1).

Skoruppa informed me that he derived already 1985 in his Ph.D. thesis [Sk] these dimension formulas for all weights by means of the Shimura trace formula.

More general results, including arbitrary Fuchsian groups, can be found in the paper [Bo2] of Borchers, Sect. 7. Most of them have been proved by the Selberg trace formula, see [Iv] and also [Fi]. The Selberg trace formula in its standard form causes the restriction that the weight is  $> 2$ . Borchers mentions that “with a bit more care this also works for weight 2”. As we mentioned, this bit more care was taken in important cases already 1985 in the thesis of Skoruppa.

The purpose of this chapter is to produce the dimension formula in all weights, for general Fuchsian groups and for arbitrary rational weights.

We consider arbitrary discrete subgroups  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})/\pm$  with finite volume of the fundamental domain. For them we consider vector valued modular forms of a rational weight  $r$ . They have the transformation property

$$f(\gamma\tau) = \gamma'(\tau)^{-r/2} v(\gamma) f(\tau),$$

where  $v(\gamma)$  is matrix-valued. We assume that all  $v(\gamma)$  are of finite order and that they can be diagonalized simultaneously for all  $\gamma$  in a subgroup of finite index  $\Gamma_0$ . We make some further weak assumption for  $v$  (see Assumption 3.1) which is fulfilled at least for arithmetic groups. These assumptions are not really necessary. In principle, one could consider arbitrary real weights  $r$  and we could take arbitrary unitary multiplier systems in the sense of [Fi]. But we found that these restrictions are convenient and they cover all cases which occur usually in the theory of modular forms.

Since we allow arbitrary rational weights, there is an ambiguity in the definition of  $\gamma'(\tau)^{-r/2}$ . One standard way to overcome this, is, to use covering groups of  $\mathrm{SL}(2, \mathbb{R})$ . Instead of this we found it convenient to use the old-fashioned method of multiplier systems, here in a matrix-valued sense. They could be also called “projective representations”. In Sect. 6 we reformulate the results for representations of the two fold metaplectic covering of  $\mathrm{SL}(2, \mathbb{Z})$  and reproduce Borchers’ formula in [Bo1].

Our proof rests on the Riemann–Roch formula for vector bundles (and not on the Selberg trace formula). The idea is to use a sufficiently small normal

subgroup  $\Gamma_0$  for which the vector bundle splits into a sum of line bundles. This gives a reduction to the well-known case of scalar valued modular forms.

## 1. Fuchsian groups and Riemann surfaces

The group  $\mathrm{SL}(2, \mathbb{R})$  can be considered as closed subset of  $\mathbb{R}^4$ . We equip it with the induced topology. It is a locally compact space with countable basis of the topology. We consider discrete subgroups  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ . Here a subset  $S$  of a topological space  $X$  is called a discrete subset if it is closed and if the induced topology is the discrete topology (every set is open). For locally compact  $X$  this means that the intersection of  $S$  with each compact subset of  $X$  is finite. A basic example of a discrete subgroup in  $\mathrm{SL}(2, \mathbb{R})$  is the group  $\mathrm{SL}(2, \mathbb{Z})$ .

We recall that each matrix  $M \in \mathrm{GL}(2, \mathbb{C})$  induces a biholomorphic transformation of the Riemann sphere onto itself:

$$M\tau = (a\tau + b)(c\tau + d)^{-1}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If  $M$  is real and has positive determinant, then  $M$  maps  $\mathbb{H}$  onto itself and it maps also the extended real line  $\mathbb{R} \cup \{\infty\}$  onto itself. In particular, the group  $\mathrm{SL}(2, \mathbb{R})$  acts on the upper half plane. Two elements of  $\mathrm{SL}(2, \mathbb{R})$  define the same transformation if and only they differ by a sign. The elements  $\pm E$  ( $E$  denotes the unit matrix) build a normal subgroup of  $\mathrm{SL}(2, \mathbb{R})$ . Hence in the following we are more interested in the factor group  $\mathrm{SL}(2, \mathbb{R})/\pm$ . But at the moment we prefer to work with matrices.

**1.1 Definition.** *An element  $M \in \mathrm{SL}(2, \mathbb{R})$  is called **elliptic** if  $|\mathrm{tr}(M)| < 2$  and **parabolic** if it is different from  $\pm E$  and if  $|\mathrm{tr}(M)| = 2$ .*

From the fixed point equation

$$M(\tau) = \tau \iff c\tau^2 + (d - a)\tau - b = 0$$

we obtain the following result.

**1.2 Lemma.** *An element  $M \in \mathrm{SL}(2, \mathbb{R})$  is elliptic if and only if it is different from  $\pm E$  and if it has a fixed point in  $\mathbb{H}$ . This fixed point is unique.*

*An element  $M \in \mathrm{SL}(2, \mathbb{R})$  is parabolic if and only if it has a unique fixed point in  $\mathbb{R} \cup \{\infty\}$ .*

We say that a subgroup  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  acts *discontinuously* if for each two compact subsets  $K_1, K_2 \subset \mathbb{H}$  the set

$$\{M \in \Gamma; \quad M(K_1) \cap K_2 \neq \emptyset\}$$

is finite. It is enough to demand this for  $K_1 = K_2$  (consider  $K_1 \cup K_2$ ). In particular, the stabilizer

$$\Gamma_a := \{M \in \Gamma; \quad Ma = a\}$$

is a finite subgroup for every  $a \in \mathbb{H}$ .

**1.3 Proposition.** *Each discrete subgroup  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  acts discontinuously.*

The proof of this Proposition and its converse can be found in [Fr1], III.2.3 or [Fr2], I.1.2. Here we just mention that this is an immediate consequence of the fact that the map

$$\mathrm{SL}(2, \mathbb{R}) \longrightarrow \mathbb{H}, \quad M \longmapsto M(i),$$

is proper. This follows from the fact the the stabilizer of  $i$  is the special orthogonal group  $\mathrm{SO}(2, \mathbb{R})$  which is a compact subgroup.

Two points  $a, b \in \mathbb{H}$  are called equivalent with respect to  $\Gamma$  if there exists  $M \in \Gamma$  such that  $b = Ma$ . It is easy to check that this is an equivalence relation. We denote by  $[a]$  the equivalence class of  $a$  and we denote by  $\mathbb{H}/\Gamma$  the set of all equivalence classes. We equip  $\mathbb{H}/\Gamma$  with the quotient topology. There is a natural projection  $\mathbb{H} \rightarrow \mathbb{H}/\Gamma$ . It is continuous and open (i.e the images of open sets are open).

**1.4 Proposition.** *Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  be a discrete subgroup. The quotient  $\mathbb{H}/\Gamma$  is a (Hausdorff) locally compact space with countable basis of the topology. Moreover the following holds.*

*For each  $a \in \mathbb{H}$  there exists an open neighborhood  $U(a)$  which is invariant under the stabilizer  $\Gamma_a$  and such that two points in  $U(a)$  are equivalent mod  $\Gamma$  if and only if they are equivalent with respect to  $\Gamma_a$ .*

For a proof we refer to [Fr2], I.1.7. □

We can define the quotient space  $U(a)/\Gamma_a$  in the obvious way. There is a natural map

$$U(a)/\Gamma_a \longrightarrow \mathbb{H}/\Gamma.$$

The condition in Proposition 1.4 says that this map is injective. Moreover, it is an open embedding. This means that its image is open and that this map induces a topological map onto this image. Hence  $\mathbb{H}/\Gamma$  looks locally (close to  $[a]$ ) like  $U(a)/\Gamma_a$ .

### Extension by cusps

Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  be a discrete subgroup. An element  $a \in \mathbb{R} \cup \{\infty\}$  is called a *cusp* of  $\Gamma$  if it is the fixed point of a parabolic element  $M \in \Gamma$ . The elements of  $\mathrm{SL}(2, \mathbb{R})$  which fix  $\infty$  are of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

Only in the case  $a = \pm 1$  they are parabolic. Hence the point  $\infty$  is a cusp if and only if  $\Gamma$  contains a matrix of the form

$$\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \neq 0.$$

Let  $\kappa \neq \infty$ . Then we can choose a matrix  $N \in \mathrm{SL}(2, \mathbb{R})$  such that  $N(\kappa) = \infty$ . For example one can take

$$N = \begin{pmatrix} 0 & 1 \\ -1 & \kappa \end{pmatrix}.$$

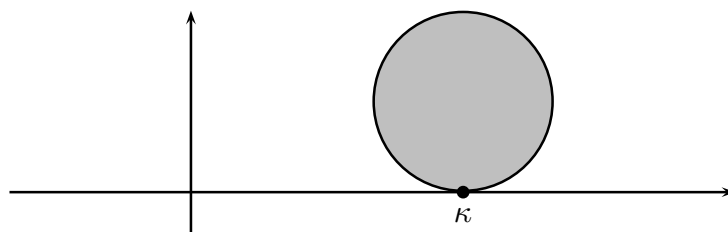
Then the conjugated group  $N\Gamma N^{-1}$  has cusp  $\infty$ . This transformation is often used to reduce properties of arbitrary cusps to the cusp  $\infty$ . We have to consider the stabilizer  $\Gamma_\kappa$  of a cusp. We have

$$N\Gamma_\kappa N^{-1} = (N\Gamma N^{-1})_\infty.$$

Hence transforming  $\kappa$  to  $\infty$  allows to restrict to  $\kappa = \infty$ . In this case the elements of  $\Gamma_\infty$  act as translations  $\tau \mapsto \tau + b$ . There exists a smallest positive  $b$ . Then every transformation of  $\Gamma_\infty$  acts as  $\tau \mapsto \tau + nb$  with some integer  $n$ . This means that the image of the group  $\Gamma_\infty$  in  $\mathrm{SL}(2, \mathbb{R})/\pm$  is a cyclic group generated by  $\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ .

In the following we denote by  $\mathbb{H}^*$  the union of  $\mathbb{H}$  and the set of cusps. Of course this depends on  $\Gamma$ . It is easy to see that a subgroup  $\Gamma_0 \subset \Gamma$  of finite index has the same cusps as  $\Gamma$  and hence leads to the same  $\mathbb{H}^*$ . The group  $\Gamma$  acts on  $\mathbb{H}^*$ .

We want to define a topology on  $\mathbb{H}^*$ . This will not be the induced topology of the Riemann sphere. Recall that in the Riemann sphere a sequence  $(a_n)$  tends to  $\infty$  if the absolute values  $|a_n|$  tend to  $\infty$ . But we want to have the following: a sequence  $(a_n)$  of points in the upper half plane should tend to  $\infty$  if the imaginary parts tend to  $\infty$ . Hence a typical neighborhood of  $\infty$  should be defined by  $\mathrm{Im} \tau > C$  (and not by  $|\tau| > C$  as in the topology of the Riemann sphere). If we apply to the set  $\mathrm{Im} \tau > C$  a transformation  $M \in \mathrm{SL}(2, \mathbb{R})$  which fixes  $\infty$  then we obtain a set of the same form (usually with different  $C$ ). In all other cases we obtain an open disk in the upper half plane which is tangent to the real axis.



By a *horocycle* at the cusp  $\kappa$  we understand such a disk if  $\kappa \neq \infty$  or a set of the form  $\mathrm{Im} \tau > C > 0$  otherwise.

**1.5 Definition.** *Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  be a discrete subgroup. A subset  $U \subset \mathbb{H}^*$  is called open if the following two conditions hold.*

- a) *The intersection  $U \cap \mathbb{H}$  is open in the usual sense.*
- b) *Assume that the cusp  $\kappa$  is contained in  $U$ . Then  $U$  contains a horocycle at the cusp  $\kappa$ .*

Hence a typical neighborhood of a cusp  $\kappa$  is the union of a horocycle at  $\kappa$  and  $\{\kappa\}$ . It is trivial that this defines a topology on  $\mathbb{H}^*$ . It is rather clear that  $\mathbb{H}$  is an open subset of  $\mathbb{H}^*$  and that the induced topology is the usual one.

