Postscript

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During the 30 years since the publication of the German original of this book, the theory of pro-p extensions was flourishing: new results were proved, new insights gained, and new conjectures emerged. We will take this opportunity to make a few comments on the origin and the history of the subject, and then briefly indicate some of the progress that has been made during the last few decades.

The Prehistory

It was Dedekind [9] who first investigated the Galois group of an infinite extension, namely of the field $\mathbb{Q}'$ of all p^nth roots of unity over the rationals $\mathbb{Q}$, apparently motivated by the fact that infinite extensions were 'off limits' for his contemporaries. In modern notation, he proved that $G = \mathrm{Gal}\,(\mathbb{Q}'/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$, and that the Galois correspondence does not give a bijection between subgroups of G and subfields of $\mathbb{Q}'/\mathbb{Q}$.

Stiemke extended Dedekind's ideal theory to infinite extensions of $\mathbb{Q}$ in his dissertation [St], which was published posthumously by Emmy Noether. This article inspired Krull [Kr1], who was able to reestablish the Galois correspondence by introducing the topology on Galois groups that now bears his name, and by restricting the correspondence to closed subgroups of G. In the same year, Krull [Kr2, Kr3] also investigated ideal theory in infinite extensions of number fields.

Number theory in infinite extensions of $\mathbb{Q}$, in particular Hilbert's theory of ramification, was then studied by Herbrand [He1, He2], Scholz [Sl] and Moriya [Mo1], and class field theory was generalized to infinite extensions by Chevalley [Ch] (see also [Mo2]), but it seems that the time was not yet ripe for asking the right questions, and the whole topic must have looked like generalization for generalizations sake to most number theorists (comparable to the lukewarm reception of Hensel's p-adic numbers before Hasse proved the first local-global principle in 1920).

Golod-Shafarevich

The theory picked up pace after World War II, and the first result that should be mentioned was proved in 1947 by Shafarevich [54]. In modern notation it states that the Galois group G_k of the maximal p-extension of a $\mathfrak{p}$-adic number field k not containing the p-th roots of unity is a free pro-p group. In 1947, however, Shafarevich formulated this result as a theorem on certain finite p-extensions of k. The structure of the Galois group G_k was studied first by Kawada [28]. He showed that if k contains the p-th roots of unity, then G_k is a pro-p group with one generating relation. Faddeev and Skopin [14] gave a new proof for Kawada's result in 1959.

It seems that this article inspired I.R. Shafarevich to study the maximal p-extension k_S of a number field k that is unramified outside of S, with the goal in mind to prove that k_∞ is infinite for certain number fields k.

Here's the author's account of the early history of his collaboration with Shafarevich:

> In the spring of 1962, Shafarevich visited East Berlin. He asked me me to provide him with a copy of this article, and succeeded in transferring its main idea to global fields, thereby finding a bound for the number of relations of the group $G_S = \mathrm{Gal}\,(k_S/k)$. Later he told me that he first got the idea that in general there may be not enough relations to make G_S finite during a performance of the Rosenkavalier in the Berliner Staatsoper that we attended. Shafarevich presented his estimate of the relation rank at the ICM in Stockholm in 1962 in his survey [57] on algebraic number fields. In that lecture he posed the problem of finding a bound for the relation rank r in terms of the generator rank d of pro-p groups, with the hope that such a bound might prove the existence of infinite class field towers.
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> Several mathematicians, in particular group theorists, must have tried to solve this problem; as Shafarevich told me later, Thompson was one of them. Nevertheless it was Shafarevich himself, in collaboration with his student Golod, who found a sufficiently strong bound. They showed in 1964 that a pro-p group is infinite if the relation rank r and the generator rank d satisfy the inequality $r \leq (d-1)^2$.
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> The proof was quite complicated but was simplified almost immediately by Vinberg [62] and Gaschütz (cf. Roquette [49]), who also improved the bound to $r \leq \frac{1}{4}d^2$. The main idea in each of the three proofs is going from the group G to the complete group algebra $\mathbb{F}_p[[G]]$ and proving results on algebras.
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> In the summer of 1962 I went to Verna in Bulgaria for holidays. During the hot afternoons I used to lay on the bed thinking about p-extensions. I had the idea that for maximal p-extensions of a number

field, all relations should come from the local relations; for restricted ramification one would have to factor out the inertia groups. Lacking stimulation, I did not pursue this any further. But in the autumn of 1962, when I learned about Shafarevich's estimate of the relation rank of G_S, I remembered these considerations and could develop the main ideas of my book in a very short time. A short version was published in the Monatsberichte der Deutschen Akademie der Wissenschaften 1963, and in the autumn of 1963 I gave talks on these results in Shafarevich's seminar in Moscow. When we met again in Oberwolfach in 1964, he had worked out a cohomological version of my results and showed me how the proofs could be formulated in an elegant way within the framework of the cohomology of pro-p groups. These ideas were taken up and refined by Höchsmann, who presented his investigations at the Brighton conference on algebraic number theory in 1965. This set the frame for the present book, which I wrote mainly while I was at the Steklov institute in Moscow during 1967-68.

