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Fold maps
and
Positive Topological Quantum Field Theories

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Abstract

The notion of positive TFT as coined by Banagl is specified by an axiomatic system based on Atiyah's original axioms for TFTs. By virtue of a general framework that is based on the concept of Eilenberg completeness of semirings from computer science, a positive TFT can be produced rigorously via quantization of systems of fields and action functionals - a process inspired by Feynman's path integral from classical quantum field theory.
The purpose of the present dissertation thesis is to investigate a new differential topological invariant for smooth manifolds that arises as the state sum of the fold map TFT, which has been constructed by Banagl as a example of a positive TFT. By eliminating an internal technical assumption on the fields of the fold map TFT, we are able to express the informational content of the state sum in terms of an extension problem for fold maps from cobordisms into the plane. Next, we use the general theory of generic smooth maps into the plane to improve known results about the structure of the state sum in arbitrary dimensions, and to determine it completely in dimension two. The aggregate invariant of a homotopy sphere, which is derived from the state sum, naturally leads us to define a filtration of the group of homotopy spheres in order to understand the role of indefinite fold lines beyond a theorem of Saeki. As an application, we show how Kervaire spheres can be characterized by indefinite fold lines in certain dimensions.

Der von Banagl geprägte Begriff einer positiven TFT wird durch ein Axiomensystem festgelegt, dem Atiyahs ursprüngliche Axiome für TFTs zugrunde liegen. Vermöge eines allgemeinen Frameworks, das auf dem Konzept der Eilenberg-Vollständigkeit von Semiringen aus der Informatik aufbaut, kann eine positive TFT mathematisch streng durch Quantisierung von Systemen von Feldern und Wirkungsfunktionalen erzeugt werden - ein Prozess, der von Feynmans Pfadintegral aus der klassischen Quantenfeldtheorie inspiriert wird.
Das Ziel der vorliegenden Dotkorarbeit besteht darin, eine neue differentialtopologische Invariante glatter Mannigfaltigkeiten zu untersuchen, die als Zustandssumme der Faltungsabbildungs-TFT, die von Banagl als Beispiel für eine positive TFT konstruiert wurde, auftritt. Durch Beseitigung einer internen technischen Annahme an die Felder der Faltungsabbildungs-TFT können wir den Informationsgehalt der Zustandssumme durch ein Ausdehnungsproblem für Faltungsabbildungen von Kobordismen in die Ebene ausdrücken. Anschließend verwenden wir die allgemeine Theorie von generischen glatten Abbildungen in die Ebene, um bestehende Resultate über das Aussehen der Zustandssumme in beliebiger Dimension zu verbesseren, und um sie in Dimension 2 vollständig zu bestimmen. Die aus der Zustandssume abgeleitete Aggregatinvariante einer Homotopiesphäre führt uns in natürlicher Weise auf die Definition einer Filtration der Gruppe von Homotopiesphären, mittels der sich die Rolle von indefiniten Faltungslinien in Anknüpfung an ein Theorem von Saeki verstehen lässt. Als Anwendung zeigen wir, wie Kervaire-Sphären in gewissen Dimensionen durch indefinite Faltungslinien ausgezeichnet werden.

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Contents

Introduction

The notion of a *topological quantum field theory* (TFT) was axiomatically coined by Atiyah [3] who succeeded in packing common underlying principles of various low-dimensional geometric-physical theories (among others, Witten's quantization of Chern-Simons theory [61]) into a single mathematical axiomatic system. Atiyah's axioms are strongly related to Segal's mathematical framework of conformal field theories [50], which is however not purely topological due to the conformal structure. Roughly speaking, an $(n+1)$-dimensional TFT assigns to any closed n-manifold M a *state module* $Z(M)$ (i.e., a finitely generated module over a fixed ground ring), and to any cobordism W^{n+1} between two closed n-manifolds a *state sum* $Z_W \in Z(\partial W)$. According to Atiyah this assignment is most notably required to obey a multiplicative *gluing axiom*. More precisely, the gluing axiom postulates for any triple (M, N, P) of closed n-manifolds the existence of a product

$$\langle \cdot, \cdot \rangle : Z(M \sqcup N) \otimes Z(N \sqcup P) \to Z(M \sqcup P)$$

such that, whenever a cobordism W results from gluing a cobordism U from M to N and a cobordism V from N to P along the common boundary part N, one has $Z_W = \langle Z_U, Z_V \rangle$. The gluing axiom is of particular value because it allows in principle to calculate the state sum of a cobordism from a decomposition into simpler ones. Note that, in contrast to homological invariants well-known in topology such as the Euler characteristic (which satisfies $\chi(W) = \chi(U) + \chi(V) - \chi(N)$ in the above situation), the gluing axiom does in general *not* require extra contributions from the common boundary part N. In a compressed form, a TFT can be formulated as a symmetric monoidal functor from the bordism category to the category of vector spaces. As emphasized by Atiyah, his axiomatic system should not be considered as rigid, but instead as a flexible theoretic framework that allows for various adaptions to concrete situations while preserving essential characterstics such as the gluing axiom. Furthermore, Atiyah points out that his axioms can be of purely mathematical interest. While originating from theoretical physics, the concept of a TFT could therefore as well stimulate the construction of new topological invariants for manifolds.

Followed by a thorough outline of the overall structure of the present thesis, the upcoming sections introduce the two main themes of the thesis, namely Banagl's concept of positive TFTs and fold maps.

Positive Topological Quantum Field Theories

In [5] Banagl presents an individual approach to the construction of certain TFTs of any dimension by involving the notion of a *semiring* from computer science. The resulting concept of a *positive* TFT follows Atiyah's original ideas with a few necessary modifiactions in the axioms. In this context, positivity means that the state sum of the theory takes values in a complete semiring, which helps to avoid set theoretic trouble that generally arises in the definition of the Feynman path integral from quantum field theory. Compared to a ring, a semiring is not required to have additive inverses, i.e. "negative" elements, which motivates the choice of the name *positive* TFT. The minimal example of a semiring that is not a ring is given by the Boolean semiring $\mathbb{B}$. This is the set $\mathbb{B} = \{0, 1\}$ equipped with addition defined by $1 + 1 = 1$ and multiplication given by $0 \cdot 0 = 0$ (where 0 and 1 serve as identity elements for addition and multiplication). Distributivity holds, but in $\mathbb{B}$ there exists no additive inverse for 1. The loss of additive inverses offers the chance to consider *complete* semirings, in which the sum of a familiy of elements indexed by an arbitrary index set can be formed in a well-defined way. However, a complete semiring that is even a ring can be shown to be trivial. The concept of a complete semiring goes back to Eilenberg [12] and incorporates ideas due to Conway from automata theory and formal languages. Inspired by Feynman's path integral, positive TQFTs can be produced through quantization. The use of Maslov's idempotent integration [34] based on Eilenberg's completeness of semirings makes this method mathematically rigorous. Positive TQFTs have the potential to motivate the construction of new invariants for smooth manifolds, an important example being the *aggregate invariant* of an exotic sphere. In contrast to Atiyah's classical axioms a positive TFT is based on a pair (Q^c, Q^m) of semirings with the same underlying additive monoid Q. The state modules $Z(M)$ are (in general not finitely generated) semialgebras over both semirings.

An $(n+1)$-dimensional positive TFT can be constructed via *quantization* from given systems of fields and action functionals. A system $\mathcal{F}$ of *fields* assigns to every closed manifold M^n and to every bordism W^{n+1} the sets of fields $\mathcal{F}(M)$ and $\mathcal{F}(W)$. (By definition, $\mathcal{F}(\emptyset)$ is a set with one element.) Fields on a bordism can be restricted to subbordisms and to codimension 1 submanifolds. Apart from a desirable behaviour under the action of homeomorphisms and under disjoint union, fields are especially required to glue under the gluing of bordisms. A system $\mathbb{T}$ of action functionals (or action exponentials) over a fixed strict monoidal category $\mathbf{C}$ axiomatizes the exponential of the action from physics that appears in the integrand of the Feynman path integral. It assigns to every bordism W a map $\mathbb{T}_W : \mathcal{F}(W) \longrightarrow \mathrm{Mor}(\mathbf{C})$ in such a way that a disjoint union of bordisms is reflected by the tensor product of morphisms in $\mathbf{C}$ and a gluing of bordisms corresponds to composition of morphisms. More precisely, it is required that $\mathbb{T}_W(f) = \mathbb{T}_U(f|_U) \otimes \mathbb{T}_V(f|_V)$ for fields on the disjoint union $W = U \sqcup V$, and $\mathbb{T}_W(f) = \mathbb{T}_U(f|_U) \circ \mathbb{T}_V(f|_V)$ for fields on the gluing $W = U \cup_N V$ of U and V along N. Furthermore, the action functional is invariant under the action of homeomorphisms. A convenient choice for $\mathbf{C}$ is the category **Vect** of real vector spaces with linear maps as morphisms. As described in [5, Section 8.1, p. 42ff], **Vect** is promoted to a strict monoidal category with unit object $\mathbb{R}$ by introducing the *Schauenburg tensor product.*

We next describe the process of quantization, which produces a positive TFT from given data $\mathcal{F}$ and $\mathbb{T}$. The first step is to construct a complete additive monoid Q from a fixed complete semiring S and the strict monoidal category $\mathbf{C}$. The elements of Q are just maps $\mathrm{Mor}(\mathbf{C}) \to S$.

By equipping Q with two different multiplications via the completeness of S, one obtains a pair (Q^c, Q^m) of complete semirings. Multiplication in Q^c is based on the composition of morphisms in $\mathbf{C}$, whereas multiplication in Q^m exploits the monoidal structure of $\mathbf{C}$. Given a bordism W let $T_W\colon \mathcal{F}(W) \longrightarrow Q$ denote the composition of $\mathbb{T}_W\colon \mathcal{F}(W) \longrightarrow \mathrm{Mor}(\mathbf{C})$ with the map $\mathrm{Mor}(\mathbf{C}) \to Q$ that assigns to every morphism in $\mathbf{C}$ its characteristic function. In analogy with the quantum Hilbert state from physics, the state module $Z(M)$ of a closed manifold M^n consists of all maps ("states") $\mathcal{F}(M) \to Q$ that satisfy a certain constraint equation. The state sum $Z_W\colon \mathcal{F}(\partial W) \to Q$ is defined on f as

$$Z_W(f) = \sum_{F \in \mathcal{F}(W,f)} T_W(F) \in Q,$$

where the sum ranges over all fields on W that extend f. The state sum is defined in analogy with the Feynman path integral from quantum field theory,

$$\int_{F \in \mathcal{F}(W,f)} e^{iS_W(F)} d\mu_W,$$

whose integrand depends on the exponential of the action S_W. The counterpart of the observation that the amplitude of the integrand is 1 is the fact that the values of T_W are characteristic functions. Note that the definition of Z_W is rigorous due to the completeness of Q, whereas the existence of a rigorous definition for the measure μ_W is in general doubtful from a mathematical point of view. It can be shown that Z_W satisfies the constraint equation and is thus an element of the state module $Z(\partial W)$. Furthermore, the state modules and state sums thus defined can be shown to satisfy all axioms of a positive TFT. In particular, the gluing axiom holds, where the definition of the product $\langle \cdot, \cdot \rangle : Z(\partial U) \otimes Z(\partial V) \to Z(\partial W)$ involves the multiplication in Q^c. For a topologically meaningful choice of fields and the action functional the state sum Z_W of the induced positive TFT can be expected to be an interesting invariant of bordisms W that can be investigated further.

Crucial for Banagl's main example of a smooth positive TFT in [4] is the general observation that cobordism groups of smooth maps with prescribed singularity type are in principle capable of distinguishing exotic smooth structures on spheres. For instance, the oriented bordism group $\mathrm{SI}(n,1)$ of codimension 1 immersions $M^n \to \mathbb{R}^{n+1}$ (with immersions of oriented $(n+1)$-cobordisms into $\mathbb{R}^n \times [0,1]$ serving as bordism relation) can detect whether a given homotopy n-sphere can be bounded by a parallelizable bordism (i.e., whether it lies in the subgroup bP_{n+1} of the group Θ_n of homotopy n-spheres) as follows. As explained in [54, Sections 2.1 and 2.2, p. 101], the group $\mathrm{SI}(n,1)$ is known to be isomorphic (via suspension and slight perturbation of representatives $M^n \to \mathbb{R}^{n+1}$) to the bordism group Ω_n^{fr} of stably (normally) framed embeddings $M^n \to \mathbb{R}^{n+k}$, which is in turn isomorphic via the Pontjagin-Thom construction to the n-th stable homotopy group π_n^S of spheres. A given homotopy sphere Σ^n can always be promoted to an element in $\mathrm{SI}(n,1)$ by choosing an immersion $f\colon \Sigma^n \to \mathbb{R}^{n+1}$. (The resulting element in $\mathrm{SI}(n,1)$ might depend on the chosen immersion.) Varying over all possible f, the corresponding element in $\mathrm{SI}(n,1) \cong \Omega_n^{fr} \cong \pi_n^S$ can by [27, Lemma 4.2, p. 510] happen to be the zero element if and only if Σ^n bounds a parallelizable cobordism. Thus, for dimensions n in which bP_{n+1} forms a proper subgroup of the group of homotopy n-spheres, this demonstrates how $\mathrm{SI}(n,1)$ could in principle be used to detect exotic spheres in the complement $\Theta_n \setminus bP_{n+1}$. Another more recent result is due to Saeki [47] and gives an isomorphism between the n-th oriented cobordism group of so-called *special generic functions* and Θ_n. The type of singularities of the maps on cobordisms that are involved here are *fold singularities*, and will be focused on in the next section. Inspired by the potential of fold maps to detect exotic smooth structures on spheres, Banagl employs in [5] certain fold maps into the plane as fields of his smooth positive TFT.

Fold Maps

A *fold map* is a smooth map $M^m \to N^n$ between smooth manifolds of dimensions $m \geq n \geq 1$ that takes around each of its critical points the form

$$(x_1, \ldots, x_m) \mapsto (y_1, \ldots, y_n) = (x_1, \ldots x_{n-1}, \pm x_n^2 \pm \cdots \pm x_m^2)$$

in suitable charts $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ centered at the critical point and its image point. Note that the normal form of a fold singularity is just the multiple suspension of the standard normal form of a Morse singularity. The singular locus of a fold map $M^m \to N^n$ is a smooth submanifold of M of dimension $n-1$. To each of its components one can assign an integer called *absolute index* that can be calculated in terms of the number of minus signs occuring in the local normal form of a fold point. The following are interesting ideas to study:

(*a*) *What does the existence of a fold map $M \to \mathbb{R}^n$ reveal about the topology of M, e.g. in terms of characteristic classes of M?*
For instance, Levine [32] proved that a closed oriented manifold of dimension > 2 admits a fold map into the plane if and only if its Euler characteristic is even. Eliashberg [13] studied fold maps between equidimensional manifolds. More recently, Saeki [49] and others studied fold maps from closed smooth 4-manifolds into $\mathbb{R}^3$.

(*b*) *Construct a fold map $M \to \mathbb{R}^n$ with desired properties such as prescribed boundary conditions, or constraints on the components of its singular locus.*
A general tool for the construction of fold maps is Eliashberg's folding theorem [14]. This is an h-principle that produces a fold map with prescribed singular locus from more algebraic data, namely certain morphisms of tangent bundles. One essential condition for this method to work is that all possible values for the absolute index are required to occur.

The present thesis is specifically focused on fold maps from cobordisms into the plane that satisfy given boundary conditions, and such that the occuring absolute indices of fold points are required to lie in a prescribed set. The main techniques to handle fold maps in this setting are Levine's elimination of cusps [32] paired with the complementary process of creating cusps, Cerf theory [9], Stein factorization [47], and recent ideas due to Gay and Kirby [28] that were motivated by the study of broken Lefschetz fibrations in symplectic geometry. As an interesting feature, fold maps fit into Banagl's framework of positive topological quantum field theories [5]. In this context, the aggregate invariant of homotopy spheres can be defined via fold lines to detect exotic smooth structures on spheres, and we will study it by means of the above techniques.

Overall Structure

The thesis is divided into three parts, all of which combine known results with new material.

Firstly, Part I discusses three examples of smooth positive TFTs in the context of singularity theory of smooth maps. They all have in common that certain smooth maps on spaces related to the given cobordism serve as fields, and the action functional is in some way related to the singular locus of a field. The turning number of regular closed curves in the plane serves as input for the positive TFT presented in Chapter 1. Although this is a very simple one-dimensional toy example, it already requires some technical effort to make the axioms hold rigorously, which is why it is presented in detail. Moreover, Chapter 2 introduces a positive TFT that arises naturally from the construction of relative Stiefel-Whitney numbers in the spirit of Stiefel [52]. However, the main focus lies on the investigation of the third example, namely Banagl's *fold map TFT* (see Chapter 3). This is the most complicated of the three examples since the existence of fields that extend given boundary conditions and have prescribed values under the action functional is related to difficult questions about the existence of fold maps with desired properties on their singular locus. On top of that, in the construction of the fold map TFT there arises the technical difficulty that an additional condition has to be imposed on a fold map to make it a fold *field*. On the one hand, this condition grants that the indispensable gluing formula holds. On the other hand, the condition imposed on fold maps is quite intransparent, which makes the computation of state sums even less accessible. The heart of Chapter 3 is the proof of Theorem 3.4.9 in Section 3.4, which essentially allows to circumvent this condition in practice. More precisely, any pattern of the singular locus of a fold map can also be produced by that of some fold field. Furthermore, it is shown in Section 3.1.6 that the informational content of the state sum can precisely be described by *state sets*, namely sets of integers that occur as the number of loops in fold maps (whose singular locus is subject to a constraint) on the given cobordism. For this purpose, we prove in Section 3.2 that, roughly speaking, all linear representations of the Brauer category are faithful. As it turns out, the other two examples of smooth positive TFTs presented in Part I are in some way related to the fold map TFT. In fact, another reason for discussing the rotation number TFT is that the additivity of its action functional under gluing of fields will be employed for the complete computation of the state sum of the 2-dimensional fold map TFT that is pursued in Chapter 5. Furthermore, see Remark 5.3.3 (ii) for a possible connection of fold fields with fields in a stable version of the relative Stiefel-Whitney number TFT.

Motivated by questions that arise from the study of the fold map TFT, the purpose of the subsequent two parts of the thesis is to go deeper into singularity theory of smooth maps, which will eventually result in new theorems about fold maps. More precisely, Part II opens with a study of fold maps (see Chapter 4) from the general perspective of the theory of Thom-Boardman singularities. From this point of view one considers generic smooth maps from cobordisms into the plane, i.e. smooth maps with only fold and cusp singularities. The major part of the material is taken from [17] and Levine's article [32]. This allows to deduce some general theorems concerning the form of the state sum of the fold map TFT in Chapter 6. These results are in accordance with Eliashberg's folding theorem [14] in that they show the stronger statement that, in the presence of all possible absolute indices, the fold locus of a fold map into the plane carries no homological information about the manifold on which the fold map is defined.

Finally, working more concretely with fold maps defined on cobordisms that are bounded by exotic spheres, Part III sheds some light on the informational content of their indefinite fold lines. For this purpose, one needs to know how to construct fold maps on cobordisms in a controlled way (see Chapter 7), and how to extend Morse functions that are defined on the two ends of a cylindrical cobordism generically (see Chapter 8). Furthermore, Chapter 9 gives a detailed review of the method of Stein factorization of certain indefinite fold maps. With all these techniques at hand, we proceed to prove our main theorem in Chapter 10 which studies a certain filtration of the group of homotopy spheres and can be seen as a continuation of Cerf theory. These insights culminate in the detection of Kervaire spheres via fold maps, a result which is of independent interest, but can as well be interpreted in terms of the *aggregate invariant*, an invariant of exotic spheres that arises naturally within the fold map TFT.

Finally, the appendix covers various fundamental subjects in differential topology, ranging from background on the technique of transversality in Appendix A over issues concerning the construction of collar and tubular neighbourhoods with specific properties in Appendix B to Morse theory in Appendix C. The main reasons for the selection of this material are to supply rigorous proofs where results are declared as folklore in the literature, and to present it in a form that is convenient for the use in the main body.

Part I

Three Examples of Smooth Positive TFTs in Singularity Theory

Chapter 1

The Turning Number TFT

We present a simple example of a 1-dimensional positive TFT defined on smooth oriented 1-dimensional bordisms. Roughly speaking, fields (see Section 1.1) on a given bordism are immersions into the plane, and a real-valued extension of the *turning number* to non-closed regular curves serves as action functional (see Section 1.2). Additivity of this action exponential under gluing of cobordisms plays a role (see Proposition 5.4.22) in the calculation of 2-dimensional state sums of Banagl's fold map TFT presented in Chapter 3.

1.1 System of Fields

In the following, W denotes a smooth oriented 1-dimensional bordism. (Ingoing and outgoing boundaries are also fixed, but suppressed in the notation. For instance, there are several different ways to consider the (oriented) interval $[a,b]$ ($a<b$) as a 1-bordism.) The term *regular curve* will always mean a smooth immersion $W \to \mathbb{R}^2$ for some bordism W.

Lemma 1.1.1. *To a regular curve $\alpha : [a,b] \to \mathbb{R}^2$ we assign the map*

$$\omega_\alpha : [a,b] \to S^1, \qquad \omega_\alpha(t) = \frac{\alpha'(t)}{||\alpha'(t)||}.$$

Given the regular curve α, the assignment ω has the following transformation properties:

(i) If $\iota : [a,b] \to [a,b]$ is given by $\iota(t) = a+b-t$, then $\omega_\beta = -\omega_\alpha \circ \iota$ for the regular curve $\beta := \alpha \circ \iota : [a,b] \to \mathbb{R}^2$.

(ii) If $\xi \colon [a',b'] \to [a,b]$ is an orientation preserving diffeomorphism, then $\omega_\beta = \omega_\alpha \circ \xi$ for the regular curve $\beta := \alpha \circ \xi \colon [a',b'] \to \mathbb{R}^2$.

(iii) If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\varphi(x,y) = (x,-y)$ (note that φ corresponds to complex conjugation under the identification $\mathbb{R}^2 \cong \mathbb{C}$), then $\omega_\beta = \varphi \circ \omega_\alpha$ for the regular curve $\beta := \varphi \circ \alpha : [a,b] \to \mathbb{R}^2$.

(iv) If $[a',b'] \subset [a,b]$, then $\omega_\alpha|_{[a',b']} = \omega_{\alpha|_{[a',b']}}$. (This is clear by definition of ω.)

Proof. (i). The chain rule implies $\beta'(t) = (\alpha \circ \iota)'(t) = \iota'(t)\alpha'(\iota(t)) = -\alpha'(\iota(t))$. Hence,

$$\omega_\beta(t) = \frac{\beta'(t)}{||\beta'(t)||} = -\frac{\alpha'(\iota(t))}{||\alpha'(\iota(t))||} = -\omega_\alpha(\iota(t)) = -(\omega_\alpha \circ \iota)(t).$$

(ii). The chain rule implies $\beta'(t) = (\alpha \circ \xi)'(t) = \xi'(t)\alpha'(\iota(t))$. Hence, using $\xi'(t) > 0$ for all t,

$$\omega_\beta(t) = \frac{\beta'(t)}{||\beta'(t)||} = \frac{\xi'(t)\alpha'(\xi(t))}{|\xi'(t)| \cdot ||\alpha'(\xi(t))||} = \frac{\alpha'(\xi(t))}{||\alpha'(\xi(t))||} = \omega_\alpha(\xi(t)) = (\omega_\alpha \circ \xi)(t).$$

(iii). Linearity of φ implies $\beta'(t) = (\varphi \circ \alpha)'(t) = \varphi(\alpha'(t))$ and

$$\omega_\beta(t) = \frac{\beta'(t)}{||\beta'(t)||} = \frac{\varphi(\alpha'(t))}{||\alpha'(t)||} = \varphi\left(\frac{\alpha'(t)}{||\alpha'(t)||}\right) = \varphi(\omega_\alpha(t)) = (\varphi \circ \omega_\alpha)(t).$$

□

Loops in a space X can either be thought of as maps $\vartheta\colon S^1 \to X$ or as maps $\theta\colon [0,1] \to X$ such that $\theta(0) = \theta(1)$. Let $\rho := e|_{[0,1]}\colon [0,1] \to S^1$ be the restriction to $[0,1]$ of the universal cover

$$e : \mathbb{R} \to S^1, \qquad e(s) = \exp(2\pi i s).$$

We will identify the quotient space of $\pi\colon [0,1] \to [0,1]/\{0,1\}$ with S^1 via the unique homeomorphism $\tilde{\rho}\colon [0,1]/\{0,1\} \xrightarrow{\cong} S^1$ such that $\tilde{\rho} \circ \pi = \rho$. With this identification the above two versions of loops in X mutually determine each other via $\theta = \vartheta \circ \rho$. Fixing the basepoint $x_0 := \rho(0) = \rho(1) = (1,0) \in S^1$, this applies in particular to loops in $\pi_1(S^1, x_0)$.

Definition 1.1.2. Based on the assignment ω of Lemma 1.1.1 we will assign to a regular curve $\alpha\colon W \to \mathbb{R}^2$ its *turning map* $\omega_\alpha\colon W \to S^1$ as follows. We distinguish the following three cases:

- If there exists an orientation preserving diffeomorphism $\psi\colon [a,b] \xrightarrow{\cong} W$, then we define

  $$\omega_\alpha := \omega_{\alpha\circ\psi} \circ \psi^{-1}\colon W \to S^1.$$

  Note that the definition of ω_α does not depend on the choice of ψ and is in particular compatible with the assignment ω of Lemma 1.1.1. In fact, a second orientation preserving diffeomorphism $\psi'\colon [a',b'] \to W$ is related to ψ via $\psi' = \psi \circ \xi$ for some orientation preserving diffeomorphism $\xi\colon [a',b'] \to [a,b]$. Hence, by Lemma 1.1.1(ii),

  $$\omega_{\alpha\circ\psi'} \circ (\psi')^{-1} = \omega_{\alpha\circ\psi\circ\xi} \circ \xi^{-1} \circ \psi^{-1} = \omega_{\alpha\circ\psi} \circ \xi \circ \xi^{-1} \circ \psi^{-1} = \omega_{\alpha\circ\psi} \circ \psi^{-1}.$$

- If there exists an orientation preserving diffeomorphism $\psi\colon S^1 \xrightarrow{\cong} W$, then we define

  $$\omega_\alpha := \lambda \circ \psi^{-1}\colon W \to S^1,$$

  where the loop $\lambda\colon S^1 \to S^1$ is uniquely determined by $\lambda \circ \rho = \omega_{\alpha\circ\psi\circ\rho}$. Note that the definition of ω_α does not depend on the choice of ψ and is in particular compatible with the assignment ω of Lemma 1.1.1. Indeed, a second orientation preserving diffeomorphism $\psi'\colon S^1 \xrightarrow{\cong} W$ is related to ψ via $\psi' = \psi\circ\eta$ for some orientation preserving diffeomorphism $\eta\colon S^1 \xrightarrow{\cong} S^1$. Let $\overline{\eta\circ\rho}\colon [0,1] \to \mathbb{R}$ be a lift of $\eta\circ\rho$ under e. Note that $\overline{\eta\circ\rho}$ restricts to an orientation preserving diffeomorphism $\xi\colon [0,1] \xrightarrow{\cong} \overline{\eta\circ\rho}([0,1]) =: [a,b]$. Noting that $\psi' \circ \rho = \psi \circ \eta \circ \rho = \psi \circ e \circ \overline{\eta\circ\rho} = \psi \circ e|_{[a,b]} \circ \xi$, Lemma 1.1.1(ii) implies that $\omega_{\alpha\circ\psi'\circ\rho} = \omega_{\alpha\circ\psi\circ e|_{[a,b]}} \circ \xi$. It follows from $\lambda\circ\rho = \omega_{\alpha\circ\psi\circ\rho}$ that $\lambda\circ e|_{[a,b]} = \omega_{\alpha\circ\psi\circ e|_{[a,b]}}$. (In fact, as $\delta := \alpha\circ\psi\circ e$ is 1-periodic, we have $\omega_{\alpha\circ\psi\circ e|_{[a,b]}}(t) = \delta'(t)/||\delta'(t)|| = \delta'(t-k)/||\delta'(t-k)|| = \omega_{\alpha\circ\psi\circ\rho}(t-k) = (\lambda\circ\rho)(t-k) = (\lambda\circ e|_{[a,b]})(t)$ for $t \in [a,b]$ and $k \in \mathbb{Z}$ such that $t-k \in [0,1]$.)

Hence, $\omega_{\alpha\circ\psi'\circ\rho} = \lambda\circ e|_{[a,b]}\circ\xi = \lambda\circ e\circ\overline{\eta\circ\rho} = \lambda\circ\eta\circ\rho$. Thus, the loop $\lambda' := \lambda\circ\eta$ is uniquely determined by $\lambda'\circ\rho = \omega_{\alpha\circ\psi'\circ\rho}$, and we obtain

$$\lambda'\circ(\psi')^{-1} = (\lambda\circ\eta)\circ(\psi\circ\eta)^{-1} = \lambda\circ\psi^{-1}.$$

- If W is not connected, then we define ω_α on every component $W_0\subset W$ by

$$\omega_\alpha|_{W_0} := \omega_{\alpha|_{W_0}}.$$

Lemma 1.1.3. *If $\alpha\colon W\to\mathbb{R}^2$ is a regular curve and $W'\subset W$ is a subbordism, then $\alpha|_{W'}$ is a regular curve, and the turning maps of α and $\alpha|_{W'}$ are related by $\omega_\alpha|_{W'} = \omega_{\alpha|_{W'}}$.*

Proof. It suffices to show that the desired equality holds on every component of W'. Let W_0' be a component of W' and let W_0 denote the component of W that contains W_0'. If $\omega_{\alpha|_{W_0}}|_{W_0'} = \omega_{\alpha|_{W_0'}}$, then the claim follows: $(\omega_\alpha|_{W'})|_{W_0'} = \omega_\alpha|_{W_0'} = (\omega_\alpha|_{W_0})|_{W_0'} = \omega_{\alpha|_{W_0}}|_{W_0'} = \omega_{\alpha|_{W_0'}} = \omega_{\alpha|_{W'}}|_{W_0'}$. In order to show that $\omega_{\alpha|_{W_0}}|_{W_0'} = \omega_{\alpha|_{W_0'}}$, we distinguish the following three cases:

- If $W_0'\cong[0,1]$ and $W_0\cong[0,1]$, then we choose an orientation preserving diffeomorphism $\psi\colon[a,b]\xrightarrow{\cong}W_0$ and note that ψ restricts on $[a',b'] := \psi^{-1}(W_0')$ to an orientation preserving diffeomorphism $\psi'\colon[a',b']\xrightarrow{\cong}W_0'$. Lemma 1.1.1(iv) implies $\omega_{\alpha|_{W_0}\circ\psi}|_{[a',b']} = \omega_{(\alpha|_{W_0}\circ\psi)|_{[a',b']}} = \omega_{\alpha|_{W_0'}\circ\psi'}$. Hence,

$$\omega_{\alpha|_{W_0}}|_{W_0'} = (\omega_{\alpha|_{W_0}\circ\psi}\circ\psi^{-1})|_{W_0'} = \omega_{\alpha|_{W_0}\circ\psi}|_{[a',b']}\circ(\psi')^{-1} = \omega_{\alpha|_{W_0'}\circ\psi'}\circ(\psi')^{-1} = \omega_{\alpha|_{W_0'}}.$$

- If $W_0'\cong[0,1]$ and $W_0\cong S^1$, then we choose an orientation preserving diffeomorphism $\psi\colon S^1\xrightarrow{\cong}W_0$ and note that $\psi\circ\rho\colon[0,1]\to W_0$ restricts on $[a',b'] := (\psi\circ\rho)^{-1}(W_0')\subset[0,1]$ to an orientation preserving diffeomorphism $\psi'\colon[a',b']\xrightarrow{\cong}W_0'$. Lemma 1.1.1(iv) implies $\omega_{\alpha|_{W_0}\circ\psi\circ\rho}|_{[a',b']} = \omega_{(\alpha|_{W_0}\circ\psi\circ\rho)|_{[a',b']}} = \omega_{\alpha|_{W_0'}\circ\psi'}$. Hence, if the loop $\lambda\colon S^1\to S^1$ is uniquely determined by $\lambda\circ\rho = \omega_{\alpha|_{W_0}\circ\psi\circ\rho}$, then

$$\omega_{\alpha|_{W_0}}|_{W_0'} = (\lambda\circ\psi^{-1})|_{W_0'} = (\lambda\circ\rho)|_{[a',b']}\circ(\psi')^{-1} = \omega_{\alpha|_{W_0'}\circ\psi'}\circ(\psi')^{-1} = \omega_{\alpha|_{W_0'}}.$$

- If $W_0'\cong S^1$, then $W_0 = W_0'$ and there is nothing to show.

□

In Theorem 1.1.7 we introduce the system $\mathcal{F}$ of fields on smooth oriented 1-bordisms W^1 and closed smooth oriented 0-dimensional manifolds M^0 (note that the underlying space M is a finite set). Intuitively, a field on the bordism W is a piecewise immersion of W into the plane such that the normalized tangent vector fields of the immersed pieces in the plane given by the orientation of W fit together continuously. A field on M assigns to each point in M a point in the plane and a unit tangent vector in the tangent space at that point.

Definition 1.1.4. Let $(\alpha,\omega)\colon W\to\mathbb{R}^2\times S^1$ be a continuous map. We say that a subbordism $W'\subset W$ is *compatible* with (α,ω) if $\alpha|_{W'}$ is a regular curve and $\omega|_{W'} = \omega_{\alpha|_{W'}}$. Moreover, we say that a finite cover $W = \bigcup_{\lambda\in\Lambda}W_\lambda$ by subbordisms $W_\lambda\subset W$ is a *compatible cover* for (α,ω) if every W_λ is compatible with (α,ω).

Example 1.1.5. The bordism $W = [-1,1]$ from $\{-1\}$ to $\{1\}$ can be obtained from gluing $W = W' \cup_N W''$, where $W' := [-1,0]$ is a bordism from $\{-1\}$ to $N := \{0\}$ and $W'' := [0,1]$ is a bordism from N to $\{1\}$. The immersions $\alpha' \colon [-1,0] \to \mathbb{R}^2$, $\alpha'(t) = (t,0)$, and $\alpha'' \colon [0,1] \to \mathbb{R}^2$, $\alpha''(t) = (2t,0)$, glue to a continuous map $\alpha = \alpha' \cup \alpha'' \colon W \to \mathbb{R}^2$ and the turning maps $\omega_\alpha \equiv (1,0)$ and $\omega_{\alpha'} \equiv (1,0)$ glue to the continuous map $\omega \equiv (1,0) \colon W \to S^1$. Note that W is not compatible with (α,ω) as α is not even differentiable at 0. However, W' and W'' are compatible with (α,ω) by construction ($\alpha|_{W'} = \alpha'$ and $\alpha|_{W''} = \alpha''$ are regular curves with $\omega|_{W'} = \omega_{\alpha'}$ and $\omega|_{W''} = \omega_{\alpha''}$), which makes $W = W' \cup W''$ a compatible cover for (α,ω). As this example shows, *gluing* of two immersions will in general not yield an immersion, which forces us to allow for *fields* (α,ω) where α is a piecewise immersion rather than an immersion (see Theorem 1.1.7). Continuity of ω is required to define the restriction of fields to subbordisms of codimension 1 in a reasonable way.

Lemma 1.1.6. *Let $(\alpha,\omega)\colon W \to \mathbb{R}^2 \times S^1$ be a continuous map. Suppose that there exists a (not necessarily finite) cover $W = \bigcup_{\lambda\in\Lambda} W_\lambda$ by subbordisms $W_\lambda \subset W$ such that every W_λ is compatible with (α,ω). If $W' \subset W$ is a subbordism such that $\alpha|_{W'}$ is a regular curve, then W' is compatible with (α,ω).*

Proof. We have to show that $\omega|_{W'} = \omega_{\alpha|_{W'}}$. As both sides are continuous, it suffices to show $\omega|_V = \omega_{\alpha|_{W'}}|_V$ for some dense subset $V \subset W'$. Setting $V_\lambda := \operatorname{int} W' \cap \operatorname{int} W_\lambda$ for all $\lambda \in \Lambda$ and $V := \bigcup_{\lambda\in\Lambda} V_\lambda$, we check that V has the desired properties:

- $\omega|_V = \omega_{\alpha|_{W'}}|_V$: It suffices to show that if $[0,1] \cong K \subset W$ is a subbordism such that $K \subset V_\lambda$ for some λ, then $\omega|_K = \omega_{\alpha|_{W'}}|_K$. This follows from Lemma 1.1.3 for the subbordisms $K \subset W_\lambda$ and $K \subset W'$: $\omega|_K = (\omega|_{W_\lambda})|_K = \omega_{\alpha|_{W_\lambda}}|_K = \omega_{\alpha|_K} = \omega_{\alpha|_{W'}}|_K$.
- V is dense in W': It suffices to show that $U \cap V \neq \emptyset$ for any open subset $\emptyset \neq U \subset W'$. Given such U, note that $U \cap \operatorname{int} W' \neq \emptyset$. (Indeed, one can use the general fact that if $\emptyset \neq \tilde{U} \subset W$ is an open subset and $\tilde{W} \subset W$ is a subbordism such that $\tilde{U} \cap \tilde{W} \neq \emptyset$, then $\tilde{U} \cap \operatorname{int} \tilde{W} \neq \emptyset$.) By the same argument, it follows from $W = \bigcup_{\lambda\in\Lambda} W_\lambda$ that $U \cap \operatorname{int} W' \cap \operatorname{int} W_{\lambda_0} \neq \emptyset$ for some index $\lambda_0 \in \Lambda$. This implies $U \cap V \supset U \cap V_{\lambda_0} = U \cap \operatorname{int} W' \cap \operatorname{int} W_{\lambda_0} \neq \emptyset$.

□

Theorem 1.1.7. *A system $\mathcal{F}$ of fields on smooth oriented 1-dimensional bordisms W and closed smooth oriented 0-dimensional manifolds M can be defined as follows. Let $\mathcal{F}(W)$ be the set of continuous maps $(\alpha,\omega)\colon W \to \mathbb{R}^2 \times S^1$ for which there exists a compatible cover of W. Let $\mathcal{F}(M)$ be the set of maps $(a,w)\colon M \to \mathbb{R}^2 \times S^1$. Restriction of fields on W to a subbordism $W' \subset W$ and to a closed (as a manifold) submanifold $N \subset W$ of codimension 1, and restrictions of fields on M to a submanifold $M' \subset M$ of codimension 0 are defined by restriction of maps.*

Proof. (FRES) *Restrictions*: It is clear that restriction $\mathcal{F}(W) \to \mathcal{F}(N)$, $(\alpha,\omega) \mapsto (\alpha,\omega)|_N := (\alpha|_N, \omega|_N)$, of fields on W to a closed (as a manifold) submanifold $N \subset W$ of codimension 1 and restriction $\mathcal{F}(M) \to \mathcal{F}(M')$, $(a,w) \mapsto (a,w)|_{M'} := (a|_M, w|_{M'})$, of fields on M to a submanifold $M' \subset M$ of codimension 0 are well-defined.

Let us check that the restriction $(\alpha,\omega)|_{W'} := (\alpha|_{W'}, \omega|_{W'})$ of a field (α,ω) on W to a subbordism $W' \subset W$ is in fact a field on W'. Given $x \in W'$, we have to show that x is contained in a subbordism $W'' \subset W'$ which is compatible with $(\alpha|_{W'}, \omega|_{W'})$. For this purpose, choose a descending sequence $W''_0 \supset W''_1 \supset \dots$ of subbordisms $[0,1] \cong W''_i \subset W'$ such that $x \in \partial W''_0$ and $\bigcap_{i\in\mathbb{N}} W''_i = \{x\}$. Choose a compatible cover $W = \bigcup_{\lambda\in\Lambda} W_\lambda$ for (α,ω). It suffices to find $i \in \mathbb{N}$ and $\lambda \in \Lambda$ such that $W'' := W''_i \subset W_\lambda$. (In fact, since $\alpha|_{W_\lambda}$ is a regular curve, Lemma 1.1.3 then implies that $\omega|_{W''} = (\omega|_{W_\lambda})|_{W''} = (\omega_{\alpha|_{W_\lambda}})|_{W''} = \omega_{\alpha|_{W''}}$, which shows that W'' is compatible with $(\alpha|_{W'}, \omega|_{W'})$.) If $x \in \operatorname{int} W_\lambda$ for some $\lambda \in \Lambda$, then one can find $i \in \mathbb{N}$ such that $W''_i \subset W_\lambda$. If $x \in \bigcup_{\lambda\in\Lambda} \partial W_\lambda$, then there exists $i \in \mathbb{N}$ such that $W''_i \cap \bigcup_{\lambda\in\Lambda} \partial W_\lambda = \{x\}$ because Λ is finite. Then one can find $\lambda \in \Lambda$ such that $W''_i \subset W_\lambda$.

It is clear that all restriction maps commute with each other.

(FHOMEO) *Action of homeomorphisms*: An orientation preserving diffeomorphism $\phi\colon W \xrightarrow{\cong} W'$ induces contravariantly a map $\phi^*\colon \mathcal{F}(W') \to \mathcal{F}(W)$ by precomposition with ϕ. It suffices to check that for some compatible cover $W' = \bigcup_{\lambda\in\Lambda} W'_\lambda$ for $(\alpha',\omega') \in \mathcal{F}(W')$, $W = \bigcup_{\lambda\in\Lambda} \phi^{-1}(W'_\lambda)$ is a compatible cover for $(\alpha,\omega) := \phi^*(\alpha',\omega') = (\alpha'\circ\phi, \omega'\circ\phi)$. By Lemma 1.1.3 we may assume that $W'_\lambda \cong [0,1]$ for all $\lambda \in \Lambda$. Given $\lambda \in \Lambda$, $W_\lambda := \phi^{-1}(W'_\lambda)$ is a subbordism of W such that $\alpha|_{W_\lambda}$ is a regular curve because $\alpha'|_{W'_\lambda}$ is a regular curve. It remains to check that $\omega|_{W_\lambda} = \omega_{\alpha|_{W_\lambda}}$. If $\phi_\lambda\colon W_\lambda \xrightarrow{\cong} W'_\lambda$ denotes the restriction of ϕ and $\psi'_\lambda\colon [0,1] \xrightarrow{\cong} W'_\lambda$ is an orientation preserving diffeomorphism, then composition yields an orientation preserving diffeomorphism $\psi_\lambda := \phi_\lambda^{-1}\circ\psi'_\lambda\colon [0,1] \xrightarrow{\cong} W_\lambda$. Hence, $\omega|_{W_\lambda} = (\omega'\circ\phi)|_{W_\lambda} = \omega'|_{W'_\lambda}\circ\phi_\lambda = \omega_{\alpha'|_{W'_\lambda}}\circ\phi_\lambda = \omega_{\alpha'|_{W'_\lambda}\circ\psi'_\lambda}\circ(\psi'_\lambda)^{-1}\circ\phi_\lambda = \omega_{\alpha|_{W_\lambda}\circ\psi_\lambda}\circ(\psi_\lambda)^{-1} = \omega_{\alpha|_{W_\lambda}}$.

Similarly, an orientation preserving diffeomorphism $\phi\colon M \xrightarrow{\cong} N$ induces contravariantly a map $\phi^*\colon \mathcal{F}(N) \to \mathcal{F}(M)$. It is clear that the induced maps ϕ^* commute with restrictions.