But this topology has some properties which look rather strange at a first glance. For example each cusp has a neighborhood which contains no other cusp besides  $\kappa$ . This means that the set of cusps is a discrete subset of  $\mathbb{H}^*$ . We also mention that  $\mathbb{H}^*$  is not locally compact.

A basic example is the group  $\mathrm{SL}(2, \mathbb{Z})$ . It is easy to see that all rational numbers and  $\infty$  are cusps. Hence  $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$  in this case and  $\mathbb{Q}$  is a discrete subset!

Every element  $M \in \Gamma$  induces a topological map from  $\mathbb{H}^*$  onto itself. It is clear, how to define the quotient space  $X_\Gamma := \mathbb{H}^*/\Gamma$  and the natural projection

$$\mathbb{H}^* \longrightarrow X_\Gamma, \quad a \longmapsto [a].$$

It is continuous and open. The following generalization of Proposition 1.4 holds.

**1.6 Proposition.** *Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$  be a discrete subgroup. The quotient  $X_\Gamma = \mathbb{H}^*/\Gamma$  is a (Hausdorff) locally compact space with countable basis of the topology. Moreover the following holds.*

*For each  $a \in \mathbb{H}^*$  there exists an open neighborhood  $U(a)$  which is invariant under the stabilizer  $\Gamma_a$  such that two points in  $U(a)$  are equivalent mod  $\Gamma$  if and only if they are equivalent with respect to  $\Gamma_a$ .*

*The set of cusp classes is discrete in  $X_\Gamma$ .*

A proof can be found in [Fr2], I.1.13. We just mention that the most involved part of the proof is the Hausdorff property of  $X_\Gamma$ . If  $\Gamma$  is a subgroup of finite index in  $\mathrm{SL}(2, \mathbb{Z})$ , the proof is easier. We refer to [Fr1], Proposition IV.14.6, for readers who are interested only in this case.  $\square$

We define a geometric structure  $\mathcal{O}$  on  $X_\Gamma$ . Let  $U \subset X_\Gamma$  be an open subset. We denote by  $\tilde{U}$  the inverse image of  $U$  in  $\mathbb{H}$ . By definition, a function  $f : U \rightarrow \mathbb{C}$  on some open subset belongs to  $\mathcal{O}$  if it is continuous and if the composition of the natural projection  $\tilde{U} \rightarrow U$  and  $f$  is a holomorphic function  $\tilde{U} \rightarrow \mathbb{C}$  in the usual sense.

**1.7 Proposition.** *The geometric structure  $\mathcal{O}$  defines a structure as Riemann surface on  $X_\Gamma$ .*



*Proof.* We have to construct for each point of  $X_\Gamma$  an open neighborhood  $U$ , such the geometric space  $(U, \mathcal{O}|_U)$  is isomorphic to an open subset  $V \subset \mathbb{C}$ , equipped with the usual holomorphic structure. Let  $a \in \mathbb{H}^*$  be a representative of this point. We choose an open neighborhood  $U(a)$  which is invariant under  $\Gamma_a$  and such that  $U(a)/\Gamma_a \rightarrow X_\Gamma$  is an open embedding. We equip  $U(a)/\Gamma_a$  in the same way with a structure as geometric space as we did for  $X_\Gamma$ . Then the open embedding induces an isomorphism of the geometric space  $U(a)/\Gamma_a$  onto its image in  $X_\Gamma$  (equipped with the restricted structure). Hence it is sufficient to show that  $U(a)/\Gamma_a$  is a Riemann surface.

First we treat the case that  $a$  is not a cusp. We consider the biholomorphic map from  $\mathbb{H}$  onto the unit disk  $\mathbb{E}$

$$\alpha : \mathbb{H} \xrightarrow{\sim} \mathbb{E}, \quad \tau \mapsto w = \frac{\tau - a}{\tau - \bar{a}}.$$

Every element  $M \in \Gamma_a$  induces a transformation of the unit disk

$$\mathbb{E} \longrightarrow \mathbb{E}, \quad w \mapsto \alpha(M(\alpha^{-1}(w))).$$

This transformation fixes the origin. By a well-known result of complex analysis it is of the form  $w \rightarrow \zeta w$  where  $|\zeta| = 1$ . Since  $\Gamma_a$  is finite we obtain a finite subgroup of the multiplicative group of complex numbers. Let  $n$  be its order. Another well-known result says that this group is the set of all  $n$ th roots of unity

$$G := \{e^{2\pi i\nu/n}; \quad 0 \leq \nu < n\}.$$

The map  $\alpha$  induces a topological map

$$\mathbb{H}/\Gamma_a \xrightarrow{\sim} \mathbb{E}/G.$$

We consider the natural projection  $\mathbb{E} \rightarrow \mathbb{E}/G$  and compare it with the map  $\mathbb{E} \rightarrow \mathbb{E}, w \mapsto w^n$ . There exists a unique bijective map  $\varphi : \mathbb{E}/G \rightarrow \mathbb{E}$  such the diagram

$$\begin{array}{ccc} & \mathbb{E} & \\ & \swarrow & \searrow \\ \mathbb{E}/G & \xrightarrow[\varphi]{\sim} & \mathbb{E} \end{array} \quad \begin{array}{c} q \\ \swarrow \\ \mathbb{E} \\ \searrow \\ q^n \end{array}$$

commutes. A trivial result of complex analysis says that a function  $f$  on the unit disk is holomorphic if and only if  $f(w^n)$  is holomorphic. This shows that the bijection  $\varphi : \mathbb{E}/G \rightarrow \mathbb{E}$  is an isomorphism of geometric spaces if one equips  $\mathbb{E}$  with the usual holomorphic structure. This shows that  $\mathbb{E}/G$  and hence  $\mathbb{H}/\Gamma_a$  is a Riemann surface.

The case of a cusp  $a$  is similar. We can assume that  $a$  is the cusp  $\infty$ . We have to show that  $\mathbb{H} \cup \{\infty\}/\Gamma_\infty$  is a Riemann surface. Again we construct an

isomorphism  $\varphi : \mathbb{H} \cup \{\infty\} / \Gamma_a \rightarrow \mathbb{E}$  of geometric spaces. Recall that there exists a real number  $b \neq 0$  that  $\Gamma_a$  acts by the translations  $\tau \mapsto \tau + nb$ ,  $n \in \mathbb{Z}$ . We take for  $\varphi$  the map

$$\varphi([\tau]) = e^{2\pi i \tau / b} \quad (= 0 \text{ for } \tau = \infty).$$

The topology of  $X_\Gamma$  has been defined in such a way that this map is topological. Moreover it is an isomorphism of geometric spaces. This shows that  $\mathbb{H} \cup \{\infty\} / \Gamma_a$  is a Riemann surface.  $\square$

A point  $a \in \mathbb{H}$  is called an *elliptic fixed point* of  $\Gamma$  if it is the fixed point of an elliptic element.

## 2. Vector valued automorphic forms

In the following we will work with the group  $\text{Aut}(\mathbb{H})$  of biholomorphic transformations of the upper half plane. As we explained there is a natural homomorphism  $\text{SL}(2, \mathbb{R}) \rightarrow \text{Aut}(\mathbb{H})$ . It is well-known that this is surjective. Hence we get an isomorphism of groups

$$\text{SL}(2, \mathbb{R}) / \pm 1 \xrightarrow{\sim} \text{Aut}(\mathbb{H}).$$

In the following we will prefer to work with these groups and not with  $\text{SL}(2, \mathbb{R})$ . Hence  $\Gamma$  will denote in the following a subgroup of  $\text{Aut}(\mathbb{H})$ . Each element of  $\text{Aut}(\mathbb{H})$  determines a matrix in  $\text{SL}(2, \mathbb{R})$  up to the sign.

Let  $\alpha : D \rightarrow D'$  be a biholomorphic mapping between two domains in the complex plane. Once for ever we choose a holomorphic logarithm  $\log \alpha'(z)$  and define then

$$j_r(\alpha, z) := \alpha'(z)^{-r/2} := e^{-r \log \alpha'(z)/2}$$

for an arbitrary real number  $r$ . (This definition is possible for complex  $r$ . For sake of simplicity we restrict here to automorphic forms of real weight.)

There holds a kind of chain rule for two biholomorphic mappings  $\alpha : D \rightarrow D'$ ,  $\beta : D' \rightarrow D''$ :

$$j_r(\beta\alpha, z) = w_r(\alpha, \beta)^{-1} j_r(\beta, \alpha z) j_r(\alpha, z).$$

Here  $w_r(\alpha, \beta)$  is a complex number of absolute value 1. For even  $r$  it is one.

Let  $f$  be a function on  $D'$ . We define the function  $f|_\alpha$  on  $D$  as

$$(f|_\alpha)(z) = (f|_\alpha)(z) = f(\alpha(z)) j_r(\alpha, z).$$

Then the chain rule reads as

$$f|(\beta\alpha) = w_r(\alpha, \beta) (f|_\beta)|_\alpha.$$

**2.1 Definition.** Let  $D \subset \mathbb{C}$  be a domain and  $\Gamma$  a group of biholomorphic transformations of  $D$ . By a (vector valued) multiplier system of weight  $r \in \mathbb{R}$  with values in a finite dimensional complex vector space  $V$  we understand a map

$$v : \Gamma \longrightarrow \mathrm{GL}(V)$$

with the following properties:

- 1)  $v(\gamma_1\gamma_2) = w_r(\gamma_1, \gamma_2)v(\gamma_1)v(\gamma_2)$ .
- 2) There exists a positive definite hermitian form on  $V$  such that the operators  $v(\gamma)$  are unitary.

Property 1) means that

$$J(\gamma, z) = j_r(\gamma, z)v(\gamma)$$

is a (vector valued) factor of automorphy, i.e.

$$J(\beta\alpha, \tau) = J(\beta, \alpha\tau)J(\alpha, \tau).$$

So it makes sense to consider functions  $f : D \rightarrow V$  with the transformation property

$$f(\gamma z) = J(\gamma, z)f(z).$$

**2.2 Lemma.** Let  $\alpha : D \rightarrow \tilde{D}$  be a biholomorphic map of domains and  $\Gamma$  a group of biholomorphic transformations of  $D$ . Then  $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$  is a group of biholomorphic transformations of  $\tilde{D}$ . Let  $v$  be a multiplier system of weight  $r$  for  $(D, \Gamma)$  then

$$\tilde{v}(\gamma) = v(\alpha^{-1}\gamma\alpha)w_r(\alpha^{-1}, \gamma)w_r(\alpha^{-1}\gamma\alpha, \alpha^{-1})$$

is a multiplier system for  $(\tilde{D}, \tilde{\Gamma})$  with corresponding automorphy factor

$$\tilde{J}(\gamma, w) = J(\alpha^{-1}\gamma\alpha, \alpha^{-1}w) = \tilde{v}(\gamma)j_r(\gamma, w)^{-1}.$$

Let  $f : D \rightarrow V$  be a function with the property  $f(\gamma z) = J(\gamma, z)f(z)$  for  $\gamma \in \Gamma$  then the transformed function  $\tilde{f} = f|_{\alpha^{-1}}$  has the transformation property  $\tilde{f}(\gamma w) = \tilde{J}(\gamma, w)f(w)$  for  $\gamma \in \tilde{\Gamma}$ .

From now on  $\Gamma$  denotes a group of biholomorphic transformations of the upper half plane  $\mathbb{H}$  such that its inverse image in  $\mathrm{SL}(2, \mathbb{R})$  is discrete. We denote by  $S \subset \mathbb{R} \cup \{\infty\}$  the set of cusps of this group and by  $\mathbb{H}^* = \mathbb{H} \cup S$  the extended upper half plane. We assume that the Riemann surface

$$X = X_\Gamma := \mathbb{H}^*/\Gamma$$

is compact. Then the set of cusp classes is finite.