Structure of G_S

Now let us mention some results on the structure of the groups G_S that have been obtained after the book had been published.

The results due to Kuzmin mentioned on page 165 have been published in [Ku1, Ku2]. He proved that G_S has cohomological dimension ≤ 2 if S contains all primes dividing p and all archimedean places, and if the base field k is totally complex if $p = 2$. Neumann [Ne1, Ne2] gave a simpler proof for this theorem and showed that the inequality (11.8) in Theorem 11.5 is actually an equality. Kuzmin [Ku3] also studied the question under which assumptions the maximal p-extension of $k_\mathfrak{p}$ for $\mathfrak{p} \in S$ is just a localization of k_S; again he assumes that S contains all primes dividing p as well as all archimedean primes. This condition is satisfied if $\mathfrak{p} \nmid p$.

In the case $\mathfrak{p}|p$, this condition is satisfied if the image of $G_{k,\mathfrak{p}}$ in G_S has infinite index; for this and related results (e.g. conditions under which G_S can be written as a free product of local groups) we refer to the monograph [NSW] by Neukirch, Schmidt & Wingberg. Let us also mention that Schmidt [Sc2] recently removed some of the problems caused by the prime 2 in connection with the structure of G_S when not all real places are in S.

Infinite Class Field Towers

The criterion of Golod-Shafarevich can be strengthened considerably in certain cases: Koch & Venkov [KV] showed that for odd primes p and complex quadratic base fields k, the condition $d(G_\varnothing) \geq 3$ implies that the p-class field tower of k is infinite.

This was generalized by Kisilevsky & Labute [KL] to general CM-fields (complex quadratic extensions of totally real number fields). Furuta [Fu] constructed fields of roots of unity with infinite p-class field towers; the criteria found by Schoof [Sch] led to much smaller examples, and later Wingberg [W1] came up with more general results that allowed him to prove that $\mathbb{Q}(\zeta_p)$ has infinite p class field tower for the prime $p = 157$ with irregularity index 2. The techniques of Schoof were taken up again by Maire [Ma2].

The result by Koch and Venkov remains true for the 'prime $p = 4$': if the class group of a complex quadratic number field has 4-rank ≥ 3, then its 2-class field tower is infinite (see Hajir [Ha2] for a simple proof); for similar results e.g. for cyclic cubic fields see Maire [Ma1].

Furuta [Fu] also showed that for every $s \ge 1$ there exist p-class field towers $K^{(n)}$ of cyclotomic fields K in which the p-class groups of every field $K^{(n)}$ have rank $\ge s$. This was improved by Hajir [Ha1], who showed that if a field K satisfies the Golod-Shafarevich criterion for infinite p-class field tower, then the rank of p-class groups of intermediate fields L are bounded below by a linear function of the degree $(L : K)$.

A general improvement of the Golod-Shafarevich inequality for p-groups seems is not possible for small values of the generator rank d: Newman and Wisliceny conjectured that for every prime p and every integer $d \ge 1$ there exists a finite pro-p group with d generators and $r = \lfloor d^2/4 \rfloor + 1$ relations, and together with Sauerbier they proved [NSaW] that this conjecture holds for $d \le 6$.

It is, however, possible to improve Golod-Shafarevich by invoking invariants other than just r and d: Let $G_1 = \mathrm{Fr}(G) = [G,G]G^p$ (this is an elementary abelian group of order p^d), put $G_2 = [G_1,G]G_1^p$ or $G_2 = [G_1,G]G^p$ according as $p = 2$ or $p > 2$, and define e by $(G_1 : G_2) = p^e$. If G is a finite p-group, then Gaschütz & Newman [GN] proved

$$r > \frac{d^2}{2} + (-1)^p \frac{d}{2} + \Big(e - (-1)^p \frac{d}{2} - \frac{d^2}{4}\Big)\frac{d}{2}.$$

Finite Class Field Towers

Group-theoretical investigations on the possible Galois groups of finite p-class field towers of quadratic number fields began with the paper of Scholz & Taussky [ST], where 3-class field towers of complex quadratic number fields were investigated. Galois groups G of the Hilbert p-class field tower of complex quadratic fields (where p is an odd prime) have the following properties: G is a pro-p group with finite abelianization G/G', and there is an automorphism$\sigma \in \mathrm{Aut}(G)$ such that $1+\sigma$ acts trivially on G/G'. Groups with these properties are called Schur σ-groups; see Miyake [Mi] and Arrigoni [Ar]. Benjamin, Lemmermeyer & Snyder [BLS1, BLS2] have computed the Galois groups of 2-class field towers of length ≤ 2 for certain families of complex quadratic number fields. Bush [Bu] has recently found the first example of a quadratic number field whose 2-class field tower has length 3. Boston & Perry [BP] have computed the Galois groups of certain finite maximal extensions of $\mathbb{Q}$ unramified outside few primes, and their work was extended by Boston & Leedham-Green [BL].