(FDISJ) *Disjoint Unions*: If $W = W' \sqcup W''$, then the map $\mathcal{F}(W) \to \mathcal{F}(W') \times \mathcal{F}(W'')$, $(\alpha,\omega) \mapsto ((\alpha,\omega)|_{W'}, (\alpha,\omega)|_{W''})$, is a bijection. In fact, any field $(\alpha,\omega) \in \mathcal{F}(W)$ is uniquely determined by its restrictions to W' and W''. Conversely, if $(\alpha',\omega') \in \mathcal{F}(W')$ and $(\alpha'',\omega'') \in \mathcal{F}(W'')$, then $(\alpha,\omega) := (\alpha'\sqcup\alpha'', \omega'\sqcup\omega'')$ defines a continuous map $W \to \mathbb{R}^2 \times S^1$. A compatible cover for (α,ω) is obviously given by taking the union of a compatible cover for (α',ω') and a compatible cover for (α'',ω'').

(FGLUE) *Gluing*: Let W' be an oriented bordism from $-M$ to N and let W' be an oriented bordism from $-N$ to P. Let $W = W' \cup_N W''$ be the oriented bordism resulting from gluing. (W is equipped with a smooth structure such that the inclusions of W' and W'' into W are smooth.) It is obvious that fields on W are uniquely determined by their restrictions to the subbordisms $W', W'' \subset W$. On the other hand, suppose that $(\alpha',\omega') \in \mathcal{F}(W')$ and $(\alpha'',\omega'') \in \mathcal{F}(W'')$ satisfy $(\alpha',\omega')|_N = (\alpha'',\omega'')|_N$. Since $\alpha'|_N = \alpha''|_N$, we can define a continuous maps $\alpha\colon W \to \mathbb{R}^2$ by requiring $\alpha|_{W'} = \alpha'$ and $\alpha|_{W''} = \alpha''$. Similarly, $\omega'|_N = \omega''|_N$ implies that there is a continuous map $\omega\colon W \to S^1$ such that $\omega|_{W'} = \omega'$ and $\omega|_{W''} = \omega''$. It remains to check that there exists a compatible cover $W = \bigcup_{\lambda\in\Lambda} W_\lambda$ for (α,ω). For this purpose we choose compatible covers $W' = \bigcup_{\lambda'\in\Lambda'} W'_{\lambda'}$ for (α',ω') and $W'' = \bigcup_{\lambda''\in\Lambda''} W''_{\lambda''}$ for (α'',ω'') and claim that $\bigcup_{\lambda'\in\Lambda'} W'_{\lambda'} \cup \bigcup_{\lambda''\in\Lambda''} W''_{\lambda''}$ (which is already a finite cover of W by subbordisms) is a compatible cover for (α,ω). This can be shown as follows. Let $\lambda' \in \Lambda'$. Then $W'_{\lambda'} \subset W$ is a subbordism on which α restricts to the regular curve $\alpha|_{W'_{\lambda'}} = (\alpha|_{W'})|_{W'_{\lambda'}} = \alpha'|_{W'_{\lambda'}}$. Moreover, $\omega|_{W'_{\lambda'}} = (\omega|_{W'})|_{W'_{\lambda'}} = \omega'|_{W'_{\lambda'}} = \omega_{\alpha'|_{W'_{\lambda'}}}$. An analogous argument holds for all $\lambda'' \in \Lambda''$.

□

1.2 System of Action Functionals

To every path $\omega\colon [a,b] \to S^1$ (with $a < b$) we associate a real number $\gamma(\omega) \in \mathbb{R}$ as follows (see [23]). If $\overline{\omega}\colon [a,b] \to \mathbb{R}$ denotes any choice of a lift of ω under the universal cover

$$e : \mathbb{R} \to S^1, \qquad e(s) = \exp(2\pi i s),$$

(i.e. $\overline{\omega}$ is a path that satisfies $e \circ \overline{\omega} = \omega$), then we set $\gamma(\omega) := \overline{\omega}(b) - \overline{\omega}(a) \in \mathbb{R}$. (Note that this difference is independent of the choice of the lift $\overline{\omega}$ since any other lift of ω under e is of the form $\overline{\omega} + k$ with $k \in \mathbb{Z}$.)

Lemma 1.2.1. *Let $\omega\colon [a,b] \to S^1$ be a path.*

(i) If $\iota : [a,b] \to [a,b]$ is given by $\iota(t) = a + b - t$, then the path $\upsilon := \omega \circ \iota : [a,b] \to S^1$ satisfies $\gamma(\upsilon) = -\gamma(\omega)$.

(ii) If $\xi\colon [a',b'] \to [a,b]$ is an orientation preserving diffeomorphism, then the path $\upsilon := \omega \circ \xi : [a',b'] \to S^1$ satisfies $\gamma(\upsilon) = \gamma(\omega)$.

(iii) If $a < c < b$, then $\gamma(\omega) = \gamma(\omega|_{[a,c]}) + \gamma(\omega|_{[c,b]})$.

Proof. (i). If $\overline{\omega}$ is a lift of ω under e, then $\overline{\upsilon} := \overline{\omega} \circ \iota$ is a lift of υ under e since $e \circ \overline{\upsilon} = e \circ \overline{\omega} \circ \iota = \omega \circ \iota = \upsilon$. Hence, $\gamma(\upsilon) = \overline{\upsilon}(b) - \overline{\upsilon}(a) = \omega(a) - \omega(b) = -\gamma(\omega)$.

(ii). If $\overline{\omega}$ is a lift of ω under e, then $\overline{\upsilon} := \overline{\omega} \circ \xi$ is a lift of υ under e since $e \circ \overline{\upsilon} = e \circ \overline{\omega} \circ \xi = \omega \circ \xi = \upsilon$. Hence, $\gamma(\upsilon) = \overline{\upsilon}(b) - \overline{\upsilon}(a) = \omega(b) - \omega(a) = \gamma(\omega)$.

(iii). Let $\overline{\omega} : [a,b] \to \mathbb{R}$ be a lift of ω under the universal cover $e\colon \mathbb{R} \to S^1$. Then $\overline{\omega|_{[a,c]}} := \overline{\omega}|_{[a,c]}$ and $\overline{\omega_{[c,b]}} := \overline{\omega}|_{[c,b]}$ are admissible lifts of $\omega|_{[a,c]}$ and $\omega|_{[c,b]}$. All in all,

$$\gamma(\omega|_{[a,c]}) + \gamma(\omega|_{[c,b]}) = \overline{\omega|_{[a,c]}}(c) - \overline{\omega|_{[a,c]}}(a) + \overline{\omega|_{[c,b]}}(b) - \overline{\omega|_{[c,b]}}(c) = \overline{\omega}|_{[c,b]}(b) - \overline{\omega}|_{[a,c]}(a) = \gamma(\omega).$$

□

Recall that the *degree* of a continuous map $\phi\colon S^1 \to S^1$ is the integer $\deg(\phi) \in \mathbb{Z}$ that is uniquely determined by $\deg(\phi)\delta = \phi_*(\delta)$ where δ denotes any generator of $H_1(S^1)$.

Lemma 1.2.2. *If $\phi\colon S^1 \to S^1$ is any loop, then $\gamma(\phi \circ \rho) = \deg(\phi)$, where $\rho = e|_{[0,1]}$.*

Proof. Fix the basepoint $x_0 := \rho(0) = \rho(1) = (1,0) \in S^1$.

It is a well-known fact (see [20, Theorem 1.7, p. 29]) that γ induces an isomorphism

$$\tilde{\gamma}\colon \pi_1(S^1, x_0) \xrightarrow{\cong} \mathbb{Z}, \quad \tilde{\gamma}([\phi]) = \gamma(\phi \circ \rho).$$

Moreover, by [20, Corollary 4.25, p. 361], the degree map induces an isomorphism

$$\tilde{\deg}\colon \pi_1(S^1, x_0) \xrightarrow{\cong} \mathbb{Z}, \quad \tilde{\deg}([\phi]) = \deg(\phi).$$

The isomorphisms $\tilde{\gamma}$ and $\tilde{\deg}$ coincide. (In fact, the calculation $\tilde{\deg}([\mathrm{id}_{S^1}]) = 1 = \gamma(\rho) = \tilde{\gamma}([\mathrm{id}_{S^1}])$ shows that $\tilde{\gamma}$ and $\tilde{\deg}$ agree on a generator of $\pi_1(S^1, x_0)$.)

Given a loop $\phi\colon S^1 \to S^1$, we choose a homotopy $h\colon [0,1]\times S^1 \to S^1$ such that $h_0 = \mathrm{id}_{S^1}$ and $h_1(\phi(x_0)) = x_0$. As $\phi = h_0 \circ \phi$ is homotopic to $\phi_1 := h_1 \circ \phi$ and $\phi_1(x_0) = x_0$, we have $\deg(\phi) = \deg(\phi_1) = \tilde{\deg}([\phi_1]) = \tilde{\gamma}([\phi_1]) = \gamma(\phi_1 \circ \rho) = \gamma(h_1 \circ \omega)$ where $\omega := \phi \circ \rho$. Let $\overline{h}\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ be a homotopy such that $e \circ \overline{h}_t = h_t \circ e$ for all t and $\overline{h}_0 = \mathrm{id}_{\mathbb{R}}$. Let $\overline{\omega}$ be a lift of ω under e. Then $\overline{h}_t \circ \overline{\omega}$ is a homotopy between $\overline{\omega}$ and $\overline{h}_1 \circ \overline{\omega}$ that lifts the homotopy $h_t \circ \omega$ between ω and $h_1 \circ \omega$ since $e \circ \overline{h}_t \circ \overline{\omega} = h_t \circ e \circ \overline{\omega} = h_t \circ \omega$ for all t. For all t we have $\gamma(h_t \circ \omega) = (\overline{h}_t \circ \overline{\omega})(1) - (\overline{h}_t \circ \overline{\omega})(0)$ which is obviously continuous in t. Moreover, $\gamma(h_t \circ \omega) \in \mathbb{Z}$ for all t since $(h_t\circ\omega)(0) = (h_t\circ\omega)(1)$ for all t (using $\rho(0) = \rho(1)$). Hence, $\gamma(h_t\circ\omega)$ is constant in t and $\gamma(h_1 \circ \omega) = \gamma(\omega)$.

□

Definition 1.2.3. To a continuous map $\omega\colon W \to S^1$ we assign its *turning number* $\gamma(\omega) \in \mathbb{R}$ (which depends on the orientation of W) as follows. We distinguish the following three cases:

- If there exists an orientation preserving diffeomorphism $\psi\colon [a,b] \xrightarrow{\cong} W$, then we define $\gamma(\omega) := \gamma(\omega \circ \psi)$. (Indeed, the definition of $\gamma(\omega)$ does not depend on the choice of ψ since a second orientation preserving diffeomorphism $\psi'\colon [a',b'] \xrightarrow{\cong} W$ is related to ψ via $\psi' = \psi \circ \xi$ for some orientation preserving diffeomorphism $\xi\colon [a',b'] \xrightarrow{\cong} [a,b]$. Hence, by Lemma 1.2.1(ii), $\gamma(\omega \circ \psi') = \gamma(\omega \circ \psi \circ \xi) = \gamma(\omega \circ \psi)$.)
- If there exists an orientation preserving diffeomorphism $\psi\colon S^1 \xrightarrow{\cong} W$, then we define $\gamma(\omega) := \deg(\omega \circ \psi)$. (In fact, the definition of $\gamma(\omega)$ does not depend on the choice of ψ since a second orientation preserving diffeomorphism $\psi'\colon S^1 \xrightarrow{\cong} W$ is related to ψ via $\psi' = \psi \circ \eta$ for some orientation preserving diffeomorphism $\eta\colon S^1 \xrightarrow{\cong} S^1$. Hence, $\deg(\omega \circ \psi') = \deg(\omega \circ \psi \circ \eta) = \deg(\omega \circ \psi)\deg(\eta) = \deg(\omega \circ \psi)$.)
- If W is not connected, then we define $\gamma(\omega) := \sum_{W_0 \in \operatorname{comp} W} \gamma(\omega|_{W_0})$.

Theorem 1.2.4. *A system $\mathbb{T}$ of $\mathbb{R}$-valued action exponentials can be defined by $\mathbb{T}_W(\alpha,\omega) := \gamma(\omega)$ for fields $(\alpha,\omega) \in \mathcal{F}(W)$ on smooth oriented 1-dimensional bordisms W.*

Proof. (TDISJ): Note that for all $(\alpha,\omega) \in \mathcal{F}(W)$,

$$\mathbb{T}_W(\alpha,\omega) = \gamma(\omega) = \sum_{W_0 \in \operatorname{comp} W} \gamma(\omega|_{W_0}) = \sum_{W_0 \in \operatorname{comp} W} \mathbb{T}_{W_0}((\alpha,\omega)|_{W_0}).$$

In particular, if $W' \sqcup W''$ is the ordered disjoint union of the bordisms W' and W'', then $\mathbb{T}_{W'\sqcup W''}(\alpha,\omega) = \mathbb{T}_{W'}((\alpha,\omega)|_{W'}) + \mathbb{T}_{W''}((\alpha,\omega)|_{W''}) = \mathbb{T}_{W''\sqcup W'}(\alpha,\omega)$ for all $(\alpha,\omega) \in \mathcal{F}(W'\sqcup W'')$.

(TGLUE): Suppose that $W = W' \cup_N W''$ is obtained by gluing a bordism W' with outgoing boundary N to a bordism W'' with ingoing boundary $-N$. We claim that $\mathbb{T}_W(\alpha,\omega) = \mathbb{T}_{W'}((\alpha,\omega)|_{W'}) + \mathbb{T}_{W''}((\alpha,\omega)|_{W''})$ for all $(\alpha,\omega) \in \mathcal{F}(W)$. Assuming that W is connected (and $N \neq \emptyset$ without loss of generality), we distinguish the following two cases for the proof:

- If $W \cong [0,1]$, then there exists an orientation preserving diffeomorphism $\psi\colon [a,b] \xrightarrow{\cong} W$. There exists $n > 0$ such that $\psi^{-1}(N) = \{c_1,\ldots,c_n\}$ where $a =: c_0 < c_1 < \cdots < c_n < c_{n+1} := b$. For all i, ψ restricts to an orientation preserving diffeomorphism $\psi_i\colon [c_i,c_{i+1}] \xrightarrow{\cong} \psi([c_i,c_{i+1}]) =: W_i$. Moreover, set $W^j := \bigsqcup_{i\equiv j(2)} W_i$ for $j \in \{0,1\}$.

Note that $\{W^0, W^1\} = \{W', W''\}$. Hence, Lemma 1.2.1(iii) implies

$$\begin{aligned}\mathbb{T}_W(\alpha,\omega) &= \gamma(\omega) = \gamma(\omega\circ\psi)\\ &= \sum_{i=0}^{n}\gamma((\omega\circ\psi)|_{[c_i,c_{i+1}]}) = \sum_{i=0}^{n}\gamma(\omega|_{W_i}\circ\psi_i)\\ &= \sum_{i=0}^{n}\gamma(\omega|_{W_i}) = \sum_{i=0}^{n}\mathbb{T}_{W_i}((\alpha,\omega)|_{W_i})\\ &\overset{(TDISJ)}{=} \mathbb{T}_{W^0}((\alpha,\omega)|_{W^0}) + \mathbb{T}_{W^1}((\alpha,\omega)|_{W^1})\\ &= \mathbb{T}_{W'}((\alpha,\omega)|_{W'}) + \mathbb{T}_{W''}((\alpha,\omega)|_{W''}).\end{aligned}$$

- If $W \cong S^1$, then there exists an orientation preserving diffeomorphism $\psi\colon S^1 \xrightarrow{\cong} W$ such that $\psi(1,0) \in N$. There exists $n > 0$ such that $(\psi\circ\rho)^{-1}(N) = \{c_0,\dots,c_n\}$ where $0 =: c_0 < \dots < c_n := 1$. For all i, $\psi\circ\rho$ restricts to an orientation preserving diffeomorphism $(\psi\circ\rho)_i\colon [c_i, c_{i+1}] \xrightarrow{\cong} (\psi\circ\rho)([c_i,c_{i+1}]) =: W_i$. Moreover, set $W^j := \bigsqcup_{i\equiv j(2)} W_i$ for $j \in \{0,1\}$. Note that $\{W^0,W^1\} = \{W',W''\}$. Hence, Lemma 1.2.1(iii) implies $\mathbb{T}_W(\alpha,\omega) = \gamma(\omega) = \deg(\omega\circ\psi) \overset{1.2.2}{=} \gamma(\omega\circ\psi\circ\rho) = \sum_{i=0}^{n-1}\gamma((\omega\circ\psi\circ\rho)|_{[c_i,c_{i+1}]}) = \sum_{i=0}^{n-1}\gamma(\omega|_{W_i}\circ(\psi\circ\rho)_i) = \sum_{i=0}^{n-1}\gamma(\omega|_{W_i}) = \sum_{i=0}^{n-1}\mathbb{T}_{W_i}((\alpha,\omega)|_{W_i}) \overset{(TDISJ)}{=} \mathbb{T}_{W^0}((\alpha,\omega)|_{W^0}) + \mathbb{T}_{W^1}((\alpha,\omega)|_{W^1}) = \mathbb{T}_{W'}((\alpha,\omega)|_{W'}) + \mathbb{T}_{W''}((\alpha,\omega)|_{W''})$.

The case of a general bordism W can be deduced from (TDISJ) and the case where W is connected as follows. For every component W_0 of W, let W_0' be the union of all components of W' that are subbordisms of W_0 and let W_0'' be the union of all components of W'' that are subbordisms of W_0. Note that $W_0 = W_0' \cup_{N_0} W_0''$ is obtained by gluing W_0' with outgoing boundary $N_0 := W_0' \cap N$ to W_0'' with ingoing boundary $N_0 = W_0'' \cap N$. Also note that $W' = \bigsqcup_{W_0\in\operatorname{comp} W} W_0'$ and $W'' = \bigsqcup_{W_0\in\operatorname{comp} W} W_0''$. Hence,

$$\begin{aligned}\mathbb{T}_W(\alpha,\omega) &\overset{(TDISJ)}{=} \sum_{W_0\in\operatorname{comp} W}\mathbb{T}_{W_0}((\alpha,\omega)|_{W_0})\\ &\overset{(TGLUE)}{=} \sum_{W_0\in\operatorname{comp} W}\mathbb{T}_{W_0'}((\alpha,\omega)|_{W_0'}) + \sum_{W_0\in\operatorname{comp} W}\mathbb{T}_{W_0''}((\alpha,\omega)|_{W_0''})\\ &\overset{(TDISJ)}{=} \mathbb{T}_{W'}((\alpha,\omega)|_{W'}) + \mathbb{T}_{W''}((\alpha,\omega)|_{W''}).\end{aligned}$$

(THOMEO): We have to show that any orientation preserving diffeomorphism $\phi\colon W \xrightarrow{\cong} W'$ satisfies $\mathbb{T}_W(\phi^*(\alpha',\omega')) = \mathbb{T}_{W'}(\alpha',\omega')$ for all $(\alpha',\omega') \in \mathcal{F}(W')$. For every component W_0 of W, ϕ restricts to an orientation preserving diffeomorphism $\phi_{W_0}\colon W_0 \xrightarrow{\cong} W_0' := \phi(W_0)$. Note that $\gamma(\omega'|_{W_0'}\circ\phi_{W_0}) = \gamma(\omega'|_{W_0'})$. (In fact, if $W_0 \cong [0,1]$, then there exists an orientation preserving diffeomorphism $\psi\colon [a,b] \xrightarrow{\cong} W_0$, and $\gamma(\omega'|_{W_0'}\circ\phi_{W_0}) = \gamma(\omega'|_{W_0'}\circ\phi_{W_0}\circ\psi) = \gamma(\omega'|_{W_0'})$. Similarly, if $W_0 \cong S^1$, then there exists an orientation preserving diffeomorphism $\psi\colon S^1 \xrightarrow{\cong} W_0$, and $\gamma(\omega'|_{W_0'}\circ\phi_{W_0}) = \gamma(\omega'|_{W_0'}\circ\phi_{W_0}\circ\psi\circ\rho) = \gamma(\omega'|_{W_0'})$.) Hence, $\mathbb{T}_W(\phi^*(\alpha',\omega')) = \mathbb{T}_W(\alpha'\circ\phi,\omega'\circ\phi) = \gamma(\omega'\circ\phi) = \sum_{W_0\in\operatorname{comp} W}\gamma((\omega'\circ\phi)|_{W_0}) = \sum_{W_0\in\operatorname{comp} W}\gamma(\omega'|_{W_0'}\circ\phi_{W_0}) = \sum_{W_0'\in\operatorname{comp} W'}\gamma(\omega'|_{W_0'}) = \gamma(\omega') = \mathbb{T}_{W'}(\alpha',\omega')$.

$\square$

Chapter 2

The Relative Stiefel-Whitney Number TFT

As explained in [5, Section 11.2, p. 55f], an interesting perspective on Novikov additivity of the signature is supplied by its reformulation in terms of a positive TFT. In this context, Novikov additivity ensures that the gluing axiom holds, which is significant for the construction pursued by Banagl in [5] of a positive TFT from a system of fields and a system of category-valued action functionals on fields via the process of quantization.

In contrast to the resulting *signature TFT*, it appears at first glance vague what it should mean for characteristic numbers to behave additively under gluing of smooth bordisms. For instance, when writing the real projective plane $\mathbb{RP}^2$ as the gluing of a small 2-disc and the Möbius strip along their common boundary S^1, it turns out that the Stiefel-Whitney number $w_2[\mathbb{RP}^2]$ is the non-trivial element of $\mathbb{Z}/2$ (see [38, p. 47]), whereas both the 2-disc and the Möbius strip obviously have vanishing second Stiefel-Whitney class. An analogous observation is made in the introduction of [4, p. 4] for the Pontrjagin number $p_1[\mathbb{CP}^2]$. These examples indicate that one should in general not attempt to define characteristic numbers of smooth manifolds with boundary in terms of their (absolute) characteristic classes if additivity of characteristic numbers under gluing of bordisms is required to hold.

Under the assumption that the middle-dimensional homology groups of the common boundary $\partial W_1 = \partial W_2$ of two oriented bordisms W_1^8 and W_2^8 vanish, Milnor [39] shows additivity of the Pontrjagin number p_1^2 under the gluing $W_1 \cup W_2$ in the course of the construction of his λ-invariant. The Pontrjagin number $p_1^2[W]$ of such a bordism W is by definition the Kronecker product of a certain relative class $\alpha \in H^8(W, \partial W)$ and the fundamental class $[W] \in H_8(W, \partial W)$. Concerning the construction of α, the homological vanishing assumption on ∂W ensures that the homomorphism $H^4(W, \partial W) \to H^4(W)$ induced by inclusion is an isomorphism. Hence, the first Pontrjagin class $p_1(W) \in H^4(W)$ can be viewed as a relative class $\beta \in H^4(W, \partial W)$, and one defines $\alpha := \beta^2$. As $H^*(W, \partial W) \to H^*(W)$ needs not be bijective for general W, one would have to associate to W *relative* characteristic classes as elements in $H^*(W, \partial W)$ from the outset to have characteristic numbers available in full generality.

The construction of characteristic classes in relative cohomology is implemented in [26] via the obstruction theoretic approach (see [38, 51]) that was historically one of the motivations to study characteristic classes. As it turns out, the definition of relative Stiefel-Whitney characteristic classes on a bordism W requires to specify *boundary conditions* as additional data, namely a suitable number of linearly independent sections of $TW|_{\partial W}$.

Thus, if one wants to capture the essence of relative Stiefel-Whitney numbers in a positive TFT, then it is natural to introduce fields on a bordism W as certain families of sections of the tangent bundle TW. (Not just over ∂W since fields must be restrictable to any subbordism.) Involving the principle of transversality, the action functional will then extract information from the singular locus of these fields. This perspective will be employed in the present chapter to introduce the $(n+1)$-dimensional *relative Stiefel-Whitney number TFT* Z_{SW}^I that is associated to the relative Stiefel-Whitney number w_I corresponding to an integer partition $I=(i_1,\ldots,i_r)$ of $n+1$. For this purpose, we will crucially exploit the geometric approach to characteristic classes pursued in [1], where characteristic classes are expressed in terms of the singular locus of generic vector bundle morphisms from certain product bundles to the tangent bundle.

The present chapter is structured as follows.

We work throughout in the category of smooth manifolds and smooth maps, and the smooth setting that is convenient for defining fields that obey the gluing axiom is carefully introduced in Section 2.1. Here, the essential technical item is that all codimension 1 submanifolds of a given smooth bordism, and in particular its boundary, are assumed to be equipped with a germ of tubular neighbourhoods (or collars).

Section 2.2 lays the technical foundations for subsequent sections by collecting aspects of transversality within the areas of Whitney stratifications (see Section 2.2.1), generic smooth vector bundle morphisms (Section 2.2.2), and the intersection product (Section 2.2.3). Of particular importance is the concept of a *transverse system* (see Definition 2.2.4) of Whitney stratified subspaces of a smooth manifold, which basically requires that the intersection of some of these subspaces is transverse to any other subspace. This notion turns out to be the appropriate geometric requirement that is needed to express the iterated cup product of characteristic classes by the intersection product.

The system of fields $\mathcal{F}$ and the system of category-valued action functionals $\mathbb{T}$ on fields that determine the positive TFT Z_{SW}^I will be specified in Section 2.3. Fields on a smooth bordism W^{n+1} (see Definition 2.3.9) are defined in Section 2.3.1 essentially as r-tuples of generic smooth bundle morphisms from product bundles on W (whose ranks depend on the chosen partition $I=(i_1,\ldots,i_r)$) to the tangent bundle TW with the additional property that the singular loci of these morphisms form a transverse system. Roughly speaking, fields on a closed manifold M^n (see Definition 2.3.2) are germs of fields on the cylinder of M such that the corresponding r bundle morphisms have no simultaneous singularities. Let $\mathbf{N}$ denote the monoidal category determined by the monoid $(\mathbb{N},+,0)$ (see [5, Lemma 4.6, p. 19]). The system $\mathbb{T}$ of $\mathbf{N}$-valued action functionals of fields is introduced in Section 2.3.2. The map $\mathbb{T}_W\colon \mathcal{F}(W)\to \mathrm{Mor}(\mathbf{N})=\mathbb{N}$ of a given bordism W (see Definition 2.3.13) assigns to a field $F\in\mathcal{F}(W)$ the (finite) number of simultaneous singularities of the corresponding r bundle morphisms.

The process of quantization is indicated in Section 2.4. The shape of the state sum Z_{SW}^I on closed bordisms inspires the definition of an $\mathbb{N}$-valued invariant $|\chi^I|$ for closed smooth manifolds (see Definition 2.4.2) that generalizes the absolute value of the Euler characteristic, $|\chi^{(n+1)}|=|\chi|$, and reduces $\bmod 2$ to the Stiefel-Whitney number $w_I[W]$ (see Corollary 2.4.3).

Finally, the project culminates in Section 2.5 in the proof the following theorem which asserts that Z^I_{SW} reproduces the characteristic number w_I on closed smooth manifolds:

Theorem 2.0.1. *If W is a closed smooth bordism, then every field $F \in \mathcal{F}(W)$ satisfies*

$$\mathbb{T}_W(F) \bmod 2 = w_I[W] \qquad \in \mathbb{Z}/2.$$

It remains to be shown in future work that the invariant $\mathbb{T}_W(F) \bmod 2$ coincides with the relative Stiefel-Whitney numbers supplied by the obstruction theoretic approach of [26].

Remark 2.0.2. For simplicity, the present chapter focuses on Stiefel-Whitney characteristic classes. It can be expected that a *relative Pontrjagin number TFT* could be implemented in an analogous way, by working with certain families of generic sections of the complexified tangent bundle of a bordism. Furthermore, one should clarify the relation to relative Pontrjagin numbers as discussed in [28, Appendix A, p. 109f].

2.1 A Smooth Setting for Positive TFTs

The desired relative Stiefel-Whitney number TFT will be constructed in Section 2.3 by specifying a system of fields and an $\mathbf{N}$-valued action functional on these fields obeying the axioms stated in [5, section 5, p. 20ff]. As pointed out in [5, Remark 5.4, p. 23], it might be necessary to adapt this axiomatic framework to the situation of interest by introducing additional data on the manifolds. In our case, working in the smooth category requires to equip codimension 1 submanifolds with germs of collars or tubular neighbourhoods. The advantage of working with smooth mapping germs is that one does not need to impose transversality conditions for fields. The purpose of the present section is to state in detail the necessary modifications of the topological setting described in [5, section 5, p. 20].

Fix an integer $n \geq 0$. An $(n+1)$-dimensional *(smooth) bordism* is a quintuple (W, M, μ, N, ν) consisting of a compact smooth manifold W^{n+1} of dimension $n+1$ with boundary $\partial W = M \sqcup N$, an M-germ μ represented by collar neighbourhoods $M \times [0, \varepsilon)$ of $M \times 0 = M$ in W, and an N-germ ν represented by collar neighbourhoods $N \times (-\varepsilon, 0]$ of $N \times 0 = N$ in W. (We say that two collar neighbourhoods $M \times [0, \varepsilon)$ and $M \times [0, \varepsilon')$ of $M \times 0 = M$ in W represent the same M-germ μ if they restrict to the same collar neighbourhood $M \times [0, \varepsilon'')$ of $M \times 0 = M$ in W for suitable $0 < \varepsilon'' < \varepsilon, \varepsilon'$, and similarly for N-germs.) A *framed (smooth) codimension* 1 *submanifold* of the bordism (W, M, μ, N, ν) is a pair (P, π), where P^n is a smooth submanifold of W of codimension 1 such that every component C of P satisfies either $C \subset \partial W$ or $C \cap \partial W = \emptyset$, and such that π is a P-germ given on components C of P by $\pi|_C = \mu|_C$ whenever $C \subset M$, $\pi|_C = \nu|_C$ whenever $C \subset N$, and the C-germ $\pi|_C$ can for $C \cap \partial W = \emptyset$ be represented by a *trivial* tubular neighbourhood $C \times (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, of $C \times 0 = C$ in W. In particular, (M, μ) and (N, ν) are framed codimension 1 submanifolds of W. The disjoint union of framed codimension 1 submanifolds is defined as $(P, \pi) \sqcup (P', \pi') = (P \sqcup P', \pi \sqcup \pi')$. The disjoint union of bordisms is given by

$$(W, M, \mu, N, \nu) \sqcup (W', M', \mu', N', \nu') = (W \sqcup W', M \sqcup M', \mu \sqcup \mu', N \sqcup N', \nu \sqcup \nu').$$

Recording germs of collar neighbourhoods allows us to canonically equip the glued manifold

$$(W, M, \mu, N, \nu) \cup_N (W', N, \nu', P, \pi) = (W \cup_N W', M, \mu, P, \pi)$$

with the smooth structure given by the requirement that $(N, \nu \cup_N \nu')$ is a framed codimension 1 submanifold of $W \cup_N W'$. A *diffeomorphism of bordisms* $\phi\colon (W, M, \mu, N, \nu) \to (W', M', \mu', N', \nu')$ is a diffeomorphism $\phi\colon W \to W'$ such that $\phi(M) = M'$, $\phi(N) = N'$, and such that there exists $\varepsilon > 0$ with $\phi|_{M \times [0,\varepsilon)} = (\phi|_M) \times \mathrm{id}_{[0,\varepsilon)}$ and $\phi|_{N \times (-\varepsilon,0]} = (\phi|_N) \times \mathrm{id}_{(-\varepsilon,0]}$. Finally, a quintuple $(W_0, M_0, \mu_0, N_0, \nu_0)$ is called a *(smooth) subbordism* of the bordism (W, M, μ, N, ν) if (M_0, μ_0) and (N_0, ν_0) are framed codimension 1 submanifolds of W, and (W_0, M_0, N_0) is a subbordism of (W, M, N) in the topological sense (i.e. W_0 is a codimension 0 submanifold of W such that for every connected component C of M_0 either $C \cap \partial W = \emptyset$ or $C \subset M$, and for every connected component C of N_0 either $C \cap \partial W = \emptyset$ or $C \subset N$). In particular, note that $(W_0, M_0, \mu_0|_{W_0}, N_0, \nu_0|_{W_0})$ is itself a bordism.

2.2 Background on Transversality

Transversality is the key concept in our implementation of the relative Stiefel-Whitney number TFT. For convenience, the present section collects background material concerning the role of transversality in the context of Whitney stratifications (see Section 2.2.1), generic smooth vector bundle morphisms (Section 2.2.2), and the intersection product (Section 2.2.3).

Throughout the present section it is assumed that X is a fixed smooth manifold (without boundary) of dimension m.

2.2.1 Whitney Stratifications

The relevance of Whitney stratifications in the context of the present chapter comes from the fact that the singular locus of a *generic* smooth vector bundle morphism is a Whitney stratified space (as explained in Section 2.2.2). For a brief introduction to Whitney stratified spaces see [6, Section 6.2, p. 127ff]. Roughly, a Whitney stratification of a closed subset $W \subset X$ is a locally finite partition of W into certain *pieces* (see [6, Definition 6.2.1, p. 128]) which are required to be locally closed smooth submanifolds of X that fit together via Whitney's condition B. (Whitney's condition B in turn implies Whitney's condition A, compare [6, Definition 6.2.2, p. 128]). The concept of transversality carries over from submanifolds to Whitney stratified subspaces of X (see [6, Definition 6.2.3, p. 130]):

Definition 2.2.1. Two Whitney stratified spaces $W_1 \subset X$ and $W_2 \subset X$ are called *transverse* in X (in short: $W_1 \pitchfork W_2$ in X), if each stratum of W_1 is transverse in X to each stratum of W_2 (in the sense of smooth submanifolds of X).

Consequently, the *transverse intersection* $W_1 \cap W_2$ of the two transverse Whitney stratified spaces $W_1 \subset X$ and $W_2 \subset X$ is again a Whitney stratified space that is canonically stratified by mutual intersections of strata.

A noteable consequence of Whitney's condition A is that transversality of a map to a Whitney stratified space is an open condition:

Proposition 2.2.2. *Let $f\colon X \to Y$ be a smooth map between smooth manifolds. Suppose that $W \subset Y$ is Whitney stratified. If $p \in X$ is a point such that f is transverse to W at p, then there exists an open neighbourhood U of p in X such that f is transverse to W on U.*

Proof. Consider the jet extension $j^1(f)\colon X \to J^1(X,Y)$ of f. By the sufficiency part of the proof of [58, §3, Lemma 1, p. 757f], $V := \{z \in J^1(X,Y);\ z \pitchfork W\}$ is an open subset of $J^1(X,Y)$. (Note that $W \subset Y$ is by our definition assumed to be a closed subset.) Hence, it follows from $j^1(f)(p) \in V$ that $U := j^1(f)^{-1}(V)$ is the desired open neighbourhood of p in X. □

Remark 2.2.3. In [58, §3, p. 756ff], Wall considers stratifications of not necessarily closed subsets X of a smooth manifold M. Therefore, in [58, §3, Lemma 1, p. 757f] closedness is an additional condition of the stratified space X. However, as pointed out in [55, Note (iv), p. 274], there are counterexamples to the closedness of X in the necessity part of [58, §3, Lemma 1, p. 757f].

Definition 2.2.4. A finite set $\{W_1, \dots, W_r\}$ of Whitney stratified subspaces of X is called a *transverse system* in X if for every permutation σ of $\{1, \dots, r\}$ the following condition holds for $s = 1, \dots r-1$:

$$(*)_s^\sigma \qquad W_{\sigma(s+1)} \pitchfork W_{\sigma(1)} \cap \dots \cap W_{\sigma(s)} \qquad \text{in } X.$$

Remark 2.2.5. It is clear that any Whitney stratified subspace W of X forms a transverse system $\{W\}$ in X. Furthermore, two Whitney stratified subspaces of X form a transverse system $\{W_1, W_2\}$ in X if and only if $W_1 \pitchfork W_2$. Given a set $\{W_1, \dots, W_r\}$ of Whitney stratified subspaces of X for $r \geq 3$, note that condition $(*)_s^\sigma$ inductively implies for $s \in \{1, \dots, r-2\}$ that the intersection $W_{\sigma(1)} \cap \dots \cap W_{\sigma(s)} \cap W_{\sigma(s+1)}$ is a Whitney stratified subspace of X. Consequently, condition $(*)_{s+1}^\sigma$ is meaningful.

Lemma 2.2.6. *Let $W, W_1, W_2 \subset X$ be Whitney stratified subspaces such that $W_1 \pitchfork W$ in X and $W_2 \pitchfork W$ in X. Then, $W_1 \pitchfork W_2 \cap W$ in X if and only if $W_2 \pitchfork W_1 \cap W$ in X.*

Proof. By symmetry of the claim it suffices to show that $W_1 \pitchfork W_2 \cap W$ in X implies that $W_2 \pitchfork W_1 \cap W$ in X. Furthermore, it suffices to consider the case that $W, W_1, W_2 \subset X$ are submanifolds. Supposing that $x \in W_2 \cap (W_1 \cap W)$ and $V \in T_xX$, one has to show that $V \in T_xW_2 + T_x(W_1 \cap W)$. It follows from $W_2 \pitchfork W$ in X that there exist $V_2 \in T_xW_2$ and $U \in T_xW$ such that $V = V_2 + U$. Moreover, it follows from $W_1 \pitchfork W_2 \cap W$ in X that there exist $V_1 \in T_xW_1$ and $U_2 \in T_x(W_2 \cap W) = T_xW_2 \cap T_xW$ such that $U = V_1 + U_2$. Hence,

$$V_1 = U - U_2 \in T_xW_1 \cap T_xW = T_x(W_1 \cap W).$$

All in all,

$$V = V_2 + U = V_2 + V_1 + U_2 = (U_2 + V_2) + V_1 \in T_xW_2 + T_x(W_1 \cap W).$$

□

The following criterion allows to extend a given transverse system in X:

Proposition 2.2.7. *Let $\{W_1, \dots, W_r\}$, $r \geq 1$, be a transverse system in X. Suppose that W_{r+1} is a Whitney stratified subspace such that $W_{r+1} \pitchfork \bigcap_{j \in J} W_j$ for every subset $J \subset \{1, \dots, r\}$. Then, $\{W_1, \dots, W_{r+1}\}$ is a transverse system in X.*

Proof. Let σ be a permutation of $\{1, \dots, r+1\}$. We have to check condition $(*)_s^\sigma$ for every $s \in \{1, \dots, r\}$. If $t \in \{1, \dots, r+1\}$ is such that $\sigma(t) = r+1$, then we distinguish between the following three cases for $s \in \{1, \dots, r\}$ to check the validity of condition $(*)_s^\sigma$:

- $s \in \{1, \dots, t-2\}$. In this case, $(*)_s^\sigma$ claims that $W_{\sigma(s+1)} \pitchfork W_{\sigma(1)} \cap \dots \cap W_{\sigma(s)}$ in X, which is true because $\{W_1, \dots, W_r\}$ is a transverse system in X by assumption.
- $s = t-1$. In this case, $(*)_{t-1}^\sigma$ claims that $W_{r+1} \pitchfork W_{\sigma(1)} \cap \dots \cap W_{\sigma(t-1)}$ in X, which is true by assumption on W_{r+1}.
- $s \in \{t, \dots, r\}$. In this case, $(*)_s^\sigma$ claims that $W_{\sigma(s+1)} \pitchfork W_{\sigma(1)} \cap \dots \cap W_{\sigma(t-1)} \cap W_{r+1} \cap W_{\sigma(t+1)} \cap \dots \cap W_{\sigma(s)}$ in X. By assumtion the Whitney stratified subspaces W_{r+1} and $W_{\sigma(s+1)}$ of X are both transverse to $W := W_{\sigma(1)} \cap \dots \cap W_{\sigma(t-1)} \cap W_{\sigma(t+1)} \cap \dots \cap W_{\sigma(s)}$

in X. Hence, it follows from Lemma 2.2.6 that $W_{r+1} \pitchfork W_{\sigma(s+1)} \cap W$ in X implies $W_{\sigma(s+1)} \pitchfork W_{r+1} \cap W$ in X.

□

2.2.2 Generic Smooth Vector Bundle Morphisms

Following the presentation of [1, Section 4, p. 1223f], let us briefly recall the notion of a generic smooth vector bundle morphism. Smooth vector bundle morphisms $\varphi\colon \zeta \to \xi$ between smooth vector bundles ζ and ξ on X correspond obviously to smooth sections $s_\varphi\colon X \to \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)$ of the smooth vector bundle $\pi\colon \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi) \to X$, and we will sometimes identify $\varphi = s_\varphi$ by abuse of notation. For any open subset $U \subset X$ let $C^\infty(U, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi))$ denote the real vector space of smooth sections of π over U. Whenever $U \subset V$ are open subsets of X, there is obviously a restriction map $C^\infty(V, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)) \to C^\infty(U, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi))$. Note also that $C^\infty(U, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)) = C^\infty(U, \operatorname{Hom}_{\mathbb{R}}(\zeta|_U,\xi|_U))$. A smooth vector bundle morphism $\varphi\colon \zeta \to \xi$ gives for every integer $j \in \{0,\dots,\operatorname{rank}\zeta\}$ rise to a subset

$$Z_j(\varphi) := \{x \in X;\ \dim_{\mathbb{R}} \ker \varphi_x = j\} \qquad \subset X.$$

What can be said about the structure of the sets $Z_j(\varphi)$, at least in the case of a "generic" vector bundle morphism $\varphi\colon \zeta \to \xi$? Given smooth vector bundles ζ and ξ on X, a key role is played by the subsets $Z_j(\tau) \subset \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)$ that belong to the *tautological bundle morphism* $\tau\colon \pi^*\zeta \to \pi^*\xi$ over the base space $\operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)$. In fact, as explained in [1, Section 4, p. 1223], the subset $Z_j(\tau) \subset \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)$ is a submanifold of codimension $j(l-k+j)$, where $k := \operatorname{rank}\zeta$ and $l := \operatorname{rank}\xi$. Moreover, it is a subbundle of $\operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)$ with fiber $Z_j(\tau)_x = \{A \in \operatorname{Hom}_{\mathbb{R}}(\zeta_x,\xi_x);\ \dim\ker A = j\}$. Furthermore, the sets $Z_p(\tau)$ with $p \geq j$ form a Whitney stratification of $\overline{Z}_j(\tau) = \bigcup_{p\geq j} Z_p(\tau)$.

Definition 2.2.8. A smooth vector bundle morphism $\varphi\colon \zeta \to \xi$ (or its corresponding section $s_\varphi \in C^\infty(X, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi))$) is called *generic* if s_φ is transverse to $Z_j(\tau)$ for all $j \geq 0$. Let $C^\infty_{\text{gen}}(X, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi)) \subset C^\infty(X, \operatorname{Hom}_{\mathbb{R}}(\zeta,\xi))$ denote the subset of generic smooth sections.

As a consequence, for every generic smooth vector bundle morphism $\varphi\colon \zeta \to \xi$ the set $Z_j(\varphi) = s_\varphi^{-1}(Z_j(\tau))$ is a submanifold of X of codimension $j(l-k+j)$ where $k := \operatorname{rank}\zeta$ and $l := \operatorname{rank}\xi$, and the subsets $Z_p(\varphi)$, $p \geq j$, form a Whitney stratification of $\overline{Z}_j(\varphi) = \bigcup_{p\geq j} Z_p(\varphi)$. In particular, the *singular locus* $\overline{Z}_1(\varphi) \subset X$ of φ is a Whitney stratified space of dimension $m-l+k-1$. (Recall that m denotes the dimension of X.) In particular:

Remark 2.2.9. Let ζ be a smooth vector bundle of rank k over X. If $\varphi\colon \zeta \to TX$ is a generic smooth vector bundle morphism, then $\overline{Z}_1(\varphi) \subset X$ is a Whitney stratified space of dimension $k-1$.

Let κ_X^k denote the product bundle of rank k over X. Given a smooth vector bundle ξ over X of rank n, it is shown in [1, Proposition 5, p. 1224f] that the singular locus $\overline{Z}_1(\varphi)$ of a generic smooth vector bundle morphism $\varphi\colon \kappa_X^k \to \xi$ possesses a canonical "desingularization"

$$\pi_\varphi\colon \tilde{Z}(\varphi) \to \overline{Z}_1(\varphi)$$

given by restriction of the projection $\pi_X \colon X \times \mathbb{RP}^{k-1} \to X$ to the $(m+k-n-1)$-dimensional smooth submanifold

$$\tilde{Z}(\varphi) := \{(x, L) \in X \times \mathbb{RP}^{k-1};\ (x, L) \subset \ker \varphi_x\} \qquad \subset X \times \mathbb{RP}^{k-1}.$$

(If X is assumed to be compact, then $\tilde{Z}(\varphi)$ is compact as well by [1, Proposition 7, p. 1225].)