In the following we fix a group  $\Gamma$ , a rational number  $r$  and a multiplier system  $\nu : \Gamma \rightarrow \text{GL}(V)$  of weight  $r$  for  $\Gamma$ . Let  $n$  be the dimension of  $V$ . We want to define for each point  $a \in \mathbb{H}^*$  an unordered  $n$ -tuple of numbers

$$\xi_1, \dots, \xi_n, \quad 0 \leq \xi_i < 1 \quad (n = \dim V).$$

We will call them the *characteristic numbers* of  $a$  (with respect to  $\Gamma, \nu, r$ ).

We start with the case where  $a \in \mathbb{H}$  is an inner point. We transform it to the origin of the unit disk  $\mathbb{E}$  by means of the transformation

$$\alpha(\tau) = \frac{\tau - a}{\tau - \bar{a}}.$$

We consider the conjugate group  $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$ . We also consider the conjugate multiplier system  $\tilde{\nu}$  and corresponding automorphy factor  $\tilde{J}$  in the sense of Lemma 2.2.

**2.3 Remark and Definition.** *For a point  $a \in \mathbb{H}$  we consider the conjugate group*

$$\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}, \quad \alpha(\tau) = \frac{\tau - a}{\tau - \bar{a}},$$

*and the transformed automorphy factor  $\tilde{J}(\gamma, w)$ . The stabilizer of the origin in  $\tilde{\Gamma}$  is generated by the transformation  $r_e(w) = e^{2\pi i/e}w$  where  $e$  is the order of the stabilizer  $\Gamma_a$ . The transformation  $R = \tilde{J}(r_e, w)$  is independent of  $w$  and has the property  $R^e = \text{id}$ . We define the characteristic numbers*

$$\xi_1, \dots, \xi_n, \quad 0 \leq \xi_i < 1,$$

*such that  $e^{2\pi i\xi_\nu}$  are the eigenvalues of  $R$ . The numbers  $e\xi_\nu$  are integral.*

The proof is rather trivial. Every element  $\gamma \in \tilde{\Gamma}$  which stabilizes the origin must be of the form  $w \mapsto \zeta w$  where  $\zeta$  is a complex number of absolute number one. The only subgroup of order  $e$  of the multiplicative group of complex numbers is the group generated by  $e^{2\pi i/e}$ . Since the derivative of  $\gamma$  is constant,  $\tilde{J}(\gamma, w)$  is independent of  $w$ . The automorphy property implies that it is a homomorphism. The image is a group of some order that divides  $e$ .  $\square$

Another way to describe the characteristic numbers is as follows.

**2.4 Remark.** *(Notations as in Remark and Definition 2.3.) Let  $\beta$  be any biholomorphic map from  $\mathbb{H}$  to  $\mathbb{E}$  with the property  $\beta(a) = 0$ . Consider in the stabilizer  $\Gamma_a$  the generator  $\gamma$  that corresponds to the element  $r_e(w) = e^{2\pi i/e}w$  in the group  $\beta\Gamma_a\beta^{-1}$ . Then the characteristic numbers  $\xi_\nu$  are defined such that  $0 \leq \xi_\nu < 1$  and that  $e^{2\pi i\xi_\nu}$  are the eigen values of  $J(\gamma, a)$ .*

**Supplement.** *The characteristic numbers depend only on the  $\Gamma$ -equivalence class of  $a$ .*

*Proof.* In the case where  $\beta$  is the transformation  $\alpha$  this follows from the formula  $\tilde{J}(\gamma, w) = J(\alpha^{-1}\gamma\alpha, \alpha^{-1}w)$  (Lemma 2.2). In the general case one has to use that a general  $\beta$  and  $\alpha$  are related by a rotation  $\beta(\tau) = \zeta\alpha(\tau)$ .

For the proof of the supplement we consider an equivalent point  $b = \gamma(a)$ . We transform  $a$  to 0 by  $\alpha$  and then use  $\beta = \gamma(a)$  to transform  $b$  to zero. For the two corresponding generators  $\gamma_a \in \Gamma_a$  and  $\gamma_b \in \Gamma_b$  we then have  $\gamma_b = \gamma\gamma_a\gamma^{-1}$ . The chain rules for  $\gamma_b\gamma$  and  $\gamma\gamma_b$  show that  $J(a, \gamma_a)$  and  $J(b, \gamma_b)$  are conjugate.  $\square$

Next we define the characteristic numbers in the case where  $a$  is the cusp  $\infty$ . The stabilizer  $\Gamma_\infty$  is generated by a translation

$$t^N(\tau) := \tau + N, \quad N > 0.$$

The matrix

$$R = J(t^N, \tau)$$

is independent of  $\tau$ . Its eigen values have absolute value 1. The characteristic numbers are defined such that  $e^{2\pi i\xi_\nu}$  are the eigenvalues of  $R$ .

We treat the case where  $a$  is an arbitrary cusp. We choose a transformation  $\alpha \in \text{Aut}(\mathbb{H})$  with the property  $\alpha(a) = \infty$ . We can consider the conjugate group  $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$  and the conjugate multiplier system  $\tilde{v}$  (and of course the same  $r$ ). The group  $\tilde{\Gamma}$  has the cusp  $\infty$ . We want to define the characteristic numbers of  $(\Gamma, r, v)$  at the cusp  $a$  to be the characteristic numbers of  $(\tilde{\Gamma}, r, \tilde{v})$  at  $\infty$ . It is easy to prove that this definition does not depend on the choice of  $\alpha$ .

**2.5 Lemma and Definition.** *Let  $\infty$  be a cusp of  $\Gamma$  and let  $t^N$  be the translation in  $\Gamma$  with smallest positive  $N$ . The characteristic numbers*

$$\xi_1, \dots, \xi_n, \quad 0 \leq \xi_i < 1,$$

*are defined such that  $e^{2\pi i\xi_\nu}$  are the eigenvalues of  $R = J(t^N, \tau)$ .*

*Let  $a$  be an arbitrary cusp and let  $\alpha \in \text{Aut}(\mathbb{H})$  be a transformation with the property  $\alpha(a) = \infty$ . We consider the conjugate group  $\tilde{\Gamma} = \alpha\Gamma\alpha^{-1}$  and the conjugate multiplier system  $\tilde{v}$ . The characteristic numbers of  $(\tilde{\Gamma}, r, \tilde{v})$  at  $\infty$  are independent of the choice of  $\alpha$ . They are called the characteristic numbers of the cusp  $a$ .*

*Proof of the second part.* We can assume that  $a = \infty$ . Then  $\alpha$  is of the form  $\alpha(\tau) = u\tau + v$ . Let  $t^N(\tau) = \tau + N$  be the generator of  $\Gamma_\infty$ . We set  $\tilde{N} = uN$ . Then  $t^{\tilde{N}} = \alpha t^N \alpha^{-1}$  and this is the generator of  $\tilde{\Gamma}_\infty$ . Taking a suitable basis of  $V$  we can assume that the matrix of  $R = J(t^N, \cdot)$  is diagonal. Then we can assume that  $V$  has dimension one and that  $R$  acts by multiplication by  $e^{2\pi i x}$  where  $x$  is the characteristic number. The function  $f(\tau) = e^{2\pi i x \tau}$  then has the property  $f(\tau + N) = J(t^N, \tau)f(\tau)$ . From the second part of Lemma 2.2 follows that the function  $\tilde{f}(\tau) = f(u\tau + v)$  has the property

$$\tilde{f}(\tau + \tilde{N}) = \tilde{J}(t^{\tilde{N}}, \tau)\tilde{f}(\tau).$$

This formula implies that  $J(t^{\tilde{N}}, \tau)$  is also the multiplication by  $e^{2\pi i x}$ .  $\square$

**2.6 Lemma.** *The characteristic numbers depend only on the  $\Gamma$ -orbit of a point  $a \in \mathbb{H}^*$ . Hence they can be considered for points  $x \in X_\Gamma$ . They can be different from  $(0, \dots, 0)$  only for cusps or elliptic fixed points.*

We want to introduce *local automorphic forms*. Let  $U \subset X_\Gamma$  be an open subset. We denote its inverse image in  $\mathbb{H}^*$  by  $\tilde{U}$ . This is a  $\Gamma$ -invariant subset. Hence we can consider all holomorphic functions  $f : \tilde{U} - S \rightarrow \mathbb{C}$  with the transformation property

$$f(\gamma\tau) = J(\gamma, \tau)f(\tau), \quad \gamma \in \Gamma.$$

Assume that  $\Gamma$  has cusp  $\infty$  and that it is contained in  $\tilde{U}$ . Then  $\tilde{U}$  contains some upper half plane  $\text{Im } \tau > C > 0$  and  $f$  has the transformation property  $f(\tau + N) = Rf(\tau)$ . Since  $R$  has finite order,  $f$  has some multiple of  $N$  as period. We call  $f$  regular at  $\infty$  if  $f$  is bounded for  $\text{Im } \tau \rightarrow \infty$  and cuspidal if it tends to 0. We can diagonalize  $R$  and describe  $f$  by components

$$f_\nu(\tau + N) = e^{2\pi i \xi_\nu} f_\nu(\tau).$$

The function  $g_\nu(\tau) = f_\nu(\tau)e^{-2\pi i \xi_\nu \tau / N}$  has period  $N$ . Hence we have a Fourier expansion

$$f_\nu(\tau) = e^{\frac{2\pi i}{N} \xi_\nu \tau} \sum_{m=-\infty}^{\infty} b_\nu(m) e^{\frac{2\pi i}{N} m \tau}.$$

We want to simplify the notations and introduce for this the symbolic notation

$$q^a := e^{2\pi i a \tau} \quad (a \in \mathbb{C}).$$

Then we can write the expansion in the simplified form

$$f_\nu(\tau) = \sum_{m \in \Lambda_\nu} a_\nu(m) q^m \quad \text{where} \quad \Lambda_\nu = \frac{1}{N}(\xi_\nu + \mathbb{Z})$$

The function  $f$  is called regular at  $\infty$  if  $a_\nu(m) \neq 0$  implies  $m \geq 0$  and cuspidal if it implies  $m > 0$ .

Using “transformation to  $\infty$ ” one can define the notions “regular” and “cuspidal” also for other cusps. It is clear that this notion does not depend on the choice of the transformation. It is also clear that this notion depends only on the  $\Gamma$ -orbit of a cusp.

We define a certain sheaf  $\mathcal{M} = \mathcal{M}_\Gamma(r, v)$  on  $X_\Gamma$ . For open  $U$  in  $X = X_\Gamma$  the space  $\mathcal{M}(U)$  consists of all local automorphic forms  $f : \tilde{U} - S \rightarrow V$  which are regular at the cusps. This defines a sheaf and even more an  $\mathcal{O}_X$ -module. For even  $r$  and the trivial one-dimensional representation  $v$  we write  $\mathcal{M}(r)$  instead of  $\mathcal{M}(r, v)$ . We also can consider the subsheaf  $\mathcal{M}^{\text{cusp}} = \mathcal{M}_\Gamma^{\text{cusp}}(r, v)$  of all cuspidal local automorphic forms. This is also an  $\mathcal{O}_X$ -module.

**2.7 Lemma.** *The sheafs  $\mathcal{M} = \mathcal{M}_\Gamma(r, v)$ ,  $\mathcal{M}^{\text{cusp}} = \mathcal{M}_\Gamma^{\text{cusp}}(r, v)$  are vector bundles of rank  $n = \dim V$ , hence coherent.*

*Proof.* We restrict to  $\mathcal{M}$  since the case  $\mathcal{M}^{\text{cusp}}$  is similar. We want to show that  $\mathcal{M}_x$  is a free  $\mathcal{O}_{X,x}$ -module for each  $x \in X_\Gamma$ .

