Gauss-Manin Connection in Disguise

How to generalize modular forms?

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Modular forms, Eisenstein series and Ramanujan equations

Definition 1. *A (full) modular form of weight k is a holomorphic function (submitted to a growth condition) $f : \mathbb{H} \subset \mathbb{C} \to \mathbb{C}$ defined on the upper half plane which satisfy*

$$f(A \bullet z) = (cz+d)^k f(z), \quad A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL(2,\mathbb{Z}). \tag{1}$$

Modular forms can be seen as functions in the space of lattices of the form which transform as $f(\lambda\Lambda) = \lambda^{-k}\Lambda$. They can be seen, also, as functions on a variable $q = e^{2\pi i z}$, since they satisfy $f(z) = f(z+1)$.

The most important modular forms are the Eisenstein series E_k. Those are given by

$$E_k(q) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \qquad k > 2, \tag{2}$$

where B_k are the Bernoulli numbers and $\sigma_k(n)$ is the sum of the k-th powers of the positive divisors of n.

Theorem 1 ([1], Chapter 1, Prop. 4)**.** *The algebra of modular forms is generated by E_4 and E_6. That is, any modular form is a polynomial in E_4 and E_6.*

The remaining series, E_2, is not a modular form, but it also satisfies an identity similar to (1). It is the first example of a **quasi-modular form**. The point of adding E_2 to the algebra generated by E_4 and E_6 is that we get the following

Theorem 2 ([1],Chapter 1, Prop. 15)**.** *The algebra generated by E_2, E_4 and E_6 is closed under differentiation. Specifically, we have:*

$$E_2' = \frac{E_2^2 - E_4}{12}, \quad E_4' = \frac{E_2E_4 - E_6}{3}, \quad E_6' = \frac{E_2E_6 - E_4^2}{2}, \tag{3}$$

where $E_k' = \frac{1}{2\pi i}\frac{dE_k}{dz} = q\frac{dE_k}{dq}$

The relations given in (3) were first discovered by S. Ramanujan and are sometimes called Ramanujan equations. They have many applications in number theory, specifically in transcendence theory. We want to give a geometric interpretation of these equations and then find analogues of the Eisenstein series in more general contexts.

Algebraic De Rham cohomology and Gauss-Manin connection

The main tool needed in order to carry out our program are the algebraic de Rham cohomology and the Gauss-Manin connection.

Definition 2. *Let $X \to T$ be a smooth family of quasi-projective varieties. Then, one can define the relative de Rham cohomology sheaf as*

$$\mathcal{H}^q_{DR}(X/S) = \boldsymbol{R}^q\pi_*\left(\Omega_{\dot{X}/S}\right). \tag{4}$$

This sheaf can be show to be locally free, and one can see it as the bundle whose fibers are given by the cohomologies $H^q_{dR}(X_t)$ of each element of the family.

The Gauss-Manin connection ∇ is defined on this bundle. We are not going to give its formal definition, but only its main property:

$$\int \nabla\sigma = d\left(\int\sigma\right) \tag{5}$$

where σ is a section of $\mathcal{H}^q_{DR}(X/S)$ (and therefore its integral is a function on T). For details, see [4].

Enhanced elliptic curves

To give a geometric interpretation of the Ramanujan equations (3), we have to deal with elliptic curves and elliptic integrals. In order to deal with both at the same time we will consider **enhanced elliptic curves**. This idea was first implemented in [5].

Definition 3. *A triple (E,α,ω), where α is a holomorphic 1-form (first piece of the Hodge filtration) and ω is not holomorphic such that $\langle\alpha,\omega\rangle = 1$ is called enhanced elliptic curve.*

Here, $\langle,\rangle$ is the usual intersection product on the algebraic de Rham cohomology. The definition above allow us to consider, the integrals of α and ω over paths in E, that is, to study elliptic integrals.

Theorem 3 ([5], Prop 5.4)**.** *The moduli space of enhanced elliptic curves is given by*

$$T = \{(t_1,t_2,t_3) \in \mathbb{C}^3 \mid 27t_3^2 - t_2^3 \neq 0\}, \tag{6}$$

where the (t_1,t_2,t_3) corresponds to the triple

$$E: y^2 = 4(x-t_1)^3 + t_2(x-t_1) + t_3 \qquad \alpha = \frac{dx}{y} \qquad \omega = x\frac{dx}{y}.$$

We now have a universal family $X \to T$ of enhanced elliptic curves and a basis of sections of the de Rham cohomology bundle. If we compute the Gauss-Manin connection in this basis (α,ω), we get an explicit matrix in terms of the differentials dt_i. After that, the Ramanujan equations from Theorem 3 make their appearance!

Theorem 4 ([5], Prop. 4.1)**.** *Let R be a vector field in T such that $\nabla_R\alpha = -\omega$ and $\nabla_R\omega = 0$. Then R is unique and it is given by*

$$\mathrm{R} = \left(t_1^2 - \frac{1}{12}t_2\right)\frac{\partial}{\partial t_1} + (4t_1t_2 - 6t_3)\frac{\partial}{\partial t_2} + \left(6t_1t_3 - \frac{1}{3}t_2^2\right)\frac{\partial}{\partial t_3} \tag{7}$$

If R is written as a system of differential equations, after multiplying t_i by some constants, we get exactly the Ramanujan equations! By looking at the locus L for which R generates the tangent space, the maps t_1, t_2, t_3 restricted to L will be the Eisenstein series after a change of coordinates! This is a sign that we can generalize modular forms by looking at functions on a suitable moduli space for each case.