Remark 2.2.10. By construction, π_φ is surjective and restricts to a bijection

$$\tilde{Z}^\circ(\varphi) := \pi_\varphi^{-1}(Z_1(\varphi)) = \tilde{Z}(\varphi) \cap \pi_X^{-1}(Z_1(\varphi)) \to Z_1(\varphi).$$

(If X is assumed to be compact, then this restriction is a diffeomorphism by [1, Proposition 7, p. 1224f].)

Lemma 2.2.11. *Suppose that* $A\colon (-\varepsilon, \varepsilon) \to \mathrm{Mat}_{m\times k}(\mathbb{R})$ *is a smooth map for some* $\varepsilon > 0$ *such that the the matrix* $A(t)\colon \mathbb{R}^k \to \mathbb{R}^m$ *has the same rank for every* $t \in (-\varepsilon, \varepsilon)$. *Given* $0 \neq v \in \ker A(0)$, *there exists* $\varepsilon' \in (0, \varepsilon)$ *and a smooth map* $\gamma\colon (-\varepsilon', \varepsilon') \to \mathbb{R}^k$ *such that* $\gamma(0) = v$ *and* $0 \neq \gamma(t) \in \ker A(t)$ *for all* $t \in (-\varepsilon', \varepsilon')$.

Proof. Let $a_i(t) \in \mathbb{R}^m$ denote the i-th column of $A(t)$ for $i = 1, \ldots, k$. If r denotes the rank of $A(0)$, then there exist $i_1, \ldots, i_r \in \{1, \ldots, k\}$ such that $\operatorname{im} A(0)$ is spanned by $a_{i_1}(0), \ldots a_{i_r}(0)$. Let $\{a_{i_1}(0), \ldots a_{i_r}(0), b_1, \ldots, b_{m-r}\}$ be an extension of the linear independent system $\{a_{i_1}(0), \ldots a_{i_r}(0)\}$ to a basis of $\mathbb{R}^m$. Then there exists $\varepsilon' \in (0, \varepsilon)$ such that $\{a_{i_1}(t), \ldots a_{i_r}(t), b_1, \ldots, b_{m-r}\}$ is a basis of $\mathbb{R}^m$ for all $t \in (-\varepsilon', \varepsilon')$.

Let $\{j_1, \ldots, j_{k-r}\} = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_r\}$. Note that $k - r \geq 1$ since $0 \neq v \in \ker A(0)$ by assumption. For every $l = 1, \ldots, k-r$ and all $t \in (-\varepsilon', \varepsilon')$ there exists a unique linear combination

$$a_{j_l}(t) = \mu_1^{(l)}(t) a_{i_1}(t) + \cdots + \mu_r^{(l)}(t) a_{i_r}(t) + \nu_1^{(l)}(t) b_1 + \cdots + \nu_{m-r}^{(l)}(t) b_{m-r}.$$

For every $l = 1, \ldots, k-r$ it follows from Cramer's rule that the coefficient functions

$$\mu_1^{(l)}, \ldots, \mu_r^{(l)}, \nu_1^{(l)}, \ldots, \nu_{m-r}^{(l)} \colon (-\varepsilon', \varepsilon') \to \mathbb{R}$$

depend smoothly on t. (Indeed, they are rational functions in the coefficients of $A(t)$.)

As $A(t)$ has rank r by assumption, and $\{a_{i_1}(t), \ldots a_{i_r}(t)\}$ is a linear independent system for all $t \in (-\varepsilon', \varepsilon')$, we conclude that $\nu_1^{(l)}(t) = \cdots = \nu_{m-r}^{(l)}(t) = 0$ for all $t \in (-\varepsilon', \varepsilon')$ and $l = 1, \ldots, k-r$.

Hence, we obtain for every $l = 1, \ldots, k-r$ a smooth map

$$w^{(l)} = (w_1^{(l)}, \ldots, w_k^{(l)}) \colon (-\varepsilon', \varepsilon') \to \mathbb{R}^k, \qquad w_i^{(l)} = \begin{cases} \mu_s^{(l)}, & i = i_s, \\ -1, & i = j_l, \\ 0, & \text{else.} \end{cases}$$

By construction, $\{w^{(1)}(t), \ldots, w^{(k-r)}(t)\}$ is a basis of $\ker A(t)$ for all $t \in (-\varepsilon', \varepsilon')$. Write

$$v = c_1 w^{(1)}(0) + \cdots + c_{k-r} w^{(k-r)}(0).$$

Then, a curve γ with the desired properties is given by

$$\gamma\colon (-\varepsilon', \varepsilon') \to \mathbb{R}^k, \quad \gamma(t) = c_1 w^{(1)}(t) + \cdots + c_{k-r} w^{(k-r)}(t).$$

□

Proposition 2.2.12. *Let $r \geq 1$ be an integer. Suppose that $\varphi_s\colon \kappa_X^{k_s} \to TX$, $s \in \{1, \ldots, r\}$, are generic smooth vector bundle morphisms. For every $s \in \{1, \ldots, r\}$ use the projection $\pi_s^{(r)}\colon X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_r-1} \to X \times \mathbb{RP}^{k_s-1}$ to define*

$$Z_s^{(r)} := (\pi_s^{(r)})^{-1}(\tilde{Z}(\varphi_s)).$$

If $\{\overline{Z}_1(\varphi_1), \ldots, \overline{Z}_1(\varphi_r)\}$ is a transverse system in X (see Definition 2.2.4), then $\{Z_1^{(r)}, \ldots, Z_r^{(r)}\}$ is a transverse system in $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_r-1}$.

Furthermore, the projection $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_r-1} \to X$ restricts to a surjective map

$$\alpha\colon Z_1^{(r)} \cap \cdots \cap Z_r^{(r)} \to \overline{Z}_1(\varphi_1) \cap \cdots \cap \overline{Z}_1(\varphi_r)$$

with the following properties:

(1) *$\alpha^{-1}(Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r)) = (\pi_s^{(r)})^{-1}(\tilde{Z}^\circ(\varphi_1)) \cap \cdots \cap (\pi_s^{(r)})^{-1}(\tilde{Z}^\circ(\varphi_r))$. (Recall the definition of $\tilde{Z}^\circ(\varphi_s)$ from Remark 2.2.10.)*

(2) *The restriction of α to the following map is a bijection:*

$$\alpha^{-1}(Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r)) \to Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r).$$

Proof. We prove the claim by induction on $r \geq 1$ with the case $r = 1$ being trivial. Suppose that the statement of the Proposition holds for some integer $r \geq 1$. Let $\varphi_s\colon \kappa_X^{k_s} \to TX$, $s \in \{1, \ldots, r+1\}$, be generic smooth vector bundle morphisms such that $\{\overline{Z}_1(\varphi_1), \ldots, \overline{Z}_1(\varphi_{r+1})\}$ is a transverse system in X. In the following we will use Proposition 2.2.7 to show inductively for $s = 1, \ldots, r+1$ that $\{Z_1^{(r+1)}, \ldots, Z_s^{(r+1)}\}$ is a transverse system in $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_s-1}$ with the case $s = 1$ being trivial. Suppose that $\{Z_1^{(r+1)}, \ldots, Z_s^{(r+1)}\}$ is a transverse system in $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_s-1}$ for some $s \in \{1, \ldots, r\}$. In order to show that $\{Z_1^{(r+1)}, \ldots, Z_{s+1}^{(r+1)}\}$ is a transverse system in $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_{r+1}-1}$, it suffices by Proposition 2.2.7 to show that $Z_{s+1}^{(r+1)} \pitchfork \bigcap_{t \in T} Z_t^{(r+1)}$ in $X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_{r+1}-1}$ for every given subset $T \subset \{1, \ldots, s\}$. For this purpose, fix such a subset $T = \{t_1, \ldots, t_l\} \subset \{1, \ldots, s\}$ where $t_1 < \cdots < t_l$. Set $S := T \cup \{s+1\} \subset \{1, \ldots, s+1\}$. Define the projections

$$\begin{aligned}
\pi_S^{(r+1)}&\colon X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_{r+1}-1} \to X \times \mathbb{RP}^{k_{t_1}-1} \times \cdots \times \mathbb{RP}^{k_{t_l}-1} \times \mathbb{RP}^{k_{s+1}-1}, \\
\pi_{s'}^S&\colon X \times \mathbb{RP}^{k_{t_1}-1} \times \cdots \times \mathbb{RP}^{k_{t_l}-1} \times \mathbb{RP}^{k_{s+1}-1} \to X \times \mathbb{RP}^{k_{s'}-1}, \qquad s' \in S.
\end{aligned}$$

Note that there is the factorization $\pi_{s'}^{(r+1)} = \pi_{s'}^S \circ \pi_S^{(r+1)}$ for all $s' \in S$. Setting

$$Z_{s'}^S := (\pi_{s'}^S)^{-1}(\tilde{Z}(\varphi_{s'})) \subset X \times \mathbb{RP}^{k_{t_1}-1} \times \cdots \times \mathbb{RP}^{k_{t_l}-1} \times \mathbb{RP}^{k_{s+1}-1}, \qquad s' \in S,$$

one obtains for every $s' \in S$ that

$$Z_{s'}^{(r+1)} = (\pi_{s'}^{(r+1)})^{-1}(\tilde{Z}(\varphi_{s'})) = (\pi_S^{(r+1)})^{-1}(Z_{s'}^S).$$

Note that $\{Z^S_{t_1},\dots,Z^S_{t_l}\}$ is a transverse system in $X\times\mathbb{RP}^{k_{t_1}-1}\times\cdots\times\mathbb{RP}^{k_{t_l}-1}\times\mathbb{RP}^{k_{s+1}-1}$ because $\{Z^{(r+1)}_1,\dots,Z^{(r+1)}_s\}$ (and hence, $\{Z^{(r+1)}_{t_1},\dots,Z^{(r+1)}_{t_l}\}$) is a transverse system in $X\times\mathbb{RP}^{k_1-1}\times\cdots\times\mathbb{RP}^{k_s-1}$ by induction hypothesis, and it suffices to show that $Z^S_{s+1}\pitchfork\bigcap_{t\in T}Z^S_t$ in $X\times\mathbb{RP}^{k_{t_1}-1}\times\cdots\times\mathbb{RP}^{k_{t_l}-1}\times\mathbb{RP}^{k_{s+1}-1}$. (Indeed, observe that if A and B are smooth manifolds and $\pi\colon A\times B\to A$ denotes the projection, then two submanifolds $M,N\subset A$ are transverse in A if and only if $\pi^{-1}(M)\pitchfork\pi^{-1}(N)$ in $A\times B$, and one has $\pi^{-1}(M\cap N)=\pi^{-1}(M)\cap\pi^{-1}(N)$.)

Given a point

$$p=(x,L_{t_1},\dots,L_{t_l},L_{s+1})\in Z^S_{t_1}\cap\cdots\cap Z^S_{t_l}\cap Z^S_{s+1}\subset X\times\mathbb{RP}^{k_{t_1}-1}\times\cdots\times\mathbb{RP}^{k_{t_l}-1}\times\mathbb{RP}^{k_{s+1}-1}$$

one has to show that any given vector

$$\begin{aligned}(V,W_{t_1},\dots,W_{t_l},W_{s+1})&\in T_p(X\times\mathbb{RP}^{k_{t_1}-1}\times\cdots\times\mathbb{RP}^{k_{t_l}-1}\times\mathbb{RP}^{k_{s+1}-1})\\&=T_xX\times T_{L_{t_1}}\mathbb{RP}^{k_{t_1}-1}\times\cdots\times T_{L_{t_l}}\mathbb{RP}^{k_{t_l}-1}\times T_{L_{s+1}}\mathbb{RP}^{k_{s+1}-1}\end{aligned}$$

is contained in $T_pZ^S_{s+1}+T_p(\bigcap_{t\in T}Z^S_t)$.

Note that $x\in\overline{Z}_1(\varphi_{t_1})\cap\cdots\cap\overline{Z}_1(\varphi_{t_l})\cap\overline{Z}_1(\varphi_{s+1})$. (In fact, if $\pi^{s'}\colon X\times\mathbb{RP}^{k_{s'}-1}\to X$ denotes the projection for $s'\in S$, then $x=\pi^{s'}(\pi^S_{s'}(p))\in\pi^{s'}(\pi^S_{s'}(Z^S_{s'}))=\pi^{s'}(\tilde Z(\varphi_{s'}))=\overline{Z}_1(\varphi_{s'})$.) Hence, for every $s'\in S$ there exists an integer $j_{s'}\geq 1$ such that $x\in Z_{j_{s'}}(\varphi_{s'})$.

As $\{\overline{Z}_1(\varphi_1),\dots,\overline{Z}_1(\varphi_r)\}$ is a transverse system in X by assumption, one can conclude that $T_xX=T_xZ_{j_{s+1}}(\varphi_{s+1})+T_x\bigcap_{t\in T}Z_{j_t}(\varphi_t)$. Therefore, there exist vectors $V_{s+1}\in T_xZ_{j_{s+1}}(\varphi_{s+1})$ and $V_T\in T_x\bigcap_{t\in T}Z_{j_t}(\varphi_t)$ such that $V=V_{s+1}+V_T$ in T_xX. For suitable $\varepsilon>0$ we fix smooth curves

$$\begin{aligned}\lambda_{s+1}\colon(-\varepsilon,\varepsilon)&\to Z_{j_{s+1}}(\varphi_{s+1}), &\lambda_{s+1}(0)=x,\lambda'_{s+1}(0)=V_{s+1},\\ \lambda_T\colon(-\varepsilon,\varepsilon)&\to\bigcap_{t\in T}Z_{j_t}(\varphi_t), &\lambda_T(0)=x,\lambda'_T(0)=V_T.\end{aligned}$$

Note that $(x,L_{s'})=\pi^S_{s'}(p)\in\pi^S_{s'}(Z^S_{s'})=\tilde Z(\varphi_{s'})$ for every $s'\in S$. By Lemma 2.2.11 there exists for every $s'\in S$ (and suitable $\varepsilon'\in(0,\varepsilon)$) a smooth curve

$$\tilde\lambda_{s'}\colon(-\varepsilon',\varepsilon')\to\tilde Z(\varphi_{s'})\qquad(\subset X\times\mathbb{RP}^{k_{s'}-1})$$

such that $\tilde\lambda_{s'}(0)=(x,L_{s'})$ and $\pi^{s'}\circ\tilde\lambda_{s'}=\lambda_{s'}|_{(-\varepsilon',\varepsilon')}$, where $\lambda_t:=\lambda_T$ for every $t\in T$. (In fact, choose $\varepsilon''\in(0,\varepsilon)$ so small that the image of $\lambda_{s'}|_{(-\varepsilon'',\varepsilon'')}$ is entirely contained in a chart $U=\mathbb{R}^m$ of X. On this chart the section $\varphi_{s'}\in\mathrm{Hom}_{\mathbb{R}}(\kappa_X^{k_{s'}},TX)$ has for every $s'\in S$ the form

$$U\to U\times\mathbb{R}(m,k_{s'}),\qquad x\mapsto(x,\phi_{s'}(x)),$$

for a suitable smooth map $\phi_{s'}\colon U\to\mathbb{R}(m,k_{s'})$, where $\mathbb{R}(q,p)$ denotes the real vector space of $(q\times p)$-matrices with real coefficients. As $\lambda_{s'}$ is a curve in $Z_{j_{s'}}(\varphi_{s'})$, it follows that the image of the smooth curve $A_{s'}:=\phi_{s'}\circ\lambda_{s'}|_{(-\varepsilon'',\varepsilon'')}$ is contained in

$$\mathbb{R}_{k_{s'}-j_{s'}}(m,k_{s'}):=\{A\in\mathbb{R}(m,k_{s'});\ \mathrm{rank}\,A=k_{s'}-j_{s'}\}.$$

Application of Lemma 2.2.11 to the smooth curve $A_{s'}\colon(-\varepsilon'',\varepsilon'')\to\mathbb{R}_{k_{s'}-j_{s'}}(m,k_{s'})$ and some

$0 \neq v_{s'} \in L_{s'} \subset \ker \phi_{s'}(x) \subset \mathbb{R}^{k_{s'}}$ yields $\varepsilon' \in (0, \varepsilon'')$ and a smooth curve $\gamma_{s'}\colon (-\varepsilon', \varepsilon') \to \mathbb{R}^{k_{s'}}$ such that $\gamma_{s'}(0) = v_{s'}$ and $0 \neq \gamma_{s'}(u) \in \ker A_{s'}(u)$ for all $u \in (-\varepsilon', \varepsilon')$. Then, the smooth curve

$$\tilde{\lambda}_{s'} := (\lambda_{s'}|_{(-\varepsilon',\varepsilon')}, \mathbb{R}\gamma_{s'})\colon (-\varepsilon', \varepsilon') \to \tilde{Z}(\varphi_{s'}) \qquad (\subset X \times \mathbb{RP}^{k_{s'}-1})$$

has the desired properties.) Set $V_t := V_T$ for every $t \in T$. For every $s' \in S$ it follows from $\pi^{s'} \circ \tilde{\lambda}_{s'} = \lambda_{s'}|_{(-\varepsilon',\varepsilon')}$ that there exists $\tilde{W}_{s'} \in T_{L_{s'}}\mathbb{RP}^{k_{s'}-1}$ such that

$$\tilde{\lambda}'_{s'}(0) = (V_{s'}, \tilde{W}_{s'}) \in T_{(x,L_{s'})}\tilde{Z}(\varphi_{s'}) \subset T_{(x,L_{s'})}(X \times \mathbb{RP}^{k_{s'}-1}) = T_xX \times T_{L_{s'}}\mathbb{RP}^{k_{s'}-1}.$$

For every $s' \in S$ it follows from $Z^S_{s'} = (\pi^S_{s'})^{-1}(\tilde{Z}(\varphi_{s'}))$ that

$$T_pZ^S_{s'} = \{Y \in T_p(X \times \mathbb{RP}^{k_{t_1}-1} \times \cdots \times \mathbb{RP}^{k_{t_l}-1} \times \mathbb{RP}^{k_{s+1}-1});\ d_p\pi^S_{s'}(Y) \in T_{(x,L_{s'})}\tilde{Z}(\varphi_{s'})\}.$$

Consequently,

$$\begin{aligned}
&(V, W_{t_1}, \ldots, W_{t_l}, W_{s+1})\\
&= (V_{s+1}, W_{t_1} - \tilde{W}_{t_1}, \ldots, W_{t_l} - \tilde{W}_{t_l}, \tilde{W}_{s+1}) + (V_T, \tilde{W}_{t_1}, \ldots, \tilde{W}_{t_l}, W_{s+1} - \tilde{W}_{s+1})\\
&\in T_pZ^S_{s+1} + \bigcap_{t\in T} T_pZ^S_t = T_pZ^S_{s+1} + T_p(\bigcap_{t\in T} Z^S_t).
\end{aligned}$$

Next, note that for every $s \in \{1, \ldots, r\}$ the projection $\pi^{(r)}\colon X \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_r-1} \to X$ factorizes as $\pi^{(r)} = \pi^s \circ \pi^{(r)}_s$, where $\pi^s\colon X \times \mathbb{RP}^{k_s-1} \to X$. Therefore, for every $s \in \{1, \ldots, r\}$,

$$\pi^{(r)}(Z^{(r)}_1 \cap \cdots \cap Z^{(r)}_r) \subset \pi^{(r)}(Z^{(r)}_s) = \pi^s(\pi^{(r)}_s(Z^{(r)}_s)) = \pi^s(\tilde{Z}(\varphi_s)) = \overline{Z}_1(\varphi_s).$$

Hence, $\pi^{(r)}$ restricts to a surjective map

$$\alpha\colon Z^{(r)}_1 \cap \cdots \cap Z^{(r)}_r \to \overline{Z}_1(\varphi_1) \cap \cdots \cap \overline{Z}_1(\varphi_r).$$

It remains to check the desired properties (1) and (2):

(1). Claim: $\alpha^{-1}(Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r)) = (\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_1)) \cap \cdots \cap (\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_r))$.

Since $\pi^{(r)} = \pi^s \circ \pi^{(r)}_s$, it is clear that

$$\begin{aligned}
&\alpha((\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_1)) \cap \cdots \cap (\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_r)))\\
&= \pi^{(r)}((\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_1)) \cap \cdots \cap (\pi^{(r)}_s)^{-1}(\tilde{Z}^\circ(\varphi_r)))\\
&= \pi^s(\tilde{Z}^\circ(\varphi_1)) \cap \cdots \cap \pi^s(\tilde{Z}^\circ(\varphi_r))\\
&= Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r).
\end{aligned}$$

Conversely, suppose that the point $p := (w, L_1, \ldots, L_r) \in Z^{(r)}_1 \cap \cdots \cap Z^{(r)}_r \subset W \times \mathbb{RP}^{k_1-1} \times \cdots \times \mathbb{RP}^{k_r-1}$ satisfies $\alpha(p) \in Z_1(\varphi_1) \cap \cdots \cap Z_1(\varphi_r)$. One has to show that $\pi^{(r)}_s(p) \in \tilde{Z}^\circ(\varphi_s)$ for every $s \in \{1, \ldots, r\}$. For this purpose, fix $s \in \{1, \ldots, r\}$. It is clear that $\pi^{(r)}_s(p) \in \pi^{(r)}_s(Z^{(r)}_s) = (\pi^{(r)}_s)^{-1}(\pi^{(r)}_s(Z^{(r)}_s)) = \tilde{Z}(\varphi_s)$. Therefore, it follows from the assumption $\pi^s(\pi^{(r)}_s(p)) = \pi^{(r)}(p) = \alpha(p) \in Z_1(\varphi_s)$ that $\pi^{(r)}_s(p) \in \tilde{Z}(\varphi_s) \cap (\pi^s)^{-1}(Z_1(\varphi_s)) = \tilde{Z}^\circ(\varphi_s)$.

(2). Claim: α restricts to a bijection

$$\alpha^{-1}(Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r)) \to Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r).$$

In order to prove surjectivity, suppose that $w\in Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r)$. For every $s\in\{1,\dots,r\}$ it follows from $\pi^s(\tilde{Z}(\varphi_s))=\overline{Z}_1(\varphi_s)$ that there exists $L_s\in\mathbb{RP}^{m-i_s}$ such that $(w,L_s)\in\tilde{Z}(\varphi_s)$. Hence, the point $p:=(w,L_1,\dots,L_r)\in W\times\mathbb{RP}^{m-i_1}\times\cdots\times\mathbb{RP}^{m-i_r}$ satisfies $\pi^{(r)}_s(p)=(w,L_s)\in\tilde{Z}(\varphi_s)$ for every $s\in\{1,\dots,r\}$, thus being an element of $(\pi^{(r)}_1)^{-1}(\tilde{Z}(\varphi_1))\cap\cdots\cap(\pi^{(r)}_r)^{-1}(\tilde{Z}(\varphi_r))=Z^{(r)}_1\cap\cdots\cap Z^{(r)}_r$. Finally, $\alpha(p)=\pi^{(r)}(p)=w\in Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r)$ implies that $p\in\alpha^{-1}(Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r))$ as required.

As far as injectivity is concerned, suppose that two points

$$p=(w,L_1,\dots,L_r),p'=(w',L'_1,\dots,L'_r)\in\alpha^{-1}(Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r))$$

have the same image under α, i.e.

$$w=\pi^{(r)}(p)=\alpha(p)=\alpha(p')=\pi^{(r)}(p')=w'\in Z_1(\varphi_1)\cap\cdots\cap Z_1(\varphi_r).$$

Given $s\in\{1,\dots,r\}$, it remains to show that $L_s=L'_s$. Property (1) implies that the points $p_s:=\pi^{(r)}_s(p)=(w,L_s)$ and $p'_s:=\pi^{(r)}_s(p')=(w,L'_s)$ are both contained in $\tilde{Z}^\circ(\varphi_s)$. Moreover, $\pi^s(p_s)=\pi^s(\pi^{(r)}_s(p))=\pi^{(r)}(p)=\alpha(p)=\alpha(p')=\pi^{(r)}(p')=\pi^s(\pi^{(r)}_s(p'))=\pi^s(p'_s)$. Finally, since π^s restricts to a bijection $\tilde{Z}^\circ(\varphi_s)\to Z_1(\varphi_s)$ by Remark 2.2.10, one obtains $(w,L_s)=p_s=p'_s=(w,L'_s)$. □

2.2.3 The Intersection Product

This section employs the intersection product $\bullet$ as defined in [7, Chapter VI.11, p. 366ff].

All homology and cohomology groups are assumed to have $\mathbb{Z}/2$-coefficients. Hence, note that every closed smooth manifold X^m possesses a fundamental class $[X]\in H_m(X)$.

Given a closed connected smooth manifold W^m, let $D=D_W\colon H^i(W)\to H_{m-i}(W)$ denote the inverse of the Poincaré isomorphism. In other words, $D[\omega]\cap[W]=[\omega]$ for all $[\omega]\in H_i(W)$.

Recall that the intersection product $\bullet\colon H_i(W)\otimes H_j(W)\to H_{i+j-m}(W)$ is by definition Poincaré dual to the cup product:

$$[\omega]\bullet[\eta]:=D^{-1}(D[\omega]\cup D[\eta]).$$

The following classical result (see [7, Theorem 11.9, p. 372]) gives a geometric interpretation of the intersection product of homology classes represented by smooth submanifolds:

Theorem 2.2.13. *Let W be a closed connected smooth manifold. If $\kappa_i\colon K_i\hookrightarrow W$, $i=1,2$, are smoothly embedded submanifolds that are transverse to each other, $K_1\pitchfork K_2$, then the intersection product of the fundamental classes $\kappa_{1*}[K_1]$ and $\kappa_{2*}[K_2]$ in $H_*(W)$ equals the fundamental class of the smoothly embedded submanifold $\kappa\colon K_1\cap K_2\hookrightarrow W$ given by the physical intersection of K_1 and K_2:*

$$\kappa_{1*}[K_1]\bullet\kappa_{2*}[K_2]=\kappa_*[K_1\cap K_2].$$

The following proposition shows that the intersection product is in some sense compatible with the fibered product of smooth product bundles:

Proposition 2.2.14. *Let X, X_1, X_2 be closed smooth manifolds. For $i = 1,2$ let $\sigma_i\colon X \times X_i \to X$, $\tau_i\colon X \times X_1 \times X_2 \to X \times X_i$ denote the projections. Then the projection $\rho\colon X \times X_1 \times X_2 \to X$ clearly satisfies $\rho = \sigma_i \circ \tau_i$ for $i = 1,2$. Suppose that $\alpha_i\colon A_i \hookrightarrow X \times X_i$ is for $i = 1,2$ a closed smooth submanifold. Let $\beta_i\colon B_i \hookrightarrow X \times X_1 \times X_2$ denote the inclusion of the closed smooth submanifold $B_i := \tau_i^{-1}(A_i)$ of $X \times X_1 \times X_2$. Then,*

$$\sigma_{1*}\alpha_{1*}[A_1] \bullet \sigma_{2*}\alpha_{2*}[A_2] = \rho_*(\beta_{1*}[B_1] \bullet \beta_{2*}[B_2]).$$

Proof. As we are working with $\mathbb{Z}/2$-coefficients, all signs that occur in the following computations are only virtual and can hence be neglected.

Consider the left-hand side of the claim:

$$\begin{aligned}
\sigma_{1*}\alpha_{1*}[A_1] \bullet \sigma_{2*}\alpha_{2*}[A_2] &= D\sigma_{2*}\alpha_{2*}[A_2] \cap \sigma_{1*}\alpha_{1*}[A_1] \\
&= \sigma_2^! D\alpha_{2*}[A_2] \cap \sigma_{1*}\alpha_{1*}[A_1] \\
&\overset{(!)}{=} \sigma_{2*}(D\alpha_{2*}[A_2] \cap \sigma_{2!}\sigma_{1*}\alpha_{1*}[A_1]) \\
&= \sigma_{2*}(\sigma_{2!}\sigma_{1*}\alpha_{1*}[A_1] \bullet \alpha_{2*}[A_2]).
\end{aligned}$$

(The equality sign marked with the shriek (!) is property (4) of [7, Proposition 14.1, p. 394] applied to the map $\sigma_2\colon X \times X_2 \to X$.)

Consider the right-hand side of the claim:

$$\begin{aligned}
\rho_*(\beta_{1*}[B_1] \bullet \beta_{2*}[B_2]) &\overset{(*)}{=} \rho_*(\tau_{1!}\alpha_{1*}[A_1] \bullet \tau_{2!}\alpha_{2*}[A_2]) \\
&= \rho_*(D\tau_{2!}\alpha_{2*}[A_2] \cap \tau_{1!}\alpha_{1*}[A_1]) \\
&= \rho_*(\tau_2^* D\alpha_{2*}[A_2] \cap D^{-1}\tau_1^* D\alpha_{1*}[A_1]) \\
&= \sigma_{2*}\tau_{2*}(\tau_2^* D\alpha_{2*}[A_2] \cap (\tau_1^* D\alpha_{1*}[A_1] \cap [X \times X_1 \times X_2])) \\
&= \sigma_{2*}(D\alpha_{2*}[A_2] \cap \tau_{2*}(\tau_1^* D\alpha_{1*}[A_1] \cap [X \times X_1 \times X_2])) \\
&\overset{(!)}{=} \sigma_{2*}(D\alpha_{2*}[A_2] \cap (\tau_2^! \tau_1^* D\alpha_{1*}[A_1] \cap [X \times X_2])) \\
&= \sigma_{2*}((D\alpha_{2*}[A_2] \cup D\tau_{2*}D^{-1}\tau_1^* D\alpha_{1*}[A_1]) \cap [X \times X_2]) \\
&= \sigma_{2*}(D(\alpha_{2*}[A_2] \bullet \tau_{2*}\tau_{1!}\alpha_{1*}[A_1]) \cap [X \times X_2]) \\
&= \sigma_{2*}(\alpha_{2*}[A_2] \bullet \tau_{2*}\tau_{1!}\alpha_{1*}[A_1]) \\
&= \sigma_{2*}(\tau_{2*}\tau_{1!}\alpha_{1*}[A_1] \bullet \alpha_{2*}[A_2]).
\end{aligned}$$

(The equality marked with $(*)$ is an application of [1, Lemma 8, p. 1226] to the projection $f := \tau_i$. The equality sign marked with the shriek (!) is property (1) of [7, Proposition 14.1, p. 394] applied to the projection $\tau_2\colon X \times X_1 \times X_2 \to X \times X_2$.)

Comparing the two sides of the claim, it remains to show that $\sigma_{2!}\sigma_{1*}\alpha_{1*}[A_1] = \tau_{2*}\tau_{1!}\alpha_{1*}[A_1]$.

Setting $[\omega] := \sigma_{1*}\alpha_{1*}[A_1] \in H_r(X)$, $r := \dim A_1$, one obtains

$$\sigma_{2!}\sigma_{1*}\alpha_{1*}[A_1] = D^{-1}\sigma_2^* D[\omega]$$

and

$$\begin{aligned}\tau_{2*}\tau_{1!}\alpha_{1*}[A_1] &= \tau_{2*}\beta_{1*}[B_1]\\ &= (\sigma_1 \times \mathrm{id}_{X_2})_*(\alpha_1 \times \mathrm{id}_{X_2})_*[A_1 \times X_2]\\ &= ((\sigma_1 \circ \alpha_1) \times \mathrm{id}_{X_2})_*(\times([A_1] \otimes [X_2]))\\ &= [\omega] \times [X_2].\end{aligned}$$

Therefore, it suffices to show that $D^{-1}\sigma_2^* D[\omega] = [\omega] \times [X_2]$ for all $[\omega] \in H_r(X)$, or, as $D\colon H_r(X) \to H^{p-r}(X)$, $p := \dim X$, is an isomorphism, $D^{-1}\sigma_2^*[\eta] = D^{-1}([\eta]) \times [X_2]$ for all $[\eta] \in H^{p-r}(X)$. In fact,

$$\begin{aligned}D^{-1}\sigma_2^*[\eta] &= \sigma_2^*[\eta] \cap [X \times X_2]\\ &= \times([\eta] \otimes 1) \cap \times([X] \otimes [X_2])\\ &\overset{(\times)}{=} \times(([\eta] \cap [X]) \otimes (1 \cap [X_2]))\\ &= D^{-1}([\eta]) \times [X_2].\end{aligned}$$

(The equality sign marked with the cross $(\times)$ is [7, Theorem 5.4, p. 337].) □

Remark 2.2.15. The equality $\sigma_{2!}\sigma_{1*}\alpha_{1*}[A_1] = \tau_{2*}\tau_{1!}\alpha_{1*}[A_1]$ should be compared to [1, Proposition 9, p. 1226], see also [1, Remark 10, p. 1227]. It might be interesting to investigate in what way Proposition 2.2.14 can be generalized to the fibered product of smooth fiber bundles $Y_i \to X$ with fiber X_i, $i = 1, 2$.

Theorem 2.2.13 and Proposition 2.2.14 directly combine to the following

Corollary 2.2.16. *Let X, X_1, X_2 be closed connected smooth manifolds. For $i = 1, 2$ let $\sigma_i\colon X \times X_i \to X$, $\tau_i\colon X \times X_1 \times X_2 \to X \times X_i$ denote the projections. Then the projection $\rho\colon X \times X_1 \times X_2 \to X$ clearly satisfies $\rho = \sigma_i \circ \tau_i$ for $i = 1, 2$. Suppose that $\alpha_i\colon A_i \hookrightarrow X \times X_i$ is for $i = 1, 2$ a closed smooth submanifold. Let $\beta_i\colon B_i \hookrightarrow X \times X_1 \times X_2$ denote the inclusion of the closed smooth submanifold $B_i := \tau_i^{-1}(A_i)$ of $X \times X_1 \times X_2$. Suppose that $B_1 \pitchfork B_2$, and let $\beta\colon B \hookrightarrow X \times X_1 \times X_2$ denote the inclusion of the smooth submanifold $B := B_1 \cap B_2$. Then,*

$$\sigma_{1*}\alpha_{1*}[A_1] \bullet \sigma_{2*}\alpha_{2*}[A_2] = \rho_*\beta_*[B].$$

2.3 System of Fields and System of Action Functionals

Let $n \geq 0$ be an integer. Fix a partition $I = (i_1, \dots, i_r)$ of $m := n+1$ (i.e. $i_1 + \dots + i_r = n+1$ and $1 \leq i_1 \leq \dots \leq i_r$). We proceed to define the system of fields $\mathcal{F}$ and the $\mathbf{N}$-valued action functional $\mathbb{T}$ of the $(n+1)$-dimensional relative Stiefel-Whitney number TFT Z^I_{SW} that corresponds to the Stiefel-Whitney number $w_I = w_{i_1} \dots w_{i_r}$.

2.3.1 System of Fields

Let M denote a closed smooth (nonempty) manifold of dimension $n = m-1$. The open subsets of $M \times \mathbb{R}$ of the form $M \times (0, \varepsilon)$, $\varepsilon > 0$, form a directed set via inclusion.

Definition 2.3.1. Let $k \in \{1, \dots, m\}$. The set of *(outgoing) k-fields* on M is

$$\mathcal{G}^k(M) := \varinjlim_{\varepsilon > 0} C^\infty_{\text{gen}}(M \times (0,\varepsilon), \operatorname{Hom}_{\mathbb{R}}(\kappa^k_{M\times\mathbb{R}}, T(M \times \mathbb{R}))).$$

Explicitly, an element $f \in \mathcal{G}^k(M)$ is represented by a generic smooth vector bundle morphism

$$\varphi\colon \kappa^k_{M\times(0,\varepsilon)} \to T(M \times (0,\varepsilon)), \qquad \varepsilon > 0.$$

Moreover, two such morphisms $\varphi\colon \kappa^k_{M\times(0,\varepsilon)} \to T(M\times(0,\varepsilon))$ and $\varphi'\colon \kappa^k_{M\times(0,\varepsilon')} \to T(M\times(0,\varepsilon'))$ represent the same element in $\mathcal{G}^k(M)$ if and only if there exists $\varepsilon'' \in \{0, \min(\varepsilon,\varepsilon')\}$ such that φ and φ' restrict to the same morphism $\kappa^k_{M\times(0,\varepsilon'')} \to T(M \times (0,\varepsilon''))$.

Definition 2.3.2. The set $\mathcal{F}(M)$ of *fields* on M consists of all r-tuples

$$f = (f_1, \dots, f_r) \in \mathcal{G}^{m-i_1+1}(M) \times \dots \times \mathcal{G}^{m-i_r+1}(M)$$

with the following property. There exist $\varepsilon > 0$ and for every $s \in \{1, \dots, r\}$ a representative $\varphi_s\colon \kappa^{m-i_s+1}_{M\times(0,\varepsilon)} \to T(M \times (0,\varepsilon))$ of $f_s \in \mathcal{G}^{m-i_s+1}(M)$ such that $\{\overline{Z}_1(\varphi_1), \dots, \overline{Z}_1(\varphi_r)\}$ is a transverse system in $M \times (0,\varepsilon)$ (see Definition 2.2.4), and $\overline{Z}_1(\varphi_1) \cap \dots \cap \overline{Z}_1(\varphi_r) = \emptyset$.

Note that if $f = (f_1, \dots, f_r) \in \mathcal{F}(M)$ satisfies the property of Definition 2.3.2 for some $\varepsilon > 0$ and some representative $\varphi_s\colon \kappa^{m-i_s+1}_{M\times(0,\varepsilon)} \to T(M\times(0,\varepsilon))$ of $f_s \in \mathcal{G}^{m-i_s+1}(M)$ for every $s \in \{1, \dots, r\}$, then f satisfies this property of Definition 2.3.2 also for any $\varepsilon' \in (0,\varepsilon)$ and the restriction $\varphi_s|_{M\times(0,\varepsilon')}$ as representative of f_s.

Definition 2.3.3. The set $\mathcal{F}_<(M)$ of *left-extendable fields* on M consists of all r-tuples

$$f = (f_1, \dots, f_r) \in \mathcal{G}^{m-i_1+1}(M) \times \dots \times \mathcal{G}^{m-i_r+1}(M)$$

with the following property. There exist $\varepsilon > 0$ and for every $s \in \{1, \dots, r\}$ a generic smooth vector bundle morphism $\varphi_s\colon \kappa^{m-i_s+1}_{M\times(-\varepsilon,\varepsilon)} \to T(M \times (-\varepsilon,\varepsilon))$ such that $\{\overline{Z}_1(\varphi_1), \dots, \overline{Z}_1(\varphi_r)\}$ is a transverse system in $M \times (-\varepsilon,\varepsilon)$, and such that $\varphi_s|_{M\times(0,\varepsilon)}$ is a representative of $f_s \in \mathcal{G}^{m-i_s+1}(M)$ for every $s \in \{1, \dots, r\}$.

Proposition 2.3.4. *We have the inclusion $\mathcal{F}_<(M) \subset \mathcal{F}(M)$.*

Proof. Fix $f = (f_1, \dots, f_r) \in \mathcal{F}_<(M)$. Let $\varepsilon > 0$ and $\varphi_s\colon \kappa^{m-i_s+1}_{M\times(-\varepsilon,\varepsilon)} \to T(M \times (-\varepsilon,\varepsilon))$, $s \in \{1,\dots,r\}$, be as in Definition 2.3.3. Hence, by Definition 2.3.2, it suffices to show that there exists $\varepsilon' \in (0,\varepsilon)$ for which the restrictions $\varphi'_s := \varphi_s|_{M\times(0,\varepsilon')}$, $s \in \{1,\dots,r\}$, satisfy $\overline{Z}_1(\varphi'_1) \cap \cdots \cap \overline{Z}_1(\varphi'_r) = \emptyset$. As explained in Remark 2.2.9, $\overline{Z}_1(\varphi_s)$ is for every $s \in \{1,\dots,r\}$ a Whitney stratified subspace of $M \times (-\varepsilon,\varepsilon)$ of dimension $m - i_s$.

As $\{\overline{Z}_1(\varphi_1), \dots, \overline{Z}_1(\varphi_r)\}$ is a transverse system in $M \times (-\varepsilon,\varepsilon)$, it follows that the intersection $Z := \overline{Z}_1(\varphi_1) \cap \cdots \cap \overline{Z}_1(\varphi_r)$ is a Whitney stratified subspace of $M \times (-\varepsilon,\varepsilon)$ of dimension $(m - i_1) + \cdots + (m - i_r) - (r-1)m = 0$. Hence, Z is a discrete (and closed) subset of $M \times (-\varepsilon,\varepsilon)$. Therefore, the intersection $Z \cap (M \times [0,\varepsilon/2])$ is finite, so the desired ε' exists. □

In the following, let (W, M, μ, N, ν) denote a (nonempty) smooth bordism of dimension $m = n+1$. Let W_∞ denote the smooth manifold $(W \setminus M) \cup_{N\times 0} N \times [0,\infty)$ (without boundary). The open subsets of W_∞ of the form $W_\varepsilon := (W \setminus M) \cup_{N\times 0} N \times [0,\varepsilon)$, $\varepsilon > 0$, form a directed set via inclusion.

Definition 2.3.5. Let $k \in \{1,\dots,m\}$. The set of *(outgoing) k-fields* on the bordism W is

$$\mathcal{H}^k(W) := \varinjlim_{\varepsilon>0} C^\infty_{\text{gen}}(W_\varepsilon, \operatorname{Hom}_{\mathbb{R}}(\kappa^k_{W_\infty}, TW_\infty)).$$

Explicitly, an element $F \in \mathcal{H}^k(W)$ is represented by a generic smooth vector bundle morphism $\Phi\colon \kappa^k_{W_\varepsilon} \to TW_\varepsilon$, $\varepsilon > 0$. Two vector bundle morphisms $\Phi\colon \kappa^k_{W_\varepsilon} \to TW_\varepsilon$ and $\Phi'\colon \kappa^k_{W_{\varepsilon'}} \to TW_{\varepsilon'}$ are equivalent if there exists $\varepsilon'' \in \{0, \min(\varepsilon,\varepsilon')\}$ such that Φ and Φ' coincide over $W_{\varepsilon''}$.

Remark 2.3.6. If $N = \emptyset$, then $\mathcal{H}^k(W)$ is just the set of all generic smooth vector bundle morphisms $F = \Phi\colon \kappa^k_{W\setminus M} \to T(W \setminus M)$. The symbol $\mathcal{H}$ is chosen in order to distinguish $\mathcal{H}^k(W)$ from $\mathcal{G}^k(W)$ when $\partial W = \emptyset$. (Note that Definition 2.3.1 also applies literally to closed smooth manifolds of dimension m.)