First we assume that  $a$  is a cusp. We can restrict to  $a = \infty$ . As explained above, the elements of  $\mathcal{M}_x$  can be considered as Fourier series of the kind

$$f_\nu(\tau) = \sum_{m \in \mathbb{Z}, m + \xi_\nu \geq 0} b_\nu(m) e^{\frac{2\pi i}{N}(m + \xi_\nu)\tau}.$$

Recall that  $0 \leq \xi_\nu < 1$ . Hence  $m + \xi_\nu \geq 0$  means the same as  $m \geq 0$ . If we map  $f_\nu(\tau)$  to

$$\sum_{m \geq 0} b_\nu(m) e^{\frac{2\pi i}{N}m\tau}$$

we get an isomorphism from  $\mathcal{M}_x$  to  $\mathcal{O}_{X,x}^n$ . This shows that  $\mathcal{M}_x$  is free. The case  $\mathcal{M}^{\text{cusp}}$  is similar.

Next we consider the case where  $a$  is an interior point. In this case we can identify  $\mathcal{M}_x$  with holomorphic functions  $f$  in a small disk around  $w = 0$  which transform as

$$f(e^{2\pi i/e}w) = Rf(w), \quad R^e = \text{id}.$$

The components of  $f$  with respect to a basis of eigenvectors satisfy

$$f_\nu(e^{2\pi i/e}w) = e^{2\pi i \xi_\nu} f_\nu(w).$$

From the Taylor expansion one can derive that  $f_\nu(w) = w^{e\xi_\nu} g_\nu(w^e)$ . The local ring  $\mathcal{O}_{X,x}$  can be identified with the ring of power series  $\mathbb{C}\{w^e\}$ . The map  $f_\nu \mapsto g_\nu$  gives an  $\mathcal{O}_{X,x}$ -linear isomorphism from  $\mathcal{M}_x$  to  $\mathcal{O}_{X,x}^n$ .

So we have shown that  $\mathcal{M}_x$  is free for all  $x$ . The statement that  $\mathcal{M}$  is a vector bundle means a little more, namely that a basis of  $\mathcal{M}_x$  gives a basis in a full neighborhood of  $x$ . This follows from the following trivial description of  $\mathcal{M}_x$  in the case where  $a \in \mathbb{H}$  is not a fixed point: let  $f_1, \dots, f_n$  be holomorphic functions in an open neighborhood of  $a$  which do not vanish at  $a$ . Denote by  $e_1, \dots, e_n$  the unit vectors. Then  $f_1 e_1, \dots, f_n e_n$  defines a basis of the module  $\mathcal{M}_x$ .

This completes the proof of Lemma 2.7. □

For later purpose we collect the local description of the sheaf  $\mathcal{M}(v, r)$  that we obtained during the proof of Lemma 2.7.

**2.8 Lemma.** *Let  $\infty$  be a cusp of  $\Gamma$ . Then we have*

$$\mathcal{M}(v, r)_{[\infty]} \cong \bigoplus e^{2\pi i \xi_\nu \tau / N} \mathcal{O}_{X_\Gamma, [\infty]} e_\nu.$$

*Let  $a \in \mathbb{H}$  be an interior point. Then we have*

$$\mathcal{M}(v, r)_{[a]} \cong \bigoplus w^{e\xi_\nu} \mathcal{O}_{X_\Gamma, [a]} e_\nu.$$

For a multiplier system  $v$  of weight  $r$  with values in the vector space  $V$  one can define the dual multiplier system  $v'$ . It is realized on  $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ . By definition  $v'(\gamma)$  is the transposed of  $v(\gamma^{-1})$ . It is easy to check that this is a multiplier system of weight  $-r$ . We mention that a multiplier system of weight  $r$  can be considered as multiplier system of weight  $r'$  for each  $r' \equiv r \pmod{2}$ .

As in the case of the sheaf  $\mathcal{M}$ , we write  $\mathcal{M}^{\text{cusp}}(r)$  instead of  $\mathcal{M}^{\text{cusp}}(r, v)$  for even  $r$  and the trivial one-dimensional multiplier system  $v$ .

**2.9 Lemma.** *The sheaf  $\mathcal{M}^{\text{cusp}}(2)$  is a canonical sheaf. The dual sheaf of  $\mathcal{M}(r, v)$  is isomorphic to  $\mathcal{M}^{\text{cusp}}(2 - r, v')$ , where  $v'$  denotes the dual multiplier system.*

*Proof.* The canonical sheaf on a compact Riemann surface is the sheaf of holomorphic differentials. Let  $\omega$  be a holomorphic differential on an open subset  $U \subset X_{\Gamma}$ . Its inverse image on  $\tilde{U} - S$  is of the form  $f(\tau)d\tau$ . The function  $f$  transforms like an automorphic form of weight two (and trivial multiplier system). Using the formula

$$2\pi idz = dq/q \quad \text{for} \quad q = e^{2\pi i\tau}$$

it is easy to show that the regularity of  $\omega$  at the cusp classes means that  $f$  is cuspidal. For the elliptic fixed points a similar argument works. We omit it.  $\square$

Next we define a pairing

$$\mathcal{M}(r, v) \times \mathcal{M}^{\text{cusp}}(2 - r, v') \longrightarrow \mathcal{M}^{\text{cusp}}(2).$$

For this we use the natural pairing

$$V \times \text{Hom}(V, \mathbb{C}) \longrightarrow \mathbb{C}, \quad \langle v, L \rangle = L(v).$$

Let  $f \in \mathcal{M}(r, v)$  and  $g \in \mathcal{M}^{\text{cusp}}(2 - r, v')$  be local automorphic forms on some  $U \subset X_{\Gamma}$ . Then  $\langle f, g \rangle$  transforms like an automorphic form of weight 2 with respect to the trivial multiplier system. It is clear that it is cuspidal. So the pairing has been defined. It has to be checked that it is non-degenerated. This can be done by a local computation at points  $x \in X_{\Gamma}$ . We restrict to the case where  $x$  is the image of the cusp  $\infty$ . Recall that – using a suitable basis of  $V$  – the elements of  $\mathcal{M}(r, v)_x$  can be identified with Fourier series

$$f_{\nu}(\tau) = \sum_{m+\xi_{\nu} \geq 0}^{\infty} a_{\nu}(m) e^{\frac{2\pi i}{N}(m+\xi_{\nu})\tau}.$$

The characteristic numbers  $y_{\nu}$  of the dual multiplier system have the property  $\xi_{\nu} + y_{\nu} \equiv 0 \pmod{1}$ . Hence – using the dual basis – the elements of  $\mathcal{M}^{\text{cusp}}(2 - r, v')_x$  can be identified with Fourier series

$$g_{\nu}(\tau) = \sum_{m-\xi_{\nu} > 0}^{\infty} b_{\nu}(m) e^{\frac{2\pi i}{N}(m-\xi_{\nu})\tau}$$



and the pairing is just  $\sum f_\nu g_\nu$ . The condition  $m + \xi_\nu \geq 0$  is equivalent to  $m \geq 0$  and the condition  $m - \xi_\nu > 0$  is equivalent to  $m \geq 1$ . Finally  $\mathcal{M}(2)_x$  can be identified with all Fourier series

$$h(\tau) = \sum_{m \geq 1} c(m) e^{\frac{2\pi i}{N} m \tau}.$$

Let  $q = e^{\frac{2\pi i}{N} \tau}$ . Using the isomorphisms

$$\begin{aligned} \mathcal{M}(r, v)_x &\xrightarrow{\sim} \mathbb{C}\{q\}^n, & f &\longmapsto \left( \sum a_\nu(m) q^m \right), \\ \mathcal{M}^{\text{cusp}}(2-r, v')_x &\xrightarrow{\sim} \mathbb{C}\{q\}^n, & f &\longmapsto \left( \sum b_\nu(m) q^{m-1} \right), \\ \mathcal{M}(2)_x &\xrightarrow{\sim} \mathbb{C}\{q\}^n, & h &\longmapsto \left( \sum c_\nu(m) q^{m-1} \right), \end{aligned}$$

the pairing gets equivalent to the standard pairing

$$\mathbb{C}\{q\}^n \times \mathbb{C}\{q\}^n \longrightarrow \mathbb{C}\{q\}, \quad \langle P, Q \rangle = \sum_{\nu} P_\nu Q_\nu,$$

which is obviously non-degenerated. □

### 3. The computation of the degree, first method

The first method to compute the degree needs some more assumptions about  $(v, r)$ , but it has the advantage to be very simple. In particular, the volume of the fundamental domain and the Gauss–Bonnet formula will not be used. For practical use as in the theory of Borcherds products this special case is good enough. Hence we will describe this method in some detail. In a later section we will sketch, how these assumptions can be avoided.

**3.1 Assumption.** *The triple  $\Gamma, v, r$  has the following property.*

- 1) *The number  $r$  is rational.*
- 2) *The matrices  $v(\gamma)$  are of finite order.*
- 3) *There exists a subgroup  $\Gamma_0 \subset \Gamma$  of finite index such that  $v(\gamma)$  can be simultaneously diagonalized for  $\gamma \in \Gamma_0$ .*
- 4) *There exists a normal subgroup of finite index  $\Gamma_0 \subset \Gamma$  such that  $\Gamma_0$  acts fixed point free on  $\mathbb{H}$  and that the characteristic numbers of all cusps with respect to  $\Gamma_0$  are zero.*

For the rest of this section we take this assumption to be granted. We restrict  $v$  to  $\Gamma_0$  and consider the sheaf

$$\mathcal{M}_0 = \mathcal{M}_{\Gamma_0}(r, v)$$

on the Riemann surface  $X_{\Gamma_0}$ . We want to compare the degrees of  $\mathcal{M}$  and  $\mathcal{M}_0$ . Let

$$\pi : X_{\Gamma_0} \longrightarrow X_{\Gamma}$$

be the natural covering. We know  $\deg \mathcal{M}^* = \deg(X_0/X) \deg \mathcal{M}$ . There is an obvious inclusion of sheaves  $\mathcal{M} \hookrightarrow \pi_* \mathcal{M}_0$ . By functoriality this induces a map

$$\pi^* \mathcal{M} \longrightarrow \mathcal{M}_0.$$

Let  $x \in X_{\Gamma_0}$ . The stalk of  $\pi^* \mathcal{M}$  is

$$(\pi^* \mathcal{M})_a \cong \mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma}, \pi(x)}} \mathcal{O}_{X_{\Gamma_0}, x}.$$

Since  $\mathcal{O}_{X_{\Gamma_0}, x}$  is a free  $\mathcal{O}_{X_{\Gamma}, \pi(x)}$ -module we see that

$$(\pi^* \mathcal{M})_{\pi(x)} \longrightarrow (\mathcal{M}_0)_x$$

is injective. Outside a finite set (images of cusps and of elliptic fixed points of  $\Gamma$ ) it is an isomorphism. So we get an exact sequence

$$0 \longrightarrow \pi^* \mathcal{M} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{K} \longrightarrow 0$$

with a skyscraper sheaf  $\mathcal{K}$ . We have to compute its degree

$$\deg \mathcal{K} = \sum_{x \in X_{\Gamma_0}} \dim \mathcal{K}_x.$$

We compute

$$\mathcal{K}_x = (\mathcal{M}_0)_x / (\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma}, \pi(x)}} \mathcal{O}_{X_{\Gamma_0}, x})$$

first in the case where  $x$  comes from an inner point  $a \in \mathbb{H}$ . Let  $w = (\tau - a)/(\tau - \bar{a})$ . Recall that we assume that  $a$  is not an elliptic fixed point of  $\Gamma_0$ . Then the local ring  $\mathcal{O}_{X_{\Gamma_0}, x}$  can be identified with the ring of power series  $\mathbb{C}\{w\}$ . The ring  $\mathcal{O}_{X_{\Gamma}, \pi(x)}$  can be identified with  $\mathbb{C}\{w^e\}$ . As in section two we take a basis of  $V$  such that all  $v(\gamma)$  are diagonal. Then we have natural isomorphisms

$$(\mathcal{M}_0)_x \cong \mathbb{C}\{w\}^n.$$

and

$$\mathcal{M}_{\pi(x)} = \prod_{\nu=1}^n w^{e\xi_{\nu}} \mathbb{C}\{w^e\}.$$

If we tensor this with  $\mathbb{C}\{w\}$  we get

$$\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{X_{\Gamma}, \pi(x)}} \mathcal{O}_{X_{\Gamma_0}, x} = \prod_{\nu=1}^n w^{e\xi_{\nu}} \mathbb{C}\{w\}.$$

This shows the following result.