Remark 1. *The locus L in the last paragraph has an interpretation based on the integrals of α and ω over integral cycles.*

Applications to Mirror Symmetry

Mirror Symmetry arose in late 80s, when physicists from string theory started studying different objects for which the quantum field theories are equivalent. Comparing computations from one side of the mirror with computations from the other yields impressive mathematical results, as the numbers of rational curves on a quintic threefold! Here we deal with the most basic example: a generic quintic on $\mathbb{P}^4$ and its mirror, which is a family of manifolds known as mirror quintic family.

Definition 4. *Let $\psi^5 \neq 1$ and let $G = \left\{(a_0,\dots,a_4) \in \mathbb{Z}_5^5 : \sum_i a_i \equiv 0 \bmod 5\right\}/\mathbb{Z}_5$, where $\mathbb{Z}_5$ is embedded diagonally. This group acts on $\mathbb{P}^4$ in the natural way: $(a_0,\dots,a_4)\bullet[x_0,\dots,x_4] \mapsto [\mu^{a_0}x_0 : \dots \mu^{a_4}x_4]$, where μ is a primitive fifth root of unit. For us, a* **mirror quinitic** *X_ψ is the resolution of singularities of the quotient*

$$\left\{[x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \mid x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0x_1x_2x_3x_4 = 0\right\}/G. \tag{8}$$

Rational curves on the quintic

Our goal is to follow the same ideas as for enhanced elliptic curves. We defined enhanced mirror quintics by considering X_ψ and a basis $\alpha_1,\dots,\alpha_4$ for the third de Rham cohomology. This basis has constant intersection product and is compatible with the Hodge filtration.

Theorem 5 ([6], Thm. 3)**.** *The moduli space of enhanced mirror quintics is given by*

$$T = \{(t_0,\dots,t_6) \in \mathbb{C}^7 \mid t_4t_5(t_0^5 - t_4) \neq 0\}. \tag{9}$$

Besides that, there exists a unique vector field R for which the Gauss-Manin connection satisfy

$$\nabla_{\mathrm{R}}(\alpha_1) = \alpha_2 \qquad \nabla_{\mathrm{R}}(\alpha_2) = Y\alpha_3 \qquad \nabla_{\mathrm{R}}(\alpha_3) = -\alpha_4 \qquad \nabla_{\mathrm{R}}(\alpha_4) = 0, \tag{10}$$

for some function Y in T. This vector field and the function Y have explicit expressions in terms of the t_i.

Writing R as differential equation and solving it considering the t_i as functions of q, we find expressions for t_i and find that Y is (up to constant) the so called Yukawa coupling, first computed in [2], which is the generating function for the counts of rational curves on the quintic, denoted by n_d.

$$Y = 5 + \sum_d n_d d^3 \frac{q^d}{1-q^d} \tag{11}$$

Disk counts with boundary on the real quintic

Another interesting problem is to consider not curves on the quintic, but holomorphic disks with boundary on the real quintic lagragian. In this case, we have to consider a slightly different situation. We need to study not only the de Rham cohomology of the mirror quintic, but the homology with boundary on a pair of rational curves. We do not get a Hodge structure, but a mixed Hodge structure.

We can still define a moduli space of mirror quintics enhanced with a basis of the de Rham cohomology with boundary with constant intersection product and compatible with the mixed Hodge structure.

Theorem 6 ([3], Thm. 2)**.** *The moduli space of relatively enhanced mirror quintics is given by*

$$T = \{(t_0,\dots,t_8) \in \mathbb{C}^9 \mid t_0t_4t_5(t_0^{10} - t_4^{10}) \neq 0\}. \tag{12}$$

Besides that, there exists a unique vector field R for which the Gauss-Manin connection satisfy

$$\nabla_{\mathrm{R}}(\alpha_0) = 0 \qquad \nabla_{\mathrm{R}}(\alpha_1) = \alpha_2 \qquad \nabla_{\mathrm{R}}(\alpha_2) = F\alpha_0 + Y\alpha_3 \qquad \nabla_{\mathrm{R}}(\alpha_3) = -\alpha_4 \qquad \nabla_{\mathrm{R}}(\alpha_4) = 0, \tag{13}$$

for some functions F and Y in T. This vector field and the functions F and Y have explicit expressions in terms of the t_i.

Again, by solving the differential equation associated with the vector field, we get that Y is the same Yukawa coupling as before and that F is (up to constant) the generating function for the disk counts, which was first predicted in [7].

$$F = \sum_d n_d^{disk} d^2 \frac{q^{d/2}}{1-q^d}$$

Remark 2. *In both cases, the locus tangent to R has a more intrinsic interpretation related to the integral of the differential forms over cycles. That's where the conditions satisfied by the Gauss-Manin connection in theorems 5 and 6 come from.*

Acknowledgements

I would like to thank my PhD supervisor, Hossein Movasati, for presenting modular forms and the Gauss-Manin connection to me. I also would like to thank IMPA for the great research ambient and the CNPq (grant 141069/2020-1) for funding my PhD studies. I also would like to thank GAeL organizers for the opportunity to present my work.

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