However, the *suspension* of an element of $\mathcal{H}^k(W)$ produces in fact an element of $\mathcal{G}^{k+1}(W)$:

Proposition 2.3.7. *Suppose that $\partial W = \emptyset$. There is a canonical injective map (*suspension map*)*

$$\mathcal{H}^k(W) \to \mathcal{G}^{k+1}(W)$$

given by assigning to every generic smooth vector bundle morphism $F\colon \kappa^k_W \to TW$ the class in $\mathcal{G}^{k+1}(W)$ represented by its suspension

$$\overline{F}\colon \kappa^{k+1}_{W\times(0,1)} \to T(W \times (0,1))$$

which is defined at $(w,t) \in W \times (0,1)$ by the homomorphism

$$\overline{F}_{(w,t)} := F_w \times \mathrm{id}_{\mathbb{R}}\colon (\kappa^{k+1}_{W\times(0,1)})_{(w,t)} = \mathbb{R}^{k+1} = \mathbb{R}^k \times \mathbb{R}^1 \to T_{(w,t)}(W \times (0,1)) = T_wW \times \mathbb{R}^1.$$

Proof. Let $F \in \mathcal{H}^k(W)$. Hence, $F\colon \kappa^k_W \to TW$ is a smooth vector bundle morphism that is generic, i.e. the corresponding section $s_F\colon W \to \operatorname{Hom}_{\mathbb{R}}(\kappa^k_W, TW)$ is transverse to $Z_j(\tau)$ for all j, where τ denotes the tautological bundle over $\operatorname{Hom}_{\mathbb{R}}(\kappa^k_W, TW)$. It suffices to show that the smooth $(k+1)$-field $\overline{F}$ is generic, i.e. the corresponding section

$$s_{\overline{F}}\colon W \times (0,1) \to \operatorname{Hom}_{\mathbb{R}}(\kappa^{k+1}_{W\times(0,1)}, T(W \times (0,1)))$$

is transverse to $Z_j(\overline{\tau})$ for all $j \in \{0, \dots, k+1\}$, where $\overline{\tau}$ denotes the tautological bundle over $\mathrm{Hom}_{\mathbb{R}}(\kappa^{k+1}_{W\times(0,1)}, T(W \times (0,1)))$. Recall that $Z_j(\overline{\tau})$ is a subbundle of $\mathrm{Hom}_{\mathbb{R}}(\kappa^{k+1}_{W\times(0,1)}, T(W \times (0,1)))$ with fibers

$$\begin{aligned} Z_j(\overline{\tau})_{(w,t)} = \{A \in \mathrm{Hom}_{\mathbb{R}}((\kappa^{k+1}_{W\times(0,1)})_{(w,t)}, T_{(w,t)}(W \times (0,1))) \\ = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{k+1}, T_w(W) \times \mathbb{R});\ \dim\ker A = j\}. \end{aligned}$$

If $\mathbb{R}(q,p)$ denotes the real vector space of $(q \times p)$-matrices with real coefficients, then [1, Lemma 2, p. 1223] states that

$$\mathbb{R}_r(q,p) := \{A \in \mathbb{R}(q,p);\ \mathrm{rank}\, A = r\}, \qquad r \in \{0, \dots, \min(q,p)\},$$

is a smooth submanifold of $\mathbb{R}(q,p)$ of codimension $(q-r)(p-r)$.

Lemma 2.3.8. *The embedding*

$$\iota_{q,p} \colon \mathbb{R}(q,p) \to \mathbb{R}(q+1,p+1), \qquad \iota_{q,p}(R) = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix},$$

is for every $r \in \{0, \dots, \min(q,p)\}$ transverse to the submanifold $\mathbb{R}_{r+1}(q+1,p+1) \subset \mathbb{R}(q+1,p+1)$, and $\iota^{-1}_{m,k}(\mathbb{R}_{r+1}(q+1,p+1)) = \mathbb{R}_r(q,p)$.

Proof. Given a matrix $R \in \mathbb{R}(q,p)$ such that $S := \iota_{q,p}(R) \in \mathbb{R}_{r+1}(q+1,p+1)$, one has to show that $T_S\mathbb{R}(q+1,p+1) = d\iota_{q,p}(T_R\mathbb{R}(q,p)) + T_S\mathbb{R}_{r+1}(q+1,p+1)$. For $(i,j) \in \{1,\dots,q+1\} \times \{1,\dots,p+1\}$ the matrices $E^{q+1,p+1}_{ij}$ with (i',j')-entry $\delta_{ii'}\delta_{jj'}$ form a basis of $\mathbb{R}(q+1,p+1) = \mathbb{R}^{(q+1)\times(p+1)} \cong T_S\mathbb{R}(q+1,p+1)$. Hence, it suffices to construct for every pair $(i,j) \in \{1,\dots,q+1\} \times \{1,\dots,p+1\}$ either a smooth path $\rho \colon (-\varepsilon,\varepsilon) \to \mathbb{R}(q,p)$ such that $\rho(0) = R$ and $d\iota_{q,p}(\rho'(0)) = E^{q+1,p+1}_{ij}$, or a smooth path $\sigma \colon (-\varepsilon,\varepsilon) \to \mathbb{R}_{r+1}(q+1,p+1)$ such that $\sigma(0) = S$ and $\sigma'(0) = E^{q+1,p+1}_{ij}$.

If $(i,j) \in \{1,\dots,q\} \times \{1,\dots,p\}$, then the smooth path $\rho \colon \mathbb{R} \to \mathbb{R}(q,p)$, $\rho(t) = R + t \cdot E^{q,p}_{ij}$, satisfies $\rho(0) = R$ and $d\iota_{q,p}(\rho'(0)) = (\iota_{q,p} \circ \rho)'(0) = \frac{d}{dt}(\iota_{q,p}(R) + t \cdot E^{q+1,p+1}_{ij})|_{t=0} = E^{q+1,p+1}_{ij}$.

If, however, $i = q+1$ or $j = p+1$, then one constructs a path σ of the above form as follows. The proof of [1, Lemma 2, p. 1223] shows that there exists a permutation $\pi \colon \mathbb{R}_{r+1}(q+1,p+1) \to \mathbb{R}_{r+1}(q+1,p+1)$ of matrix entires (given by a change of rows and columns) leaving the last column $(i,p+1)$ and the last row $(q+1,j)$ fixed, and such that a chart of $\pi(S)$ in $\pi(\mathbb{R}_{r+1}(q+1,p+1))$ is given by matrices of the form

$$(*) \qquad \begin{pmatrix} CB & C \\ AB & A \end{pmatrix},$$

where $A \in \mathbb{R}(r+1,r+1)$ is invertible, and $B \in \mathbb{R}(r+1,p-r)$, $C \in \mathbb{R}(q-r,r+1)$, and there exists an invertible matrix $A_0 \in \mathbb{R}(r,r)$ and matrices $B_0 \in \mathbb{R}(r,p-r)$ and $C_0 \in \mathbb{R}(q-r,r)$ such that

$$\pi(S) = \begin{pmatrix} C_0B_0 & C_0 & 0 \\ A_0B_0 & A_0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If $i=q+1$ or $j=p+1$, then define the smooth path

$$\sigma\colon(-1,1)\to\mathbb{R}_{r+1}(q+1,p+1),\qquad \sigma(t)=\pi^{-1}(\pi(S)+t\cdot E_{ij}^{q+1,p+1}).$$

We distinguish between the following four cases for (i,j) to verify that $\sigma(t)\in\mathbb{R}_{r+1}(q+1,p+1)$ for every $t\in(-1,1)$:

- If $(i,j)\in\{q-r+1,\dots,q+1\}\times\{p+1\}$, then $\pi(\sigma(t))$ is of the form $(*)$ for the choice

$$A=\begin{pmatrix}A_0&0\\0&1\end{pmatrix}+t\cdot E_{i-q+r,r+1}^{r+1,r+1},\quad B=\begin{pmatrix}B_0\\0\end{pmatrix},\quad C=\begin{pmatrix}C_0&0\end{pmatrix}.$$

  Note that A is invertible also for $i=q+1$ since $t\neq-1$.
- If $(i,j)\in\{q+1\}\times\{p-r+1,\dots,p\}$, then $\pi(\sigma(t))$ is of the form $(*)$ for the choice

$$A=\begin{pmatrix}A_0&0\\0&1\end{pmatrix}+t\cdot E_{r+1,j-p+r}^{r+1,r+1},\quad B=\begin{pmatrix}B_0\\-t\cdot(B_0)_{j-p+r}\end{pmatrix},\quad C=\begin{pmatrix}C_0&0\end{pmatrix},$$

  where $(B_0)_{j-p+r}$ denotes the $(j-p+r)$-th row of the matrix B_0.
- If $(i,j)\in\{q+1\}\times\{1,\dots,p-r\}$, then $\pi(\sigma(t))$ is of the form $(*)$ for the choice

$$A=\begin{pmatrix}A_0&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}B_0\\0\end{pmatrix}+t\cdot E_{r+1,j}^{r+1,p-r},\quad C=\begin{pmatrix}C_0&0\end{pmatrix}.$$

- If $(i,j)\in\{1,\dots,q-r\}\times\{p+1\}$, then $\pi(\sigma(t))$ is of the form $(*)$ for the choice

$$A=\begin{pmatrix}A_0&0\\0&1\end{pmatrix},\quad B=\begin{pmatrix}B_0\\0\end{pmatrix},\quad C=\begin{pmatrix}C_0&0\end{pmatrix}+t\cdot E_{i,r+1}^{q-r,r+1}.$$

It is clear that $\sigma(0)=S$ and $\sigma'(0)=E_{ij}^{q+1,p+1}$. (Recall that the permutation π leaves the last column $(i,p+1)$ and the last row $(q+1,j)$ fixed.) ☐

By construction of $\overline{F}$ it is clear that $s_{\overline{F}}(W\times(0,1))\cap Z_{k+1}(\overline{\tau})=\emptyset$. For $j\in\{0,\dots,k\}$, it suffices to check $s_{\overline{F}}\pitchfork Z_j(\overline{\tau})$ on all local charts on $W\times(0,1)$. Let $\mathbb{R}^m$ be any local coordinate system on W. Then s_F is locally in this coordinate system a smooth section of the form

$$s\colon\mathbb{R}^m\to\mathbb{R}(m,k)\times\mathbb{R}^m,\qquad s(w)=(\sigma(w),w),$$

that is for all $j\in\{0,\dots,k\}$ transverse to $\mathbb{R}_{k-j}(m,k)\times\mathbb{R}^m$. Consequently, in the corresponding coordinate system $\mathbb{R}^m\times(0,1)$ on $W\times(0,1)$, $s_{\overline{F}}$ is given by the section

$$\begin{aligned}\overline{s}\colon\mathbb{R}^m\times(0,1)&\to\mathbb{R}(m+1,k+1)\times\mathbb{R}^m\times(0,1),\\ \overline{s}(w,t)&=(\iota_{m,k}(\sigma(w)),w,t)=:(\overline{\sigma}(w),w,t),\end{aligned}$$

and the claim is that $\overline{s}$ is transverse to $\mathbb{R}_{k-j+1}(m+1,k+1)\times\mathbb{R}^m\times(0,1)$ for all $j\in\{0,\dots,k\}$.

Since a smooth map $\varphi\colon A\to B$ is transverse to a submanifold $M\subset B$ if and only if the map $A\to A\times B$, $a\mapsto(a,\varphi(a))$, is transverse to $A\times M$, we know that the smooth map $\sigma\colon\mathbb{R}^m\to\mathbb{R}(m,k)$ is transverse to the submanifold $\mathbb{R}_{k-j}(m,k)\subset\mathbb{R}(m,k)$, and have to show that the smooth map $\overline{\sigma}=\iota_{m,k}\circ\sigma\colon\mathbb{R}^m\to\mathbb{R}(m+1,k+1)$ is transverse to the submanifold

$\mathbb{R}_{k-j+1}(m+1,k+1) \subset \mathbb{R}(m+1,k+1)$. This is a direct consequence of Lemma 2.3.8. □

Definition 2.3.9. The set $\mathcal{F}(W)$ of *fields* on the bordism W consists of all r-tuples

$$F = (F_1, \dots, F_r) \in \mathcal{H}^{m-i_1+1}(W) \times \cdots \times \mathcal{H}^{m-i_r+1}(W)$$

with the following property. There exists $\varepsilon > 0$ and for every $s \in \{1, \dots, r\}$ a representative $\Phi_s \colon \kappa_{W_\varepsilon}^{m-i_s+1} \to TW_\varepsilon$ of $F_s \in \mathcal{H}^{m-i_s+1}(W)$ such that $\{\overline{Z}_1(\Phi_1), \dots, \overline{Z}_1(\Phi_r)\}$ is a transverse system in W_ε, and such that the r-tuple $(\Phi_1|_{M\times(0,\varepsilon)}, \dots, \Phi_r|_{M\times(0,\varepsilon)})$ represents an element in $\mathcal{F}(M) \subset \mathcal{G}^{m-i_1+1}(M) \times \cdots \times \mathcal{G}^{m-i_r+1}(M)$.

Let us check the axioms that are required for fields in our smooth setting (compare [5, p. 20f]):

(FRES). *Restrictions*: All restrictions of fields are defined by restriction of representatives. Note that our implementation of fields on bordisms treats incoming and outgoing boundaries asymmetrically. It follows from Definition 2.3.9 that there exists a restriction map $\mathcal{F}(W) \to \mathcal{F}(M)$ of fields on W to the incoming boundary. The restriction map $\mathcal{F}(W) \to \mathcal{F}(N)$ to fields on the outgoing boundary exists by Proposition 2.3.4. For the same reasons one obtains a restriction map $\mathcal{F}(W) \to \mathcal{F}(P)$ to framed codimension 1 submanifolds (P, π) of W. In order to describe the restriction map $\mathcal{F}(W) \to \mathcal{F}(W_0)$ of fields on W to a subbordism $(W_0, M_0, \mu_0, N_0, \nu_0)$ of (W, M, μ, N, ν) one just has to note that $(W_0)_\varepsilon$ is canonically an open subset of W_ε for sufficiently small $\varepsilon > 0$. There is also a restriction map $\mathcal{F}(M) \to \mathcal{F}(M_0)$ for codimension 0 submanifolds $M_0 \subset M$. All restriction maps commute with each other since they are all defined by restriction of representatives.

(FHOMEO). *Action of diffeomorphisms*: A diffeomorphism $\alpha \colon M \to M'$ induces contravariantly a bijection $\alpha^* \colon \mathcal{F}(M') \to \mathcal{F}(M)$ as follows. Given an element $f' = (f'_1, \dots, f'_r) \in \mathcal{F}(M')$, choose $\varepsilon > 0$ and for every $s \in \{1, \dots, r\}$ a representative $\varphi'_s \colon \kappa_{M'\times(0,\varepsilon)}^{m-i_s+1} \to T(M'\times(0,\varepsilon))$ of f'_s with the properties of Definition 2.3.2. Then the element $\alpha^*(f') \in \mathcal{F}(M)$ is represented by the r-tuple $(\varphi_1, \dots, \varphi_r)$, where the generic smooth vector bundle morphism $\varphi_s \colon \kappa_{M\times(0,\varepsilon)}^{m-i_s+1} \to T(M \times (0,\varepsilon))$ is for every $s \in \{1, \dots, r\}$ given by $\varphi_s = (d(\alpha \times \mathrm{id}_{(0,\varepsilon)}))^{-1} \circ \varphi'_s \circ (\alpha \times \mathrm{id}_{(0,\varepsilon)} \times \mathrm{id}_{\mathbb{R}^{m-i_s+1}})$. Note that $(\mathrm{id}_M)^* = \mathrm{id}_{\mathcal{F}(M)}$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$ for a diffeomorphism $\beta \colon M' \to M''$. A diffeomorphism $\phi \colon (W, M, \mu, N, \nu) \to (W', M', \mu', N', \nu')$ of smooth bordisms (i.e., a diffeomorphism $\phi \colon W \to W'$ such that $\phi(M) = M'$, $\phi(N) = N'$, and such that there exists $\varepsilon > 0$ with $\phi|_{M\times[0,\varepsilon)} = (\phi|_M) \times \mathrm{id}_{[0,\varepsilon)}$ and $\phi|_{N\times(-\varepsilon,0]} = (\phi|_N) \times \mathrm{id}_{(-\varepsilon,0]}$) induces contravariantly a bijection $\phi^* \colon \mathcal{F}(W') \to \mathcal{F}(W)$ as follows. Given an element $F' = (F'_1, \dots, F'_r) \in \mathcal{F}(W')$, choose $\varepsilon > 0$ and for every $s \in \{1, \dots, r\}$ a representative $\Phi'_s \colon \kappa_{W'_\varepsilon}^{m-i_s+1} \to TW'_\varepsilon$ of F'_s with the properties of Definition 2.3.9. Note that the diffeomorphism $\phi \colon W \to W'$ induces in an obvious way a diffeomorphism $\phi_\varepsilon \colon W_\varepsilon \to W'_\varepsilon$. Then the element $\phi^*(F') \in \mathcal{F}(W)$ is represented by the r-tuple $(\Phi_1, \dots, \Phi_r)$, where the generic smooth vector bundle morphism $\Phi_s \colon \kappa_{W_\varepsilon}^{m-i_s+1} \to TW_\varepsilon$ is for every $s \in \{1, \dots, r\}$ given by $\Phi_s = (d\phi_\varepsilon)^{-1} \circ \Phi'_s \circ (\phi_\varepsilon \times \mathrm{id}_{\mathbb{R}^{m-i_s+1}})$. Note that $(\mathrm{id}_W)^* = \mathrm{id}_{\mathcal{F}(W)}$ and $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ for a diffeomorphism $\psi \colon (W', M', \mu', N', \nu') \to (W'', M'', \mu'', N'', \nu'')$. It is easy to show that these induced maps commute with the resriction maps of (FRES).

(FDISJ). *Disjoint unions*: Given smooth bordisms (W, M, μ, N, ν) and (W', M', μ', N', ν'), the product of restrictions $\mathcal{F}(W \sqcup W') \to \mathcal{F}(W) \times \mathcal{F}(W')$ is a bijection, i.e. any field on $W \sqcup W'$ is uniquely determined by its restrictions to W and W', and a field on W together with a field on W' give rise to a field on $W \sqcup W'$. Similarly, $F(M \sqcup N) \to F(M) \times F(N)$ is a bijection in dimension n.

(FGLUE). *Gluing*: Let $(W, M, \mu, P, \pi) = (W', M, \mu, N, \nu') \cup_N (W'', N, \nu'', P, \pi)$ denote the smooth bordism obtained by gluing two smooth bordisms (W', M, μ, N, ν') and (W'', N, ν'', P, π) along their common boudary part N. Then the map

$$\mathcal{F}(W) \to \{(F', F'') \in \mathcal{F}(W') \times \mathcal{F}(W''); F'|_N = F''|_N\}, \qquad F \mapsto (F|_{W'}, F|_{W''}),$$

is a bijection. (Note that the map is well defined since $(N, \nu' \cup \nu'')$ is a framed codimension 1 submanifold of (W, M, μ, P, π), and $(F|_{W'})|_N = F|_N = (F|_{W''})|_N$ by (FRES).)

Proposition 2.3.10. *For all boundary conditions $f \in \mathcal{F}(M)$ and $g \in \mathcal{F}(N)$, we have*

$$g \in \mathcal{F}_<(N) \qquad \Leftrightarrow \qquad \mathcal{F}(W, f, g) := \{F \in \mathcal{F}(W);\ F|_M = f,\ F|_N = g\} \neq \emptyset.$$

Proof. It is obvious from Definition 2.3.3 that $\mathcal{F}(W, f, g) \neq \emptyset$ implies $g \in \mathcal{F}_<(N)$.

Conversely, suppose that $g \in \mathcal{F}_<(N)$. Choose $\varepsilon > 0$ such that $M \times (0, 2\varepsilon) \subset W_\varepsilon$ and $N \times (-\varepsilon, \varepsilon) \subset W_\varepsilon$ are disjoint, and such that there exist for every $s \in \{1, \ldots, r\}$

- a generic smooth vector bundle morphism

$$\varphi_s \in C^\infty_{\mathrm{gen}}(M \times (0, 2\varepsilon), \mathrm{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$$

  representing $f_s \in \mathcal{G}^{m-i_s+1}(M)$ such that $\{\overline{Z}_1(\varphi_1), \ldots, \overline{Z}_1(\varphi_r)\}$ is a transverse system in $M \times (0, 2\varepsilon)$, and such that $\overline{Z}_1(\varphi_1) \cap \cdots \cap \overline{Z}_1(\varphi_r) = \emptyset$ (see Definition 2.3.2), and
- a generic smooth vector bundle morphism

$$\gamma_s \in C^\infty_{\mathrm{gen}}(N \times (-\varepsilon, \varepsilon), \mathrm{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$$

  such that $\{\overline{Z}_1(\gamma_1), \ldots, \overline{Z}_1(\gamma_r)\}$ is a transverse system in $N \times (-\varepsilon, \varepsilon)$, and such that $\gamma_s|_{N \times (0,\varepsilon)}$ is a representative of $g_s \in \mathcal{G}^{m-i_s+1}(N)$ (see Definition 2.3.3).

It suffices to construct for every $s \in \{1, \ldots, r\}$ a generic smooth section

$$\Phi_s \in C^\infty_{\mathrm{gen}}(W_\varepsilon, \mathrm{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$$

such that $\Phi_s|_{M \times (0,\varepsilon)} = \varphi_s|_{M \times (0,\varepsilon)}$ and $\Phi_s|_{N \times (0,\varepsilon)} = \gamma_s|_{N \times (0,\varepsilon)}$ for all $s \in \{1, \ldots, r\}$, and such that $\{\overline{Z}_1(\Phi_1), \ldots, \overline{Z}_1(\Phi_r)\}$ is a transverse system in W_ε. Hence, if F_s denotes the class of Φ_s in $\mathcal{H}^{m-i_s+1}(W)$, then $F := (F_1, \ldots, F_r)$ defines an element in $\mathcal{F}(W, f, g) \neq \emptyset$ as required.

By Proposition A.4.3 it is possible to construct for every $s \in \{1, \ldots, r\}$ separately a generic smooth section Φ_s that extends φ_s and γ_s as required. However, in order to achieve in addition that $\{\overline{Z}_1(\Phi_1), \ldots, \overline{Z}_1(\Phi_r)\}$ is a transverse system in W_ε, one has to construct the section Φ_s by induction on $s \in \{1, \ldots, r\}$. Suppose that $\Phi_1, \ldots, \Phi_s$ have been constructed for some $s \in \{1, \ldots, r\}$ such that $\{\overline{Z}_1(\Phi_1), \ldots \overline{Z}_1(\Phi_s)\}$ is a transverse system in W_ε (which is no condition for $s = 1$). If $s < r$, then we explain in the following how to construct a generic smooth section Φ_{s+1} that extends φ_{s+1} and γ_{s+1} as required, and such that $\{\overline{Z}_1(\Phi_1), \ldots, \overline{Z}_1(\Phi_{s+1})\}$ is a transverse system in W_ε.

As $\{\overline{Z}_1(\Phi_1), \ldots, \overline{Z}_1(\Phi_s)\}$ is a transverse system in W_ε by assumption, one obtains for every subset $J \subset \{1,\ldots,s\}$ a Whitney stratified subspace $Z_J \subset W_\varepsilon$ (where $Z_\emptyset := W_\varepsilon$) via

$$Z_J := \cap_{j\in J}\overline{Z}_1(\Phi_j).$$

Define an open subset $U \subset W_\varepsilon$ by

$$U := W_\varepsilon \setminus (M\times(0,\varepsilon] \sqcup N\times[0,\varepsilon)).$$

In the following, we will use Proposition A.4.3 to produce inductively for $i = s+1,\ldots,0$ a compact subset $C_i \subset U$ and a generic smooth section

$$\Phi^{(i)}_{s+1} \in C^\infty_{\text{gen}}(W_\varepsilon \setminus C_i, \text{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$$

with the following properties:

$(1)_i$ $\Phi^{(i)}_{s+1}|_{W_\varepsilon\setminus U} = \varphi_{s+1}|_{M\times(0,\varepsilon]} \sqcup \gamma_{s+1}|_{N\times[0,\varepsilon)}$,
$(2)_i$ $\overline{Z}_1(\Phi^{(i)}_{s+1}) \pitchfork Z_J$ in W_ε for all subsets $J \subset \{1,\ldots,s\}$,
$(3)_i$ $C_i \cap Z_J = \emptyset$ for all subsets $J\subset\{1,\ldots,s\}$ with (at least) i elements.

Note that $\Phi_{s+1} := \Phi^{(0)}_{s+1}$ will have the desired properties. (Indeed, property $(3)_0$ implies that $C_0 = \emptyset$ since $Z_\emptyset = W_\varepsilon$. Thus, $\Phi_{s+1} \in C^\infty_{\text{gen}}(W_\varepsilon, \text{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$. Furthermore, $\Phi_{s+1}|_{M\times(0,\varepsilon)} = \varphi_{s+1}|_{M\times(0,\varepsilon)}$ and $\Phi_{s+1}|_{N\times(0,\varepsilon)} = \gamma_{s+1}|_{N\times(0,\varepsilon)}$ follows from property $(1)_0$. Finally, $\{\overline{Z}_1(\Phi_1),\ldots,\overline{Z}_1(\Phi_{s+1})\}$ is a transverse system in W_ε, which follows from property $(2)_0$ and Proposition 2.2.7 because $\{\overline{Z}_1(\Phi_1),\ldots,\overline{Z}_1(\Phi_s)\}$ is by induction hypothesis a transverse system in W_ε.)

Initially, for $i = s+1$, fix some $\varepsilon' \in (\varepsilon, 2\varepsilon)$ and define a compact subset $C_{s+1} \subset U$ by

$$C_{s+1} := W_\varepsilon \setminus (M\times(0,\varepsilon') \sqcup N\times(\varepsilon-\varepsilon',\varepsilon)).$$

Furthermore, define a generic smooth section

$$\Phi^{(s+1)}_{s+1} \in C^\infty_{\text{gen}}(W_\varepsilon \setminus C_{s+1}, \text{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon}, TW_\varepsilon))$$

by $\Phi^{(s+1)}_{s+1}|_{M\times(0,\varepsilon')} = \varphi_{s+1}|_{M\times(0,\varepsilon')}$ and $\Phi^{(s+1)}_{s+1}|_{N\times(\varepsilon-\varepsilon',\varepsilon)} = \gamma_{s+1}|_{N\times(\varepsilon-\varepsilon',\varepsilon)}$.

Note that property $(1)_{s+1}$ holds by construction and property $(3)_{s+1}$ is tautological. Property $(2)_{s+1}$ follows from $\overline{Z}_1(\Phi^{(s+1)}_{s+1}) = \overline{Z}_1(\varphi_{s+1}) \cap M\times(0,\varepsilon') \sqcup \overline{Z}_1(\gamma_{s+1}) \cap N\times(\varepsilon-\varepsilon',\varepsilon)$ since $\{\overline{Z}_1(\varphi_1),\ldots,\overline{Z}_1(\varphi_r)\}$ is a transverse system in $M\times(0,2\varepsilon)$ and $\{\overline{Z}_1(\gamma_1),\ldots,\overline{Z}_1(\gamma_r)\}$ is a transverse system in $N\times(-\varepsilon,\varepsilon)$.

Next, suppose that $\Phi^{(i)}_{s+1}$ has been constructed for some $i\in\{1,\ldots,s+1\}$. In the following, we explain the construction of $\Phi^{(i-1)}_{s+1}$.

By property $(3)_i$ it is possible to choose an open subset $U_{i-1} \subset W_\varepsilon$ such that $C_i \subset U_{i-1}$, $\overline{U_{i-1}}^{W_\varepsilon} \subset U$, and $U_{i-1} \cap Z_J = \emptyset$ for all subsets $J \subset \{1,\ldots,s\}$ with (at least) i elements. Therefore,

$$Z_{i-1} := U_{i-1} \cap \bigcup_{J\subset\{1,\ldots,s\},|J|=i-1} Z_J$$

is a Whitney stratified subspace of U_{i-1}. (Note that $(U_{i-1}\cap Z_J)\cap(U_{i-1}\cap Z_{J'}) = U_{i-1}\cap Z_{J\cup J'} = \emptyset$ for all subsets $J, J' \subset \{1,\ldots,s\}$ with $i-1$ elements such that $J \neq J'$.)

Let $\pi\colon \operatorname{Hom}_{\mathbb{R}}(\kappa_{W_\varepsilon}^{m-i_s+1}, TW_\varepsilon) \to W_\varepsilon$ denote the projection map. Note that

$$\tilde{Z}_{i-1} := \overline{Z}_1(\tau) \cap \pi^{-1}(Z_{i-1})$$

is (by transversality of $\overline{Z}_1(\tau)$ and $\pi^{-1}(Z_{i-1})$ in $\operatorname{Hom}_{\mathbb{R}}(\kappa_{W_\varepsilon}^{m-i_s+1}, TW_\varepsilon)$) a Whitney stratified subspace of $\pi^{-1}(U_{i-1})$.

Choose an open subset $\tilde{U}_{i-1} \subset W_\varepsilon$ such that $C_i \subset \tilde{U}_{i-1}$ and $\overline{\tilde{U}_{i-1}}^{W_\varepsilon} \subset U_{i-1}$.

Application of Proposition A.4.3 to $\Phi_{s+1}^{(i)}|_{U_{i-1}\setminus C_i}$ and $C_i \subset \tilde{U}_{i-1}$ yields a smooth section

$$\tilde{\Phi}_{s+1}^{(i-1)} \in C^\infty(U_{i-1}, \operatorname{Hom}_{\mathbb{R}}(\kappa_{W_\varepsilon}^{m-i_s+1}, TW_\varepsilon))$$

that is transverse to $\tilde{Z}_{i-1}$ and satisfies $\tilde{\Phi}_{s+1}^{(i-1)}|_{U_{i-1}\setminus\tilde{U}_{i-1}} = \Phi_{s+1}^{(i)}|_{U_{i-1}\setminus\tilde{U}_{i-1}}$. (In order to show that $\Phi_{s+1}^{(i)} \pitchfork \tilde{Z}_{i-1}$ on $U_{i-1}\setminus C_i$, one employs the following

Lemma 2.3.11. *Let $f\colon X \to Y$ be a smooth map between smooth manifolds, and let $B \subset Y$ be a submanifold such that $f \pitchfork B$. Then, for any submanifold $A \subset X$, the following statements are equivalent:*

(i) $A \pitchfork f^{-1}(B)$.
(ii) The smooth map $F = (\mathrm{id}_X, f)\colon X \to X \times Y$ is transverse to $A \times B$.

The statement of the Lemma generalizes directly to the case of Whitney stratified subspaces $A \subset X$ and $B \subset Y$.

Proof. $(ii) \Rightarrow (i)$. Given $x \in A \cap f^{-1}(B)$ and $V \in T_xX$, one has to show that $V \in T_xA + T_xf^{-1}(B)$. Since $F(x) = (x, f(x)) \in A \times B$ and $(V,0) \in T_xX \times T_{f(x)}Y = T_{F(x)}(X\times Y)$, it follows from (ii) that there exist $V_X \in T_xX$ and $(W_A, W_B) \in T_xA \times T_{f(x)}B$ such that

$$(V,0) = d_xF(V_X) + (W_A, W_B) = (V_X + W_A, d_xf(V_X) + W_B).$$

In particular, $V = V_X + W_A$, where $W_A \in T_xA$, and $V_X \in T_xf^{-1}(B)$ since $d_xf(V_X) = -W_B \in T_{f(x)}B$. (Here we have used that $T_xf^{-1}(B) = \{U \in T_xX;\ d_xf(U) \in T_{f(x)}B\}$.)

$(i) \Rightarrow (ii)$. Given $x \in X$ with $F(x) = (x, f(x)) \in A \times B$ and $(W_X, W_Y) \in T_xX \times T_{f(x)}Y = T_{F(x)}(X \times Y)$, one has to show that $(W_X, W_Y) \in d_xF(T_xX) + T_{F(x)}(A\times B)$.

Since $f(x) \in B$ and $W_Y \in T_{f(x)}Y$, it follows from $f \pitchfork B$ that there exist $V_X \in T_xX$ and $W_B \in T_{f(x)}B$ such that $W_Y = d_xf(V_X) + W_B$.

Furthermore, since $x \in A \cap f^{-1}(B)$ and $W_X - V_X \in T_xX$, (i) implies that there exist $V_A \in T_xA$ and $V_B \in T_xf^{-1}(B)$ such that $W_X - V_X = V_A + V_B$. Hence, the sum of $d_xF(V_X + V_B) \in d_xF(T_xX)$ and $(V_A, W_B - d_xf(V_B)) \in T_xA \times T_{f(x)}B = T_{F(x)}(A \times B)$ is given by

$$d_xF(V_X+V_B)+(V_A, W_B-d_xf(V_B)) = ((V_X+V_B)+V_A, d_xf(V_X+V_B)+W_B-d_xf(V_B)) = (W_X, W_Y).$$

□

It suffices to show that every $w \in U_{i-1}\setminus C_i$ has an open neighbourhood U_w such that $\Phi_{s+1}^{(i)} \pitchfork \tilde{Z}_{i-1}$ on U_w. Given $w \in U_{i-1}\setminus C_i$, choose an open neighbourhood U_w over which the bundle π trivializes, i.e. there exists a vector bundle isomorphism $\pi^{-1}(U_w) \cong U_w \times \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^{m-i_s+1}, \mathbb{R}^m)$.

Under this isomorphism, $\overline{Z}_1(\tau)\cap\pi^{-1}(U_w)$ corresponds to $U_w\times\cup_{j\geq 1}\mathbb{R}_j(m,m-i_s+1)$, and the section $\Phi^{(i)}_{s+1}|_{U_w}$ is of the form $(\mathrm{id}_{U_w},\phi)\colon U_w\to U_w\times\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{m-i_s+1},\mathbb{R}^m)$ for some smooth map $\phi\colon U_w\to\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{m-i_s+1},\mathbb{R}^m)$. Genericity of $\Phi^{(i)}_{s+1}$ implies that $(\mathrm{id}_{U_w},\phi)\pitchfork U_w\times\cup_{j\geq 1}\mathbb{R}_j(m,m-i_s+1)$. The latter is equivalent to $\phi\pitchfork\cup_{j\geq 1}\mathbb{R}_j(m,m-i_s+1)$. Property $(2)_i$ implies that $\overline{Z}_1(\Phi^{(i)}_{s+1}|_{U_w})\pitchfork Z_{i-1}\cap U_w$ in U_w. Consequently, $\phi^{-1}(\mathbb{R}_j(m,m-i_s+1))\pitchfork Z_{i-1}\cap U_w$. Therefore, by the implication $(i)\Rightarrow(ii)$ of Lemma 2.3.11 applied to the smooth map $f:=\phi$ from $X:=U_w$ to $Y:=\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^{m-i_s+1},\mathbb{R}^m)$ which is transverse to $B:=\cup_{j\geq 1}\mathbb{R}_j(m,m-i_s+1)$ and satisfies $f^{-1}(B)\pitchfork A$ for $A:=Z_{i-1}\cap U_w$, one obtains that (id_{U_w},ϕ) is transverse to $A\times B$. This corresponds to the desired statement that $\Phi^{(i)}_{s+1}|_{U_w}$ is transverse to $\tilde{Z}_{i-1}\cap\pi^{-1}(U_w)$.)

It follows from $\tilde{\Phi}^{(i-1)}_{s+1}\pitchfork\tilde{Z}_{i-1}$ that $\tilde{\Phi}^{(i-1)}_{s+1}$ is on Z_{i-1} transverse to $\overline{Z}_1(\tau)\cap\pi^{-1}(Z_J)$ for all subsets $J\subset\{1,\ldots,s\}$ with at most $i-1$ elements. By Proposition 2.2.2 there exists an open neighbourhood V_{i-1} of Z_{i-1} in U_{i-1} such that $\tilde{\Phi}^{(i-1)}_{s+1}$ is on V_{i-1} transverse to $\overline{Z}_1(\tau)\cap\pi^{-1}(Z_J)$ for all subsets $J\subset\{1,\ldots,s\}$ with at most $i-1$ elements. In particular, the choice $J=\emptyset$ implies that $\tilde{\Phi}^{(i-1)}_{s+1}$ is transverse to $\overline{Z}_1(\tau)$ on V_{i-1}. Consequently, the implication $(ii)\Rightarrow(i)$ of Lemma 2.3.11 implies that $\overline{Z}_1(\tilde{\Phi}^{(i-1)}_{s+1}|_{V_{i-1}})\pitchfork Z_J$ for all subsets $J\subset\{1,\ldots,s\}$ with at most $i-1$ elements.

Let us define the required pair $(C_{i-1},\Phi^{(i-1)}_{s+1})$. Set $C_{i-1}:=\overline{\tilde{U}_{i-1}}^{W_\varepsilon}\setminus V_{i-1}$, which is a compact subspace of U. Note that $W_\varepsilon\setminus C_{i-1}=(W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon})\cup V_{i-1}$. Define the section

$$\Phi^{(i-1)}_{s+1}\in C^\infty_{\mathrm{gen}}(W_\varepsilon\backslash C_{i-1},\mathrm{Hom}_{\mathbb{R}}(\kappa^{m-i_s+1}_{W_\varepsilon},TW_\varepsilon)),\qquad \Phi^{(i-1)}_{s+1}(w)=\begin{cases}\tilde{\Phi}^{(i-1)}_{s+1}(w), & w\in V_{i-1},\\ \Phi^{(i)}_{s+1}(w), & w\in W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}.\end{cases}$$

(Indeed, $\Phi^{(i-1)}_{s+1}$ is a generic smooth section because the two generic smooth sections $\tilde{\Phi}^{(i-1)}_{s+1}|_{V_{i-1}}$ and $\Phi^{(i)}_{s+1}|_{W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}}$ agree by construction on the intersection $V_{i-1}\cap(W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon})=V_{i-1}\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}\subset U_{i-1}\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}$.)

It remains to check the desired properties $(1)_{i-1}$, $(2)_{i-1}$ and $(3)_{i-1}$. Property $(1)_{i-1}$ follows from property property $(1)_i$ and $\Phi^{(i-1)}_{s+1}|_{W_\varepsilon\backslash U}=\Phi^{(i)}_{s+1}|_{W_\varepsilon\backslash U}$ since $\overline{\tilde{U}_{i-1}}^{W_\varepsilon}\subset U$. To check property $(2)_{i-1}$, suppose that $w\in W_\varepsilon\setminus C_{i-1}$, i.e. $w\in V_{i-1}$ or $w\in W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}$. Fix a subset $J\subset\{1,\ldots,s\}$. If $w\in V_{i-1}$, then the claim follows since $\overline{Z}_1(\tilde{\Phi}^{(i-1)}_{s+1}|_{V_{i-1}})\pitchfork Z_J$ whenever J has at most $i-1$ elements, and $V_{i-1}\cap Z_J\subset U_{i-1}\cap Z_J=\emptyset$ whenever J has at least i elements. Furthermore, if $w\in W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}$, then $\Phi^{(i-1)}_{s+1}|_{W_\varepsilon\backslash\overline{\tilde{U}_{i-1}}^{W_\varepsilon}}=\Phi^{(i)}_{s+1}|_{W_\varepsilon\backslash\overline{\tilde{U}_{i-1}}^{W_\varepsilon}}$, $W_\varepsilon\setminus\overline{\tilde{U}_{i-1}}^{W_\varepsilon}\subset W_\varepsilon\setminus C_i$, and property $(2)_i$ imply that $\overline{Z}_1(\Phi^{(i-1)}_{s+1})\pitchfork Z_J$ at w. Finally, property $(3)_{i-1}$ holds since $C_{i-1}\cap Z_J\subset Z_{i-1}\setminus V_{i-1}=\emptyset$ for all subsets $J\subset\{1,\ldots,s\}$ with $i-1$ elements. □

2.3.2 System of Action Functionals

Let $\mathbf{N}$ denote the (small strict) monoidal category determined by the commutative monoid $(\mathbb{N},+,0)$ (see [5, Lemma 4.6, p. 19]). Thus, $\mathrm{Ob}\,\mathbf{N}=\{I\}$, $\mathrm{End}_{\mathbf{N}}(I)=\mathbb{N}=\{0,1,2,\ldots\}$, $I\otimes I=I$, and all $a,b\in\mathbb{N}$ satisfy $a\circ b=a+b=a\otimes b$.

Lemma 2.3.12. *Suppose that $F\in\mathcal{F}(W)$ is a field on a smooth bordism (W,M,μ,N,ν) that is represented by an r-tuple of generic smooth vector bundle morphisms $(\Phi_1,\ldots,\Phi_r)$ on W_ε, $\varepsilon>$*

0, *with the properties of Definition 2.3.9. Then,* $\overline{Z}_1(\Phi_1)\cap\cdots\cap\overline{Z}_1(\Phi_r) = Z_1(\Phi_1)\cap\cdots\cap Z_1(\Phi_r)$. *Furthermore, for sufficiently small* $\varepsilon > 0$, *the intersection* $\overline{Z}_1(\Phi_1)\cap\cdots\cap\overline{Z}_1(\Phi_r)$ *is a finite subset of* W *that does not depend on the choice of the representative* $(\Phi_1,\dots,\Phi_r)$ *of* F.

Proof. First of all, note that $\overline{Z}_1(\Phi_s)$ is for $s\in\{1,\dots,r\}$ a Whitney stratified subspace of W of dimension $m-i_s$ by Remark 2.2.9. Consequently, as $\{\overline{Z}_1(\Phi_1),\dots,\overline{Z}_1(\Phi_r)\}$ is a transverse system in W, the intersection $Z := \overline{Z}_1(\Phi_1)\cap\cdots\cap\overline{Z}_1(\Phi_r)$ is a Whitney stratified subspace of W of dimension

$$(m-i_1)+\cdots+(m-i_r)-(r-1)m = m-(i_1+\cdots+i_r) = 0.$$

It follows that $\overline{Z}_1(\Phi_1)\cap\cdots\cap\overline{Z}_1(\Phi_r) = Z_1(\Phi_1)\cap\cdots\cap Z_1(\Phi_r)$ because any other intersection of strata $Z_{l_1}(\Phi_1)\cap\cdots\cap Z_{l_r}(\Phi_r)$ with $l_s \geq 1$ for every $s\in\{1,\dots,r\}$ has dimension strictly smaller 0 and is hence empty. Moreover, Z is a discrete and closed subset of W_ε. The assumption on the behaviour of Φ_s near the boundaries of W implies that Z is contained in a compact subset of W_ε that is independent of ε for suitably small $\varepsilon > 0$. ☐

The previous Lemma permits the construction of the system of $\mathbf{N}$-valued action functionals $\mathbb{T}$.

Definition 2.3.13. Let (W,M,μ,N,ν) be a smooth bordism. The evaluation of the $\mathbf{N}$-valued action functional $\mathbb{T}_W$ on a field $F\in\mathcal{F}(W)$ is defined by

$$\mathbb{T}_W(F) := |\overline{Z}_1(\Phi_1)\cap\cdots\cap\overline{Z}_1(\Phi_r)| \qquad \in\{0,1,2,\dots\},$$

where F is represented by an r-tuple $(\Phi_1,\dots,\Phi_r)$ of generic smooth vector bundle morphisms on W_ε with the properties of Definition 2.3.9, and $\varepsilon > 0$ is sufficiently small.

Let us verify the required axioms for the system of action functionals (compare [5, p. 25]):

(TDISJ) Given two smooth bordisms (W,M,μ,N,ν) and (W',M',μ',N',ν'), one has

$$\mathbb{T}_{W\sqcup W'}(F) = \mathbb{T}_W(F|_W)\otimes\mathbb{T}_{W'}(F|_{W'})$$

for all $F\in\mathcal{F}(W\sqcup W')$ because $\otimes$ in $\mathbf{N}$ corresponds to addition of natural numbers.

(TGLUE) Let $(W,M,\mu,P,\pi) = (W',M,\mu,N,\nu')\cup_N(W'',N,\nu'',P,\pi)$ denote the smooth bordism obtained by gluing two smooth bordisms (W',M,μ,N,ν') and (W'',N,ν'',P,π) along their common boudary part N. Then,

$$\mathbb{T}_W(F) = \mathbb{T}_W(F|_{W''})\circ\mathbb{T}_W(F|_{W'})$$

for all $F\in\mathcal{F}(W)$ because $\circ$ in $\mathbf{N}$ corresponds to addition of natural numbers.

(THOMEO) If $\phi\colon (W,M,\mu,N,\nu)\to(W',M',\mu',N',\nu')$ is a diffeomorphism of smooth bordisms, then for any field $F\in\mathcal{F}(W')$, one has

$$\mathbb{T}_W(\phi^*F) = \mathbb{T}_{W'}(F)$$

under the bijection $\phi^*\colon\mathcal{F}(W')\to\mathcal{F}(W)$ of (FHOMEO).