**3.2 Lemma.** *Let  $\pi : X_{\Gamma_0} \rightarrow X_{\Gamma}$  be the natural projection. We consider a point  $x \in X_{\Gamma_0}$  which is the image of an inner point  $a \in \mathbb{H}$  and Let  $v$  be a multiplier system of weight  $r$  for  $\Gamma$ . We denote by*

$$\sigma(a) = \xi_1 + \cdots + \xi_n$$

*the sum of the characteristic numbers at  $a$ . Then the formula*

$$\dim \mathcal{K}_x = e\sigma(a), \quad \mathcal{K} = \mathcal{M}_{\Gamma_0}(v, r)/\pi^* \mathcal{M}_{\Gamma}(v, r),$$

*holds.*

Now we consider the case that  $x \in X_{\Gamma_0}$  is the image of the cusp  $\infty$ . Analogously to  $N$  for  $\Gamma$ , we denote by  $N_0$  the smallest positive number such that  $\tau \mapsto \tau + N_0$  is in  $\Gamma_0$ . The number  $N_0/N$  is integral. We set

$$q = e^{\frac{2\pi i}{N_0}\tau}.$$

Then the local ring of  $X_{\Gamma_0}$  at  $x$  is  $\mathbb{C}\{q\}$  and the local ring of  $X_{\Gamma}$  at  $\pi(x)$  is  $\mathbb{C}\{q^{N_0/N}\}$ . The stalk of  $\mathcal{M} = \mathcal{M}_{\Gamma}(v, r)$  at  $\pi(x)$  is (after diagonalization)

$$\mathcal{M}_{\pi(x)} = \prod_{\nu=1}^n e^{\frac{2\pi i}{N}\xi_{\nu}\tau} \mathbb{C}\{q^{N_0/N}\}.$$

We get

$$\mathcal{M}_{\pi(x)} \otimes_{\mathcal{O}_{\Gamma, \pi(x)}} \mathcal{O}_{X_{\Gamma_0}, x} = \prod_{\nu=1}^n e^{\frac{2\pi i}{N}\xi_{\nu}\tau} \mathbb{C}\{q\}.$$

The characteristic numbers  $y_1, \dots, y_n$  of with respect to  $\Gamma_0$  are defined by

$$y_{\nu} \equiv (N_0/N)\xi_{\nu} \pmod{1}, \quad 0 \leq y_{\nu} < 1,$$

or, using the Gauss bracket,

$$y_{\nu} = (N_0/N)\xi_{\nu} - [(N_0/N)\xi_{\nu}].$$

Hence the stalk of  $\mathcal{M}_0 = \mathcal{M}_{\Gamma_0}(v, r)$  at  $x$  is

$$\mathcal{M}_{0,x} = \prod_{\nu=1}^n e^{\frac{2\pi i}{N_0}y_{\nu}\tau} \mathbb{C}\{q\}.$$

This shows

$$\dim \mathcal{K}_x = \sum_{\nu=1}^n [(N_0/N)\xi_{\nu}].$$

Recall that we assume that the characteristic numbers of the cusp  $\infty$  with respect to  $\Gamma_0$  are zero. Then  $(N_0/N)\xi_{\nu}$  is integral. We also mention that  $N_0/N$  is the index of  $\Gamma_{0,\infty}$  in  $\Gamma_{\infty}$ . Hence we get

$$\dim \mathcal{K}_x = \sum_{\nu=1}^n [\Gamma_{\infty} : \Gamma_{0,\infty}] \xi_{\nu}.$$

We recall that the characteristic numbers for a point  $x \in X_{\Gamma}$  depend on  $(\Gamma, v, r)$ . To point out this dependency we will write sometimes  $\xi_{\nu} = \xi_{\nu}(v, r)$  and  $\sigma(x) = \sigma_{\Gamma}(x, v, r)$  for their sum.

**3.3 Lemma.** *Let  $\pi : X_{\Gamma_0} \rightarrow X_{\Gamma}$  be the natural projection. We consider a point  $x \in X_{\Gamma_0}$  which is the image of a cusp  $a$ . Let  $v$  be a multiplier system of weight  $r$  for  $\Gamma$ . We denote by  $\sigma(a) = \sigma_{\Gamma}(a, r)$  the sum of the characteristic numbers. Then the formula*

$$\dim \mathcal{K}_x = [\Gamma_a : \Gamma_{0,a}] \sigma(a), \quad \mathcal{K} = \mathcal{M}_{\Gamma_0}(v, r) / \pi^* \mathcal{M}_{\Gamma}(v, r),$$

holds.

We use the notation

$$\pi : X_{\Gamma_0} \longrightarrow X_{\Gamma}$$

for the canonical map. We get a formula for the degree of  $\mathcal{K}$ .

**3.4 Proposition.** *The formula*

$$\deg \mathcal{K} = [\Gamma : \Gamma_0] \sum_{x \in X_{\Gamma}} \sigma(x)$$

holds. Here  $\sigma(x) = \sigma_{\Gamma}(x, v, x)$  is the sum of the characteristic numbers at (a representative of)  $x$ .

We now get the link between the degrees of  $\mathcal{M}_{\Gamma}(v, r)$  and  $\mathcal{M}_{\Gamma_0}(v, r)$ . The covering degree of  $\pi : X_{\Gamma_0} \rightarrow X_{\Gamma}$  equals the index  $[\Gamma : \Gamma_0]$ . Using 5) from Theorem 6.4.3 we get the following formula.

$$[\Gamma : \Gamma_0] \deg \mathcal{M}_{\Gamma}(v, r) = \deg \mathcal{M}_{\Gamma_0}(v, r) - [\Gamma : \Gamma_0] \sum_{x \in X_{\Gamma}} \sigma(x).$$

The group  $\Gamma_0$  can be chosen small enough such that the multiplier system is diagonal, that it acts fixed point free on  $\mathbb{H}$  and that the characteristic numbers of the cusps are zero. Then  $\mathcal{M}_{\Gamma_0}(v, r)$  is a direct sum of line bundles and we are reduced to the well-known case  $V = \mathbb{C}$  which has been treated at various places in the literature. For sake of completeness we repeat shortly the argument. Since every line-bundle has a meromorphic section (i.e. a meromorphic automorphic form)  $f$ . We associate to  $f$  a divisor  $D = (f)$ . such that  $\mathcal{O}_D$  is isomorphic to  $\mathcal{M}_{\Gamma_0}(v, r)$ . If  $x \in X_{\Gamma_0}$  is the image of an inner point  $a \in \mathbb{H}$ , then  $D(x)$  is the usual order of  $f$  at  $a$ . Let  $a$  be the cusp  $\infty$ . Since the characteristic number are zero, we can consider  $f$  as a holomorphic function in  $q = e^{\frac{2\pi i}{N_0} \tau}$  and we define  $D(x)$  to be the order of this function at  $q = 0$ . For an arbitrary cusp we use "transformation to  $\infty$ ". It is easy to check that the order is independent of the choice of the transformation and even more that it depends only one the  $\Gamma_0$ -orbit of  $a$ . Let  $m$  be a natural number. Then on has  $(f^m) = m(f)$  (since the characteristic numbers of the cusps vanish). We can take  $m$  such that  $mr$  is even and such the multiplier system of  $f$  is trivial. Now we can compare with modular forms of weight two.

$$\deg \mathcal{M}_{\Gamma_0}(v, r) = \frac{r}{2} \deg \mathcal{M}(2).$$

We use that  $\mathcal{M}^{\text{cusp}}(2)$  is a canonical bundle and that the degree of the canonical bundle is  $2g_0 - 2$ . We obtain that the degree of  $\mathcal{M}(2)$  is  $2g_0 - 2 + h_0$  where  $h_0$  denotes the number of cusp classes of  $\Gamma_0$ . Collecting together we obtain the following formula.

**3.5 Proposition.** *The formula*

$$\deg \mathcal{M}_\Gamma(v, r) = \frac{rn}{2[\Gamma : \Gamma_0]}(2g_0 - 2 + h_0) - \sum_{x \in X_\Gamma} \sigma_\Gamma(x, v, r).$$

*holds. Here  $g_0$  denotes the genus of  $X_{\Gamma_0}$ . The rank of  $v$  is denoted by  $n$  and  $h_0$  denotes the number of cusp classes of  $\Gamma_0$ .*

We want to express the formula above in data of the group  $\Gamma$  alone. For this we apply to use the Riemann–Hurwitz ramification formula VI.7.5 to the natural projection  $\pi : X_{\Gamma_0} \rightarrow X_\Gamma$ . It states

$$g_0 - 1 = \deg(\pi)(g - 1) + \frac{1}{2} \sum_{a \in X_{\Gamma_0}} (\text{Ord}(\pi, a) - 1).$$

The degree of  $\pi : X_{\Gamma_0} \rightarrow X_\Gamma$  equals the index  $[\Gamma : \Gamma_0]$ . Since  $\Gamma_0$  is a normal subgroup of  $\Gamma$  the order at a point  $a \in X_{\Gamma_0}$  depends only on its image  $b \in X_\Gamma$ . We denote this order by  $e(b)$ . So the ramification formula can be written as

$$\frac{g_0 - 1}{[\Gamma : \Gamma_0]} = (g - 1) + \frac{1}{2} \sum_{b \in X_\Gamma} \left(1 - \frac{1}{e(b)}\right).$$

**3.6 Lemma.** *Let  $\tilde{b} \in \mathbb{H}^*$  be a representative of  $b \in X_\Gamma$ . Then*

$$e(b) = [\Gamma_{\tilde{b}} : \Gamma_{0, \tilde{b}}].$$

*In particular, if  $b$  is not a cusp, then  $e(b) = \#\Gamma_{\tilde{b}}$  which is independent on the choice of  $\Gamma_0$ .*

The number of inverse points of a given cusp class  $b \in X_\Gamma$  is  $[\Gamma_0 : \Gamma]/e(b)$ . Hence we have

$$h_0 = \sum_{b \in X_\Gamma \text{ cusp}} \frac{[\Gamma_0 : \Gamma]}{e(b)}.$$

Now Proposition 3.7 can be rewritten in the following form.

**3.7 Proposition.** *The degree formula can be written as*

$$\deg \mathcal{M}_\Gamma(v, r) = rn \left( g - 1 + \frac{h}{2} + \frac{1}{2} \sum_{b \in X_\Gamma \text{ not cusp}} \left(1 - \frac{1}{e(b)}\right) \right) - \sum_{x \in X_\Gamma} \sigma_\Gamma(x, v, r).$$

## 4. The dimension formula

In this section we assume that the assumption 3.1 is true.

The Riemann–Roch formula states

$$\chi(\mathcal{M}_\Gamma(v, r)) = \deg(\mathcal{M}_\Gamma(v, r)) + \text{Rank}(\mathcal{M}_\Gamma(v, r))(1 - g).$$

We are more interested in the spaces of automorphic forms

$$[\Gamma, v, r] := H^0(X_\Gamma, \mathcal{M}_\Gamma(v, r)).$$

The Serre dual space is the subspace of cusp forms of  $[\Gamma, v', 2 - r]$ . In the case  $r > 2$  this space vanishes.