Odlyzko's Discriminant Bounds

In the 1930s there was hope that improvements to Minkowski's discriminant bounds would lead to a proof that class field towers terminate; nowadays, class field towers are used to show that the best known discriminant bounds (due to Odlyzko) are close to best possible. The first constructions of infinite class field towers with this goal in mind were obtained by Martinet [Mt2, Mt3] and Schmithals [Sh]; Martinet's records were improved recently by Hajir & Maire [HM]. Martinet [Mt1] also showed how to use Odlyzko bounds to construct fields with class number 1 and large degree (up to 116).

Applications

Let p be a fixed prime, S be a finite set of places of a number field K containing all places above p, and let $G = G_S$ denote the Galois group of the maximal pro-p extension unramified outside of S. Define $G_1 = [G, G]$, ..., $G_{n+1} = [G, G_n]$.

The structure of G/G_1 is determined by class field theory, Fröhlich [15] determined the structure of G/G_2 for $K = \mathbb{Q}$, Ullom & Watt [UW] did the same for cyclotomic fields $\mathbb{Q}(\zeta_p)$ for regular primes p, and Movahhedi & Nguyen-Quang-Do [MT] for 'p-rational' number fields. Komatsu [Kom] studies this problem for real quadratic number fields K with class number not divisible by p and fundamental units ε with $\varepsilon^{p-1} \equiv 1 \bmod p^2$.

A particularly nice application of the results in this book was given by Bölling [Boe]: Callahan [Ca] had conjectured that $r_3(K) = r_3(k) - 1$, where $r_3(K)$ denotes the 3-rank of the class group of a non-normal cubic field K, and where $r_3(k)$ is the 3-rank of the class group of the quadratic subfield of the normal closure of $K/\mathbb{Q}$. This conjecture was later proved independently by Gerth and Gras. Bölling generalized this to extensions $K/\mathbb{Q}$ of odd prime degree p whose normal closure has degree $2p$ by proving the inequalities

$$r_p(k) - 1 \le r_p(K) \le (r_p(k) - 1)\frac{p-1}{2}.$$

Galois groups G of the maximal p-extension of $\mathbb{Q}$ unramified outside of two primes p and q have two generators and relation rank 1. The structure of G/G_2 is given in Thm. 11.10; Markshaitis [Mk1, Mk2] determined G/G_3 in the case $q \equiv 1 \bmod p^2$.

Fontaine-Mazur Conjecture

According to the Fontaine-Mazur conjecture, G_S is either finite or non-analytic if S does not contain primes dividing p. In all cases where the infinitude of G_S can be proved by the Golod-Shafarevich inequality, one can also prove that G_S is not analytic. In fact, if the generator rank $d(G_S)$ of G_S is at least 2, the Golod-Shafarevich inequality $d(G_S) \leq \frac{1}{4}\, d(G_S)^2$ for the relation rank $r(G_S)$ implies that G_S is infinite and even non-analytic. Since analytic groups are characterized by the property of having finite rank (the open subgroups have bounded generator rank), the Fontaine-Mazur conjecture predicts that the p-rank of the class group of the layers of an infinite subquotient of k_S/k tends to infinity whenever S is away from p (see Hajir [Ha1]).

While the Fontaine-Mazur conjecture remains out of reach at present, there are articles investigating some of its consequences; see Boston [Bo1, Bo2], Koch [Ko], Nomura [No1, No2, No3], and Wingberg [W2], as well as the contributions in du Sautoy, Segal, & Shalev [SSS].

Textbooks

Nowadays there are many textbooks discussing profinite groups from various points of view. For general topological groups there are Lutz [Lu] and Higgins [Hi]. Pontryagin duality for locally compact abelian groups is discussed by Morris [Mor].

For profinite groups we have the notes by Ribes [Ri] based on lectures by Neukirch, which have recently been reprinted; there are also new editions of Serre's Cohomologie Galoisienne [53]. The excellent report [Po] edited by Poitou is unfortunately out of print and would deserve more attention. The notes by Shatz [Sha] provide an introduction to the cohomology of profinite groups using spectral sequences; Fried & Jarden [FJ] also discuss profinite groups. Detailed introductions to profinite groups were given by Ribes & Zalesskii [RZ] as well as Wilson [Wil]. Dixon, du Sautoy Mann & Segal [DSMS] discuss analytic pro-p groups, and to complete the list, let us mention [NSW] once more. Finally Jarden's survey [Ja] gives a nice summary of results and open problems in infinite Galois theory.

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