By Theorem 2.0.1 the parity of $\mathbb{T}_W(F)$ is a Stiefel-Whitney number of the closed smooth bordism W and hence independent of the choice of the field $F \in \mathcal{F}(W)$. Moreover, bordism invariance of Stiefel-Whitney numbers implies that the parity of $\mathbb{T}_W$ is a bordism invariant. The following Proposition indicates a geometric proof of these properties of $\mathbb{T}_W$ without using Theorem 2.0.1.

Proposition 2.3.14. *If V is an $(m+1)$-dimensional smooth manifold, then*

$$\mathbb{T}_{\partial V} \equiv 0 \pmod 2.$$

Proof. Since $W := \partial V$ is an m-dimensional closed smooth bordism, Remark 2.3.6 implies that $\mathcal{H}^k(W) = C^\infty_{\mathrm{gen}}(\kappa^k_W, TW)$ for all $k \in \{1, \dots, m\}$. Hence, fields $F \in \mathcal{F}(W)$ (see Definition 2.3.9) are just r-tuples $F = (F_1, \dots, F_r)$ of generic smooth vector bundle morphisms

$$F_s \colon W \to \mathrm{Hom}_{\mathbb{R}}(\kappa_W^{m-i_s+1}, TW)$$

such that $\{\overline{Z}_1(F_1), \dots, \overline{Z}_1(F_r)\}$ is a transverse system in W. By Proposition 2.3.7 the suspension $\overline{F}_s \colon \kappa^{(m+1)-i_s+1}_{W\times(0,1)} \to T(W \times (0,1))$ represents for every $s \in \{1, \dots, r\}$ an element

$$g_s \in \mathcal{G}^{(m+1)-i_s+1}(\partial V).$$

Choosing a W-germ ω of collar neighbourhoods of W in V, one can interpret $(V, W, \omega, \emptyset, \nu_\emptyset)$ as an $(m+1)$-dimensional smooth bordism with incoming boundary W and empty outgoing boundary. By arguments analogous to the proof of Proposition 2.3.10 there exists for every $s \in \{1, \dots, r\}$ an element

$$G_s \in \mathcal{H}^{(m+1)-i_s+1}(V) = C^\infty_{\mathrm{gen}}(\kappa^{(m+1)-i_s+1}_{V\setminus W}, T(V \setminus W))$$

that extends g_s, and such that $\{\overline{Z}_1(G_1), \dots, \overline{Z}_1(G_r)\}$ is a transverse system in $V \setminus W$.

By Remark 2.2.9 the dimension of the Whitney stratified subspace $\overline{Z}_1(G_s) \subset V \setminus W$ is $m - i_s + 1$. Hence, the dimension of the Whitney stratified subspace $\overline{Z}_1(G_1) \cap \dots \cap \overline{Z}_1(G_r) \subset V \setminus W$ is

$$\begin{aligned}
&(m - i_1 + 1) + \dots + (m - i_r + 1) - (r-1)(m+1) \\
&= rm - (i_1 + \dots + i_r) + r - (r-1)(m+1) \\
&= rm - m + r - (r-1)(m+1) = rm - m + r - rm - r + m + 1 = 1.
\end{aligned}$$

If $W \times (0, \varepsilon)$ is a representative of ω in $V \setminus W$ for sufficiently small $\varepsilon > 0$, then the intersection

$$(W \times (0,\varepsilon)) \cap \overline{Z}_1(G_1) \cap \dots \cap \overline{Z}_1(G_r) = (W \times (0,\varepsilon)) \cap Z_1(G_1) \cap \dots \cap Z_1(G_r) = Z \times (0, \varepsilon)$$

is the suspension of $Z := \overline{Z}_1(F_1) \cap \dots \cap \overline{Z}_1(F_r) \subset W$. Hence, if $\overline{Z}_1(G_1) \cap \dots \cap \overline{Z}_1(G_r)$ was a 1-dimensional *manifold*, then the cardinality of Z would automatically be even. However, $\overline{Z}_1(G_1) \cap \dots \cap \overline{Z}_1(G_r)$ might have 0-dimensional singularities. Nevertheless, passing to the desingularizations $\tilde{Z}(G_s) \to \overline{Z}_1(G_s)$, $s \in \{1, \dots, r\}$, and the submanifolds

$$Z_s^{(r)} := (\pi_s^{(r)})^{-1}(\tilde{Z}(G_s)) \subset (V \setminus W) \times \mathbb{RP}^{(m+1)-i_1+1} \times \dots \times \mathbb{RP}^{(m+1)-i_r+1},$$

it is easy to check that the intersection $Z^{(r)} := Z_1^{(r)} \cap \dots \cap Z_r^{(r)}$ is a 1-dimensional smooth submanifold, and that the projection

$$(W \times (0,\varepsilon)) \times \mathbb{RP}^{(m+1)-i_1+1} \times \cdots \times \mathbb{RP}^{(m+1)-i_r+1} \to W \times (0,\varepsilon)$$

restricts to a diffeomorphism

$$Z^{(r)} \cap ((W \times (0,\varepsilon)) \times \mathbb{RP}^{(m+1)-i_1+1} \times \cdots \times \mathbb{RP}^{(m+1)-i_r+1}) \to Z \times (0,\varepsilon).$$

(In fact, use that $\tilde{Z}(G_s) \subset (V \setminus W) \times \mathbb{RP}^{(m+1)-i_1+1}$ has dimension $(m+1)-i_s$. Furthermore, the construction of the suspension in Proposition 2.3.7 implies that $\tilde{Z}(G_s) \cap ((W \times (0,\varepsilon)) \times \mathbb{RP}^{(m+1)-i_s+1})$ is the cylinder on

$$\tilde{Z}(F_s) \times \mathbb{RP}^{m-i_s+1} \subset \tilde{Z}(F_s) \times \mathbb{RP}^{(m+1)-i_s+1},$$

and its projection to $\overline{Z}_1(G_s) \cap ((W \times (0,\varepsilon))$ is given by the suspension of the projection $\tilde{Z}(F_s) \times \mathbb{RP}^{m-i_s+1} \to \overline{Z}_1(F_s)$. Finally, these statements can easily be carried over to the iterated product of projective spaces.) □

Corollary 2.3.15. *If* (W, M, μ, N, ν) *is a smooth bordism and* $f \in \mathcal{F}(M)$, $g \in \mathcal{F}(N)$, *then* $\mathbb{T}_W(F_1) \equiv \mathbb{T}_W(F_2) \pmod 2$ *for any two fields* $F_1, F_2 \in \mathcal{F}(W; f, g)$.

Proof. The idea is to consider extensions $\tilde{F}_1$ and $\tilde{F}_2$ of F_1 and F_2 on the double DW of W that agree on the other half. (If f is not left-extendable, then one chooses a representative of f defined on some cylinder $M \times (0,\varepsilon)$, and defines a new boundary condition $f' \in \mathcal{F}_<(M)$ by shifting the values of the interval $(0,\varepsilon)$ slightly to the left. This modification has no effect on the evaluation of $\mathbb{T}_W$.) Since DW is the boundary of an $(m+1)$-dimensional smooth manifold, Proposition 2.3.14 implies that $\mathbb{T}_{DW}(\tilde{F}_1)$ and $\mathbb{T}_{DW}(\tilde{F}_2)$ are even. Now additivity of the action functional under gluing of two copies of W along the common boundary implies that $\mathbb{T}_W(F_1)$ and $\mathbb{T}_W(F_2)$ have the same parity. □

2.4 Quantization

Given a partition $I = (i_1, \ldots, i_r)$ of $n+1$, the process of *quantization* described in [5, Section 6, p. 28] can now be applied to the system of fields $\mathcal{F}$ and the system of $\mathbf{N}$-valued action functionals $\mathbb{T}$ defined in the previous section, resulting in the $(n+1)$-dimensional *relative Stiefel-Whitney number TFT* Z^I_{SW}. Let (W, M, μ, N, ν) be a smooth bordism. As usual, the evaluation of $(Z^I_{SW})_W$ on boundary conditions $(f, g) \in \mathcal{F}(M) \times \mathcal{F}(N)$ is given by

$$(Z^I_{SW})_W(f,g) = \sum_{F \in \mathcal{F}(W;f,g)} T_W(F),$$

where T_W is defined in terms of $\mathbb{T}_W$, and the sum is well-defined since it takes place in a complete semiring. (For more details, see [5, Section 6, p. 28ff], and in particular [5, p. 30].)

Remark 2.4.1. Consider the partition $I = (i_1) = (m)$ of m. In this case fields on a bordism W are represented by generic smooth vector fields φ on W_ε, and $\overline{Z}_1(\varphi)$ is a finite number of points. Suppose that W is oriented and closed. Then the equality of the self-intersection number of the zero section of TW to the Euler characteristic $\chi(W)$ implies that the absolute value $|\chi(W)|$ is a lower bound for $\mathbb{T}_W$. According to [7, p. 382] it can be shown that intersection points of complementary sign can be cancelled. This point of view is also presented in [15, p. 5]. Hence, the lower bound $|\chi(W)|$ for $\mathbb{T}_W$ is indeed realized. The lower bound of $\mathbb{T}_W$ for a general partition remains to be investigated.

The previous Remark inspires the following

Definition 2.4.2. Let (W, M, μ, N, ν) be a smooth bordism. Given boundary conditions $(f, g) \in \mathcal{F}(M) \times \mathcal{F}_<(N)$ (compare Proposition 2.3.4), define

$$|\chi^I_W|(f,g) := \min_{F \in \mathcal{F}(W;f,g)} \mathbb{T}_W(F) \qquad \in \mathbb{N}.$$

Corollary 2.4.3. *The invariant $W \mapsto |\chi^I_W|$ is a diffeomorphism invariant on closed smooth manifolds W. Furthermore, Theorem 2.0.1 implies that, for closed W, $|\chi^I_W|$ reduces* $\bmod 2$ *to the Stiefel-Whitney number $w_I[W]$. Whenever W is closed, the previous Remark 2.4.1 states that the invariant $|\chi^{(n+1)}_W|$ coincides with the absolute value of the Euler characteristic of W.*

The complete calculation of Z^I_{SW} remains an open problem.

2.5 Proof of Theorem 2.0.1

Without loss of generality one may assume that W is connected.

Suppose that $F = (F_1, \dots, F_r) \in \mathcal{F}(W)$ (see Definition 2.3.9). By Remark 2.3.6, the assumption $\partial W = \emptyset$ implies that every F_s, $s \in \{1, \dots, r\}$, is a generic smooth vector bundle morphism $F_s = \Phi_s \colon \kappa_W^{m-i_s+1} \to TW$, and $\{\overline{Z}_1(\Phi_1), \dots, \overline{Z}_1(\Phi_r)\}$ is a transverse system in W.

Let $\zeta_s \colon Z_s := \tilde{Z}(\Phi_s) \subset W \times \mathbb{RP}^{m-i_s}$ denote the desingularization of $\overline{Z}_1(\Phi_s)$. Let $\pi_s \colon W \times \mathbb{RP}^{m-i_s} \to W$ be the projection. Hence, $\pi_s(Z_s) = \overline{Z}_1(\Phi_s)$.

For every $s \in \{1, \dots, r\}$ define the projection maps

$$\pi_{s'}^{(s)} \colon W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s} \to W \times \mathbb{RP}^{m-i_{s'}}, \qquad s' \in \{1, \dots, s\}.$$

Moreover, let $\pi^{(s)} := \pi_{s'} \circ \pi_{s'}^{(s)} \colon W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s} \to W$ (this is independent of the chosen $s' \in \{1, \dots, s\}$). For every $s' \in \{1, \dots, s\}$ we define the smoothly embedded submanifold

$$\zeta_{s'}^{(s)} \colon Z_{s'}^{(s)} := (\pi_{s'}^{(s)})^{-1}(Z_{s'}) \hookrightarrow W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s}.$$

Proposition 2.2.12 implies that $\{Z_1^{(s)}, \dots, Z_s^{(s)}\}$ is a transverse system im $W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s}$ for every $s \in \{1, \dots, r\}$. We prove by induction on $s \in \{1, \dots, r\}$ that the smoothly embedded submanifold

$$\zeta^{(s)} \colon Z^{(s)} := Z_1^{(s)} \cap \cdots \cap Z_s^{(s)} \hookrightarrow W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s}$$

satisfies

$$\pi_*^{(s)} \zeta_*^{(s)} [Z^{(s)}] = \pi_{1*}\zeta_{1*}[Z_1] \bullet \cdots \bullet \pi_{s*}\zeta_{s*}[Z_s] \qquad (*)_s.$$

Concerning the induction basis $s = 1$, the claim $(*)_1$ obviously holds since $\pi_1^{(1)} = \mathrm{id}_{W \times \mathbb{RP}^{m-i_1}}$ implies that

$$\pi_*^{(1)} \zeta_*^{(1)} [Z^{(1)}] = \pi_{1*}\pi_{1*}^{(1)} \zeta_{1*}^{(1)} [Z_1^{(1)}] = \pi_{1*}\zeta_{1*}[Z_1].$$

Next, supposing that $(*)_s$ holds for some $s \in \{1, \dots, r-1\}$, one has to conclude that $(*)_{s+1}$ is valid. Indeed, the right-hand side of $(*)_{s+1}$ is given as

$$\begin{aligned} &(\pi_{1*}\zeta_{1*}[Z_1] \bullet \cdots \bullet \pi_{s*}\zeta_{s*}[Z_s]) \bullet \pi_{s+1*}\zeta_{s+1*}[Z_{s+1}] \\ &\overset{(*)_s}{=} \pi_*^{(s)} \zeta_*^{(s)} [Z^{(s)}] \bullet \pi_{s+1*}\zeta_{s+1*}[Z_{s+1}] \\ &\overset{\textit{Corollary } 2.2.16}{=} \pi_*^{(s+1)} \zeta_*^{(s+1)} [Z^{(s+1)}]. \end{aligned}$$

(Let us explain how Corollary 2.2.16 is applied in the last equality. Consider

$$\begin{aligned} X &:= W, \\ X_1 &:= \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s}, \\ X_2 &:= \mathbb{RP}^{m-i_{s+1}}, \\ A_1 &:= Z^{(s)}, \\ A_2 &:= Z_{s+1}. \end{aligned}$$

Then, $\sigma_1 = \pi^{(s)}$, $\sigma_2 = \pi_{s+1}$, τ_1 is the projection

$$\pi^{(s+1)}_{(s)} \colon \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_{s+1}} \to \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_s},$$

$\tau_2 = \pi^{(s+1)}_{s+1}$, $\rho = \sigma_2 \circ \tau_2 = \pi^{(s+1)}$, and $\alpha_1 = \zeta^{(s)}$, $\alpha_2 = \zeta_{s+1}$. Furthermore,

$$\begin{aligned} B_1 &= (\tau_1)^{-1}(A_1) = (\pi^{(s+1)}_{(s)})^{-1}(Z^{(s)}) \\ &= (\pi^{(s+1)}_{(s)})^{-1}(Z^{(s)}_1 \cap \cdots \cap Z^{(s)}_s) \\ &= (\pi^{(s+1)}_{(s)})^{-1}(Z^{(s)}_1) \cap \cdots \cap (\pi^{(s+1)}_{(s)})^{-1}(Z^{(s)}_s) \\ &= (\pi^{(s+1)}_{(s)})^{-1}((\pi^{(s)}_1)^{-1}(Z_1)) \cap \cdots \cap (\pi^{(s+1)}_{(s)})^{-1}((\pi^{(s)}_s)^{-1}(Z_s)) \\ &= (\pi^{(s)}_1 \circ \pi^{(s+1)}_{(s)})^{-1}(Z_1) \cap \cdots \cap (\pi^{(s)}_s \circ \pi^{(s+1)}_{(s)})^{-1}(Z_s) \\ &= (\pi^{(s+1)}_1)^{-1}(Z_1) \cap \cdots \cap (\pi^{(s+1)}_s)^{-1}(Z_s) \\ &= Z^{(s+1)}_1 \cap \cdots \cap Z^{(s+1)}_s \end{aligned}$$

with inclusion $\beta_1 \colon B_1 \hookrightarrow W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_{s+1}}$. Moreover,

$$B_2 = (\tau_2)^{-1}(A_2) = (\pi^{(s+1)}_{s+1})^{-1}(Z_{s+1}) = Z^{(s+1)}_{s+1}$$

with inclusion $\beta_2 = \zeta^{(s+1)}_{s+1}$. Note that $B_1 \pitchfork B_2$ because $\{Z^{(s)}_1, \ldots, Z^{(s)}_r\}$ is a transverse system im $W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_r}$. One has $B := B_1 \cap B_2 = Z^{(s+1)}_1 \cap \cdots \cap Z^{(s+1)}_s \cap Z^{(s+1)}_{s+1} = Z^{(s+1)}$ with inclusion $\beta = \zeta^{(s+1)}$. All in all, Corollary 2.2.16 yields

$$\pi^{(s)}_* \zeta^{(s)}_* [Z^{(s)}] \bullet \pi_{s+1*}\zeta_{s+1*}[Z_{s+1}] = \sigma_{1*}\alpha_{1*}[A_1] \bullet \sigma_{2*}\alpha_{2*}[A_2] = \rho_*\beta_*[B] = \pi^{(s+1)}_* \zeta^{(s+1)}_* [Z^{(s+1)}].)$$

This completes the proof of $(*)_s$ for all $s \in \{1, \ldots, r\}$.

By [1, Theorem 11, p. 1227] and [1, Theorem 15, p. 1233] we have

$$w_{i_s} = D\pi_{s*}\zeta_{s*}[Z_s].$$

Therefore,

$$\begin{aligned} w_I[W] = w_{i_1} \ldots w_{i_r}[W] &= \varepsilon_*((D\pi_{1*}\zeta_{1*}[Z_1] \cup \cdots \cup D\pi_{r*}\zeta_{r*}[Z_r]) \cap [W]) \\ &= \varepsilon_*(D(\pi_{1*}\zeta_{1*}[Z_1] \bullet \cdots \bullet \pi_{r*}\zeta_{r*}[Z_r]) \cap [W]) \\ &= \varepsilon_*(\pi_{1*}\zeta_{1*}[Z_1] \bullet \cdots \bullet \pi_{r*}\zeta_{r*}[Z_r]) \\ &\overset{(*)_r}{=} \varepsilon_*(\pi^{(r)}_* \zeta^{(r)}_* [Z^{(r)}]) \\ &= \varepsilon_*[Z^{(r)}] = |Z^{(r)}| \bmod 2. \end{aligned}$$

Finally, it suffices to show that $Z^{(r)}$ and $\overline{Z}_1(\Phi_1) \cap \cdots \cap \overline{Z}_1(\Phi_r)$ have the same cardinality. For this purpose, one just has to note that the projection $\pi^{(r)} \colon W \times \mathbb{RP}^{m-i_1} \times \cdots \times \mathbb{RP}^{m-i_r} \to W$ restricts to a bijection $Z^{(r)} \to \overline{Z}_1(\Phi_1) \cap \cdots \cap \overline{Z}_1(\Phi_r)$. This is just the map α from Proposition 2.2.12, and it is a bijection by property (2) of the same Proposition because $\overline{Z}_1(\Phi_1) \cap \cdots \cap \overline{Z}_1(\Phi_r) = Z_1(\Phi_1) \cap \cdots \cap Z_1(\Phi_r)$ by Lemma 2.3.12.

This completes the proof of Theorem 2.0.1.

Chapter 3

Banagl's Fold Map TFT

3.1 Review of Definitions and Outline of New Results

Based on the general method of quantization Banagl [4] introduces a concrete positive TFT defined on smooth cobordisms of any given dimension $n \geq 2$. In the following sections the basic definitions from [4] are presented in condensed form since they are crucial as a reference for subsequent sections. Furthermore, we give an outline of the results of Chapter 3 in order to motivate the approaches pursued in Part II and Part III.

Let $n \geq 2$ be an integer.

3.1.1 Cobordisms

Closed $(n-1)$-dimensional smooth manifolds will be referred to as M, N, P etc. in the following. Fix an integer $D \geq 2n+1$. In Chapter 3 the notation $M \subset \mathbb{R}^D$ always means that M is smoothly embedded in $\mathbb{R}^D$, and that every connected component of M is contained in a hyperplane of the form $\{k\} \times \mathbb{R}^{D-1}$ where $k \in \{0, 1, 2, \dots\}$.

Definition 3.1.1. A *cobordism* from M to N is a compact smooth n-dimensional manifold with boundary $W \subset [0,1] \times \mathbb{R}^D$ (smoothly embedded) with the following properties:

(1) $\partial W = M \coprod N$, where $M \subset \mathbb{R}^D = \{0\} \times \mathbb{R}^D$ and $N \subset \mathbb{R}^D = \{1\} \times \mathbb{R}^D$,
(2) $W \setminus \partial W \subset (0,1) \times \mathbb{R}^D$,
(3) there exists $0 < \varepsilon < \frac{1}{2}$ such that $W \cap [0, \varepsilon] \times \mathbb{R}^D = [0, \varepsilon] \times M$ and $W \cap [1-\varepsilon, 1] \times \mathbb{R}^D = [1-\varepsilon, 1] \times N$ are product embeddings (any such ε will be called *cylinder scale*),
(4) every connected component of W is contained in a set of the form $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$ where $k \in \{0, 1, 2, \dots\}$.

The requirement that cobordisms are embedded in a high-dimensional Euclidean space can always be achieved due to a variant of Whitney's embedding theorem for smooth manifolds with boundary. However, the advantage of this assumption is that any cobordism $W \subset [0,1] \times \mathbb{R}^D$ is naturally equipped with a *time function* $\omega \colon W \to [0,1]$ given by projection to the first coordinate.

3.1.2 Fold Maps and System of Fields

Suppose that W is a cobordism from M to N. The definition of fields on W is based on the notion of fold map into the plane $\mathbb{R}^2 \cong \mathbb{C}$. The actual definition of a fold map between smooth manifolds without boundary is given in Section 3.3.2. By Section 3.3.4 fold maps can equivalently be defined by the local normal form of their singular points, namely

$$(t, x_1, ..., x_{n-1}) \mapsto \left(t, -(x_1^2 + ... + x_i^2) + x_{i+1}^2 + ... + x_{n-1}^2\right).$$

The singular locus of a fold map turns out to be a 1-dimensional submanifold.

What do we mean by a fold map $W \to \mathbb{R}^2$ (where the source manifold is allowed to have a boundary)?

Definition 3.1.2. A smooth map $F\colon W \to \mathbb{R}^2$ is called *fold map* if it extends to a fold map $\tilde{F}\colon ((-\varepsilon, 0] \times M) \cup W \cup ([1, 1+\varepsilon) \times N) \to \mathbb{R}^2$ for some $\varepsilon > 0$.

Note the technical subtlety that a condition on the singular locus $S(F)$ of F like $S(F) \pitchfork \partial W$ is to be read as $S(\tilde{F}) \pitchfork \partial W$ for some (and hence, any) extension $\tilde{F}$ of of F as in Definition 3.1.2.

Remark 3.1.3. The conditions that make a smooth map a fold map can be reformulated in terms of properties of its jet extensions (see Remark 4.3.12). These are pointwise conditions that are shown in Lemma 4.3.9 to be open (i.e., they extend to a neighbourhood of a point at which they hold). Hence, it would suffice to require in Definition 3.1.2 that these conditions hold on W for one (and hence any) smooth extension $\tilde{F}\colon ((-\varepsilon, 0] \times M) \cup W \cup ([1, 1+\varepsilon) \times N) \to \mathbb{R}^2$ of F.

For every regular value $t \in [0, 1]$ of the time function $\omega\colon W \to [0, 1]$ the preimage $\omega^{-1}(t)$ is a codimension 1 submanifold of W.

Definition 3.1.4. Given a fold map $F\colon W \to \mathbb{C}$, let

$$\pitchfork(F) = \left\{t \in [0,1];\ t \text{ is a regular value of } \omega \text{ and } S(F) \pitchfork \omega^{-1}(t)\right\} \quad \subset [0,1].$$

Definition 3.1.5. A fold map $F\colon W \to \mathbb{C}$ has *generic imaginary parts* over $t \in [0, 1]$ if the restriction $\operatorname{Im} \circ F| : S(F) \cap \omega^{-1}(t) \to \mathbb{R}$ is injective. Let

$$\operatorname{GenIm}(F) = \{t \in [0,1];\ F \text{ has generic imaginary parts over } t\} \quad \subset [0,1].$$

For given $k \in \{0, 1, 2, \dots\}$ let $F(k)$ denote the restriction of a fold map $F\colon W \to \mathbb{C}$ to the part of W that lies in $[0, 1] \times \{k\} \times \mathbb{R}^{D-1}$:

$$F(k) = F| : W \cap [0,1] \times \{k\} \times \mathbb{R}^{D-1} \to \mathbb{C}.$$

Fields on W are now fold maps $F\colon W \to \mathbb{C}$ with certain properties concerning the subsets $\pitchfork(F(k))$ and $\operatorname{GenIm}(F(k))$ of $[0, 1]$:

Definition 3.1.6. A *fold field* on W is a fold map $F\colon W \to \mathbb{C}$ so that for all $k \in \{0, 1, 2, ...\}$ the following conditions hold:

(1) $0, 1 \in \pitchfork(F(k)) \cap \operatorname{GenIm}(F(k))$.

(2) $\mathrm{GenIm}(F(k))$ is residual in $[0,1]$.

Condition (1) is needed for the definition of the action functional $\mathbb{S}\colon \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{Br})$. Condition (2) is crucial for the proof of the indispensable gluing theorem of [4, Section 7.7, p. 57ff].

Let $\mathcal{F}(W)$ denote the set of all fold fields on W. If $W = \emptyset$, then one puts $\mathcal{F}(W) = \{*\}$ (set with a single element). Fields on closed $(n-1)$-dimensional manifolds are introduced in Section 3.1.4, which completes the definition of the system $\mathcal{F}$ of fields.

3.1.3 Brauer Category and System of Action Functionals

Let **Vect** denote the category of real vector spaces with linear maps as morphisms. As described in [5, Section 8.1, p. 42ff], the introduction of the *Schauenburg tensor product* makes **Vect** a strict monoidal category with unity object $\mathbb{R}$. Based on the construction of [5, Section 8.2, p.43ff] the system $\mathbb{T}$ of action functionals is for every cobordism W defined as

$$\mathbb{T}_W : \mathcal{F}(W) \xrightarrow{\mathbb{S}_W} \mathrm{Mor}(\mathbf{Br}) \xrightarrow{Y} \mathrm{Mor}(\mathbf{Vect}).$$

Here, the *Brauer category* **Br** (see Definition 3.1.7) is a strict monoidal small category, $\mathbb{S}$ is a system of **Br**-valued action functionals, and $Y\colon \mathbf{Br} \to \mathbf{Vect}$ denotes a linear representation of the Brauer category (i.e. an appropriate strict monoidal functor). Roughly speaking, $\mathbb{S}$ takes the fold lines of a field on W and interprets these as a morphisms of the Brauer category.

Definition 3.1.7. The *Brauer category* **Br** consists of the following objects and morphisms:

- The objects are given by $[0], [1], [2], \ldots$, where $[0] = \emptyset$ denotes the empty set, and $[m]$ is the set $\{1, \ldots, m\} \subset \mathbb{R}$ for all integers $m > 0$. In the following, $[m]$ will be identified with the 0-dimensional submanifold $\{1, \ldots, m\} \times \{0\} \times \{0\} \subset \mathbb{R}^3$.
- A morphism $[m] \to [m']$ between two given objects in **Br** is represented by a compact smooth 1-dimensional manifold $W \subset [0,1] \times \mathbb{R}^3$ (smoothly embedded), such that properties (1), (2) and (3) from Definition 3.1.1 hold, where $M = [m]$, $N = [m']$ and $n = 1$. Two such manifolds W and W' represent the same morphism in **Br** if there exists a smooth isotopy on $[0,1] \times \mathbb{R}^3$ which maps W to W' and restricts to the identity map near $\{0,1\} \times \mathbb{R}^3$. A representativefor the composition of two morphisms $[m] \to [m']$ (represented by $W \subset [0,1] \times \mathbb{R}^3$) and $[m'] \to [m'']$ (represented by $W' \subset [0,1] \times \mathbb{R}^3$) can be constructed as follows. First one glues W along $\{1\} \times [m'] \subset \{1\} \times \mathbb{R}^3$ with the translation of W' to $[1,2] \times \mathbb{R}^3$, and then one rescales $[0,2] \times \mathbb{R}^3$ to $[0,1] \times \mathbb{R}^3$. The identity morphism $\mathrm{id}_{[0]}$ is represented by $W = \emptyset$. For an integer $m > 0$ $\mathrm{id}_{[m]}$ is represented by $W = [0,1] \times [m]$.

There exists a morphism between two objects $[m]$ and $[m']$ in **Br** if and only if $m + m'$ is even. The morphisms of the Brauer category run between a finite number of *ordered* points. Thus, they contain more combinatorial information than, for instance, the morphisms in the category of 1-cobordisms between 0-dimensional compact manifolds.

The Brauer category is a strict monoidal category when equipped with with unity object $[0]$ and the tensor product $\otimes\colon \mathbf{Br} \times \mathbf{Br} \to \mathbf{Br}$ which is introduced as follows. On the object level, let $[m] \otimes [m'] = [m + m']$. The tensor product $\varphi \otimes \psi : [m + p] \to [m' + p']$ of two morphisms $\varphi : [m] \to [m']$ (represented by $W \subset [0,1] \times \mathbb{R}^3$) and $\psi : [p] \to [p']$ (represented by $W' \subset [0,1] \times \mathbb{R}^3$) is represented by the "stacking" of W' above W.

An important morphism in **Br** is the *loop* $\lambda : [0] \to [0]$ which is represented by a smooth embedding of the circle $S^1 \hookrightarrow (0,1) \times \mathbb{R}^3$. Given two objects $[m]$ and $[m']$ in **Br**, let $\mathrm{OP}_{m,m'} \subset \mathrm{Hom}_{\mathbf{Br}}([m],[m'])$ denote the (finite!) subset of *open* morphisms $[m] \to [m']$, i.e. morphisms that are represented by some W without closed components. For every morphism $\varphi : [m] \to [m']$ in **Br** there exists a uniquely determined number $l \geq 0$ and a uniquely determined morphism $\varphi_0 : [m] \to [m']$ in $\mathrm{OP}_{m,m'}$ such that $\varphi = \lambda^{\otimes l} \otimes \varphi_0$. (Here, $\lambda^{\otimes l}$ denotes the l-fold tensor product $\lambda \otimes \cdots \otimes \lambda$.)

Let us proceed to describe the system $\mathbb{S}$ of **Br**-valued action functionals. Let W be a cobordism from M to N. If W is empty, set $\mathbb{S}(*) = \mathrm{id}_{[0]}$. Next suppose that $W \neq \emptyset$ is contained in a set of the form $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$ with $k \in \{0,1,2,\dots\}$. Let $F\ (= F(k)) \in \mathcal{F}(W)$ be a field on W. By property (1) of Definition 3.1.6 we have $0,1 \in\pitchfork (F)$, so that $S(F) \cap M$, and $S(F) \cap N \subset W$ is a 0-dimensional compact submanifold. Let m denote the cardinality of $S(F) \cap M$, and let m' denote the cardinality of $S(F) \cap N$. We aim at the definition of a Brauer morphism $\mathbb{S}(F)\colon [m] \to [m']$. By property (1) of Definition 3.1.6 F has generic imaginary parts over 0 and 1, i.e. each of the restrictions of $\mathrm{Im} \circ F : W \to \mathbb{R}$ to $S(F) \cap M$ and $S(F) \cap N$ is injective. Consequently, there exist numberings $S(F) \cap M = \{p_1, \dots, p_m\}$ and $S(F) \cap N = \{q_1, \dots, q_{m'}\}$ such that $(\mathrm{Im} \circ F)(p_i) < (\mathrm{Im} \circ F)(p_j) \Leftrightarrow i < j$ and $(\mathrm{Im} \circ F)(q_i) < (\mathrm{Im} \circ F)(q_j) \Leftrightarrow i < j$. A well-defined morphism $\varphi\colon [m] \to [m']$ in $\mathrm{OP}_{m,m'}$ is represented by connecting for every non-closed component c of $S(F)$ the two points in $\{0\} \times [m] \cup \{1\} \times [m']$ that correspond to the boundary points of c by a suitable arc in $[0,1] \times \mathbb{R}^3$. Now let $\mathbb{S}(F) = \lambda^{\otimes l} \otimes \varphi$, where l denotes the number of closed components of $S(F)$. Finally, for arbitrary $W \neq \emptyset$, let $\mathbb{S}(F) = \bigotimes_{k=0}^{\infty} \mathbb{S}(F(k))$. (The tensor product is finite since W is compact.) Note that the definition of $\mathbb{S}(F)$ makes actually only use of condition (1) from Definition 3.1.6. Condition (2) is specifically designed for the proof of the gluinng theorem.

The construction of a suitable linear representation $Y\colon \mathbf{Br} \to \mathbf{Vect}$ is not pursued here. Nevertheless, it is convenient to record a few observations. Being a monoidal functor, Y preserves the unity objects, $Y([0]) = \mathbb{R}$. Application to the loop λ results in a scalar $\widehat{\lambda} = Y(\lambda) \in \mathrm{Hom}_{\mathbf{Vect}}(Y([0]), Y([0])) = \mathbb{R}$. If $V = Y([1])$, then $Y([m]) = Y([1]^{\otimes m}) = V^{\otimes m}$ for all integers $m > 0$. Given $[m], [m'] \in \mathrm{Ob}(\mathbf{Br})$, let

$$H_{m,m'} = Y(\mathrm{Hom}_{\mathbf{Br}}([m], [m'])) \quad \subset \mathrm{Hom}_{\mathbf{Vect}}(V^{\otimes m}, V^{\otimes m'}).$$

How do the elements of $H_{m,m'}$ look like? Every $\varphi \in \mathrm{Hom}_{\mathbf{Br}}([m],[m'])$ has a unique representation of the form $\varphi = \lambda^{\otimes l} \otimes \varphi_0$ with $l \in \mathbb{N}$ and $\varphi_0 \in \mathrm{OP}_{m,m'}$. Thus, Damit ist $Y(\varphi) = Y(\lambda^{\otimes l} \otimes \varphi_0) = \widehat{\lambda}^l Y(\varphi_0)$. In consequence, the following (well-defined) map is surjective:

$$\mathbb{N} \times Y(\mathrm{OP}_{m,m'}) \to H_{m,m'}, \quad (l, y) \mapsto \widehat{\lambda}^l y.$$

Y can be constructed in such a way that it is *faithful on loops*: Any two morphisms φ and ψ in **Br** that satisfy $Y(\varphi) = Y(\psi)$ have the same number of loops. If Y is faithful on loops, then the above map turns out to be a bijection. (In fact, $\widehat{\lambda}^l Y(\varphi_0) = \widehat{\lambda}^{l'} Y(\varphi_0')$ implies $Y(\lambda^{\otimes l} \otimes \varphi_0) = Y(\lambda^{\otimes l'} \otimes \varphi_0')$, so $l = l'$. Therefore, $\widehat{\lambda}^l (Y(\varphi_0) - Y(\varphi_0')) = 0$. Under the assumption $\widehat{\lambda} = 0$ one would obtain $Y(\lambda^{\otimes 1} \otimes \varphi_0) = 0 = Y(\lambda^{\otimes 2} \otimes \varphi_0)$ in contradiction to Y's faithfulness on loops. Consequently, $\widehat{\lambda} \neq 0$, and $Y(\varphi_0) = Y(\varphi_0')$.)

3.1.4 Boundary Conditions

To every closed $(n-1)$-dimensional smooth manifold $M \subset \mathbb{R}^D$ one assigns the set $\mathcal{F}(M)$ of fields on M, which have the role of boundary conditions, as follows. If M is non-empty, then set

$$\mathcal{F}(M) = \{f \in \mathcal{F}([0,1] \times M);\ \mathbb{S}(f) = \mathrm{id} \in \mathrm{Mor}\,\mathbf{Br}\}.$$

In the case that $M = \emptyset$ let $\mathcal{F}(M) = \{*\}$ (set with a single element).

3.1.5 State Module and State Sum

Recall that the *Boolean monoid* $(\mathbb{B}, +, 0)$ is the set $\mathbb{B} = \{0, 1\}$ equipped with addition given by $1 + 1 = 1$ (0 serves as identity element). One can consider $\mathbb{B}$ as $\mathbb{N}$-semimodule in a natural way. One necessarily has $0 \cdot b = 0$ and $1 \cdot b = b$ for all $b \in \mathbb{B}$. Since $b + b = b$ for all $b \in \mathbb{B}$, it follows that $m \cdot b = b$ for all integers $m > 0$.) Let $\mathbb{N}[\tau]$ denote the polynomial semiring in one variable over the semiring $\mathbb{N}$. The monoid $\mathbb{B}[[q]]$ of formal power series in one variable over $\mathbb{B}$ is a $\mathbb{N}[\tau]$-semimodule, where τ acts via formal multiplication with q.

Let $\mathrm{FM}(H_{m,m'})$ denote the free commutative monoid generated by $H_{m,m'}$. If Y is faithful on loops, then $H_{m,m'}$ can be identified with $\mathbb{N} \times Y(\mathrm{OP}_{m,m'})$, and one obtains the following isomorphism of monoids:

$$\begin{aligned} \mathrm{FM}(H_{m,m'}) = \bigoplus_{(l,y) \in H_{m,m'}} \mathbb{N} \quad &\xrightarrow{\cong} \quad \bigoplus_{y \in Y(\mathrm{OP}_{m,m'})} \mathbb{N}[\tau], \\ (m_{(l,y)})_{(l,y) \in H_{m,m'}} \quad &\mapsto \quad \Big(m_{(0,y)} \tau^0 + m_{(1,y)} \tau^1 + \ldots \Big)_{y \in Y(\mathrm{OP}_{m,m'})}. \end{aligned}$$

The $\mathbb{N}[\tau]$-semimodule structure of the monoid $\bigoplus_{y \in Y(\mathrm{OP}_{m,m'})} \mathbb{N}[\tau]$ on the right induces by the above isomorphism the structure of a $\mathbb{N}[\tau]$-semimodule on the monoid $\mathrm{FM}(H_{m,m'})$. Multiplication with τ on the right-hand side corresponds in $\mathrm{FM}(H_{m,m'})$ to multiplication with $\widehat{\lambda}$ on the generators of $H_{m,m'}$. One defines

$$Q(H_{m,m'}) = \mathrm{FM}(H_{m,m'}) \otimes_{\mathbb{N}[\tau]} \mathbb{B}[[q]] \cong \bigoplus_{y \in Y(\mathrm{OP}_{m,m'})} \mathbb{B}[[q]].$$

The image $Y(\varphi) \in H_{m,m'}$ of the morphism $\varphi = \lambda^{\otimes l} \otimes \varphi_0\colon [m] \to [m']$ in $\mathbf{Br}$ corresponds in $Q(H_{m,m'})$ to the element with entry $q^l \in \mathbb{B}[[q]]$ at $Y(\varphi_0)$, and zero elsewhere. Therefore, q is also called *loop parameter*.

Note that $Q(H_{m,m'})$ is a complete idempotent $\mathbb{N}[\tau]$-semimodule since the same is true for $\mathbb{B}[[q]]$. One now defines the complete idempotent $\mathbb{N}[\tau]$-semimodule

$$Q = \prod_{m,m' \in \mathbb{N}} Q(H_{m,m'}).$$

Composition and tensor product of linear maps in suitable $H_{m,m'}$ can be used to equip Q with two products $\cdot$ and $\times$. With respect to each of these products Q becomes a complete idempotent semiring.

Let $M, N \subset \mathbb{R}^D$ be closed $(n-1)$-dimensional smooth manifolds.

Definition 3.1.8. A *state* is a map $\mathcal{F}(M) \to Q$. The *state module* $Z(M)$ of M is defined as the set of all states, $Z(M) = \{\mathcal{F}(M) \to Q\}$.

Let X be a closed smooth manifold, and let $a < b$ and $a' < b'$ be real numbers. For two smooth functions $f\colon [a,b] \times X \to \mathbb{C}$ and $g\colon [a',b'] \times X \to \mathbb{C}$ the notation $f \approx g$ means that there exists a diffeomorphism $\xi\colon [a,b] \to [a',b']$ with $\xi(a) = a'$, and such that $f(t,x) = g(\xi(t),x)$ for all $(t,x) \in [a,b] \times X$.

Let now W be a cobordism from M to N with cylinder scale ε.

Definition 3.1.9. Given a boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, let

$$\begin{aligned}\mathcal{F}(W; f_M, f_N) = \{F \in \mathcal{F}(W);\ \forall k \in \mathbb{N}\ \exists \varepsilon(k), \varepsilon'(k) \in (0,\varepsilon) :\\ F|_{[0,\varepsilon(k)] \times M(k)} \approx f_M(k), F|_{[1-\varepsilon'(k),1] \times N(k)} \approx f_N(k)\}.\end{aligned}$$

Definition 3.1.10. The *state sum* $Z_W\colon \mathcal{F}(M) \times \mathcal{F}(N) \to Q$ is given by

$$Z_W(f_M, f_N) := \sum_{F \in \mathcal{F}(W; f_M, f_N)} Y\mathbb{S}(F) \otimes 1 \quad \in Q.$$

It is shown in [4] that Z defines a positive TFT. Of course, the required axioms are to be adjusted to the category of smooth manifolds. In particular, the gluing axiom holds, as well as a time consistent version of diffeomorphism invariance.

3.1.6 State Sets

What is the informational content of $Z_W(f_M, f_N)$? If m and m' are chosen such that $\mathbb{S}(f_M) = \mathrm{id}_{[m]}$ and $\mathbb{S}(f_N) = \mathrm{id}_{[m']}$, then $Z_W(f_M, f_N) \in Q(H_{m,m'})$. For every $y \in Y(\mathrm{OP})$, $Z_W(f_M, f_N)$ is given by a power series $\sum_{l=0}^{\infty} c_l q^l$, for which the coefficient $c_l \in \mathbb{B}$ takes the value 1 precisely if there exists a fold field $F\colon W \to \mathbb{C}$ extending the boundary condition (f_M, f_N) in such a manner that $\mathbb{T}_W(F) = \widehat{\lambda}^l y$. We will show in Theorem 3.2.6 that Y can (and will) always be chosen in such a way that Y restricts to a bijection $\mathrm{OP} \to Y(\mathrm{OP})$. Thus, the statement $\mathbb{T}_W(F) = \widehat{\lambda}^l y$ is equivalent to $\mathbb{S}(F) = \lambda^{\otimes l} \otimes \varphi$, where $\varphi \in \mathrm{OP}$ is uniquely determined by $Y(\varphi) = y$. (In fact, writing $\mathbb{S}(F) = \lambda^{\otimes l'} \otimes \varphi'$ for suitable $l' \in \mathbb{N}$ and $\varphi' \in \mathrm{OP}$, one concludes from

$$Y(\lambda^{\otimes l'} \otimes \varphi') = \mathbb{T}_W(F) = \widehat{\lambda}^l y = Y(\lambda^{\otimes l} \otimes \varphi)$$

that $l = l'$ and $Y(\varphi') = Y(\varphi)$ by faithfulness of Y on loops. But then, $\varphi = \varphi'$ because Y restricts to a bijection $\mathrm{OP} \to Y(\mathrm{OP})$.)

Definition 3.1.11. The family of state sets for a boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ is given by

$$L_W(f_M, f_N; \varphi) := \{l \in \mathbb{N};\ \exists F \in \mathcal{F}(W; f_M, f_N) : \mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l}\}, \qquad \varphi \in \mathrm{OP}_{m_S, n_S}.$$

In consequence of our argument above, the informational content of $Z_W(f_M, f_N)$ is precisely encoded in the family $\{L_W(f_M, f_N; \varphi)\}_\varphi$ of state sets. Thus, we have reduced the problem of calculation of state sums to state sets.

Without loss of generality we may assume that W is a *simple* cobordism. (By Definition 3.4.1 this means that $W = W(k)$ for some $k \in \mathbb{N}$.) Indeed, the state sum Z_W for an arbitrary cobordism W can then be calculated from the state sums $Z_{W(k)}$ of the simple cobordisms $W(k)$, $k \in \mathbb{N}$, as shown in [4, Theorem 7.22, page 55].