In the case  $r = 2$  there is a difference depending on the fact whether  $\Gamma$  has a cusp or not. We first mention that  $v$  is a representation (homomorphism) in this case. Modular forms of weight 0 are constants. Hence  $[\Gamma, v', 0]$  is just the space of  $v'$ -invariants of  $V$ . This is isomorphic to the space of  $v$ -invariants of  $V$ . We denote the space of invariants by  $V^v$ . Since constant cusp forms are zero if there is a cusp, we obtain the following dimension formula.

**4.1 Theorem.** *In the case  $r > 2$  we have*

$$\begin{aligned} \dim[\Gamma, v, r] = & rn \left( g - 1 + \frac{h}{2} + \frac{1}{2} \sum_{b \in X_\Gamma \text{ not cusp}} \left( 1 - \frac{1}{e(b)} \right) \right) \\ & + n(1 - g) - \sum_{x \in X_\Gamma} \sigma(x). \end{aligned}$$

Here  $g$  is the genus of  $X_\Gamma$ . The dimension of  $V$  is  $n$  and  $\sigma(x) = \sigma_\Gamma(x, v, r)$  is the sum of the characteristic numbers. The number  $e(b)$  is the order of the stabilizer of a representative of  $b$  in  $\mathbb{H}$ .

**Supplement.** *When  $\Gamma$  has a cusp then this formula remains true in the case  $r = 2$ . Otherwise one has to add  $\dim V^v$  to the right hand side.*

We denote by  $[\Gamma, v, r]_0$  the subspace of cusp forms of  $[\Gamma, v, r]$ . This is the space of global sections of the sheaf  $\mathcal{M}^{\text{cusp}} = \mathcal{M}^{\text{cusp}}(\Gamma, v, r)$ . Since the quotient  $\mathcal{M}/\mathcal{M}^{\text{cusp}}$  is a skyscraper sheaf we have

$$\chi(\mathcal{M}) - \chi(\mathcal{M}^{\text{cusp}}) = \sum_{x \in X_\Gamma, \text{ cusp}} \dim(\mathcal{M}_x/\mathcal{M}_x^{\text{cusp}}).$$

Recall that  $\mathcal{M}_x$  is given by Fourier series with summation over integers  $m$  such that  $m + \xi_\nu(r) \geq 0$  and in the subspace  $\mathcal{M}_x^{\text{cusp}}$  the summation is restricted to  $m + \xi_\nu(r) > 0$ . There is only a difference if the characteristic number  $\xi_\nu$  is zero. We see

$$\chi(\mathcal{M}^{\text{cusp}}) = \chi(\mathcal{M}) - \sum_{x \in X_\Gamma, \text{ cusp}} \#\{\nu; \xi_\nu(r) = 0\}.$$

**4.2 Remark.** *Assume that  $\Gamma$  has cusp  $\infty$ . The number*

$$\#\{\nu; \xi_\nu = 0\}$$

*equals the dimension of the subspace of invariants of  $V$  under the transformations  $J(\gamma, \tau)$ ,  $\gamma \in \Gamma_\infty$ . (These transformations do not depend on  $\tau$ .)*

Finally we formulate the dimension formula for the space of cusp forms. In the case of weight 2 we have to be careful, since

$$\chi(\mathcal{M}^{\text{cusp}}(\Gamma, v, 2)) = \dim[\Gamma, v, 2]_0 - \dim[\Gamma, v', 0].$$

In the case of an even weight,  $v$  is a representation and  $[\Gamma, v', 0]$  can be identified with the space of invariants of  $v'$ . We obtain the following result.

**4.3 Theorem.** *Assume that  $\Gamma$  has at least one cusp. In the case  $r > 2$  the dimension of the space of cusp forms is*

$$\dim[\Gamma, v, r]_0 = \dim[\Gamma, v, r] - \sum_{x \in X_\Gamma, \text{ cusp}} \#\{\nu; \xi_\nu(r) = 0\}.$$

*In the case  $r = 2$  we have to add  $\dim V^v$  to the right hand side.*

## 5. The full modular group

We specialize the dimension formula to the group  $\Gamma = \text{SL}(2, \mathbb{Z})/\pm$ . As usual it acts on the upper half plane by  $(a\tau + b)(c\tau + d)^{-1}$ . In this case  $g = 0$  and  $h = 1$ . We have two classes of elliptic fixed points of order  $e = 2$  resp.  $e = 3$ . In the dimension formula we get

$$\sum_{b \in X_\Gamma \text{ not cusp}} \left(1 - \frac{1}{e(b)}\right) = \left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) = \frac{7}{6}.$$

So the dimension formula gives

$$\dim[\Gamma, v, r] = \frac{rn}{12} + n - \sum_{x \in X_\Gamma} \sigma(x).$$

We use the usual generators

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Representatives of the elliptic fixed points are  $i$  and  $\zeta_3 = -1/2 + i\sqrt{3}/2$ . The elements in their stabilizers which correspond to the rotation with factor  $e^{2\pi i/e}$  can be computed easily as  $S$  resp.  $(ST)^{-1}$ .

Let  $A$  be a complex matrix of finite order. The eigenvalues are roots of unity which we can write in the form

$$\lambda = \exp(2\pi i\alpha) \text{ with } 0 \leq \alpha < 1.$$

We use the notation

$$\alpha(A) = \sum_{\lambda} \alpha,$$

where  $\lambda$  runs through all eigenvalues (counted with multiplicity).

The contributions of the characteristic numbers in the dimension formula can be written as

$$\sum_{x \in X_{\Gamma}} (\xi_1(r) + \cdots + \xi_n(r)) = \alpha(J(S, i)) + \alpha(J((ST)^{-1}, \zeta_3)) + \alpha(J(T, \cdot)).$$

(The function  $J(T, \tau)$  is independent of  $\tau$ .)

**5.1 Theorem.** *In case of the full modular group the dimension formula is valid for  $r \geq 2$  (including  $r = 2$ ) and reads as*

$$\dim[\Gamma, \varrho, r] = \frac{rn}{12} + n - \alpha(J(S, i)) - \alpha(J((ST)^{-1}, \zeta_3)) - \alpha(J(T, \cdot)).$$

For the subspace of cusp forms one has

$$\dim[\Gamma, v, r]_0 = \dim[\Gamma, v, r] - \dim V^{J(T, \cdot)} + \begin{cases} 0 & \text{if } r > 2, \\ \dim V^e & \text{if } r = 2. \end{cases}$$

We treat a simple example just to get a feeling how the formula works. The weight  $r$  is assumed to be even and we consider the case of a trivial multiplier system. This means  $J(\gamma, \tau) = (c\tau + d)^{r/2}$ . So we get

$$J(S, i) = e^{2\pi ir/4}$$

and

$$J((ST)^{-1}, \zeta_3) = e^{2\pi ir/6}.$$

The sum of both is

$$\begin{cases} 0 & \text{for } r \equiv 0 \pmod{12}, \\ 7/6 & \text{for } r \equiv 2 \pmod{12}, \\ 1/3 & \text{for } r \equiv 4 \pmod{12}, \\ 1/2 & \text{for } r \equiv 6 \pmod{12}, \\ 2/3 & \text{for } r \equiv 8 \pmod{12}, \\ 5/6 & \text{for } r \equiv 10 \pmod{12}. \end{cases}$$

Using the table above, one gets

$$\begin{cases} \left[ \frac{r}{12} \right] & \text{if } r \equiv 2 \pmod{12}, \\ \left[ \frac{r}{12} \right] + 1 & \text{else.} \end{cases}$$

This formula is true for all even  $r > 0$  (also for  $r = 2$ ).



## 6. The metaplectic group

There is a different way to express multiplier systems using the metaplectic group. We recall this concept briefly in the case of half-integral weight. The metaplectic group

$$\mathrm{Mp}(2, \mathbb{R}) \longrightarrow \mathrm{SL}(2, \mathbb{R})$$

can be described as the set of all pairs  $(M, J)$ , where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$  and where  $J = \sqrt{c\tau + d}$  is one of the two holomorphic square roots of the function  $c\tau + d$  on the upper half plane  $\mathbb{H}$ . The group law is

$$(M, \sqrt{c\tau + d})(M', \sqrt{c'\tau + d'}) = (MM', \sqrt{c'\tau + d'}\sqrt{cM'\tau + d}).$$

One knows that  $\mathrm{Mp}(2, \mathbb{Z})$  is generated by

$$T = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad S = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right), \quad \mathrm{Re} \tau > 0,$$

and that the relations

$$S^2 = (ST)^3 = Z, \quad Z = \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i \right), \quad Z^4 = 1$$

are defining ones.

Let

$$\varrho : \mathrm{Mp}(2, \mathbb{Z}) \longrightarrow \mathrm{GL}(V)$$

a representation of  $\mathrm{Mp}(2, \mathbb{Z})$  on some finite dimensional complex vector space. Let  $r$  be an integer or a half integer ( $2r \in \mathbb{Z}$ ). An (entire) modular form of weight  $r$  with respect to  $\varrho$  is holomorphic function  $f : \mathbb{H} \rightarrow V$  with the transformation law

$$f(M\tau) = \sqrt{c\tau + d}^{2r} \varrho(M)f(\tau) \quad \text{for all } (M, \sqrt{c\tau + d}) \in \mathrm{Mp}(2, \mathbb{Z})$$

and such that  $f$  is bounded for  $\mathrm{Im} \tau \geq 1$ .

We denote by  $[\mathrm{Mp}(2, \mathbb{Z}), r, \varrho]$  the space of all entire modular forms.

Let  $V_0 \subset V$  be the subspace on which  $\varrho(-E, i)$  ( $E$  denotes the unit matrix) acts by multiplication with  $e^{-\pi i r} = i^{-2r}$ . It is quite clear that the values of  $f$  are contained in  $V_0$  and that  $V_0$  is invariant under  $\mathrm{Mp}(2, \mathbb{Z})$ . Let  $\gamma \in \mathrm{SL}(2, \mathbb{Z})/\pm$  a modular transformation. We choose a pre-image  $(M, \sqrt{c\tau + d}) \in \mathrm{Mp}(2, \mathbb{C})$  and define the operator

$$J(\gamma, \tau)a = \sqrt{c\tau + d}^{2r} \varrho(M)a \quad \text{for } a \in V_0.$$

This is independent of the choice of the pre-image (since we restrict to  $V_0$ ). By trivial reason

$$J(\gamma, \tau)\alpha'(\tau)^{r/2}$$

is a  $V_0$ -valued multiplier system  $v$  and Assumption 3.1 is satisfied. The space of automorphic forms  $[\mathrm{SL}(2, \mathbb{Z})/\pm, r, v]$  of weight  $r$  with respect to this multiplier system coincides with  $[\mathrm{Mp}(2, \mathbb{Z}), \varrho, r]$ . Hence the dimension formula gives the following result.

**6.1 Theorem.** *Let  $\varrho : \text{Mp}(2, \mathbb{Z}) \rightarrow \text{GL}(V)$  be a representation on a finite dimensional vector whose image is finite. Let  $V_0$  be the biggest subspace of  $V$  where  $\varrho(-E, i)$  acts by multiplication with  $e^{-\pi i r}$ . We denote by  $d$  the dimension of  $V_0$ . Then one has for  $r \geq 2$  (including  $r = 2$ )*

$$\dim[\text{Mp}(2, \mathbb{Z}), \varrho, r] = \frac{rd}{12} + d - \alpha(e^{\pi i r/2} \varrho(S)) - \alpha\left(\left(e^{\pi i r/3} \varrho(ST)\right)^{-1}\right) - \alpha(\varrho(T)).$$

The invariants  $\alpha$  have to be taken with respect to the action on  $V_0$ .

For the subspace of cusp forms one has

$$\dim[\text{Mp}(2, \mathbb{Z}), \varrho, r]_0 = \dim[\text{Mp}(2, \mathbb{Z}), \varrho, r] - \dim V_0^{\varrho(T)} + \begin{cases} 0 & \text{if } r > 2, \\ \dim V_0^{\varrho} & \text{if } r = 2. \end{cases}$$

The operator  $e^{\pi i r/2} \varrho(S)$  (considered on  $V_0$ ) has order 2 and  $\left(e^{\pi i r/3} \varrho(ST)\right)^{-1}$  has order 3. Their  $\alpha$ -invariants can be computed very easily by means of the following lemma.

**6.2 Lemma.** *Let  $A$  be a  $d \times d$ -matrix. We have*

$$\alpha(A) = \begin{cases} \frac{d}{4} - \frac{\text{tr}(A)}{4} & \text{if } A^2 = E \\ \frac{d}{3} - \frac{1}{3} \text{Re}(\text{tr}(A^{-1})) + \frac{1}{3\sqrt{3}} \text{Im}(\text{tr}(A^{-1})) & \text{if } A^3 = E. \end{cases}$$

## Appendix. Generalizations of the dimension formula

We sketch another method for the computation of the degree which gives the dimension formula for arbitrary *real* weight and arbitrary multiplier systems in the sense of Definition 2.1. This method is more involved and uses hermitian metrics on bundles and their Chern forms. It is related to the Gauss–Bonnet formula.

### 7. Chern forms

We describe a different method for the computation of the degree of a line bundle. This uses more general theory which we do not want to develop here since the cases we could treat so far are sufficient usually. So we will keep very short and expect a reader which has some experience and is willing to fill the gaps himself.

We want to consider *hermitian forms* on a vector bundle  $\mathcal{M}$ ,

$$\mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{C}_X^\infty.$$

By definition this is a family of maps

$$\langle \cdot, \cdot \rangle : \mathcal{M}(U) \times \mathcal{M}(U) \longrightarrow \mathcal{C}_X^\infty(U) \quad (U \subset X \text{ open}),$$

compatible with restriction such that the rules

$$\langle s_1 + s_2, t \rangle = \langle s_1, t \rangle + \langle s_2, t \rangle, \quad \langle fs, t \rangle = f \langle s, t \rangle, \quad \langle g, f \rangle = \overline{\langle f, g \rangle}$$

are satisfied. Here  $s, s_1, s_2, t$  are sections of  $\mathcal{M}$  and  $f$  is a differentiable function. The expressions  $\langle s, s \rangle$  are real functions. We call the hermitian form *positive semidefinit* if  $\langle s, s \rangle$  is nowhere negative.

We want to define what it means that a hermitian form is *positive definit*. For this have to introduce the value  $s(a)$  of a section  $s \in \mathcal{M}(U)$  at a point  $a \in U$ . We consider the  $\mathbb{C}$ -vector space

$$\mathcal{M}(a) := \mathcal{M} \otimes_{\mathcal{O}_{X,a}} \mathbb{C}.$$

Here  $\mathbb{C}$  is considered as module over the ring  $\mathcal{O}_{X,a}$  through the natural homomorphism  $\mathcal{O}_{X,a} \rightarrow \mathbb{C}$  (evaluation at  $a$ ). If  $\mathcal{M}$  has rank  $n$ , then  $\mathcal{M}(a)$  is an  $n$ -dimensional vector space. We define  $s(a)$  to be the image of  $s_a$  under the natural map

$$\mathcal{M}_a \longrightarrow \mathcal{M}(a), \quad s \longmapsto a \otimes 1.$$

In the special case  $\mathcal{M} = \mathcal{O}_X$  we can identify  $\mathcal{O}_X(a)$  and  $\mathbb{C}$  and then  $f(a)$  for a section  $f \in \mathcal{O}_X(U)$  is just the value of the function  $f$  at the point  $a$ .

Now we can define when a positive semidefinite hermitian form is called definite. The condition is that for each section  $s \in \mathcal{M}(U)$  and each point  $a \in U$  we have

$$\langle s, s \rangle(a) = 0 \implies s(a) = 0.$$

We are mainly interested in hermitian forms on line bundles. In this case the definition of positive definiteness is still simpler. So let  $\mathcal{L}, \langle \cdot, \cdot \rangle$  be a line bundle which has been equipped with a positive definite hermitian form. Let  $U \subset X$  an open subset such that  $\mathcal{L}|_U$  is trivial and let  $s \in \mathcal{L}(U)$  be a generator. This means  $\mathcal{L}(V) = s\mathcal{O}(V)$  for all open  $V \subset U$ . We consider the function  $\langle s, s \rangle$ . Positive definiteness means that this function is positive everywhere. Hence we can consider the differential form of degree two  $\partial\bar{\partial} \log \langle s, s \rangle$ . We claim that this differential form is independent on the choice of the generator  $s$  and hence all these differential forms for varying  $U$  glue to a global differential form on  $X$ .

**7.1 Proposition.** *Let  $\mathcal{L}, \langle \cdot, \cdot \rangle$  be a line bundle which has been equipped with a positive definite hermitian form. There exists a unique differential form  $\omega \in A^2(X)$  such that for each generator  $s$  of  $\mathcal{L}|_U$  on any open subset the equality*

$$\omega|_U = \frac{1}{2\pi i} \partial\bar{\partial} \log \langle s, s \rangle$$

*holds.*

*Proof.* If  $s$  is a generator then each other generator is of the form  $fs$  with a holomorphic function without zeros. The operator  $\bar{\partial}$  kills holomorphic functions. Similarly the operator  $\partial$  kills the complex conjugate of a holomorphic function. Another rule which can be verified is  $\partial\bar{\partial}h = -\bar{\partial}\partial h$  for differentiable functions  $h$ . Hence  $\partial\bar{\partial}$  kills holomorphic functions and complex conjugates of holomorphic functions. Locally the function  $\log(f\bar{f})$  is the sum of a holomorphic function and the complex conjugate of a holomorphic function. This shows

$$\partial\bar{\partial} \log(f\bar{f}) = 0.$$

We obtain

$$\bar{\partial}\partial \log(f\bar{f}h) = \bar{\partial}\partial \log h$$

for positive differentiable functions  $h$ . So the differential form is independent on the choice of the generator  $s$ .  $\square$

We call  $\omega$  a *Chern form* of the line bundle  $\mathcal{L}$ . It depends on the choice of the hermitian form. Chern forms can be used to compute the degree of a line bundle.

**7.2 Theorem.** *Let  $\omega$  be a Chern form of a line bundle  $X$ . Then the formula*

$$\deg(\mathcal{L}) = \int_X \omega$$

*holds.*

We indicate the proof. First one shows that the integral is independent of the choice of the Chern form. Then one shows that the theorem is true for a covering  $X \rightarrow Y$  if it is true for  $Y$ . So it is sufficient to verify it for the Riemann sphere and a point divisor, for example the divisor  $(\infty)$ . In this case everything can be made explicit.  $\square$

As a warm-up we apply Theorem 7.2 in the simplest case that  $X = X_\Gamma$  where  $\Gamma$  has no elliptic and parabolic fixed points. Then  $\mathbb{H}/\Gamma$  is compact and the natural projection  $\mathbb{H} \rightarrow \mathbb{H}/\Gamma$  is locally biholomorphic. Holomorphic (resp. differentiable) functions on an open subset of  $\mathbb{H}/\Gamma$  are in one-to-one correspondence to  $\Gamma$ -invariant holomorphic (resp. differentiable) functions on the inverse image in  $\mathbb{H}$ . We consider now for even  $r$  the trivial multiplier system  $v$ . Hence we study the bundle  $\mathcal{M}(r)$  which describes the transformation law

$$f(\gamma\tau) = (c\tau + d)^{2r} f(\tau), \quad \gamma \leftrightarrow \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If  $f, g$  are two functions with this transformation formula, then

$$\langle f, g \rangle := f(\tau) \overline{g(\tau)} \operatorname{Im} \tau^r$$

is  $\Gamma$  invariant and defines a differentiable function on some open subset of  $\mathbb{H}/\Gamma$ . This gives a positive definite hermitian form on  $\mathcal{M}(r)$ . We compute its Chern form. Since the operators  $\partial$  and  $\bar{\partial}$  are compatible with pull-back with respect to holomorphic mappings, we obtain that the Chern form can be identified with the  $\Gamma$ -invariant differential form

$$\partial\bar{\partial} \log(y^r) = \frac{r}{2\pi} \frac{dx \wedge dy}{y^2}.$$

Now we have a new formula for the degree of  $\mathcal{M}(r)$ . In Proposition we obtained the expression  $2r(p-1)$ . Comparing it with the new computation we obtain (from Theorem 7.2)

$$\frac{1}{2\pi} \int_{\mathbb{H}/\Gamma} \frac{dx dy}{y^2} = 2g - 2.$$

This is a special case of the *Gauss Bonnet formula* from differential geometry.

## 8. The computation of the degree, second method

Now we allow that  $\Gamma$  contains elliptic and parabolic fixed points. We make some comments about integration on  $\mathbb{H}/\Gamma$ . Let  $D \subset \mathbb{C}$  be an open domain and  $\Gamma$  a group of biholomorphic transformations acting discontinuously. Let  $\pi : D \rightarrow D/\Gamma$  be the natural projection. There exists a discrete subset  $S \subset D$  such that  $\Gamma$  acts without fixed points on  $D - S$ . Let  $\omega$  be a nowhere negative top differential form on  $X = (D - S)/\Gamma$ . This defines a measure on  $X$ . This measure extends to a measure on  $D/\Gamma$  such that the image of  $S$  gets measure 0. In particular, we can talk about the volume of  $D/\Gamma$  which may be finite or not. The volume can be computed with the help of a fundamental domain. We briefly explain what this means. We consider subsets  $F \subset D$  with the following property. We assume that  $F$  is the closure of its interior and that the boundary has measure 0. We say that  $F$  is a *fundamental set* if  $D$  is the union of all  $\gamma(D)$ ,  $\gamma \in \Gamma$ . It is called a *fundamental domain* if, in addition, two different points in the interior of  $F$  are not inequivalent mod  $\Gamma$ . For a fundamental domain  $F$  the formula

$$\int_{F/\Gamma} \omega = \int_F \pi^* \omega$$

holds for trivial reason. If  $F$  is only a fundamental set then still the inequality “ $\leq$ ” holds.

**8.1 Lemma.** *Let  $\Gamma$  be subgroup of  $\text{Aut}(\mathbb{H})$  such that its inverse image in  $\text{SL}(2, \mathbb{R})$  is discrete and such that  $\mathbb{H}^*/\Gamma$  is compact. The volume of  $\mathbb{H}/\Gamma$  with respect to the measure  $dx dy/y^2$  is finite.*

*Proof.* We construct a suitable fundamental set. Let  $\infty$  be a cusp of  $\Gamma$ . By a *cuspidal sector* at  $\infty$  we understand a domain of the form  $\text{Im } \tau \geq C$ ,  $|\text{Re } \tau| \leq C'$  where  $C, C'$  are positive. The notion of a cuspidal sector at an arbitrary cusp is defined in the usual way by transforming it to  $\infty$ . From the compactness of  $\mathbb{H}^*/\Gamma$  one can deduce that there exists a fundamental set which is the union of a compact set in  $\mathbb{H}$  and finitely many cuspidal sectors ([Fr2], Proposition I.1.15). So we are reduced to the statement that the cuspidal sector at  $\infty$  has finite volume with respect to  $dx dy/y^2$ . This is easy to check.  $\square$

We want now to construct a hermitian form on  $\mathcal{M}(r, v)$  in the general (scalar valued) case. Of course we could try to take

$$\langle f, g \rangle := f(\tau) \overline{g(\tau)} \text{Im } \tau^r$$

also in this case. But this causes problems at the elliptic fixed points and at the cusps. At the elliptic fixed points the following problem arises. The transformation law  $f(\gamma\tau) = J(\gamma, \tau)f(\tau)$  implies that  $f(a) = 0$  od  $a$  is the fixed point of a  $\gamma$  with the property  $J(\gamma, a) \neq 1$ . Hence  $\langle f, f \rangle$  would vanish at such fixed points. But this would contradict to the definiteness. The situation in

the cusps is still worse. Let  $\infty$  be cusp, then  $\langle f, f \rangle \operatorname{Im} \tau^r$  can not be expected to extend to a  $\mathcal{C}^\infty$ -function at  $[\infty]$ . Hence we have to modify the definition of  $\langle f, g \rangle$  close to an elliptic fixed point or a cusp.