The computation of state sets becomes accessible through Lemma 3.4.11 which implies that

$$L_W(f_M, f_N; \varphi) = \{l \in \mathbb{N};\ \exists F \in \mathcal{F}^{\text{pre}}(W; f_M, f_N) : \mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l}\}$$

for all $\varphi \in \mathrm{OP}_{m_S, n_S}$, where for the simple cobordism W we have by Definition 3.4.10

$$\mathcal{F}^{\text{pre}}(W; f_M, f_N) := \{F \colon W \to \mathbb{C} \text{ fold map; } \exists \varepsilon, \varepsilon' \in (0, \varepsilon_W) \colon F|_{[0,\varepsilon]\times M} \approx f_M, F|_{[1-\varepsilon',1]\times N} \approx f_N\}.$$

For the study of state sets in the future one can always assume that W is connected. (In fact, if W is simple but not connected, say $W = W_1 \sqcup \cdots \sqcup W_c$ with W_i connected for all i and equipped with cylinder scale ε_W, then

$$L_W(f_M, f_N; \varphi) = \{l_1 + \cdots + l_c;\ l_i \in L_{W_i}(f_{M_i}, f_{N_i}; \varphi_i)\}.$$

Here we have defined $M_i := M \cap \partial W_i$, $f_{M_i} = f_M|_{[0,1]\times M_i}$ and $N_i := N \cap \partial W_i$, $f_{N_i} = f_N|_{[0,1]\times N_i}$, and the open Brauer morphism $\varphi_i \colon [m_i] \to [n_i]$ denotes the obvious restriction of φ, where m_i and n_i denote the number of components of $S(f_{M_i})$ and $S(f_{N_i})$, respectively. If φ_i does not exist for some i, then we set $L_{W_i}(f_{M_i}, f_{N_i}; \varphi_i) = \emptyset$.)

The determination of these sets requires more input from the singularity theory of fold maps, and motivates the material presented in Part II, where the state sum for any two-dimensional cobordism is entirely computed in Chapter 5, and some general consequences of the results of Chapter 6 for the state sum of higher-dimensional cobordisms are discussed in Section 6.3.

3.1.7 The Aggregate Invariant

An important application of the theory presented so far is the one on the exotic spheres of dimension $n \geq 5$. Such a sphere is a smooth manifold Σ^n that is homeomorphic but not diffeomorphic to the standard sphere S^n.

Suppose M is a closed smooth n-dimensional manifold being homeomorphic to S^n. For a map $f \colon M \to \mathbb{R}$ let $\overline{f} : [0,1] \times M \to \mathbb{C}$ denote the suspension $(x,t) \mapsto (f(x), t)$. The suspension of Morse functions M with precisely two critical points are of great interest:

$$C_2(M) := \{\overline{f}_M : [0,1] \times M \to \mathbb{R};\ f_M : M \to \mathbb{R} \text{ is a Morse function with precisely two critical points}\} \quad \subset \mathcal{F}(M).$$

Let $\mathrm{Cob}(S^n, M)$ denote the set of all cobordisms $W \subset \mathbb{R}^D$ from S^n to M.

Definition 3.1.12. The *aggregate invariant* from [5, Definition 6.11, p. 38] is in our situation

$$\mathfrak{A}(M) := \sum_{\overline{f}_M \in C_2(M)} \sum_{W \in \mathrm{Cob}(S^n, M)} Z_W(\overline{f}_S, \overline{f}_M) \quad \in Q(H_{2,2}) \cong \bigoplus_{y \in Y(\mathrm{OP}_{2,2})} \mathbb{B}\,[[q]]\,.$$

Corollary 10.4 in [4, p. 85] implies that the invariant $\mathfrak{A}$ detects exotic differentiable structures on n-spheres:

Theorem 3.1.13. *Suppose that Σ^n is an exotic n-sphere. Then, $\mathfrak{A}(\Sigma^n) \neq \mathfrak{A}(S^n)$ holds in the semiring Q.*

An exotic n-sphere can hence be distinguished from the standard n-sphere. In proving this statement results by Saeki [47] are used. On the other hand, it is worth noticing that the gluing theorem for the state sum provides in principle a certain predictability of this invariant.

The further study of the aggregate invariant motivates the material of Part III, which is also of independent interest. The results obtained there have in turn consequences for the computation of the aggregate invariant. In fact, we will return to the aggregate invariant in Section 10.5, where it is shown in Proposition 10.1.5 that the informational content of the aggregate invariant can be encoded in a natural way in a map of the form $\Theta_n \to \mathbb{N}$. Furthermore, by Corollary 10.5.1, this map takes the value 1 on all non-trivial elements of the subgroup $bP_{n+1} \subset \Theta_n$. In particular, it follows from $bP_8 = \Theta_7$ that the aggregate invariant cannot distinguish between individual exotic 7-spheres.

3.2 The Brauer Category and its Linear Representations

We assume familiarity with the basic properties of the Brauer category and its linear representations as reviewed in Section 3.1.3. For more details, in particular on its representations, we refer to [4, Section 2, p. 7 ff].

Let V be a finite dimensional real vector space of dimension d and let (i, e) be a duality structure on V (see [4, Definition 2.5, p. 9]). By [4, Theorem 2.18], there exists a uniquely determined symmetric strict monoidal functor

$$Y \colon \mathbf{Br} \to \mathbf{Vect},$$

such that $Y([1]) = V$ and duality is preserved, that is, $Y(i_1) = i$ and $Y(e_1) = e$. Since Y is required to be symmetric, we also have $Y(b_{1,1}) = b$, where $b \colon V \otimes V \to V \otimes V$ is the braiding automorphism induced by $v \otimes w \mapsto w \otimes v$. (Note that $\otimes$ always denotes the Schauenburg tensor product, which turns **Vect** into a strict monoidal category.)

Recall that Y is called *faithful on loops*, if any two Brauer morphisms $\phi, \psi \in \mathrm{Hom}_{\mathbf{Br}}([m], [n])$ with $Y(\phi) = Y(\psi)$ contain the same number of loops.

Remark 3.2.1. Assume $d \geq 2$. If $\phi \colon [m] \to [n]$ and $\psi \colon [m'] \to [n']$ are morphisms in **Br** such that $Y(\phi) = Y(\psi)$, then we have $m = m'$ and $n = n'$. (In fact, these equalities follow from $V^{\otimes m} = V^{\otimes m'}$ and $V^{\otimes n} = V^{\otimes n'}$ for $\dim V = d \geq 2$.) Thus, Y is faithful on loops if and only if any two morphisms ϕ and ψ in **Br** with $Y(\phi) = Y(\psi)$ contain the same number of loops. (This is the original definition given in [4, Proposition 2.21].)

Y is called a *faithful functor*, if any two morphisms $\phi, \psi \in \mathrm{Hom}_{\mathbf{Br}}([m], [n])$ with $Y(\phi) = Y(\psi)$ are equal, $\phi = \psi$. The main result of this chapter states that Y is a faithful functor for $d \geq 2$ (and for any duality structure (i, e)). This is the content of Theorem 3.2.6. In particular, Y is injective on the set $\mathrm{OP}_{m,n}$ of loop-free (or "open") Brauer morphisms $[m] \to [n]$ for $d \geq 2$. Hence, throughout the present section, we will assume that $d \geq 2$.

We fix a basis $v_1, \ldots, v_d$ of V. For every positive integer r, we equip $V^{\otimes r}$ with the lexicographically ordered basis $\{v_{i_1} \otimes \cdots \otimes v_{i_r};\ i_1, \ldots, i_r \in \{1, \ldots, d\}\}$. With respect to these bases, the matrix representation of a tensor product of linear maps is the *Kronecker product* of the matrix representations of these maps. More precisely, if a, a', b, b' are positive integers and $A \colon V^{\otimes a} \to V^{\otimes a'}$ and $B \colon V^{\otimes b} \to V^{\otimes b'}$ are linear maps, then the matrix representation of $A \otimes B \colon V^{\otimes (a+b)} \to V^{\otimes (a'+b')}$ is given by the Kronecker product of the matrix representations of A and B.

Let us first study the behaviour of Y on isomorphisms in $\mathbf{Br}$. Note that if $\iota\colon [m] \to [n]$ is an isomorphism in $\mathbf{Br}$, then $m = n$. Moreover, for every $m \in \mathbb{N}$ there is an obvious correspondence between isomorphisms $[m] \to [m]$ and permutations of the set $\{1, \ldots, m\}$. If $\iota\colon [m] \to [m]$ is an isomorphism in $\mathbf{Br}$, then the corresponding permutation of $\{1, \ldots, m\}$ will also be denoted by ι.

Lemma 3.2.2. *If $\iota\colon [m] \to [m]$ is an isomorphism in $\mathbf{Br}$, then*

$$Y(\iota)(w_1 \otimes \cdots \otimes w_m) = w_{\iota^{-1}(1)} \otimes \cdots \otimes w_{\iota^{-1}(m)} \quad \text{for all} \quad w_1, \ldots, w_m \in V.$$

In particular, $Y(\iota)$ permutes the basis elements of $V^{\otimes m}$. Thus, the matrix representation of the linear isomorphism $Y(\iota)\colon V^{\otimes m} \to V^{\otimes m}$ is a permutation matrix.

Proof. The permutation of $\{1, \ldots, m\}$ which is associated with ι can be written as the composition of *adjacent* transpositions. Hence, the isomorphism ι can be written as the composition of isomorphisms $[m] \to [m]$ of the form

$$\delta_u := 1_{[1]} \otimes \cdots \otimes 1_{[1]} \otimes b_{1,1} \otimes 1_{[1]} \otimes \cdots \otimes 1_{[1]}, \qquad u \in \{1, \ldots, m-1\},$$

where $b_{1,1}\colon [2] \to [2]$ is the u-th factor in δ_u. Note that δ_u corresponds to the permutation of $\{1, \ldots, m\}$ given by $\delta_u(u) = u{+}1$, $\delta_u(u{+}1) = u$ and $\delta_u(j) = j$ for all $j \in \{1, \ldots, m\} \backslash \{u, u{+}1\}$.

It suffices to show the following statements:

(i) The claim holds for the isomorphisms $\delta_u\colon [m] \to [m]$, $u \in \{1, \ldots, m-1\}$.

(ii) If the claim holds for two isomorphisms $\alpha, \beta\colon [m] \to [m]$, then it also holds for their composition $\beta \circ \alpha\colon [m] \to [m]$.

(i). Let $w_1, \ldots, w_m \in V$. It follows from $Y(b_{1,1}) = b$ that

$$\begin{aligned} Y(\delta_u)(w_1 \otimes \cdots \otimes w_m) &= (1_V \otimes \cdots \otimes 1_V \otimes b \otimes 1_V \otimes \cdots \otimes 1_V)(w_1 \otimes \cdots \otimes w_m) \\ &= w_1 \otimes \cdots \otimes w_{u-1} \otimes b(w_u \otimes w_{u+1}) \otimes w_{u+2} \otimes \cdots \otimes w_m \\ &= w_1 \otimes \cdots \otimes w_{u-1} \otimes w_{u+1} \otimes w_u \otimes w_{u+2} \otimes \cdots \otimes w_m \\ &= w_{\delta_u^{-1}(1)} \otimes \cdots \otimes w_{\delta_u^{-1}(m)}. \end{aligned}$$

(ii). Let $w_1, \ldots, w_m \in V$. Setting $w'_i := w_{\alpha^{-1}(i)}$ for all $i \in \{1, \ldots, m\}$, we obtain

$$\begin{aligned} Y(\beta \circ \alpha)(w_1 \otimes \cdots \otimes w_m) &= Y(\beta)(Y(\alpha)(w_1 \otimes \cdots \otimes w_m)) \\ &= Y(\beta)(w_{\alpha^{-1}(1)} \otimes \cdots \otimes w_{\alpha^{-1}(m)}) \\ &= Y(\beta)(w'_1 \otimes \cdots \otimes w'_m) = w'_{\beta^{-1}(1)} \otimes \cdots \otimes w'_{\beta^{-1}(m)} \\ &= w_{\alpha^{-1}(\beta^{-1}(1))} \otimes \cdots \otimes w_{\alpha^{-1}(\beta^{-1}(m))} \\ &= w_{(\beta\circ\alpha)^{-1}(1)} \otimes \cdots \otimes w_{(\beta\circ\alpha)^{-1}(m)}. \end{aligned}$$

□

An immediate consequence of Lemma 3.2.2 is that Y is faithful on isomorphisms in **Br** for $d \geq 2$:

Corollary 3.2.3. *Assume $d \geq 2$. If $\iota_1, \iota_2 \colon [m] \to [m]$ are two isomorphisms in* **Br** *such that $Y(\iota_1) = Y(\iota_2)$, then $\iota_1 = \iota_2$.*

Proof. Let us assume that $\iota_1 \neq \iota_2$. Then, there exists $t \in \{1, \dots, m\}$ such that $\iota_1(t) \neq \iota_2(t)$. Set $v^i := v_1 \otimes \cdots \otimes v_1 \otimes v_2 \otimes v_1 \otimes \cdots \otimes v_1 \in V^{\otimes m}$, where v_2 is the i-th factor in v^i. (The existence of v_2 is ensured by the assumption $d \geq 2$.) Note that if $\iota \colon [m] \to [m]$ is an isomorphism, then we have $Y(\iota)(v^i) = v^{\iota(i)}$ by Lemma 3.2.2. (In fact, if $w_i := v_2$ and $w_j := v_1$ for $j \neq i$, then $w_{\iota^{-1}(k)} = v_2$ for $k = \iota(i)$ and $w_{\iota^{-1}(k)} = v_1$ for $k \neq \iota(i)$. Hence, $Y(\iota)(v^i) = Y(\iota)(w_1 \otimes \cdots \otimes w_m) = w_{\iota^{-1}(1)} \otimes \cdots \otimes w_{\iota^{-1}(m)} = v^{\iota(i)}$ by Lemma 3.2.2.) Therefore, we obtain $v^{\iota_1(t)} = Y(\iota_1)(v^t) = Y(\iota_2)(v^t) = v^{\iota_2(t)}$ in $V^{\otimes m}$. This is a contradiction to the linear independence of $v^{\iota_1(t)}$ and $v^{\iota_2(t)}$ for $\iota_1(t) \neq \iota_2(t)$. Hence, $\iota_1 = \iota_2$. □

The idea of the proof of the following result is taken from [42, Proposition 5.8, p. 70].

Proposition 3.2.4. *Assume $d \geq 2$. If Y is faithful on loops, then Y is a faithful functor.*

Proof. Let $\phi, \psi \in \mathrm{Hom}_{\mathbf{Br}}([m],[n])$ be two morphisms in **Br** such that $Y(\phi) = Y(\psi)$. We have to show that $\phi = \psi$. The main idea is to compare the morphisms ϕ and ψ by pre- and post-composing them with suitable Brauer morphisms. Exploiting the facts that Y is faithful on loops by assumption and faithful on isomorphisms by Corollary 3.2.3, we will be able to compare these compositions after the application of the strict monoidal functor Y.

As Y is faithful on loops, we may assume that ϕ and ψ are loop-free. (In fact, it follows from $Y(\phi) = Y(\psi)$ that ϕ and ψ have the same number of loops. As $Y(\lambda) = d$, we obtain $Y(\phi_0) = Y(\psi_0)$, where ϕ_0 and ψ_0 are the loop-free parts of ϕ and ψ. Hence, by what we will show, $\phi_0 = \psi_0$. Since ϕ and ψ have the same number of loops, we also have $\phi = \psi$.)

We represent ϕ and ψ by compact smoothly embedded 1-manifolds $W(\phi), W(\psi) \subset [0,1] \times \mathbb{R}^3$. Note that all components of $W(\phi)$ and $W(\psi)$ are intervals with endpoints in $\{0,1\} \times \mathbb{R}^3$, because ϕ and ψ are loop-free by assumption. It suffices to show that for every component C of $W(\phi)$ there exists a component C' of $W(\psi)$ such that C and C' have the same endpoints in $\{0,1\} \times \mathbb{R}^3$, that is, $C \cap (\{0,1\} \times \mathbb{R}^3) = C' \cap (\{0,1\} \times \mathbb{R}^3)$. Let C be a component of $W(\phi)$. We distinguish the following cases, where case (III) occupies the rest of the proof:

Case (I): Both endpoints of C are in $0 \times M[m] \times 0 \times 0$ ($\subset 0 \times \mathbb{R}^3$), say $C \cap (0 \times \mathbb{R}^3) = \{(0,a,0,0), (0,b,0,0)\}$. Let C' denote the unique component of $W(\psi)$ such that $(0,a,0,0) \in C'$. We construct a Brauer morphism $\xi \in \mathrm{Hom}_{\mathbf{Br}}([m-2],[m])$ by connecting the points of $0 \times M[m-2] \times 0 \times 0$ and $1 \times M[m] \times 0 \times 0$ by smooth arcs in $[0,1] \times \mathbb{R}^3$ in the following manner. Connect $(1,a,0,0)$ and $(1,b,0,0)$ by an arc. Fix any bijection $\omega \colon \{1, \dots, m-2\} \to \{1, \dots, m\} \setminus \{a,b\}$. For all $i \in \{1, \dots, m-2\}$ we connect $(0,i,0,0)$ and $(1,\omega(i),0,0)$ by an arc. Assume that $(0,b,0,0) \notin C'$. Then, by construction, $\phi \circ \xi$ contains one loop, whereas $\psi \circ \xi$ is obviously loop-free. However, we obtain $Y(\phi \circ \xi) = Y(\phi) \circ Y(\xi) = Y(\psi) \circ Y(\xi) = Y(\psi \circ \xi)$, which is a contradiction to the assumption that Y is faithful on loops. Hence, $(0,b,0,0) \in C'$ and $C \cap (\{0,1\} \times \mathbb{R}^3) = C' \cap (\{0,1\} \times \mathbb{R}^3)$.

Case (II): Both endpoints of C are in $1 \times M[n] \times 0 \times 0$ ($\subset 1 \times \mathbb{R}^3$). Analogous to case (I), one can show that there exists a component C' of $W(\psi)$ such that $C \cap (\{0,1\} \times \mathbb{R}^3) = C' \cap (\{0,1\} \times \mathbb{R}^3)$.

Case (III): One endpoint of C is in $0 \times M[m] \times 0 \times 0$ and the other is in $1 \times M[n] \times 0 \times 0$, say $C \cap (0 \times \mathbb{R}^3) = \{(0,a,0,0)\}$ and $C \cap (1 \times \mathbb{R}^3) = \{(1,b,0,0)\}$.

Let p be the number of components of $W(\phi)$ whose endpoints are both in $0 \times \mathbb{R}^3$. We construct a Brauer morphism $\sigma \in \mathrm{Hom}_{\mathbf{Br}}([m-2p],[m])$ by connecting the points of $0 \times M[m-2p] \times 0 \times 0$ and $1 \times M[m] \times 0 \times 0$ by smooth arcs in $[0,1] \times \mathbb{R}^3$ in the following manner. For every component C_0 of $W(\phi)$ whose endpoints are both in $0 \times \mathbb{R}^3$, say $C_0 \cap (0 \times \mathbb{R}^3) = \{(0,c,0,0),(0,d,0,0)\}$, we connect $(1,c,0,0)$ and $(1,d,0,0)$ by an arc. Let $X_\sigma \subset \{1,\dots,m\} = M[m]$ denote the set of the remaining $m-2p$ points which have not been realized in $1 \times M[m] \times 0 \times 0$ as the endpoint of an arc. Fix any bijection $\omega_\sigma \colon \{1,\dots,m-2p\} = M[m-2p] \to X_\sigma$. For all $i \in M[m-2p]$, we connect $(0,i,0,0)$ and $(1,\omega_\sigma(i),0,0)$ by an arc.

Analogously, let q be the number of components of $W(\phi)$ whose endpoints are both in $1 \times \mathbb{R}^3$. We construct a Brauer morphism $\tau \in \mathrm{Hom}_{\mathbf{Br}}([n],[n-2q])$ by connecting the points of $0 \times M[n] \times 0 \times 0$ and $1 \times M[n-2q] \times 0 \times 0$ by smooth arcs in $[0,1] \times \mathbb{R}^3$ in the following manner. For every component C_0 of $W(\phi)$ whose endpoints are both in $1 \times \mathbb{R}^3$, say $C_0 \cap (1 \times \mathbb{R}^3) = \{(1,c,0,0),(1,d,0,0)\}$, we connect $(0,c,0,0)$ and $(0,d,0,0)$ by an arc. Let $X_\tau \subset \{1,\dots,n\} = M[n]$ denote the set of the remaining $n-2q$ points which have not been realized in $0 \times M[n] \times 0 \times 0$ as the endpoint of an arc. Fix any bijection $\omega_\tau \colon \{1,\dots,n-2q\} = M[n-2q] \to X_\tau$. For all $j \in M[n-2q]$, we connect $(0,\omega_\tau(j),0,0)$ and $(1,j,0,0)$ by an arc.

By construction, the Brauer morphism $\tau \circ \phi \circ \sigma \colon [m-2p] \to [n-2q]$ can be written as the tensor product of a Brauer isomorphism $\iota_\phi \colon [m-2p] \to [n-2q]$ and $(p+q)$ loops:

$$\tau \circ \phi \circ \sigma = \lambda^{p+q} \otimes \iota_\phi. \tag{3.1}$$

Choose $i \in M[m-2p]$ and $j \in M[n-2q]$ such that $\omega_\sigma(i) = a$ and $\omega_\tau(j) = b$. We represent ι_ϕ by a compact smoothly embedded 1-manifold $W(\iota_\phi) \subset [0,1] \times \mathbb{R}^3$. By construction, there exists a component of $W(\iota_\phi)$ which connects the points $(0,i,0,0)$ and $(1,j,0,0)$.

Given $c,d \in M[m]$, there exists a component C_0 of $W(\phi)$ whose endpoints are $(0,c,0,0)$ and $(0,d,0,0)$ if and only if there exists a component C_0' of $W(\psi)$ which has these endpoints. (Indeed, this follows from case (I) and the symmetric result which is obtained by interchanging the roles of ϕ and ψ in case (I).) Analogously, by case (II) and its symmetric counterpart, given $c,d \in M[n]$, there exists a component C_0 of $W(\phi)$ whose endpoints are $(1,c,0,0)$ and $(1,d,0,0)$ if and only if there exists a component C_0' of $W(\psi)$ which has these endpoints. Hence, by construction, the Brauer morphism $\tau \circ \psi \circ \sigma \colon [m-2p] \to [n-2q]$ can also be written as the tensor product of a Brauer isomorphism $\iota_\psi \colon [m-2p] \to [n-2q]$ and $(p+q)$ loops:

$$\tau \circ \psi \circ \sigma = \lambda^{p+q} \otimes \iota_\psi. \tag{3.2}$$

Let C' denote the unique component of $W(\psi)$ such that $(0,a,0,0) \in C'$. Then, the other endpoint of C' is in $1 \times M[n] \times 0 \times 0$, say $C' \cap (1 \times \mathbb{R}^3) = \{(1,b',0,0)\}$. (Otherwise, both endpoints would lie in $0 \times M[m] \times 0 \times 0$. Therefore, there would exist a component of $W(\phi)$ with the same endpoints. However, this contradicts the existence of C.) Choose $j' \in M[n-2q]$ such that $\omega_\tau(j') = b'$. We represent ι_ψ by a compact smoothly embedded 1-manifold $W(\iota_\psi) \subset [0,1] \times \mathbb{R}^3$. Then, there exists a component of $W(\iota_\psi)$ which connects the points $(0,i,0,0)$ and $(1,j',0,0)$.

Application of the monoidal functor Y to equations (3.1) and (3.2) yields $d^{p+q} Y(\iota_\phi) = Y(\tau \circ$

$\phi \circ \sigma) = Y(\tau \circ \psi \circ \sigma) = d^{p+q} Y(\iota_\psi)$. Using Corollary 3.2.3, it follows from the assumption $d \geq 2$ and $Y(\iota_\phi) = Y(\iota_\psi)$ that $\iota_\phi = \iota_\psi$. Therefore, $W(\iota_\phi)$ and $W(\iota_\psi)$ represent the same Brauer isomorphism. In particular, $(0, i, 0, 0)$ is connected with $(1, j, 0, 0) = (1, j', 0, 0)$. Hence, $j = j'$ and $b = \omega_\tau(j) = \omega_\tau(j') = b'$. This shows that $(1, b, 0, 0) \in C'$ and $C \cap (\{0, 1\} \times \mathbb{R}^3) = C' \cap (\{0, 1\} \times \mathbb{R}^3)$. □

The next crucial step is to compute the preimage under Y of scalar square matrices for $d \geq 2$:

Proposition 3.2.5. *Assume $d \geq 2$. If $\phi \in \mathrm{Hom}_{\mathbf{Br}}([m], [m])$ satisfies $Y(\phi) = \mu \cdot 1_{V^{\otimes m}}$ for some $\mu \in \mathbb{R}$, then there exists $l \in \mathbb{N}$ such that $\mu = d^l$ and $\phi = \lambda^{\otimes l} \otimes 1_{[m]}$.*

Proof. The usual normal form of ϕ is given by $\phi = \lambda^{\otimes l} \otimes (\beta \circ \phi_0 \circ \alpha)$, where $l \in \mathbb{N}$, α and β are isomorphisms $[m] \to [m]$, and $\phi_0 = 1_{[m-2q]} \otimes e_1^{\otimes q} \otimes i_1^{\otimes q}$ for some $q \in \mathbb{N}$ with $2q \leq m$. As $Y(\lambda) = Y(e_1 \circ i_1) = e \circ i = \mathrm{Tr}(i, e) = \dim V = d$ by [4, Proposition 2.9], we obtain $Y(\phi) = d^l \cdot (Y(\beta) \circ Y(\phi_0) \circ Y(\alpha))$. Setting $\gamma := \beta^{-1} \circ \alpha^{-1} \colon [m] \to [m]$, the assumption $Y(\phi) = \mu \cdot 1_{V^{\otimes m}}$ reads

$$\frac{\mu}{d^l} \cdot Y(\gamma) = Y(\phi_0). \tag{3.3}$$

Let us assume that $q > 0$. Then, the matrix representation of $Y(\phi_0) = 1_{V^{\otimes(m-2q)}} \otimes e^{\otimes q} \otimes i^{\otimes q}$ contains at least one column with at least two nonzero entries. (In fact, since (i, e) is a duality structure on V, the symmetric $(d \times d)$-matrix $\mathrm{Mat}(e) = (e(v_j \otimes v_k))_{j,k}$ is invertible by [4, Proposition 2.6], and its inverse is given by $\mathrm{Mat}(i) = (i_{jk})_{j,k}$, where $i(1) = \sum_{j,k} i_{jk} v_j \otimes v_k$. Thus, it follows from $d \geq 2$ that the $(d \times d)$-matrix $\mathrm{Mat}(e)$ *contains at least two nonzero entries.* Hence, the $(1 \times d^2)$-matrix corresponding to the linear map $Y(e_1) = e \colon V \otimes V \to \mathbb{R}$ (in the usual basis of $V \otimes V = V^{\otimes 2}$) also contains at least two nonzero entries. Furthermore, the matrix representation of $e^{\otimes q}$, which is the $(1 \times d^{2q})$-matrix given by the q-th Kronecker power of the matrix representation of e, also contains at least two nonzero entries, since $q > 0$. Analogously, the $(d^{2q} \times 1)$-matrix representing $i^{\otimes q}$ contains at least two nonzero entries. Therefore, the Kronecker product $e^{\otimes q} \otimes i^{\otimes q}$ contains at least one column with at least two nonzero entries. Hence, the same is true for the matrix representation of $1_{V^{\otimes(m-2q)}} \otimes e^{\otimes q} \otimes i^{\otimes q}$.) On the other hand, it follows from Lemma 3.2.2 that $Y(\gamma)$ is represented by a permutation matrix, because $\gamma \colon [m] \to [m]$ is an isomorphism. In particular, every column of $Y(\gamma)$ has exactly one nonzero entry, which yields a contradiction in equation (3.3). Consequently, $q = 0$. This implies $\phi_0 = 1_{[m]}$ and $Y(\phi_0) = 1_{V^{\otimes m}}$. Thus, it follows from equation (3.3) that $\mu = d^l$ and $Y(\gamma) = 1_{V^{\otimes m}} = Y(1_{[m]})$. Because of the assumption $d \geq 2$, we can apply Corollary 3.2.3 to the Brauer isomorphisms $\gamma, 1_{[m]} \colon [m] \to [m]$ to obtain $\gamma = 1_{[m]}$. All in all, $\phi = \lambda^{\otimes l} \otimes (\beta \circ \phi_0 \circ \alpha) = \lambda^{\otimes l} \otimes (\beta \circ 1_{[m]} \circ \alpha) = \lambda^{\otimes l} \otimes (\beta \circ \gamma \circ \alpha) = \lambda^{\otimes l} \otimes 1_{[m]}$. □

As a sharpening of [4, Proposition 2.21], we finally prove the following

Theorem 3.2.6. *Assume* $d \geq 2$. *Then,* $Y\colon \mathbf{Br} \to \mathbf{Vect}$ *is a faithful functor.*

Proof. By Proposition 3.2.4, it suffices to show that Y is faithful on loops. Assume that $Y(\phi) = Y(\psi)$ for two given morphisms $\phi, \psi \in \mathrm{Hom}_{\mathbf{Br}}([m],[n])$ in $\mathbf{Br}$. The usual normal form of ϕ is given by $\phi = \lambda^{\otimes l} \otimes (\beta \circ \phi_0 \circ \alpha)$, where $l \in \mathbb{N}$, $\alpha\colon [m] \to [m]$ and $\beta\colon [n] \to [n]$ are isomorphisms, and $\phi_0 = 1_{[m-2p]} \otimes e_1^{\otimes p} \otimes i_1^{\otimes q}$ for some $p, q \in \mathbb{N}$ with $2p \leq m$, $2q \leq n$ and $m - 2p = n - 2q$. Analogously, we can write $\psi = \lambda^{\otimes l'} \otimes (\beta' \circ \psi_0 \circ \alpha')$, where $l' \in \mathbb{N}$, $\alpha'\colon [m] \to [m]$ and $\beta'\colon [n] \to [n]$ are isomorphisms, and $\psi_0 = 1_{[m-2p']} \otimes e_1^{\otimes p'} \otimes i_1^{\otimes q'}$ for some $p', q' \in \mathbb{N}$ with $2p' \leq m$, $2q' \leq n$ and $m - 2p' = n - 2q'$. We have to show that ϕ and ψ have the same number of loops, that is, $l = l'$. In the following, we will only show $l \leq l'$. Then, $l = l'$ follows by symmetry.

We will use Proposition 3.2.5 to reduce the assumption $Y(\phi) = Y(\psi)$ to equation (3.4), which is a statement in the Brauer category. Define $a := \alpha^{-1} \circ (1_{[m-2p]} \otimes i_1^{\otimes p}) \in \emptyset\,\mathrm{Hom}_{\mathbf{Br}}([m-2p],[m])$ and $b := (1_{[n-2q]} \otimes e_1^{\otimes q}) \circ \beta^{-1} \in \mathrm{Hom}_{\mathbf{Br}}([n],[n-2q])$. Then, one calculates

$$\begin{aligned}
b \circ \phi \circ a &= (1_{[0]} \otimes b) \circ (\lambda^{\otimes l} \otimes (\beta \circ \phi_0 \circ \alpha)) \circ (1_{[0]} \otimes a) \\
&= \lambda^{\otimes l} \otimes (b \circ \beta \circ \phi_0 \circ \alpha \circ a) \\
&= \lambda^{\otimes l} \otimes ((1_{[n-2q]} \otimes e_1^{\otimes q}) \circ \phi_0 \circ (1_{[m-2p]} \otimes i_1^{\otimes p})) \\
&= \lambda^{\otimes l} \otimes ((1_{[m-2p]} \otimes 1_{[0]} \otimes e_1^{\otimes q}) \circ (1_{[m-2p]} \otimes e_1^{\otimes p} \otimes i_1^{\otimes q}) \circ (1_{[m-2p]} \otimes i_1^{\otimes p} \otimes 1_{[0]})) \\
&= \lambda^{\otimes l} \otimes (1_{[m-2p]} \circ 1_{[m-2p]} \circ 1_{[m-2p]}) \otimes (1_{[0]} \circ e_1^{\otimes p} \circ i_1^{\otimes p}) \otimes (e_1^{\otimes q} \circ i_1^{\otimes q} \circ 1_{[0]}) \\
&= \lambda^{\otimes l} \otimes 1_{[m-2p]} \otimes (e_1 \circ i_1)^{\otimes p} \otimes (e_1 \circ i_1)^{\otimes q} \\
&= \lambda^{\otimes l+p+q} \otimes 1_{[m-2p]}.
\end{aligned}$$

Applying the monoidal functor Y to the previous equation and using $Y(\phi) = Y(\psi)$, we obtain

$$Y(b \circ \psi \circ a) = Y(b \circ \phi \circ a) = d^{l+p+q} \cdot 1_{V^{\otimes(m-2p)}}.$$

Since $d \geq 2$ and $b \circ \psi \circ a \in \mathrm{Hom}_{\mathbf{Br}}([m-2p],[m-2p])$, it follows from Proposition 3.2.5 that

$$b \circ \psi \circ a = \lambda^{\otimes(l+p+q)} \otimes 1_{[m-2p]}. \tag{3.4}$$

It suffices to show that $l' + p + q$ is an upper bound for the number of loops contained in the composition $b \circ \psi \circ a$. (Indeed, then it follows from equation (3.4) that $l + p + q \leq l' + p + q$. Thus, $l \leq l'$.)

Setting $\psi_0' := \beta^{-1} \circ \beta' \circ \psi_0 \circ \alpha' \circ \alpha^{-1}$, $a_0 := 1_{[m-2p]} \otimes i_1^{\otimes p}$ and $b_0 := 1_{[n-2q]} \otimes e_1^{\otimes q}$, we have

$$\begin{aligned}
b \circ \psi \circ a &= ((1_{[n-2q]} \otimes e_1^{\otimes q}) \circ \beta^{-1}) \circ (\lambda^{\otimes l'} \otimes (\beta' \circ \psi_0 \circ \alpha')) \circ (\alpha^{-1} \circ (1_{[m-2p]} \otimes i_1^{\otimes p})) \\
&= \lambda^{\otimes l'} \otimes ((1_{[n-2q]} \otimes e_1^{\otimes q}) \circ \psi_0' \circ (1_{[m-2p]} \otimes i_1^{\otimes p})) \\
&= \lambda^{\otimes l'} \otimes (b_0 \circ \psi_0' \circ a_0).
\end{aligned}$$

It suffices to show that the number of loops in $b_0 \circ \psi_0' \circ a_0$ is $\leq p + q$.

We choose 1-manifolds $W_0 \subset [0,1] \times \mathbb{R}^3$, $W' \subset [1,2] \times \mathbb{R}^3$ and $W_1 \subset [2,3] \times \mathbb{R}^3$ which represent (up to translations along the first coordinate) the Brauer morphisms a_0, ψ_0' and b_0 respectively. Then, $b_0 \circ \psi_0' \circ a_0$ is represented (after reparametrization of the first coordinate) by the union

$W := W_0 \cup W' \cup W_1 \subset [0,3] \times \mathbb{R}^3$.

For a 1-manifold $X \subset [s,t] \times \mathbb{R}^3$ which represents some morphism in **Br**, let $X\{e\}$ (respectively, X{i}) be the set of components of X whose endpoints are both contained in $s \times \mathbb{R}^3$ (respectively, in $t \times \mathbb{R}^3$). Moreover, denote by $X\{1\}$ the set of components of X which have one endpoint in $s \times \mathbb{R}^3$ and the other one in $s \times \mathbb{R}^3$. Finally, let $X\{\lambda\}$ be the set of closed components of X.

Note that the number of loops in $b_0 \circ \psi_0' \circ a_0$ is given by the cardinality of $W\{\lambda\}$, and we have to show that this number is $\leq p+q$. By definition of a_0 and b_0, $|W_0\{i\}| = p$ and $|W_1\{e\}| = q$. Hence, it suffices to construct an injective map $W\{\lambda\} \to W_0\{i\} \cup W_1\{e\}$.

Let $L \in W\{\lambda\}$ be a closed component of W. The intersections $L \cap W_0$, $L \cap W'$ and $L \cap W_1$ can be written as the disjoint union of components of W_0, W' and W_1 respectively. It follows from $L \cap (0 \times \mathbb{R}^3) = \emptyset$, $W_0\{e\} = \emptyset$ and $W_0\{\lambda\} = \emptyset$ that $L \cap W_0$ is a disjoint union of elements of $W_0\{i\}$. Analogously, it follows from $L \cap (3 \times \mathbb{R}^3) = \emptyset$, $W_1\{i\} = \emptyset$ and $W_1\{\lambda\} = \emptyset$ that $L \cap W_1$ is a disjoint union of elements of $W_1\{e\}$. Moreover, L has nonempty intersection with $W_0 \sqcup W_1$. (In fact, ψ_0' is a loop-free Brauer morphism, being the composition of the loop-free Brauer morphism ψ_0 and Brauer isomorphisms. Therefore, W' does not contain any closed components: $W\{\lambda\} = \emptyset$. Hence, L cannot be entirely contained in W'. Thus, $L \cap W_0 \sqcup W_1 \neq \emptyset$.) Hence, we can pick an element of $W_0\{i\} \cup W_1\{e\}$ which is contained in L. This defines a map $W\{\lambda\} \to W_0\{i\} \cup W_1\{e\}$. By construction, this map is injective. (Indeed, assume that $L, L' \in W\{\lambda\}$ are mapped to the same element $C \in W_0\{i\} \cup W_1\{e\}$. Then, $\emptyset \neq C \subset L \cap L'$ implies $L = L'$.) $\square$

3.3 Background on Fold Maps

In Definition 3.3.1 we recall the central notion of a fold map between smooth manifolds without boundary. Fold maps are sometimes called "submersions with folds", see [17, Definition III.4.1(a)].

Proposition 3.3.4 is a local key observation, which is a direct application of [17, Proposition II.4.3]. It generalizes the case $q = 1$ which is considered in [17, Proposition II.6.4]. Proposition 3.3.4 is used to prove one direction of Proposition 3.3.5, which is a characterization of fold maps by a local normal form. Moreover, it can be used to prove Lemma 3.3.6, which shows that the local normal form of a fold map into the plane can be locally perturbed in such a way that its fold locus remains unchanged, whereas its image in the plane can be perturbed in a controllable way. This turns out to be essential for the construction of fold fields (see Section 3.4.1) and stable fold maps (see Section 3.4.2) from given fold maps.

3.3.1 Jet Manifolds

In the following, let $n \geq q \geq 1$ be integers and let M^n and Q^q be smooth manifolds without boundary.

For an integer $k \geq 0$, let $J^k(M,Q)$ denote the set of k-jets from M to Q, see [17, Definition II.2.1]. Moreover, let $j^k(f)\colon M \to J^k(M,Q)$ denote the k-jet extension of a smooth map $f\colon M \to Q$. By [17, Theorem II.2.7(1)], $J^k(M,Q)$ is a smooth manifold whose dimension can be expressed in terms of n, q and k. By [17, Theorem II.2.7(4)], the k-jet extension $j^k(f)\colon M \to J^k(M,Q)$ of f is a smooth map.

Let us describe $J^1(M,Q)$ more explicitly. As a set, $J^1(M,Q)$ is the space of all triples (x,y,A), where $x \in M$, $y \in Q$ and $A\colon T_xM \to T_yQ$ is $\mathbb{R}$-linear. Moreover, the 1-jet extension of a smooth map $f\colon M \to Q$ is given by

$$j^1(f)\colon M \to J^1(M,Q), \qquad j^1(f)(x) = (x, f(x), df(x)).$$

The smooth structure on $J^1(M,Q)$ is explicitly defined as follows. If U is an open subset of $\mathbb{R}^n$ and V is an open subset of $\mathbb{R}^q$, then

$$J^1(U,V) = U \times V \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q),$$

where $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^{q\times n} \cong \mathbb{R}^{qn}$ is the real vector space of real $q \times n$-matrices. Thus, $J^1(U,V)$ can be considered as an open subset of $J^1(\mathbb{R}^n, \mathbb{R}^q) \cong \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{nq} = \mathbb{R}^{n+q+nq}$. If $\alpha : U' \to U$ is a chart on M and $\beta : V' \to V$ is a chart on Q, then the bijection

$$\tau_{\alpha\beta}\colon J^1(U',V') \to J^1(U,V), \qquad \tau_{\alpha\beta}(x,y,A) = (\alpha(x), \beta(y), d\beta(y) \circ A \circ (d\alpha(x))^{-1}),$$

is required to be a chart on $J^1(M,Q)$. (In particular, the subset $J^1(U',V') \subset J^1(M,Q)$ is required to be open in $J^1(M,Q)$.) This yields a well-defined smooth structure on $J^1(M,Q)$.

For an integer $0 \leq r \leq q$, let $S_r(M,Q)$ denote the set of points (x,y,A) in $J^1(M,Q)$, such that A drops rank by r:

$$S_r(M,Q) := \Big\{(x,y,A) \in J^1(M,Q);\ \operatorname{corank} A = r\Big\}.$$

By [17, Theorem II.5.4], $S_r(M,Q)$ is a submanifold of $J^1(M,Q)$ with $\operatorname{codim} S_r(M,Q) = r(n-q+r)$. (Note that $q := \min\{n,q\}$.) For instance, if U is an open subset of $\mathbb{R}^n$ and V is an open subset of $\mathbb{R}^q$, then

$$S_r(U,V) = U \times V \times L^r(\mathbb{R}^n,\mathbb{R}^q),$$

where $L^r(\mathbb{R}^n,\mathbb{R}^q) := \{A \in \operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^q);\ \operatorname{corank} A = r\}$ is a submanifold of $\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^q)$ of codimension $r(n-q+r)$ by [17, Proposition II.5.3].

For a smooth map $f : M \to Q$ and an integer $0 \le r \le q$, the set of points $x \in M$ such that $df(x)$ drops rank by r is given by $S_r(f) := j^1(f)^{-1}(S_r(M,Q)) \subset M$. In particular, $S_0(f)$ is the set of nonsingular points of f and $S(f) := \cup_{r=1}^{q} S_r(f)$ is the set of singular points of f.

3.3.2 Definition of Fold Maps

Definition 3.3.1. A smooth map $f\colon M \to Q$ is called a *fold map*, if

$(fm1)$ $j^1(f) \pitchfork S_1(M,Q)$.
$(fm2)$ $S(f) = S_1(f)$.
$(fm3)$ $S_1(f) \subset M$ is a submanifold of dimension $q-1$, and

$$T_xS_1(f) + \ker D_xf = T_xM \qquad \text{for all } x \in S_1(f).$$

Remark 3.3.2. Let $f\colon M \to Q$ be a smooth map.

(*i*) If f satisfies $(fm1)$, then $S_1(f) = j^1(f)^{-1}(S_1(M,Q)) \subset M$ is a submanifold of codimension $n-q+1$ by [17, Theorem II.4.4]. Therefore, if $(fm1)$ holds, then the first part of $(fm3)$ is automatically satisfied.

(*ii*) f satisfies $(fm2)$ if and only if $j^1(f)(M) \subset S_0(M,Q) \cup S_1(M,Q)$. This means that for all singular points of f the differential drops rank by 1, i.e. D_xf has rank $q-1$ for all $x \in S(f)$.

(*iii*) Suppose that $S_1(f) \subset M$ is a submanifold of dimension $q-1$. In particular, $\dim T_xS_1(f) = q-1$ for all $x \in S_1(f)$. Moreover, for all $x \in S_1(f)$, we have $\operatorname{corank} D_xf = 1$, and thus

$$\dim\ker D_xf = n - \dim\operatorname{im} D_xf = n - (q - \operatorname{corank} D_xf) = n-q+1.$$

Thus, $\dim T_xS_1(f) + \dim\ker D_xf = n = \dim T_xM$ for all $x \in S_1(f)$. Hence, if $S_1(f) \subset M$ is a submanifold of dimension $q-1$, then the following statements are equivalent:

- f satisfies $(fm3)$.
- $T_xS_1(f) \cap \ker D_xf = 0$ for all $x \in S_1(f)$.
- The restriction $f|_{S_1(f)}\colon S_1(f) \to Q$ is an immersion.

In particular, if f satisfies $(fm3)$, then the sum in $(fm3)$ is a direct sum.

Remark 3.3.3. It follows from Definition 3.3.1 that the restriction of a fold map to open subsets is again a fold map. (In particular, condition $(fm1)$ is a local condition, which can be seen as follows. If $f\colon M \to Q$ restricts to a map $f_0\colon U \to V$ between open subsets, then $J^1(U,V)$ is an open subset of $J^1(M,Q)$ and $S_1(U,V) = S_1(M,Q) \cap J^1(U,V)$. On top of that,

$j^1(f)\colon M \to J^1(M,Q)$ restricts to $j^1(f_0)\colon U \to J^1(U,V)$. Thus, one can conclude that for every point $p \in U$, $j^1(f_0)$ is transversal to $S_1(U,V)$ at p in $J^1(U,V)$ if and only if $j^1(f)$ is transversal to $S_1(M,Q)$ at p in $J^1(M,Q)$.)

Furthermore, pre-composition and post-composition of a fold map with diffeomorphisms are again fold maps.

3.3.3 Determination of Fold Maps

Let $n \geq q \geq 1$ be integers. Assume that $f\colon X \to \mathbb{R}$ is a smooth function which is defined on an open subset $X \subset \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1} = \mathbb{R}^n$. In the following, we will use the notation $p = (t,x) \in \mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ for points $p \in \mathbb{R}^n$. Given a point $(t,x) \in X$, let f_t denote the restriction of f to the nonempty open subset $X_t := \{x' \in \mathbb{R}^{n-q+1};\ (t,x') \in X\} \subset \mathbb{R}^{n-q+1}$. The following lemma presents a criterion for the map $F\colon X \to \mathbb{R}^q$, $F(t,x) = (t, f_t(x))$, to be a fold map. Concerning notation, the Jacobians of f with respect to the first and second factor of $\mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ will be denoted by

$$D^t f\colon X \to \mathbb{R}^{1\times(q-1)} = \mathbb{R}^{q-1}, \qquad D^t_p f = (\partial_{t_1} f(p) \ \cdots \ \partial_{t_{q-1}} f(p)),$$
$$D^x f\colon X \to \mathbb{R}^{1\times(n-q+1)} = \mathbb{R}^{n-q+1}, \qquad D^x_p f = (\partial_{x_1} f(p) \ \cdots \ \partial_{x_{n-q+1}} f(p)).$$

Proposition 3.3.4. *If $X \subset \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ is an open subset and $f\colon X \to \mathbb{R}$ is a smooth function, then for the smooth map*

$$F\colon X \to \mathbb{R}^q, \qquad p = (t,x) \mapsto F(t,x) = (t, f(p)),$$

the following statements hold:

(a) The singular set of F is given by $S(F) = S_1(F) = (D^x f)^{-1}(0)$. Hence, F satisfies $(fm2)$.

(b) For all $p_0 \in S_1(F)$ the following statements are equivalent:

(i) $j^1(F)$ is transversal to $S_1(X,\mathbb{R}^q) \subset J^1(X,\mathbb{R}^q)$ at p_0.

(ii) $D^x f : X \to \mathbb{R}^{1\times(n-q+1)} = \mathbb{R}^{n-q+1}$ is a submersion at p_0.

In other words, $\{p \in S_1(F);\ j^1(F) \pitchfork S_1(X,\mathbb{R}^q) \text{ at } p\} = S_1(F) \cap S_0(D^x f)$.

In particular, F satisfies $(fm1)$ if and only if $S_1(F) \subset S_0(D^x f)$.

(c) The map F is a fold map if and only if $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x f$ and the restriction $F|_S\colon S \to \mathbb{R}^q$ to the $(q-1)$-dimensional submanifold $S := S(F) = (D^x f)^{-1}(0) \subset X$ is an immersion.

(d) The map F is a fold map if and only if the Hessian $H_x(f_t)$ is non-degenerate for every point $p \in (D^x f)^{-1}(0)$.

Proof. (a). The Jacobian of F at a point $p = (t,x) \in X \subset \mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ is given by

$$D_p F = \begin{pmatrix} I_{q-1} & 0 \\ D^t_p f & D^x_p f \end{pmatrix}.$$

The rank of the matrix $D_p F$ is at least $q-1$, since its first $q-1$ lines are linearly independent in $\mathbb{R}^n$. Thus, $S_1(F) = S(F)$. Moreover, the last line of the matrix $D_p F$ is a linear combination of the first $q-1$ lines at a point $p \in X$ if and only if $D^x_p f = 0$. Thus, $(D^x f)^{-1}(0) = S_1(F)$.

(b). If $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q) := \mathbb{R}^{q\times n} \cong \mathbb{R}^{qn}$ is the space of $q \times n$-matrices with real coefficients, then

$$J^1(X, \mathbb{R}^q) = X \times \mathbb{R}^q \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q),$$

and $j^1(F)$ is given by

$$j^1(F)\colon X \to J^1(X, \mathbb{R}^q), \qquad j^1(F)(p) = (p, F(p), D_pF).$$

Moreover, if $L^1(\mathbb{R}^n, \mathbb{R}^q)$ denotes the submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q)$ which consists of all real $q \times n$-matrices of corank 1,

$$L^1(\mathbb{R}^n, \mathbb{R}^q) := \{H \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q);\ \mathrm{corank}\, H = 1\} \subset \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q),$$

then the submanifold $S_1(X, \mathbb{R}^q) \subset J^1(X, \mathbb{R}^q)$ is given by

$$S_1(X, \mathbb{R}^q) = X \times \mathbb{R}^q \times L^1(\mathbb{R}^n, \mathbb{R}^q) \quad \subset \quad J^1(X, \mathbb{R}^q).$$

Let $p_0 \in S_1(F)$. In order to show the equivalence $(i) \Leftrightarrow (ii)$, we will apply [17, Lemma II.4.3, p. 52] to X, $Y := J^1(X, \mathbb{R}^q)$, $W := S_1(X, \mathbb{R}^q)$ and the map $j^1(F)\colon X \to Y$. (Note that $j^1(F)(p_0) \in W$, because $p_0 \in S_1(F) = j^1(F)^{-1}(S_1(X, \mathbb{R}^q)) = j^1(F)^{-1}(W)$.) For this purpose, one defines an open neighbourhood U of $j^1(F)(p_0)$ in Y and a submersion $\phi : U \to \mathbb{R}^{n-q+1}$ such that $W \cap U = \phi^{-1}(0)$ as follows. The open subset $U' := \{H \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q);\ \det(H_{ij})_{1\le i,j\le q-1} \neq 0\}$ of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^q)$ gives rise to the open subset $U := X \times \mathbb{R}^q \times U'$ of $Y = J^1(X, \mathbb{R}^q)$. Note that U contains the image of $j^1(F)$ because the left upper $(q-1) \times (q-1)$-matrix of D_pF is the unit matrix I_{q-1} for all $p \in X$. In particular, $j^1(F)(p_0) \in U$. Next, define the submersion

$$\phi\colon U \to \mathbb{R}^{1\times(n-q+1)} = \mathbb{R}^{n-q+1}, \qquad \phi\left(p, y, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = D - CA^{-1}B,$$

where the block matrix consists of $A \in \mathrm{GL}_{q-1}(\mathbb{R})$, $B \in \mathbb{R}^{(q-1)\times(n-q+1)}$, $C \in \mathbb{R}^{1\times(q-1)}$ and $D \in \mathbb{R}^{1\times(n-q+1)}$. (Note that ϕ is indeed a submersion because for fixed values of p, y, A, B and C, it restricts to the diffeomorphism

$$\mathbb{R}^{1\times(n-q+1)} \to \mathbb{R}^{1\times(n-q+1)}, \qquad D \mapsto D - CA^{-1}B,$$

where $\mathbb{R}^{1\times(n-q+1)}$ is considered as a submanifold of U via $D \mapsto \left(p, y, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right)$.)

By [17, Lemma II.5.2] and since $\{H \in U';\ \mathrm{rank}\, H = q-1\} = U' \cap L^1(\mathbb{R}^n, \mathbb{R}^q)$, we have

$$\phi^{-1}(0) = X \times \mathbb{R}^q \times \{H \in U';\ \mathrm{rank}\, H = q-1\} = U \cap S_1(X, \mathbb{R}^q) = U \cap W.$$

By [17, Lemma II.4.3], $j^1(F)$ is transversal to W at p_0 (which is statement (i)) if and only if

$$X \to \mathbb{R}^{1\times(n-q+1)} = \mathbb{R}^{n-q+1}, \qquad p \mapsto \phi(j^1(F)(p)) = D_p^x f,$$

is a submersion at p_0 (which is statement (ii)). (Note that this map is well-defined, because the image of $j^1(F)$ is contained in U.) This shows the equivalence of statements (i) and (ii).

Equivalently,

$$\Big\{p \in S_1(F);\ j^1(F) \pitchfork S_1(X,\mathbb{R}^q) \text{ at } p\Big\} = S_1(F) \cap S_0(D^x f).$$

Finally, F satisfies $(fm1)$ if and only if $j^1(F)$ is transversal to $S_1(X,\mathbb{R}^q)$ at all points in $j^1(F)^{-1}(S_1(X,\mathbb{R}^q)) = S_1(F)$. By the above result, this is equivalent to $S_1(F) = S_1(F) \cap S_0(D^x f)$, which is furthermore equivalent to $S_1(F) \subset S_0(D^x f)$.

(c). Note that F is a fold map if and only if F satisfies $(fm1)$ and $(fm3)$ because $(fm2)$ is automatically satisfied by part (a). By part (b), F satisfies $(fm1)$ if and only if $S_1(F) \subset S_0(D^x f)$. The latter statement holds if and only if $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x f$, because $(D^x f)^{-1}(0) = S_1(F)$ by part (a). If these equivalent conditions are satisfied, then $S := (D^x f)^{-1}(0) = S_1(F)$ is a $(q-1)$-dimensional submanifold of X. By Remark 3.3.2$\,(iii)$ one can conclude that F satisfies $(fm3)$ if and only if the restriction $F|_S : S \to \mathbb{R}^q$ is an immersion.

(d). We make use of part (c).

First, note that $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x f\colon X \to \mathbb{R}^{n-q+1}$ if and only if the Jacobian of $D^x f$, which is given at $p \in X$ by

$$D_p(D^x f) = \begin{pmatrix} \partial_{t_1}\partial_{x_1} f(p) & \dots & \partial_{t_{q-1}}\partial_{x_1} f(p) & \\ \vdots & & \vdots & H_x(f_t) \\ \partial_{t_1}\partial_{x_{n-q+1}} f(p) & \dots & \partial_{t_{q-1}}\partial_{x_{n-q+1}} f(p) & \end{pmatrix},$$

has maximal rank $n-q+1$ for all $p \in (D^x f)^{-1}(0)$. In this case, $S := (D^x f)^{-1}(0)$ is a $(q-1)$-dimensional submanifold of X, and the tangent space of S at some point $p \in S$ is given by the $(q-1)$-dimensional vector subspace $T_pS = \ker D_p(D^x f) \subset \mathbb{R}^n$.

Moreover, note that $\ker D_pF = 0 \times \mathbb{R}^{n-q+1}$ holds for all points $p \in (D^x f)^{-1}(0)$ because the Jacobian of F at $p \in (D^x f)^{-1}(0)$ is given by

$$D_pF = \begin{pmatrix} I_{q-1} & 0 \\ D_p^t f & 0 \end{pmatrix}.$$

Therefore, if $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x f$, then D_pF is injective on T_pS for a given point $p \in S$ if and only if

$$0 = T_pS \cap \ker D_pF = \ker D_p(D^x f) \cap (0 \times \mathbb{R}^{n-q+1}).$$

Equivalently, $H_x(f_t)$ is non-degenerate, where $p = (t,x) \in S$. (In fact, for all $v \in \mathbb{R}^{(n-q+1)\times 1}$, we have $\begin{pmatrix} 0 \\ v \end{pmatrix} \in \ker D_p(D^x f)$ if and only if $H_x(f_t)v = 0$. This implies that $H_x(f_t)$ is degenerate if and only if $0 \neq \ker D_p(D^x f) \cap (0 \times \mathbb{R}^{n-q+1})$.)

All in all, if $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x f$ and $F|_S$ is an immersion, then $H_x(f_t)$ is non-degenerate for all $p = (t,x) \in S$. Conversely, if $H_x(f_t)$ is non-degenerate for all $p = (t,x) \in S$, then $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x F$ (since $D_p(D^x F)$ has maximal rank for all $p \in S = (D^x f)^{-1}(0)$), and D_pF is injective on T_pS for all $p \in S$. □

3.3.4 Local Normal Form of Fold Maps

For every integer $0 \leq i \leq n-q+1$, define the quadratic form

$$\lambda_i \colon \mathbb{R}^{n-q+1} \to \mathbb{R}, \qquad \lambda_i(x) = -\left(x_1^2 + ... + x_i^2\right) + x_{i+1}^2 + ... + x_{n-q+1}^2.$$

Proposition 3.3.5. *A smooth map $f : M^n \to Q^q$ is a fold map if and only if for every $p \in S(f)$ there exist local coordinates $(t, x) \in \mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ centered at p and $(y_1, ..., y_q)$ centered at $f(p)$, such that, for a suitable integer $0 \leq i \leq n-q+1$, f takes the form*

$$(t, x) \mapsto (t, \lambda_i(x)).$$

Proof. If $f : M^n \to Q^q$ is a fold map, then f has the required normal form by [17, Theorem III.4.5]. Conversely, suppose that f has the local normal form described above around all of its singular points. By Remark 3.3.3, it suffices to show that every map of the form

$$g \colon X \to \mathbb{R}^q, \qquad p = (t, x) \mapsto (t, \lambda_i(x)),$$

where $X \subset \mathbb{R}^n = \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ is an open neighbourhood of 0, is a fold map. For this purpose, we apply Proposition 3.3.4(c), using the function

$$h \colon X \to \mathbb{R}, \qquad p = (t, x) \mapsto \lambda_i(x).$$

The Jacobian map $D^x h \colon X \to \mathbb{R}^{1 \times (n-q+1)} = \mathbb{R}^{n-q+1}$ is given at $p \in X$ by

$$D_p^x h = \begin{pmatrix} \pm 2x_1 & \cdots & \pm 2x_{n-q+1} \end{pmatrix}.$$

Note that $D^x h$ is a submersion, since for all $p \in X$ we have

$$D_p(D^x h) = \begin{pmatrix} 0 & \operatorname{diag}(\pm 2, ..., \pm 2) \end{pmatrix}.$$

In particular, $0 \in \mathbb{R}^{n-q+1}$ is a regular value of $D^x h$, and

$$S := D^x h^{-1}(0) = X \cap (\mathbb{R}^{q-1} \times \left\{x \in \mathbb{R}^{n-q+1};\ \pm 2x_1 = ... = \pm 2x_{n-q+1} = 0\right\}) = X \cap (\mathbb{R}^{q-1} \times 0)$$

is a $(q-1)$-dimensional submanifold of X. Finally, the restriction $g|S \colon S \to \mathbb{R}^q$ is of the form $p = (t, 0) \mapsto (t, \lambda_i(0)) = (t, 0)$ and hence an immersion. □

3.3.5 Perturbation of Fold Lines

From now on, we specialize to $n \geq q = 2$ and $Q = \mathbb{R}^2$.

For any integer $0 \leq i \leq n-1$, the local normal form for fold maps into the plane,

$$\Lambda_i \colon \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^2, \qquad \Lambda_i(t,x) = (t, \lambda_i(x)),$$

is a fold map with fold line $S(\Lambda_i) = \mathbb{R} \times 0$ by Proposition 3.3.5.
Given two smooth functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ with compact support, define

$$\Delta \colon \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^2, \qquad \Delta(t,x) = (0, \alpha(t)\beta(\|x\|^2)).$$

Obviously, the perturbation $\widetilde{\Lambda}_i$ of Λ_i by Δ,

$$\widetilde{\Lambda}_i := \Lambda_i + \Delta \colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^2, \qquad \widetilde{\Lambda}_i(t,x) = (t, \lambda_i(x) + \alpha(t)\beta(\|x\|^2)),$$

agrees with Λ_i outside of a compact subset of $\mathbb{R}^n$.

The following Lemma 3.3.6 shows that, for suitable choices of α and β, the perturbed map $\widetilde{\Lambda}_i$ is still a fold map with the same fold locus as Λ_i, $S(\widetilde{\Lambda}_i) = S(\Lambda_i) = \mathbb{R} \times 0$. Note that the absolute index $\max\{i, n-1-i\}$ of the (connected) fold line $S(\Lambda_i)$ is conserved under the perturbation, since the absolute index is constant along components of the fold locus, and the modification is performed on a compact set. However, the image of the fold locus $S(\Lambda_i) = \mathbb{R} \times 0$ in the plane is perturbed. In fact, the perturbation is determined by α and $\beta(0)$:

$$(\widetilde{\Lambda}_i)(t,0) = (t, \alpha(t)\beta(0)), \qquad t \in \mathbb{R}.$$

Note that this is just the graph of $t \mapsto \alpha(t)\beta(0)$.

Lemma 3.3.6. *If $|\alpha(t)\beta'(r)| < 1$ for all $(t,r) \in \mathbb{R}^2$, then $\Lambda_i + \Delta$ is a fold map with one fold line $S(\Lambda_i + \Delta) = S(\Lambda_i)$ ($= \mathbb{R} \times 0$).*

Proof. We apply Proposition 3.3.4 to $X := \mathbb{R}^n$, $g := \Lambda_i + \Delta$ and

$$h \colon X = \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}, \qquad (t,x) \mapsto \lambda_i(x) + \alpha(t)\beta(\|x\|^2).$$

The Jacobian $D^x h \colon X \to \mathbb{R}^{1 \times (n-1)} = \mathbb{R}^{n-1}$ at $p = (t,x) \in X = \mathbb{R} \times \mathbb{R}^{n-1}$ is given by

$$D^x_p h = \Big(2x_1(\pm 1 + \alpha(t)\beta'(\|x\|^2)) \quad \cdots \quad 2x_{n-1}(\pm 1 + \alpha(t)\beta'(\|x\|^2)) \Big).$$

Since $|\alpha(t)\beta'(r)| < 1$ for all $(t,r) \in \mathbb{R}^2$ by assumption, one can conclude that

$$S := S(\Lambda_i + \Delta) = D^x h^{-1}(0) = \mathbb{R} \times 0 \subset \mathbb{R} \times \mathbb{R}^{n-1}.$$

The Jacobian $D(D^x h) \colon X \to \mathbb{R}^{(n-1) \times n}$ at $p = (t,x) \in X = \mathbb{R} \times \mathbb{R}^{n-1}$ is given by

$$D_p(D^x h) = \begin{pmatrix} 2x_1\alpha'(t)\beta'(\|x\|^2) & \ddots & \vdots & \iddots \\ \vdots & \dots & 2\varepsilon_{kl}(\pm 1 + \alpha(t)\beta'(\|x\|^2)) + 4x_k x_l \alpha(t)\beta''(\|x\|^2) & \dots \\ 2x_{n-1}\alpha'(t)\beta'(\|x\|^2) & \iddots & \vdots & \ddots \end{pmatrix}.$$

For points $p = (t, 0) \in S = \mathbb{R} \times 0$ this reduces to

$$D_p(D^x h) = \begin{pmatrix} 0 & 2(\pm 1 + \alpha(t_0)\beta'(0)) & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 2(\pm 1 + \alpha(t_0)\beta'(0)) \end{pmatrix}.$$

Since $|\alpha(t)\beta'(r)| < 1$ for all $(t, r) \in \mathbb{R}^2$ by assumption, one can conclude that the matrix $D_p(D^x h)$ has maximal rank $n-1$ for all points $p \in S$. This shows in particular that $0 \in \mathbb{R}^{n-1}$ is a regular value of $D^x h$. Furthermore, the restriction of $g = \Lambda_i + \Delta$ to the 1-dimensional submanifold $S = \mathbb{R} \times 0$ of $X = \mathbb{R} \times \mathbb{R}^{n-1}$ is given by

$$g|_S \colon S \to X, \qquad (t, 0) \mapsto (\Lambda_i + \Delta)(t, 0) = (t, \alpha(t)\beta(0)),$$

which is obviously an immersion. □

Proposition 3.3.7. *Let $F \colon U \to V$ be a fold map, where U is an n-dimensional smooth manifold without boundary and V is an open subset of $\mathbb{R}^2$. Let $p_0 \in S(F)$ be such that $F(p_0) \neq F(p)$ for all $p \in S(F) \setminus \{p_0\}$. Then there exists a fold map $\widetilde{F} \colon U \to V$ with the following properties:*

(1) There exists a compact subset $K \subset U$ such that $\widetilde{F}|_{U \setminus K} = F|_{U \setminus K}$.
(2) $S(\widetilde{F}) = S(F)$.
(3) $F(p_0) \notin \widetilde{F}(S(F))$.

Proof. By Proposition 3.3.5, there exists a chart $\phi \colon U_0 \to U_1 \subset \mathbb{R}^n$ around p_0 in U and a chart $\psi \colon V_0 \to V_1 \subset \mathbb{R}^2$ around $F(p_0)$ in V, such that $\phi(p_0) = 0 \in U_1$, $F(U_0) \subset V_0$ and $\psi(F(p_0)) = 0 \in V_1$, and there exists an integer $0 \leq i \leq n-1$, such that for all $(t, x) \in U_1 \subset \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ we have $\psi(F(\phi^{-1}(t, x))) = \Lambda_i(t, x)$ $(= (t, \lambda_i(x)))$. In particular, $\Lambda_i(U_1) \subset V_1$.

Since $0 \in U_1$, one can choose $\delta > 0$ and $\rho > 0$ such that $K_1 := [-\delta, \delta] \times \{x \in \mathbb{R}^{n-1};\ \|x\|^2 \leq \rho\} \subset U_1$. Choose a smooth function $\beta \colon \mathbb{R} \to \mathbb{R}$ such that $\beta(0) = 1$ and $\beta(r) = 0$ for $|r| \geq \rho$. Choose $R > 0$ such that $|\beta(r)| \leq R$ and $|\beta'(r)| \leq R$ for all $r \in \mathbb{R}$. Choose $d \in (0, 1]$ such that $\|y - z\| \geq d$ for all $y \in \Lambda_i(K_1)$ and all $z \in \mathbb{R}^2 \setminus V_1$. (This is possible, since $\Lambda_i(K_1)$ and $\mathbb{R}^2 \setminus V_1$ are disjoint subsets of the metric space $(\mathbb{R}^2, \|\cdot\|)$, where $\Lambda_i(K_1)$ is compact and $\mathbb{R}^2 \setminus V_1$ is a closed subset.) Choose a smooth function $\alpha \colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) \neq 0$, $\alpha(t) = 0$ for $|t| \geq \delta$ and $|\alpha(t)| < \frac{d}{R}$ for all $t \in \mathbb{R}$.

Since $|\alpha(t)\beta'(r)| \leq |\alpha(t)|R < d \leq 1$ for all $(t, r) \in \mathbb{R}^2$, it follows from Lemma 3.3.6 that the perturbation

$$\widetilde{\Lambda}_i \colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^2, \qquad \widetilde{\Lambda}_i(t, x) = (t, \lambda_i(x) + \alpha(t)\beta(\|x\|^2)),$$

of Λ_i is a fold map such that $S(\widetilde{\Lambda}_i) = S(\Lambda_i)$. Note that $\widetilde{\Lambda}_i(U_1) \subset V_1$. (In fact, assume that there exists a point $(t, x) \in U_1$ such that $\widetilde{\Lambda}_i(t, x) \in \mathbb{R}^2 \setminus V_1$. Then it follows from $\widetilde{\Lambda}_i(U_1 \setminus K_1) = \Lambda_i(U_1 \setminus K_1) \subset V_1$ that $(t, x) \in K_1$. But then, by choice of d, we obtain the contradiction $d \leq \|\Lambda_i(t, x) - \widetilde{\Lambda}_i(t, x)\| = \|(0, \alpha(t)\beta(\|x\|^2))\| = |\alpha(t)\beta(\|x\|^2)| \leq |\alpha(t)|R < d$.) Thus, $\widetilde{\Lambda}_i$ induces a fold map

$$\widetilde{F}_0 \colon U_0 \to V_0, \qquad \widetilde{F}_0(p) = \psi^{-1}(\widetilde{\Lambda}_i(\phi(p))).$$

Note that $S(\widetilde{F}_0) = \phi^{-1}(S(\widetilde{\Lambda}_i) \cap U_1) = \phi^{-1}(S(\Lambda_i) \cap U_1) = S(F) \cap U_0 = S(F|_{U_0})$.

The compact subset $K := \phi^{-1}(K_1) \subset U_0$ gives rise to the open covering $U = U_0 \cup (U \setminus K)$. Since $\widetilde{\Lambda}_i|_{U_1 \setminus K_1} = \Lambda_i|_{U_1 \setminus K_1}$ by construction, we have $\widetilde{F}_0|_{U_0 \setminus K} = F|_{U_0 \setminus K}$. This shows that the fold maps $\widetilde{F}_0$ and $F|_{U \setminus K}$ agree on the intersection $U_0 \cap (U \setminus K) = U_0 \setminus K$ and can thus be assembled to a fold map

$$\widetilde{F} \colon U \to V, \qquad \widetilde{F}(p) = \begin{cases} \widetilde{F}_0(p), & \text{for } p \in U_0, \\ F(p), & \text{for } p \notin U_0. \end{cases}$$

Let us check that $\widetilde{F}$ satisfies the claimed properties:

(1). We have $U \setminus K = (U \setminus U_0) \cup (U_0 \setminus K)$. By definition, $\widetilde{F}|_{U \setminus U_0} = F|_{U \setminus U_0}$. Moreover, $\widetilde{F}|_{U_0 \setminus K} = \widetilde{F}_0|_{U_0 \setminus K} = F|_{U_0 \setminus K}$. Thus, $\widetilde{F}|_{U \setminus K} = F|_{U \setminus K}$.

(2). As $U = U_0 \cup (U \setminus K)$, we have $S(\widetilde{F}) = S(\widetilde{F}|_{U_0}) \cup S(\widetilde{F}|_{U \setminus K}) \overset{(1)}{=} S(\widetilde{F}_0) \cup S(F|_{U \setminus K}) = S(F|_{U_0}) \cup S(F|_{U \setminus K}) = S(F)$.

(3). We have $\widetilde{F}(S(F)) = \widetilde{F}(S(F) \setminus K) \cup \widetilde{F}(S(F) \cap U_0)$. Since $p_0 = \phi^{-1}(0) \in \phi^{-1}(K_1) = K$, we have $F(p_0) \neq F(p)$ for all $p \in S(F) \setminus K$. Thus, $F(p_0) \notin F(S(F) \setminus K) \overset{(1)}{=} \widetilde{F}(S(F) \setminus K)$. Moreover, it follows from $\alpha(0) \neq 0$ and $\beta(0) = 1$ that

$$(0,0) \notin \{(t, \alpha(t));\ t \in \mathbb{R}\} = \widetilde{\Lambda}_i(\mathbb{R} \times \{0\}) = \widetilde{\Lambda}_i(S(\Lambda_i)) \supset \widetilde{\Lambda}_i(S(\Lambda_i) \cap U_1).$$

Thus, it follows from $\psi(F(p_0)) = (0,0)$ and $\widetilde{\Lambda}_i(U_1) \subset V_1 = \psi(V_0)$ that

$$F(p_0) = \psi^{-1}((0,0)) \notin \psi^{-1}(\widetilde{\Lambda}_i(S(\Lambda_i) \cap U_1)) = \widetilde{F}_0(\phi^{-1}(S(\Lambda_i) \cap U_1)) = \widetilde{F}(S(F) \cap U_0).$$

All in all, $F(p_0) \notin \widetilde{F}(S(F) \setminus K) \cup \widetilde{F}(S(F) \cap U_0) = \widetilde{F}(S(F))$.

□

3.4 Turning Fold Maps into Fold Fields

In the following, we cope with local modifications of fold maps on a given cobordism W. "Local" refers to the fact that all modifications of a given fold map on W will take place in local coordinates on W or in a tubular neihbourhood of a submanifold of W. In particular, no topological information about W is incorporated in the constructions.

We discuss several constructions of stable fold maps and (stable) fold fields from certain fold maps. Theorem 3.4.9 states that if $F\colon W \to \mathbb{C}$ is a fold map on the cobordism W such that $S(F)$ is transversal to ∂W and $\mathrm{Im} \circ F\colon W \to \mathbb{R}$ is injective on certain subsets of $S(F) \cap \partial W$ (such an F will be called *fold pre-field*, see Definition 3.4.3), then there exists a *fold field* $\widetilde{F}\colon W \to \mathbb{C}$ such that F and $\widetilde{F}$ agree in a neighbourhood of ∂W and induce the same Brauer morphism. The proof modifies the given fold pre-field F in two steps. The first step consists of a slight perturbation of the image of the fold locus $F(S(F)) \subset \mathbb{C}$ which does not affect the fold locus $S(F)$ itself. In the second step, the perturbed fold map is precomposed with a suitable diffeomorphism $W \to W$ to obtain the desired fold field. In consequence of Theorem 3.4.9, it is shown in Proposition 3.4.12 that the value of the state sum Z_W on arbitrary boundary conditions can also be calculated by admitting in the defining sum all fold *maps* (not only fold *fields*) which satisfy the given boundary conditions. Theorem 3.4.14 asserts that if F is a fold pre-field on W which is stable in a neighbourhood of ∂W, then there exists a stable fold pre-field $\widetilde{F}$ such that F and $\widetilde{F}$ agree in a neighbourhood of ∂W and induce the same Brauer morphism. Moreover, an analysis of the proof of Theorem 3.4.9 shows that if one starts with a *stable* fold pre-field on W, then the construction can be adapted in such a way that the resulting fold field is also stable (see Theorem 3.4.15).

3.4.1 Construction of Fold Fields form Fold Maps

In the following, $W \subseteq [a,b] \times \mathbb{R}^D$ denotes an n-dimensional cobordism from M to N with time function $\omega\colon W \to [a,b]$ (where $a < b$ are real numbers) and cylinder scale $\varepsilon_W > 0$. (The term *cobordism* is always used in the sense of Definition 3.1.1.) Using a reparametrization of the embedding $W \subseteq [a,b] \times \mathbb{R}^D$, we can assume that $[a,b] = [0,1]$. Set $W_A := \omega^{-1}(A)$ for subsets $A \subset [0,1]$ and $W_t := W_{\{t\}}$ for a one-point set $\{t\} \subset [0,1]$. For $k \in \mathbb{N}$ we write $W(k) := W \cap [0,1] \times \{k\} \times \mathbb{R}^{D-1}$. (By definition, $W(k)$ is the union of connected components of W that lie in the slice $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$.)

Definition 3.4.1. The cobordism W is called *simple*, if $W(k) = W$ for some $k \in \mathbb{N}$. (This means that W is entirely contained in some slice $[0,1] \times \{k\} \times \mathbb{R}^{D-1}$, $k \in \mathbb{N}$.)

Recall the definition of a fold field on W from Definition 3.1.6:

Definition 3.4.2. A *fold field* on a simple cobordism W is a fold map $F\colon W \to \mathbb{C}$ such that

$(ff1)$ $0, 1 \in \pitchfork(F) \cap \mathrm{GenIm}(F)$.
$(ff2)$ $\mathrm{GenIm}(F)$ is residual in $[0,1]$.

A fold field on an arbitrary cobordism W is a fold map $F\colon W \to \mathbb{C}$ such that the restriction $F(k)\colon W(k) \to \mathbb{C}$ of F to $W(k)$ is a fold field on $W(k)$ for all $k \in \mathbb{N}$.

Analogously, fold *pre*-fields are now defined by only requiring property (ff1):

Definition 3.4.3. A *fold pre-field* on a simple cobordism W is a fold map $F\colon W \to \mathbb{C}$ such that (ff1) holds. A fold pre-field on an arbitrary cobordism W is a fold map $F\colon W \to \mathbb{C}$ such that the fold map $F(k)\colon W(k) \to \mathbb{C}$ is a fold pre-field for all $k \in \mathbb{N}$. The collection of all fold pre-fields on a nonempty cobordism W is denoted by $\mathcal{F}^{\mathrm{pre}}(W) \subset C^\infty(W, \mathbb{C})$. Moreover, we set $\mathcal{F}^{\mathrm{pre}}(\emptyset) = \{*\}$, a set with one element.

Note that for any cobordism W, we have $\mathcal{F}(W) \subset \mathcal{F}^{\mathrm{pre}}(W)$. Moreover, as the construction of the action functional $\mathbb{S}\colon \mathcal{F}(W) \to \mathrm{Mor}(\mathbf{Br})$ as explained in Section 3.1.3 does not use property (ff2), it extends literally to the construction of an assignment $\mathbb{S}\colon \mathcal{F}^{\mathrm{pre}}(W) \to \mathrm{Mor}(\mathbf{Br})$.

An inspection of the proof shows that [4, Lemma 7.11] can be reformulated for fold pre-fields:

Lemma 3.4.4. *Let W be a simple cobordism, $F \in \mathcal{F}^{\mathrm{pre}}(W)$, and suppose that $t \in (0,1) \cap \pitchfork (F) \cap \mathrm{GenIm}(F)$. Let $F_{\leq t}\colon W_{\leq t} \to \mathbb{C}$ and $F_{\geq t}\colon W_{\geq t} \to \mathbb{C}$ denote the restrictions of F to $W_{\leq t} := W \cap [0,t] \times \mathbb{R}^D$, $W_{\geq t} := W \cap [t,1] \times \mathbb{R}^D$, respectively. Then $F_{\leq t} \in \mathcal{F}^{\mathrm{pre}}(W_{\leq t})$, $F_{\geq t} \in \mathcal{F}^{\mathrm{pre}}(W_{\geq t})$, and the Brauer morphism identity $\mathbb{S}(F_{\geq t}) \circ \mathbb{S}(F_{\leq t}) = \mathbb{S}(F)$ holds.*

As the proof of [4, Lemma 7.18] shows, Brauer invariance can be reformulated for fold pre-fields:

Lemma 3.4.5. *Suppose that two cobordisms $W \subset [0,1] \times \mathbb{R}^D$ and $W' \subset [a,b] \times \mathbb{R}^D$ are related by a diffeomorphism $\alpha\colon W' \to W$ of the form $\alpha(t,x) = (\tau(t), x)$, $(t,x) \in W'$, $t \in [a,b]$, $x \in \mathbb{R}^D$, where $\tau\colon [a,b] \to [0,1]$ is a diffeomorphism with $\tau(a) = 0$. Given $F \in \mathcal{F}^{\mathrm{pre}}(W)$, the composition $F_\alpha := F \circ \alpha$ satisfies $F_\alpha \in \mathcal{F}^{\mathrm{pre}}(W')$, $S(F_\alpha) = \alpha^{-1}(S(F))$, and the Brauer invariance $\mathbb{S}(F_\alpha) = \mathbb{S}(F)$.*

The following defines clearly an equivalence relation on the collection of fold pre-fields on W:

Definition 3.4.6. Two fold pre-fields $F, G \in \mathcal{F}^{\mathrm{pre}}(W)$ are *equivalent*, written $F \sim_W G$, if $F|_U = G|_U$ on a suitable neighbourhood U of ∂W in W and $\mathbb{S}(F) = \mathbb{S}(G)$.

The following Lemma yields another version of Brauer invariance for fold pre-fields:

Lemma 3.4.7. *Let W be a cobordism and let $\Phi : W \to W$ be a diffeomorphism such that $\Phi(x) = x$ for all x in suitable neighbourhood of ∂W in W. Given $F \in \mathcal{F}^{\mathrm{pre}}(W)$, the composition $F_\Phi := F \circ \Phi$ satisfies $F_\Phi \in \mathcal{F}^{\mathrm{pre}}(W)$, $S(F_\Phi) = \Phi^{-1}(S(F))$, and $F_\Phi \sim_W F$.*

Proof. There exists an open neighbourhood U of ∂W in W such that $F_\Phi(x) = (F \circ \Phi)(x) = F(x)$ for all $x \in U$. Thus, $F_\Phi|_U = F|_U$. F_Φ is a fold map, since the precomposition of a fold map with a diffeomorphism is again a fold map. As Φ is a diffeomorphism, we have $S(F_\Phi) = \Phi^{-1}(S(F))$. For all $k \in \mathbb{N}$ it follows from $0, 1 \in \pitchfork (F(k)) \cap \mathrm{GenIm}(F(k))$ and $F_\Phi(k)|_{U \cap W(k)} = F(k)|_{U \cap W(k)}$ that $0, 1 \in \pitchfork (F_\Phi(k)) \cap \mathrm{GenIm}(F_\Phi(k))$. Therefore, $F_\Phi \in \mathcal{F}^{\mathrm{pre}}(W)$. Moreover, it follows from $F_\Phi|_U = F|_U$ and $S(F_\Phi) = \Phi^{-1}(S(F))$ that $\mathbb{S}(F_\Phi) = \mathbb{S}(F)$. Consequently, $F_\Phi \sim_W F$. □

The following notation will be used frequently in the proofs of Theorem 3.4.9 and Theorem 3.4.14. Let $\mathrm{Reg}(\omega_{S(F)})$ be the set of regular values of the restriction $\omega_{S(F)} := \omega|_{S(F)}\colon S(F) \to [0,1]$. If $0 \leq u < v \leq 1$ and $u, v \in \mathrm{Reg}(\omega_{S(F)})$, then $S(F) \cap W_{[u,v]} = (\omega_{S(F)})^{-1}([u,v])$ is the disjoint union of a finite number of circles and intervals. Let $C_{[u,v]}$ be the set of components of $S(F) \cap W_{[u,v]}$ which are diffeomorphic to the circle S^1 and let $I_{[u,v]}$ be the set of components of $S(F)\cap W_{[u,v]}$ which are diffeomorphic to the interval $[0,1]$. Note that if $[u,v]\subset \mathrm{Reg}(\omega_{S(F)})$, then $C_{[u,v]}=\emptyset$, and $\omega\colon W\to[0,1]$ restricts to a diffeomorphism $\omega_T\colon T\xrightarrow{\cong}[u,v]$ for every $T\in I_{[u,v]}$. Moreover, set $C:=C_{[0,1]}$ and $I:=I_{[0,1]}$.

Lemma 3.4.8. *Let $F\colon W\to\mathbb{C}$ be a fold map such that $0,1\in\pitchfork(F)$. (Equivalently, $S(F)\pitchfork\partial W$.) Then there exists $\varepsilon_0\in(0,\varepsilon_W)$ such that for all $\varepsilon\in(0,\varepsilon_0)$ the following properties hold:*

($\varepsilon1$) $\omega\colon W\to[0,1]$ *restricts to a diffeomorphism*

$$T\xrightarrow{\cong}\omega(T)$$

for all $T\in I_{[0,\varepsilon]}$ and all $T\in I_{[1-\varepsilon,1]}$.

($\varepsilon2$) $C_{[0,\varepsilon]}=\emptyset$ *and* $C_{[1-\varepsilon,1]}=\emptyset$.

In particular, $S\subset W_{(\varepsilon,1-\varepsilon)}$ for all $S\in C$.

Proof. Since $0,1\in\pitchfork(F)$ and $\pitchfork(F)$ is open in $[0,1]$ by [4, Corollary 7.8], there exists $\varepsilon\in(0,\varepsilon_W)$ such that $[0,\varepsilon]\cup[1-\varepsilon,1]\subset\pitchfork(F)$. By [4, Lemma 7.7], we have $\pitchfork(F)\subset\mathrm{Reg}(\omega_{S(F)})$. Thus, it follows from $[0,\varepsilon]\subset\pitchfork(F)\subset\mathrm{Reg}(\omega_{S(F)})$ that $C_{[0,\varepsilon]}=\emptyset$ and that ω restricts to a diffeomorphism $\omega_T\colon T\xrightarrow{\cong}[0,\varepsilon]$ for every $T\in I_{[0,\varepsilon]}$. Analogously, $C_{[1-\varepsilon,1]}=\emptyset$, and ω restricts to a diffeomorphism $\omega_T\colon T\xrightarrow{\cong}[1-\varepsilon,1]$ for every $T\in I_{[1-\varepsilon,1]}$. This shows ($\varepsilon1$) and ($\varepsilon2$). Moreover, we have $S\subset W_{(\varepsilon,1-\varepsilon)}$ for all $S\in C$. □

Theorem 3.4.9. *If $F\in\mathcal{F}^{\mathrm{pre}}(W)$, then there exists $G\in\mathcal{F}(W)$ such that $F\sim_W G$.*

Proof. Let $F\in\mathcal{F}^{\mathrm{pre}}(W)$. Let us first assume that W is a simple cobordism. Thus, by Definition 3.4.3, we know that $F\colon W\to\mathbb{C}$ is a fold map and $0,1\in\pitchfork(F)\cap\mathrm{GenIm}(F)$.

Using $0,1\in\pitchfork(F)$ and Lemma 3.4.8, we can choose $\varepsilon\in(0,\varepsilon_W)$ such that ($\varepsilon1$) and ($\varepsilon2$) hold. Therefore, for every $S\in I$ there exists a unique subset $\{T_S^-,T_S^+\}\subset I_{[0,\varepsilon]}\cup I_{[1-\varepsilon,1]}$ (where $T_S^-\neq T_S^+$) such that $S\cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}=T_S^-\cup T_S^+$. Furthermore, for every $T\in I_{[0,\varepsilon]}\cup I_{[1-\varepsilon,1]}$ there exist a unique $S\in I$ such that $T\in\{T_S^-,T_S^+\}$.

Set $f:=\mathrm{Im}\circ F\colon W\to\mathbb{R}$. For every $S\in C\cup I$ we denote the restriction of f to S by $f_S\colon S\to\mathbb{R}$. For every $S\in C$ the compact interval $f(S)\subset\mathbb{R}$ has positive length. (Indeed, if $f(S)=\{t\}$ for some $t\in\mathbb{R}$, then F restricts to a submersion $S^1\cong S\to\mathbb{R}\times\{t\}\cong\mathbb{R}$. This is impossible, since S^1 is a closed manifold.) As $S(F)\cap W_{\{0,1\}}$ is a finite set, one can choose for every $S\in C$ a nonempty open subset $U_S\subset\mathbb{R}$ such that $U_S\subset f(S)\setminus f(S(F)\cap W_{\{0,1\}})$ and $U_S\cap U_{S'}=\emptyset$ for all $S\neq S'$ in C. As the set $\mathrm{Reg}(f_S)$ of regular values of f_S is dense in $\mathbb{R}$ by the Morse-Sard theorem [22, Chapter 3, Theorem 1.3], we have $\mathrm{Reg}(f_S)\cap U_S\neq\emptyset$ for all $S\in C$. Thus, we can choose for every $S\in C$ a point $b_S\in S$ such that $f(b_S)\in\mathrm{Reg}(f_S)$, $f(b_S)\notin f(S(F)\cap W_{\{0,1\}})$ and $f(b_S)\neq f(b_{S'})$ for all $S\neq S'$ in C.

It follows from $f(b_S)\notin f(S(F)\cap W_{\{0,1\}})$ that we can choose $\varepsilon\in(0,\varepsilon_W)$ so small that $f(b_S)\notin f(T)$ for all $S\in C$ and all $T\in I_{[0,\varepsilon]}\cup I_{[1-\varepsilon,1]}$. Moreover, since $0,1\in\mathrm{GenIm}(F)$, we can choose $\varepsilon\in(0,\varepsilon_W)$ so small that $f(T)\cap f(T')=\emptyset$ for all $T\neq T'$ in $I_{[0,\varepsilon]}\cup I_{[1-\varepsilon,1]}$ such that $\omega(T)=\omega(T')$.