We modify the definition of the hermitian metric as follows. We will construct a certain  $\Gamma$ -invariant positive differentiable function  $\phi$  on  $\mathbb{H} - S$  where  $S$  denotes the set of elliptic fixed points. With this function we will modify the definition of the hermitian metric as follows. Let  $U \subset X_\Gamma$  be open and  $\tilde{U}$  its inverse image in  $\mathbb{H}$ . Then we will define

$$\langle f, g \rangle := \langle f, g \rangle_0 := f(\tau) \overline{g(\tau)} \operatorname{Im} \tau^r \cdot \phi(\tau).$$

This is a  $\Gamma$ -invariant function on  $\tilde{U}$  and can be considered as a function  $\tilde{U}/\Gamma$ . This is an open subset of  $U$  but cusp classes may be missing. So we will have to make the choice of  $\phi$  in such a way that  $\phi$  extends to a continuous function on the whole  $U$  and even more, this function should be differentiable. And we want to have the that the modified  $\langle \cdot, \cdot \rangle$  is positive definit.

Assume that such a function has been constructed. Then we get the formula

$$\deg \mathcal{M}(v, r) = \frac{1}{2\pi} \int_{\mathbb{H}/\Gamma} \frac{dx dy}{y^2} + \frac{1}{2\pi i} \int_{\mathbb{H}/\Gamma} \partial \bar{\partial} \log \phi(\tau).$$

To construct a suitable  $\phi$  we choose for each point  $x \in X_\Gamma$  which is the image of an elliptic fixed point or cusp a small open neighborhood such that these are pairwise disjoint. We will construct  $\phi$  such that its support of  $\phi(\tau) - 1$  is contained in the union of the  $U(x)$ . The construction of  $\phi$  can be done in each  $U(x)$  separately.

We treat first that  $x = [a]$  is the image of an elliptic fixed point  $a$ . We use the chart  $w = (\tau - a)(\tau - \bar{a})^{-1}$  as has been described in Remark and Definition 2.3. The neighborhood is defined by  $|w| < r$  where  $r$  is sufficiently small. We have to make use of the characteristic number  $\xi$ ,  $0 \leq \xi < 1$ . From its definition follows that the function

$$s = w^{e\xi} \quad (e = \#\Gamma_a)$$

defines a section of  $\mathcal{M}(v, r)$  in  $U(a)$ . It is clear that this is a generating section. The expression  $\bar{s} s i \tau^r$  vanishes at  $a$ . This is reason why we have to introduce the modifying function  $\varphi$ . Close to  $a$  it should agree with  $(\bar{w} w)^{-e\xi}$ . So we have to demand:

*For an elliptic fixed point  $a$  the function  $\phi$  should be defined such that it is differentiable function such that  $\phi(\tau) - 1$  has compact support on  $U(a)$  and such that it agrees with  $(\bar{w} w)^{-e\xi}$  on a small neighborhood of  $a$ .*

The elliptic fixed point gives an extra contribution to the degree, namely

$$\int_{U(a)/\Gamma_a} \partial \bar{\partial} \log \phi.$$

This integral agrees with

$$\frac{1}{e} \int_{U(a)} \partial \bar{\partial} \log \phi.$$

This can be computed as follows.

**8.2 Lemma.** *Let  $r > 0$  and let  $\varphi$  be a differentiable function with compact support on the disk  $|w| < r$ . Assume that  $\varphi$  agrees on a small neighborhood of the origin with the function  $(\bar{w}w)^\xi$  for some real number  $\xi$ . Then the formula*

$$\frac{1}{2\pi i} \int_{|w| < r} \partial \bar{\partial} \log \phi = \xi$$

holds.

*Proof.* The proof is an application of the Stokes formula. First we notice that  $\partial \bar{\partial} = d\bar{\partial}$ . This follows from the rule  $\bar{\partial}^2 = 0$ . Then we consider for small  $\varepsilon$  the domain  $\varepsilon < |w| < r$ . (“Small” means that  $2\varepsilon < r$  and that  $\phi$  agrees with  $\bar{w}w^\xi$  for  $|w| < 2\varepsilon$ .) To this domain we can apply the formula of Stokes.

$$\int_{|w| < r} \partial \bar{\partial} \log \phi = \oint_{|w|=\varepsilon} \bar{\partial} \log \phi.$$

The circle line  $|w| = \varepsilon$  has to be oriented such that the integration area is on its left. This means that we have to take the clockwise orientation (opposite to the usual mathematical orientation). We set  $w = u + iv$  and compute

$$\frac{\partial}{\partial \bar{w}} \log \phi = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \log(u^2 + v^2)^\xi = \frac{\xi}{\bar{w}}.$$

Hence we obtain

$$\oint_{|w|=\varepsilon} \bar{\partial} \log \phi = \xi \oint_{|w|=\varepsilon} \frac{d\bar{w}}{\bar{w}}.$$

To compute the integral of  $d\bar{w}/\bar{w}$  we use the general formula for line integrals  $\overline{\int \eta} = \int \bar{\eta}$  and we use that the integral of  $dw/w$  is  $-2\pi i$  due to the “wrong” orientation of the circle line. The integral is independent of  $\varepsilon$ . So there is no problem to take the limit  $\varepsilon \rightarrow 0$  which completes the proof of Lemma 8.2.  $\square$

One should compare this with the degree formula 3.7. There is a term  $\sigma(\xi)$  whose occurrence – for elliptic fixed points – now finds a new explanation.

Next we treat the case that  $a$  is cusp. We can assume that it is the cusp  $\infty$ . Then we take the neighborhood in the form  $\text{Im } \tau > C$  with sufficiently big  $C$ . The stabilizer  $\Gamma_\infty$  is generated by a substitution  $\tau \mapsto \tau + N$ . The function  $q^{\xi/N}$  is a generating section. Hence in this case it looks natural to define the compensating function  $\phi(\tau)$  for  $\tau > C$  such that it is periodic and such that



for very big  $\text{Im } \tau$  it agrees with  $\bar{q}q^{-\xi/N}$  and, finally, that  $\phi(\tau) - 1$ , considered as a function of  $q^{1/N}$ , it has compact support. But this is not correct. The reason is that have also the function  $y^r$  which is increasing fir  $y \rightarrow \infty$ . This function has to by compensated too. Hence the correct choice of  $\phi$  is as follows.

*For the cusp  $\infty$  the function  $\phi$  should be defined such that it is differentiable periodic function for  $\text{Im } \tau > C$  such that  $\phi(\tau) - 1$ , considered as function  $q^{1/N}$ , has compact support and such that it agrees with*

$$(\bar{q}q)^{-\xi/N} \log y^{-r}$$

for very big  $\text{Im } \tau$ .

The computation is now essentially the same as in the case of an elliptic fixed point. There only occurs an extra contribution

$$\oint_{|q|=\varepsilon} \bar{\partial} y^r.$$

It is easy to evaluate this integral and to show that it tends to 0 for  $\varepsilon \rightarrow 0$ . Collecting together we obtain (in the scalar valued case) the degree formula

$$\text{deg } \mathcal{M}(v, r) = \frac{1}{2\pi} \int_{\mathbb{H}/\Gamma} \frac{dx dy}{y^2} - \sum_a \xi(a).$$

For example for even  $r$  and trivial  $v$  it must coincide with the degree formula 3.7 that we derived with our first method. Comparing them gives the following variant od the Gauss–Bonnet formula.

**8.3 Theorem.**

$$\frac{1}{2\pi} \int_{\mathbb{H}/\Gamma} \frac{dx dy}{y^2} = 2g - 2 + h + \sum_{a \in \mathbb{H}/\Gamma \text{ (no cusp)}} \left(1 - \frac{1}{e(a)}\right).$$

As an example we can take the modular group  $\text{SL}(2, \mathbb{Z})/\pm$ . The volume of the fundamental domain  $|\text{Re } \tau| \leq 1, |\tau| \geq 1$  computes as  $\pi/3$ . There is one cusp class, and there are two classes of elliptic fixed points of order 2 and 3. The genus is zero.

**The vector valued case**

We want to extend the degree formala to the case of a vector valued multiplier system  $(v, r)$ . For this we want to use the fact that a vector bundle  $\mathcal{M}$  and its determinant (highest exterior power)  $\bigwedge^n \mathcal{M}$  have the same degree (Remark VI.4.6). The determinant of  $\mathcal{M}(v, r)$  is related to the scalar valued multiplier system  $(\det v, rn)$ . To explain this, we assume that  $V = \mathbb{C}^n$ . The vector

valued modular forms are written as columns. Then we can consider the  $\mathcal{O}_{X_\Gamma}$  multilinear map

$$\mathcal{M}(v, r)^n \longrightarrow \mathcal{M}(\det v, rn), \quad (f_1, \dots, f_n) \longmapsto \det(f_1, \dots, f_n).$$

This is an alternating multilinear map of  $\mathcal{O}_{X_\Gamma}$ -modules which induces a map  $\bigwedge^n \mathcal{M}(v, r) \rightarrow \mathcal{M}(\det v, rn)$ . This map is injective. Hence we obtain an exact sequence

$$0 \longrightarrow \bigwedge^n \mathcal{M}(v, r) \longrightarrow \mathcal{M}(\det v, rn) \longrightarrow \mathcal{K} \longrightarrow 0.$$

We compute  $\mathcal{K}_a$ . The computation will show that this is 0 if  $a$  is not the image of an elliptic fixed point or a cusp. We explain the computation in the case that  $a$  is the image of the cusp  $\infty$  and leave the other cases to the reader. We can assume that  $v(t_N)$  is diagonalized for the generating translation. Then  $\mathcal{M}_a(v, r)$  is generated by  $q^{\xi_\nu/N} e_\nu$ . Taking the determinant we get the generator

$$q^{\sigma(a)/N}, \quad \sigma(a) = \xi_1 + \dots + \xi_n.$$

But the generator for the line bundle  $\mathcal{M}(\det v, rn)$  is

$$q^{(\sigma(a) - [\xi_1 + \dots + \xi_n])/N}.$$

Here we use the Gauss bracket  $[x] = \max\{\nu \in \mathbb{Z}; \nu \leq x\}$ . Notice that  $x - [x]$  is the representative of the coset  $x + \mathbb{Z}$  in  $[0, 1)$ . So we see

$$\dim \mathcal{K}_a = [\xi_1 + \dots + \xi_n].$$

As we mentioned a similar computation works for all points  $a$ . This shows

$$\deg(\mathcal{K}) = \sum_a [\sigma(a)].$$

The short exact sequence gives

$$\deg \mathcal{M}(v, r) = \deg \mathcal{M}(\det v, rn) - \sum_a [\sigma(a)].$$

Now we can insert the degree formula for the scalar valued case.

**8.4 Theorem.** *The degree formula that we derived under the special assumptions for  $(v, r)$  in Proposition 3.7 holds for arbitrary real weight and multiplier system  $v$ . In particular, the dimension formula in Theorem 4.1*

$$\begin{aligned} \dim[\Gamma, v, r] = & rn \left( g - 1 + \frac{h}{2} + \frac{1}{2} \sum_{b \in X_\Gamma \text{ not cusp}} \left( 1 - \frac{1}{e(b)} \right) \right) \\ & + n(1 - g) - \sum_{x \in X_\Gamma} \sigma(x). \end{aligned}$$

for  $r > 2$  (and with the same correction in the case  $r = 2$ ) is true in the general case.

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