Given $S\in I$, we would like to choose points $b_S^\pm\in T_S^\pm\cap W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ such that $f(b_S^\pm)\in\mathrm{Reg}(f_S)$. However, the image $f(T_S^\pm)$ might consist of only a single point. Therefore, in construction I, we use Proposition 3.3.7 to perturb F on a finite number of compact subsets of $W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$, such that the resulting fold map $F_1 \in \mathcal{F}^{\mathrm{pre}}(W)$ inherits all properties of F, and, in addition, $f_1(T) \subset \mathbb{R}$ (where $f_1 := \mathrm{Im} \circ F_1$) is a compact interval of positive length for all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$.

CONSTRUCTION I. There exist a fold map $F_1 \colon W \to \mathbb{C}$ and a compact subset $K_1 \subset W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ with the following properties (where $f_1 := \mathrm{Im} \circ F_1$):

(1) $F_1|_{W \setminus K_1} = F|_{W \setminus K_1}$.
(2) $S(F_1) = S(F)$.
(3) $f_1(b_S) \notin f_1(T)$ for all $S \in C$ and all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$.
(4) $f_1(T) \cap f_1(T') = \emptyset$ for all $T, T' \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ with $T \neq T'$ and $\omega(T) = \omega(T')$.
(5) The compact interval $f_1(T) \subset \mathbb{R}$ has positive length for all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$.

Let Ω be the set of all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ such that $f(T) = \{t_T\}$ for some $t_T \in \mathbb{R}$. Thus, if $T \in \Omega$, then F restricts to an immersion $T \to \mathrm{Im}^{-1}(t_T) = \mathbb{R} \times \{t_T\}$. Hence,

$(*)$ F restricts to an embedding $T \to \mathbb{R} \times \{t_T\}$ for every $T \in \Omega$.

By $(*)$, we can choose for every $T \in \Omega$ a point $p_T \in T \cap W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ such that $F(p_T) \neq F(p_{T'})$ for all $T, T' \in \Omega$ with $T \neq T'$. For every $T \in \Omega$ we choose an open neighbourhood V_T of $F(p_T)$ in $\mathbb{C}$ such that

$(V1)$ $f(b_S) \notin \mathrm{Im}(V_T)$ for all $S \in C$ and $T \in \Omega$.
$(V2)$ $\mathrm{Im}(V_T) \cap f(T') = \emptyset$ for all $T \in \Omega$ and all $T' \in I_{\omega(T)}$ with $T' \neq T$.
$(V3)$ $V_T \cap V_{T'} = \emptyset$ for all $T, T' \in \Omega$ with $T \neq T'$.
$(V4)$ $\mathrm{Im}(V_T) \cap \mathrm{Im}(V_{T'}) = \emptyset$ for all $T, T' \in \Omega$ with $T \neq T'$ and $\omega(T) = \omega(T')$.

(In fact, fix $T \in \Omega$. Since $f(b_S) \notin f(T)$ for all $S \in C$, it follows that $V_T^1 := \mathrm{Im}^{-1}(\mathbb{R} \setminus \{f(b_S)\}_{S \in C})$ is an open neighbourhood of $F(p_T)$ in $\mathbb{C}$ such that $(V1)$ holds. Moreover, since $f(T) \cap f(T') = \emptyset$ for all $T' \in I_{\omega(T)}$ with $T' \neq T$, it follows that $V_T^2 := \mathrm{Im}^{-1}(\mathbb{R} \setminus \{f(T')\}_{T' \in I_{\omega(T)} \setminus \{T\}})$ is an open neighbourhood of $F(p_T)$ in $\mathbb{C}$ such that $(V2)$ holds. Furthermore, as the points $F(p_{T'})$, $T' \in \Omega$, are pairwise distinct, there exist open neighbourhoods $V_{T'}^3$ of $F(p_{T'})$ in $\mathbb{C}$ for all $T' \in \Omega$ such that $(V3)$ holds. Additionally, since $f(T) \cap f(T') = \emptyset$ for all $T, T' \in \Omega$ with $T \neq T'$ and $\omega(T) = \omega(T')$, there exist open neighbourhoods V_T^4 of $F(p_T)$ in $\mathbb{C}$ such that $(V4)$ holds. Finally, the neighbourhoods $V_T := V_T^1 \cap V_T^2 \cap V_T^3 \cap V_T^4$ satisfy all required properties.)

For every $T \in \Omega$, define $U_T := F^{-1}(V_T) \cap W_{(0,\varepsilon)\cup(1-\varepsilon,1)} \cap W_{\omega(T)}$. By construction, U_T is an open neighbourhood of p_T in $W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ such that $F(U_T) \subset V_T$. Thus, F restricts to a fold map $F_T \colon U_T \to V_T$. The fold locus of F_T is given by

$(**)$ $S(F_T) = S(F) \cap U_T = T \cap U_T$ for all $T \in \Omega$.

(In fact, it follows from $U_T \subset W_{\omega(T)}$ that

$$S(F_T) = S(F) \cap U_T = S(F) \cap W_{\omega(T)} \cap U_T = \bigcup_{T' \in I_{\omega(T)}} T' \cap U_T.$$

Thus, it suffices to show that $T' \cap U_T = \emptyset$ for all $T' \in I_{\omega(T)}$ with $T \neq T'$. Indeed, for such T', we have $f(T' \cap U_T) \subset f(T') \cap f(U_T) = f(T') \cap \mathrm{Im}(F(U_T)) \subset f(T') \cap \mathrm{Im}(V_T) = \emptyset$ by $(V2)$.)

Note that $F_T(p_T) \neq F_T(p)$ for all $p \in S(F_T) \setminus \{p_T\}$. (In fact, by $(*)$, $F(p_T) \neq F(p)$ for all $p \in T \setminus \{p_T\}$. In particular, by $(**)$, $F_T(p_T) \neq F_T(p)$ for all $p \in (T \setminus \{p_T\}) \cap U_T = S(F_T) \setminus \{p_T\}$.) Application of Proposition 3.3.7 to F_T yields a fold map $\widetilde{F}_T \colon U_T \to V_T$ with the following properties:

(1') There exists a compact subset $K_T \subset U_T$ such that $\widetilde{F}_T|_{U_T \setminus K_T} = F_T|_{U_T \setminus K_T}$.
(2') $S(\widetilde{F}_T) = S(F_T)$.
(3') $F_T(p_T) \notin \widetilde{F}_T(S(F_T))$.

Define $K_1 := \bigsqcup_{T \in \Omega} K_T$ and $U_1 := \bigsqcup_{T \in \Omega} U_T$. (Note that the pairwise disjointness of the U_T follows from $(V3)$.) By (1'), K_1 is compact, and $K_1 \subset U_1 \subset W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$. Consider the open covering $W = U_1 \cup (W \setminus K_1)$. The fold maps $\widetilde{F}_1 := \bigsqcup_{T \in \Omega} \widetilde{F}_T$ on U_1 and $F|_{W \setminus K_1}$ on $W \setminus K_1$ agree by (1') on the open subset $U_1 \cap (W \setminus K_1) = U_1 \setminus K_1 = \bigsqcup_{T \in \Omega} U_T \setminus K_T$ of W. Thus, we can assemble these maps to obtain the fold map

$$F_1 \colon W \to \mathbb{C}, \qquad F_1(p) = \begin{cases} \widetilde{F}_1(p), & \text{for } p \in U_1, \\ F(p), & \text{for } p \notin U_1. \end{cases}$$

We consider the sets $F_1(T)$ for $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$. If $T \notin \Omega$, then $T \cap U_{T'} = T \cap S(F) \cap U_{T'} = T \cap T' \cap U_{T'} = \emptyset$ by $(**)$ for every $T' \in \Omega$. Therefore, $T \cap U_1 = \emptyset$. Hence, we obtain

(i) $F_1(T) = F(T)$ for all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ with $T \notin \Omega$.

If $T \in \Omega$, then we write $T = (T \cap U_T) \cup (T \setminus U_T)$. We have $F_1(T \cap U_T) = \widetilde{F}_T(T \cap U_T) \subset V_T$. Moreover, $(T \setminus U_T) \cap U_{T'} = (T \setminus U_T) \cap S(F) \cap U_{T'} = (T \setminus U_T) \cap T' \cap U_{T'} = \emptyset$ by $(**)$ for all $T' \in \Omega$. Therefore, $(T \setminus U_T) \cap U_1 = \emptyset$. Hence, $F_1(T \setminus U_T) = F(T \setminus U_T) \subset F(T)$. All in all,

(ii) $F_1(T) = F_1(T \cap U_T) \cup F(T \setminus U_T) \subset V_T \cup F(T)$ for all $T \in \Omega$.

Let us check that F_1 satisfies the claimed properties:

(1). We have $W \setminus K_1 = (W \setminus U_1) \cup (U_1 \setminus K_1)$, where $U_1 \setminus K_1 = \bigsqcup_{T \in \Omega} U_T \setminus K_T$. By definition, F_1 and F agree on $W \setminus U_1$. Moreover, for every $T \in \Omega$ we have $F_1|_{U_T \setminus K_T} = \widetilde{F}_T|_{U_T \setminus K_T} = F|_{U_T \setminus K_T}$ by (1'). Thus, F_1 and F also agree on $U_1 \setminus K_1$. Hence, $F_1|_{W \setminus K_1} = F|_{W \setminus K_1}$.

(2). As $W = U_1 \cup (W \setminus K_1)$, we have $S(F_1) = \bigsqcup_{T \in \Omega} S(F_1|_{U_T}) \cup S(F_1|_{W \setminus K_1})$. We have $S(F_1|_{U_T}) = S(\widetilde{F}_T) = S(F_T) = S(F|_{U_T})$ by (2'). Moreover, it follows from (1) that $S(F_1|_{W \setminus K_1}) = S(F|_{W \setminus K_1})$. Thus, $S(F_1) = \bigsqcup_{T \in \Omega} S(F|_{U_T}) \cup S(F|_{W \setminus K_1}) = S(F)$.

(3). Let $S \in C$ and $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$. Since $K_1 \subset W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ and $\omega(S) \subset (\varepsilon, 1-\varepsilon)$, it follows from (1) that $f_1(b_S) = f(b_S)$. If $T \notin \Omega$, then it follows from (i) that $f_1(b_S) = f(b_S) \notin f(T) = f_1(T)$. If $T \in \Omega$, then it follows from (ii) that $f_1(T) \subset \operatorname{Im}(V_T) \cup f(T)$. Since $f(b_S) \notin \operatorname{Im}(V_T)$ by $(V1)$, and $f(b_S) \notin f(T)$, it follows that $f_1(b_S) = f(b_S) \notin f_1(T)$.

(4). Let $T, T' \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$, $T \neq T'$, $\omega(T) = \omega(T')$. There are three cases to consider:

- $T, T' \notin \Omega$. It follows from (i) that $f_1(T) \cap f_1(T') = f(T) \cap f(T') = \emptyset$.
- $T \notin \Omega$ and $T' \in \Omega$. It follows from (i) that $f_1(T) = f(T)$ and from (ii) that $f_1(T') \subset \operatorname{Im}(V_{T'}) \cup f(T')$. Hence, $f_1(T) \cap f_1(T') \subset (f(T) \cap \operatorname{Im}(V_{T'})) \cup (f(T) \cap f(T')) = \emptyset$ by $(V2)$.
- $T, T' \in \Omega$. It follows from (ii) that $f_1(T) \subset \operatorname{Im}(V_T) \cup f(T)$ and $f_1(T') \subset \operatorname{Im}(V_{T'}) \cup f(T')$. Hence, $f_1(T) \cap f_1(T') \subset (\operatorname{Im}(V_T) \cap \operatorname{Im}(V_{T'})) \cup (\operatorname{Im}(V_T) \cap f(T')) \cup (f(T) \cap \operatorname{Im}(V_{T'})) \cup (f(T) \cap f(T')) = \emptyset$ by $(V2)$ and $(V4)$.

(5). Let $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$. If $T \notin \Omega$, then, by (i), $f_1(T) = f(T)$. By definition of Ω, this is a compact interval of positive length. Now assume that $T \in \Omega$. In this case, we note that $F(p_T) \notin F_1(T)$. (Indeed, write $F_1(T) = F_1(T \setminus U_T) \cup F_1(T \cap U_T) = F(T \setminus U_T) \cup \widetilde{F}_T(S(F_T))$. We have $F(p_T) \notin F(T \setminus U_T)$, because $p_T \in U_T \cap T$ and F is injective on T by $(*)$. But by (3'), we also have $F(p_T) = F_T(p_T) \notin \widetilde{F}_T(S(F_T))$.) Let x and x' be the endpoints of the interval

T. It follows from $U_T \subset W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ and $\omega(x), \omega(x') \in \{0, \varepsilon, 1-\varepsilon, 1\}$ that $x, x' \notin U_T$. Thus, $f_1(\{x,x'\}) = f(\{x,x'\}) = \{t_T\}$. In particular, $t_T \in f_1(T)$. Assume that $f_1(T) = \{t_T\}$. Then, using $F(p_T) \notin F_1(T)$, we have $F_1(T) \subset \mathbb{R} \times \{t_T\} \setminus \{F(p_T)\}$. As $F_1(T)$ is connected, it follows that $F_1(x) = F(x)$ and $F_1(x') = F(x')$ lie in the same component of $\mathbb{R} \times \{t_T\} \setminus \{F(p_T)\}$ (note that $F(p_T) \in \mathbb{R}\times\{t_T\}$). But this is impossible, since F restricts to an embedding $T \to \mathbb{R}\times\{t_T\}$, which implies that $F(x)$ and $F(x')$ lie in different components of $\mathbb{R} \times \{t_T\} \setminus \{F(p_T)\}$. Thus, the compact interval $f_1(T)$ consists not only of the point t_T.

This completes construction I.

For all $S \in C$ we have $S \subset W_{(\varepsilon,1-\varepsilon)}$ and for all $S \neq S'$ in C we have $f(b_S) \neq f(b_{S'})$. Thus, it follows from (1) that f_1 restricts to an injective map $\{b_S\}_{S\in C} \to \mathbb{R}$. For every $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ we use (5) to choose a nonempty open subset $U_T \subset \mathbb{R}$ such that $U_T \subset f_1(T) \setminus \{f_1(b_S);\ S \in C\}$. Moreover, we may assume that $U_T \cap U_{T'} = \emptyset$ for all $T \neq T'$ in $I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$. As $\mathrm{Reg}((f_1)_S)$ is dense in $\mathbb{R}$ for every $S \in I$ by the Morse-Sard theorem (where $(f_1)_S \colon S \to \mathbb{R}$ denotes the restriction of $f_1 \colon W \to \mathbb{R}$ to S), we have $\mathrm{Reg}((f_1)_S) \cap U_T \neq \emptyset$ for all $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$. Thus, for every $S \in I$ we can choose points $b_S^\pm \in T_S^\pm \cap W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$ such that $f_1(b_S^\pm) \in \mathrm{Reg}((f_1)_S) \cap U_{T_S^\pm}$. By construction, f_1 restricts to an injective map $\{b_S^-, b_S^+\}_{S\in I} \cup \{b_S\}_{S\in C} \to \mathbb{R}$.

Let $C_+ \subset C$ be the subset of all $S \in C$ which are not contained in a single slice W_t for some $t \in [0,1]$. For every $S \in C_+$ we proceed in the following way. As the compact interval $\omega(S)$ consists of more than one point, there exists a nonempty open subset $V_S \subset \mathbb{R}$ such that $V_S \subset \omega(S) \setminus \{\omega(b_S)\}$. As $\pitchfork(F)$ is dense in $[0,1]$ by [4, Corollary 7.8], we have $\pitchfork(F) \cap V_S \neq \emptyset$. Thus, one can choose a point $a_S \in S \setminus \{b_S\}$ such that $\omega(a_S) \in \pitchfork(F)$. We fix an immersion $\alpha_S \colon [-1,1] \to S$, such that $\alpha_S(-1) = \alpha_S(0) = \alpha_S(1) = a_S$ and α_S restricts to diffeomorphisms $\alpha_S^- \colon (-1,0) \xrightarrow{\cong} S \setminus \{a_S\}$ and $\alpha_S^+ \colon (0,1) \xrightarrow{\cong} S \setminus \{a_S\}$. Define $x_S^- := (\alpha_S^-)^{-1}(b_S) \in (-1,0)$ and $x_S^+ := (\alpha_S^+)^{-1}(b_S) \in (0,1)$. Note that α_S restricts to a diffeomorphism $(x_S^-, x_S^+) \xrightarrow{\cong} S \setminus \{b_S\}$.

For every $S \in I$ we proceed in the following way. It follows from $\varepsilon, 1-\varepsilon \in \pitchfork(F)$ and $S \cap W_{\{\varepsilon,1-\varepsilon\}} \neq \emptyset$ that there exists a nonempty open subset $V_S \subset \mathbb{R}$ such that $V_S \subset \omega(S) \cap (\varepsilon, 1-\varepsilon)$. As $\pitchfork(F)$ is dense in $[0,1]$ by [4, Corollary 7.8], we have $\pitchfork(F) \cap V_S \neq \emptyset$. Thus, one can choose a point $a_S \in S \cap W_{(\varepsilon,1-\varepsilon)}$ such that $\omega(a_S) \in \pitchfork(F)$. We fix a diffeomorphism $\alpha_S \colon [-1,1] \xrightarrow{\cong} S$, such that $\alpha_S(0) = a_S$. Thus, there exist unique $t_S^- \in (-1,0)$ and $t_S^+ \in (0,1)$ such that $T_S^- = \alpha_S([-1,t_S^-])$ and $T_S^+ = \alpha_S([t_S^+,1])$. Define $x_S^- := \alpha_S^{-1}(b_S^-) \in (-1, t_S^-)$ and $x_S^+ := \alpha_S^{-1}(b_S^+) \in (t_S^+, 1)$.

By construction, $f_1 \circ \alpha_S$ is nonsingular at $x_S^\pm$ for every $S \in C_+ \cup I$. Thus, for every $S \in C_+ \cup I$ we can choose $y_S^- \in (x_S^-, 0)$ and $y_S^+ \in (0, x_S^+)$ such that f_1 is injective on $B_S^- := \alpha_S([x_S^-, y_S^-])$ and on $B_S^+ := \alpha_S([y_S^+, x_S^+])$. For all $S \in I$ we may assume that $y_S^- \in (x_S^-, t_S^-)$ and $y_S^+ \in (t_S^+, x_S^+)$. As f_1 restricts to an injective map $\{b_S^-, b_S^+\}_{S\in I} \cup \{b_S\}_{S\in C} \to \mathbb{R}$, we may assume that f_1 restricts to an injective map $\bigcup_{S\in C_+\cup I}(B_S^- \cup B_S^+) \to \mathbb{R}$.

Construction II. There exist a diffeomorphism $\Psi \colon W \xrightarrow{\cong} W$, a compact subset $L \subset W_{(\varepsilon,1-\varepsilon)}$, and for every $S \in C_+ \cup I$ points $z_S^- \in (y_S^-, 0)$ and $z_S^+ \in (0, y_S^+)$, such that the following properties hold (where $\Psi_S := \Psi \circ \alpha_S$):

(Ψ1) $\Psi(p) = p$ for all $p \in W \setminus L$.
(Ψ2) $S(F) \cap L \subset \bigcup_{S\in C_+\cup I} \alpha_S((y_S^-, y_S^+))$.
(Ψ3) For every $S \in C_+ \cup I$ we have $\Psi_S([z_S^-, z_S^+]) \subset W_{\omega(a_S)}$.

Let $S \in C_+ \cup I$. By construction, the point $a_S \in S \cap W_{\omega(a_S)}$ satisfies $\omega(a_S) \in \pitchfork(F)$. In particular, $W_{\omega(a_S)}$ is a $(n-1)$-dimensional smooth submanifold of W and $S \pitchfork W_{\omega(a_S)}$ at a_S. By [17, Lemma III.3.10], S and $W_{\omega(a_S)}$ can be simultaneously linearized in a neighbourhood of a_S. More precisely, there exists an open neighborhood X_S of a_S in $W_{(\varepsilon,1-\varepsilon)}$ and a diffeomorphism $\psi_S\colon X_S \xrightarrow{\cong} \mathbb{R}^n$ such that $H_1 := \psi_S(X_S \cap S)$ is a 1-dimensional vector subspace of $\mathbb{R}^n$ and $H_2 := \psi_S(X_S \cap W_{\omega(a_S)})$ is a $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$, such that $H_1 \oplus H_2 = \mathbb{R}^n$. After a base change we may assume that $H_1 = 0 \times \mathbb{R}$ and $H_2 = \mathbb{R}^{n-1} \times 0$ in $\mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$. The charts $\psi_S\colon X_S \to \mathbb{R}^n$ can be chosen for all $S \in C_+ \cup I$ in such a way that $X_S \cap X_{S'} = \emptyset$ for all $S \neq S'$ in $C_+ \cup I$.

Let $S \in C_+ \cup I$. Choose $R_S > 0$ such that $S(F) \cap L_S \subset \alpha_S((y_S^-, y_S^+))$, where L_S denotes the compact subset $\psi_S^{-1}(\{x \in \mathbb{R}^n;\ ||x|| \leq R_S\}) \subset W_{(\varepsilon,1-\varepsilon)}$. Choose a smooth map $\rho_S\colon [0,\infty) \to \mathbb{R}$ such that $\rho_S(r) = 1$ for $r \in [0, \frac{R_S}{2}]$ and $\rho_S(r) = 0$ for $r \geq R_S$. Using the notation $x = (x_1, ..., x_n) = (y, x_{n-1}, x_n)$ for points $x \in \mathbb{R}^n$, where $y = (x_1, ..., x_{n-2}) \in \mathbb{R}^{n-2}$, and the identification $\mathbb{R}^2 \xrightarrow{\cong} \mathbb{C}$, $(x_{n-1}, x_n) \mapsto z = x_{n-1} + ix_n \in \mathbb{C}$, we define the smooth map

$$\gamma_S\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \gamma_S(x) = (y, e^{i\frac{\pi}{2}\rho_S(||x||)}z).$$

(Note that γ_S is clearly smooth in all points $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, if $x \in \mathbb{R}^n$ and $||x|| \leq \frac{R_S}{2}$, then $\gamma_S(x) = (y, e^{i\frac{\pi}{2}}z) = (y, iz) = (y, -x_n, x_{n-1})$. Thus, γ_S is also smooth in $x = 0$.) Obviously, γ_S is a diffeomorphism, whose inverse is given by the smooth map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto (y, e^{-i\frac{\pi}{2}\rho_S(||x||)}z)$.

Consider the open covering $W = (\bigsqcup_{S\in C_+\cup I} X_S) \cup (W \setminus \{L_S\}_{S\in C_+\cup I})$, where the open subsets $X_S \subset W$ are pairwise disjoint, and $L_S \subset X_S$ are compact subsets. For every $S \in C_+ \cup I$, the diffeomorphism $\psi_S^{-1} \circ \gamma_S \circ \psi_S\colon X_S \xrightarrow{\cong} X_S$ restricts to the identity map on $X_S \setminus L_S$. (In fact, if $p \in X_S \setminus L_S$, then $||\psi_S(p)|| > R_S$ and $\rho_S(||\psi_S(p)||) = 0$. Hence, $(\psi_S^{-1} \circ \gamma_S \circ \psi_S)(p) = (\psi_S^{-1} \circ \psi_S)(p) = p$.) Thus, the diffeomorphism $\sqcup_S(\psi_S^{-1}\circ\gamma_S\circ\psi_S)\colon \bigsqcup_{S\in C_+\cup I} X_S \xrightarrow{\cong} \bigsqcup_{S\in C_+\cup I} X_S$ agrees with the identity map on the open subset $(\bigsqcup_{S\in C_+\cup I} X_S)\cap(W\setminus\{L_S\}_{S\in C_+\cup I}) = \bigsqcup_{S\in C_+\cup I} X_S\setminus L_S$ of W. Thus, we obtain a diffeomorphism

$$\Psi\colon W \xrightarrow{\cong} W, \qquad \Psi(p) = \begin{cases} (\psi_S^{-1} \circ \gamma_S \circ \psi_S)(p), & \text{if } p \in X_S \text{ for some } S \in C_+ \cup I, \\ p, & \text{if } p \notin X_S \text{ for all } S \in C_+ \cup I. \end{cases}$$

Let us check that Ψ satisfies the desired properties:

$(\Psi 1)$. By construction, Ψ is the identity map outside the compact subset $L := \bigcup_{S\in C_+\cup I} L_S \subset W_{(\varepsilon,1-\varepsilon)}$.

$(\Psi 2)$. We have $S(F)\cap L = S(F)\cap(\bigcup_{S\in C_+\cup I} L_S) = \bigcup_{S\in C_+\cup I}(S(F)\cap L_S) \subset \bigcup_{S\in C_+\cup I} \alpha_S((y_S^-, y_S^+))$.

$(\Psi 3)$. Let $S \in C_+ \cup I$. The interval $J_S := 0 \times [-\frac{R_S}{2}, \frac{R_S}{2}] \subset H_1$ satisfies $\psi_S^{-1}(J_S) \subset L_S$ and $\psi_S^{-1}(J_S) \subset \psi_S^{-1}(H_1) = X_S \cap S \subset S(F)$. Therefore, $\psi_S^{-1}(J_S) \subset S(F) \cap L_S \subset \alpha_S((y_S^-, y_S^+))$. As $\psi_S^{-1}(0) = a_S$, we have $(\psi_S \circ \alpha_S)^{-1}(\partial J_S) \subset (y_S^-, 0) \cup (0, y_S^+)$. Define $z_S^- := (\psi_S \circ \alpha_S)^{-1}(\partial J_S) \cap (y_S^-, 0)$ and $z_S^+ := (\psi_S \circ \alpha_S)^{-1}(\partial J_S) \cap (0, y_S^+)$. Then, $\alpha_S([z_S^-, z_S^+]) = \psi_S^{-1}(J_S) \subset X_S$. Moreover, $\gamma_S(y, x_{n-1}, x_n) = (0, -x_n, 0) \in \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$ for all points $(y, x_{n-1}, x_n) = (0, 0, x_n) \in J_S$. Thus, $\gamma_S(J_S) \subset H_2$ and $\psi_S^{-1}(\gamma_S(J_S)) \subset \psi_S^{-1}(H_2) = X_S \cap W_{\omega(a_S)} \subset W_{\omega(a_S)}$. Finally, $\Psi_S([z_S^-, z_S^+]) = \Psi(\alpha_S([z_S^-, z_S^+])) = \Psi(\psi_S^{-1}(J_S)) = (\psi_S^{-1} \circ \gamma_S)(J_S) \subset W_{\omega(a_S)}$.

This completes construction II.

Construction III. There exist a diffeomorphism $\Phi\colon W \xrightarrow{\cong} W$ and a compact subset $P \subset W_{(0,1)}$ with the following properties:

(Φ1) $\Phi(p) = p$ for all $p \in W \setminus P$.
(Φ2) $\Psi(S(F)) \cap P \subset \bigcup_{S \in C_+ \cup I} \Psi_S((x_S^-, x_S^+))$.
(Φ3) For every $S \in C_+ \cup I$ we have $\Phi(\Psi_S([y_S^-, y_S^+])) \subset \Psi_S([z_S^-, z_S^+])$.
(Φ4) For every $S \in C_+ \cup I$ we have $\Phi(\Psi(S)) = \Psi(S)$.
(Φ5) For every $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ we have $\Phi(T) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} \subset T$.

Let $S \in C_+ \cup I$. We choose a tubular neighbourhood of $\Psi(S)$ in W. (This is possible by [22, Chapter 4, Theorem 6.2], because $\Psi(S)$ is a neat submanifold of W.) As $\Psi_S((x_S^-, x_S^+)) \subset W_{(0,1)}$ is an embedded interval, any vector bundle on $\Psi_S((x_S^-, x_S^+))$ is trivial. Therefore, the tubular neighbourhood of $\Psi(S)$ in W yields an embedding $\phi_S\colon (x_S^-, x_S^+) \times \mathbb{R}^{n-1} \to W_{(0,1)}$ such that $\phi_S(t, 0) = \Psi_S(t)$ for all $t \in (x_S^-, x_S^+)$ and $Y_S := \phi_S((x_S^-, x_S^+) \times \mathbb{R}^{n-1})$ is an open subset of $W_{(0,1)}$. The embeddings $\phi_S\colon (x_S^-, x_S^+) \times \mathbb{R}^{n-1} \to W_{(0,1)}$ can be chosen for all $S \in C_+ \cup I$ such that $Y_S \cap Y_{S'} = \emptyset$ for all $S \neq S'$ in $C_+ \cup I$.

Let $S \in C_+ \cup I$. We choose points $w_S^- \in (x_S^-, y_S^-)$ and $w_S^+ \in (y_S^+, x_S^+)$ and a diffeomorphism $\tau_S\colon \mathbb{R} \xrightarrow{\cong} \mathbb{R}$ such that $\tau_S([y_S^-, y_S^+]) \subset [z_S^-, z_S^+]$ and $\tau_S(t) = t$ for all $t \in \mathbb{R} \setminus (w_S^-, w_S^+)$. Thus, the smooth map

$$\sigma_S\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \qquad \sigma_S(s,t) = s\tau_S(t) + (1-s)t,$$

satisfies $\sigma_S(s,t) = t$ for all $s \in \mathbb{R}$ and $t \in \mathbb{R} \setminus (w_S^-, w_S^+)$. Choose $r_S > 0$ such that the compact subset $P_S := \phi_S([w_S^-, w_S^+] \times \{x \in \mathbb{R}^{n-1};\ ||x|| \leq r_S\}) \subset Y_S \subset W_{(0,1)}$ satisfies $\Psi(S(F)) \cap P_S = \Psi_S([w_S^-, w_S^+])$. Choose a smooth function $\rho_S\colon [0,\infty) \to \mathbb{R}$ such that $\rho_S(r) = 1$ for $r \in [0, \frac{r_S}{2}]$, $\rho_S(r) = 0$ for $r \geq r_S$, and $\rho_S(r) \in [0,1]$ for all $r \in [0,\infty)$. Define the smooth map

$$\eta_S\colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R} \times \mathbb{R}^{n-1}, \qquad \eta_S(t,x) = (\sigma_S(\rho_S(||x||), t), x).$$

(η_S is clearly smooth at all points in $\mathbb{R} \times (\mathbb{R}^{n-1} \setminus \{0\})$. Moreover, if $(t,x) \in \mathbb{R} \times \{x \in \mathbb{R}^{n-1};\ ||x|| \leq \frac{r_S}{2}\}$, then $\eta_S(t,x) = (\sigma_S(1,t), x) = (\tau_S(t), x)$. Thus, η_S is also smooth at all points in $\mathbb{R} \times 0$.) It follows from $\rho_S(r) \in [0,1]$ for all $r \in [0,\infty)$ that η_S is bijective. (Note that σ_S restricts to a diffeomorphism $\{s\} \times \mathbb{R} \xrightarrow{\cong} \mathbb{R}$ for every $s \in [0,1]$. In fact, it follows from $s \in [0,1]$ and $\tau_S'(t) > 0$ that $\partial_t \sigma_S(s,t) = s\tau_S'(t) + (1-s) > 0$ for all $t \in \mathbb{R}$.) Moreover, η_S is a diffeomorphism. (Note that $\det d\eta_S(t,x) = \partial_t \sigma_S(\rho_S(||x||), t) > 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}$.) Note that η_S restricts to a diffeomorphism $(x_S^-, x_S^+) \times \mathbb{R}^{n-1} \xrightarrow{\cong} (x_S^-, x_S^+) \times \mathbb{R}^{n-1}$. (In fact, if $t \in \mathbb{R} \setminus (x_S^-, x_S^+)$, then $\tau_S(t) = t$ and $\sigma_S(\rho_S(||x||), t) = t$ for all $x \in \mathbb{R}^{n-1}$.)

Consider the open covering $W = (\bigsqcup_{S \in C_+ \cup I} Y_S) \cup (W \setminus \{P_S\}_{S \in C_+ \cup I})$, where the open subsets $Y_S \subset W$ are pairwise disjoint, and $P_S \subset Y_S$ are compact subsets. For every $S \in C_+ \cup I$, the diffeomorphism $\phi_S \circ \eta_S \circ \phi_S^{-1}\colon Y_S \xrightarrow{\cong} Y_S$ restricts to the identity map on $Y_S \setminus P_S$. (In fact, every point $p \in Y_S \setminus P_S$ is of the form $\phi_S^{-1}(p) = (t,x) \in (x_S^-, x_S^+) \times \mathbb{R}^{n-1}$, where $t \notin [w_S^-, w_S^+]$ or $||x|| > r_S$. If $t \notin [w_S^-, w_S^+]$, then $\tau_S(t) = t$, $\sigma_S(\rho_S(||x||), t) = t$ and $(\phi_S \circ \eta_S \circ \phi_S^{-1})(p) = (\phi_S \circ \eta_S)(t,x) = \phi_S(t,x) = p$. Moreover, if $||x|| > r_S$, then $\rho_S(||x||) = 0$ and $(\phi_S \circ \eta_S \circ \phi_S^{-1})(p) = (\phi_S \circ \eta_S)(t,x) = \phi_S(\sigma_S(0,t), x) = \phi_S(t,x) = p$.) Thus, the diffeomorphism $\sqcup_S(\phi_S \circ \eta_S \circ \phi_S^{-1}\colon \bigsqcup_{S \in C_+ \cup I} Y_S \xrightarrow{\cong} \bigsqcup_{S \in C_+ \cup I} Y_S$ agrees with the identity map on the open subset $(\bigsqcup_{S \in C_+ \cup I} Y_S) \cap (W \setminus \{P_S\}_{S \in C_+ \cup I}) = \bigsqcup_{S \in C_+ \cup I} Y_S \setminus P_S = (\bigsqcup_{S \in C_+ \cup I} Y_S) \setminus P$ of W, where we have defined the compact subset $P := \bigcup_{S\in C_+\cup I} P_S \subset W_{(0,1)}$. Thus, we obtain a diffeomorphism

$$\Phi\colon W \xrightarrow{\cong} W, \qquad \Phi(p) = \begin{cases} (\phi_S \circ \eta_S \circ \phi_S^{-1})(p), & \text{if } p \in Y_S \text{ for some } S \in C_+ \cup I, \\ p, & \text{if } p \notin Y_S \text{ for all } S \in C_+ \cup I. \end{cases}$$

By construction, the following holds for every $S \in C_+ \cup I$ and all $t \in [x_S^-, x_S^+]$:

$$(*) \qquad (\Phi \circ \Psi_S)(t) = \Phi(\phi_S(t,0)) = (\phi_S \circ \eta_S)(t,0) = \phi_S(\sigma_S(1,t),0) = \phi_S(\tau_S(t),0) = (\Psi_S \circ \tau_S)(t).$$

Let us check that Φ and P satisfy the desired properties:

$(\Phi 1)$. This is clear by construction.

$(\Phi 2)$. This follows from $\Psi(S(F)) \cap P = \Psi(S(F)) \cap (\bigcup_{S\in C_+\cup I} P_S) = \bigcup_{S\in C_+\cup I}(\Psi(S(F)) \cap P_S)$ and $\Psi(S(F)) \cap P_S = \Psi_S([w_S^-, w_S^+]) \subset \Psi_S((x_S^-, x_S^+))$ for all $S \in C_+ \cup I$.

$(\Phi 3)$. Let $S \in C_+ \cup I$. By construction, we have $\tau_S([y_S^-, y_S^+]) \subset [z_S^-, z_S^+]$. Thus, we obtain from $(*)$ that $\Phi(\Psi_S([y_S^-, y_S^+])) = \Psi_S(\tau_S([y_S^-, y_S^+])) \subset \Psi_S([z_S^-, z_S^+])$.

$(\Phi 4)$. Let $S \in C_+ \cup I$. By $(\Phi 1)$, we have $\Phi(\Psi(S) \setminus P) = \Psi(S) \setminus P$. Moreover, $\Psi(S) \cap P = \Psi(S) \cap P_S = \Psi_S([w_S^-, w_S^+])$. (Note that for every $S' \in C_+ \cup I$ we have $\Psi(S) \cap P_{S'} \subset \Psi(S(F)) \cap P_{S'} = \Psi_{S'}([w_{S'}^-, w_{S'}^+]) \subset \Psi(S')$. Thus, if $S \neq S'$, then $\Psi(S) \cap P_{S'} \subset \Psi(S) \cap \Psi(S') = \emptyset$. Hence, $\Psi(S) \cap P = \Psi(S) \cap \bigcup_{S'\in C_+\cup I} P_{S'} = \Psi(S) \cap P_S$.) As the diffeomorphism τ_S restricts to the identity map on $\mathbb{R} \setminus [w_S^-, w_S^+]$, we have $\tau_S([w_S^-, w_S^+]) = [w_S^-, w_S^+]$. Thus, we obtain from $(*)$ that $\Phi(\Psi(S) \cap P) = \Phi(\Psi_S([w_S^-, w_S^+])) = \Psi_S(\tau_S([w_S^-, w_S^+])) = \Psi_S([w_S^-, w_S^+]) = \Psi(S) \cap P$.

$(\Phi 5)$. Let $S \in I$. For all $t, t' \in \mathbb{R}$, $t < t'$, we have $\tau_S(t) < \tau_S(t')$. Thus, it follows from $t_S^- < 0 < t_S^+ < y_S^+$ that $\tau_S(t_S^-) < \tau_S(y_S^+)$. By choice of τ_S, we have $\tau_S(y_S^+) \leq z_S^+$. Moreover, $z_S^+ < t_S^+$. (In fact, if we assume that $z_S^+ \geq t_S^+$, then $\alpha_S(z_S^+) \in W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. Thus, using $(\Psi 1)$ and $(\Psi 3)$, we obtain $\alpha_S(z_S^+) = \Psi(\alpha_S(z_S^+)) = \Psi_S(z_S^+) \in W_{\omega(a_S)}$. This leads to the contradiction $\alpha_S(z_S^+) \in W_{\omega(a_S)} \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} = \emptyset$.) All in all, we obtain $\tau_S(t_S^-) < \tau_S(y_S^+) \leq z_S^+ < t_S^+$. Hence, $\tau_S([-1, t_S^-]) \subset [-1, t_S^+)$. Consequently, $\tau_S([-1, t_S^-]) \cap ([-1, t_S^-] \cup [t_S^+, 1]) \subset [-1, t_S^-]$. Application of Ψ_S and $(*)$ yields $\Phi(T_S^-) \cap (T_S^- \cup T_S^+) \subset T_S^-$. Note that $\Phi(T_S^-) \cap (T_S^- \cup T_S^+) = \Phi(T_S^-) \cap (\Psi(S) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}) = \Phi(T_S^-) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. Analogously, one can show that $\Phi(T_S^+) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} \subset T_S^+$.

This completes construction III.

The diffeomorphism $\Xi := \Phi \circ \Psi\colon W \xrightarrow{\cong} W$ and the subset $Q := L \cup \Psi^{-1}(P) \subset W_{(0,1)}$ satisfy the following properties (where $\Xi_S := \Xi \circ \alpha_S$):

- $(\Xi 1)$ Ξ is the identity map outside the compact subset $Q \subset W_{(0,1)}$.
- $(\Xi 2)$ $\Xi(S(F)) \cap Q \subset \bigcup_{S\in C_+\cup I} \Xi_S((x_S^-, x_S^+))$.
- $(\Xi 3)$ For every $S \in C_+ \cup I$ we have $\Xi_S([y_S^-, y_S^+]) \subset W_{\omega(a_S)}$.
- $(\Xi 4)$ For every $S \in C \cup I$ we have $\Xi(S) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} \subset \Xi(S \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]})$.
- $(\Xi 5)$ For every $T \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ we have $\Xi(T) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} \subset T$.

Let us check the claimed properties:

$(\Xi 1)$. This follows from $(\Psi 1)$ and $(\Phi 1)$.

$(\Xi 2)$. This follows from $(\Psi 2)$ and $(\Phi 2)$.

$(\Xi 3)$. This follows from $(\Psi 3)$ and $(\Phi 3)$.

$(\Xi 4)$. Let $S \in C \cup I$ and $p \in \Xi(S) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. We have to show that the point $q := \Xi^{-1}(p) \in S$ lies in $W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. By $(\Phi 4)$, there exists $q' \in S$ such that $p = \Xi(q) = \Psi(q')$. Hence, it follows from $p \in W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$ and $(\Psi 1)$ that $\Psi(p) = p = \Psi(q')$ and thus $p = q' \in S$. In particular, we obtain from $p \in S \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$ that $S \in I$. Next, it follows from $p \in W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$ and $(\Xi 3)$ that $q \notin \alpha_S([y_S^-, y_S^+])$. Moreover, we have $y_S^- \in (x_S^-, t_S^-)$ and $y_S^+ \in (t_S^+, x_S^+)$, as $S \in I$. Finally, $q \in \alpha_S([-1, y_S^-) \cup (y_S^+, 1]) \subset \alpha_S([-1, t_S^-] \cup [t_S^+, 1]) \subset W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$.

$(\Xi 5)$. This follows from $(\Psi 1)$ and $(\Phi 5)$.

Define $G := (F_1)_{\Xi^{-1}}$. By construction I, we have $F_1 \in \mathcal{F}^{\mathrm{pre}}(W)$. By (1) and (2) we have $F_1 \sim_W F$. By $(\Xi 1)$ and Lemma 3.4.7, we obtain $G = (F_1)_{\Xi^{-1}} \in \mathcal{F}^{\mathrm{pre}}(W)$, $S(G) = S((F_1)_{\Xi^{-1}}) = \Xi(S(F_1)) \overset{(2)}{=} \Xi(S(F))$, and $G = (F_1)_{\Xi^{-1}} \sim_W F_1 \sim_W F$. It remains to show (ff2), which states that $\mathrm{GenIm}(G)$ is residual in $[0,1]$.

$[0,1] \setminus (\{\omega(a_S)\}_{S \in C_+ \cup I} \cup \{\omega(b_S)\}_{S \in C \setminus C_+})$ is residual in $[0,1]$, since a finite number of points is deleted. Let $t \in [0,1] \setminus (\{\omega(a_S)\}_{S \in C_+ \cup I} \cup \{\omega(b_S)\}_{S \in C \setminus C_+})$. It suffices to show that $g := \mathrm{Im} \circ G$ restricts to an injective map $S(G) \cap W_t \to \mathbb{R}$. Assume that $p, p' \in S(G) \cap W_t$ and $p \neq p'$. We have to show that $g(p) \neq g(p')$. Since $S(G) = \Xi(S(F))$, we can write $p = \Xi(q)$ and $p' = \Xi(q')$ for $q \neq q'$ in $S(F)$. Since $g = \mathrm{Im} \circ G = \mathrm{Im} \circ (F_1)_{\Xi^{-1}} = \mathrm{Im} \circ F_1 \circ \Xi^{-1} = f_1 \circ \Xi^{-1}$, it suffices to show that $f_1(q) \neq f_1(q')$.

Choose $S, S' \in C \cup I$ such that $q \in S$ and $q' \in S'$. Then it follows from $t \notin \{\omega(b_S)\}_{S \in C \setminus C_+}$ that $S, S' \in C_+ \cup I$. (Indeed, assume $S \in C \setminus C_+$. Since $q \in S$, it follows from $(\Xi 1)$ and $(\Xi 2)$ that $p = \Xi(q) = q$. Hence, we obtain $t = \omega(p) = \omega(q) = \omega(S) = \omega(b_S)$, a contradiction to $t \notin \{\omega(b_S)\}_{S \in C \setminus C_+}$.) Thus, we can choose $u, u' \in [-1,1]$ such that $q = \alpha_S(u)$ and $q' = \alpha_{S'}(u')$. If $S \in C_+$, then we may assume without loss of generality that $u \in [x_S^-, x_S^+]$. Analogously, if $S' \in C_+$, then we may assume that $u' \in [x_{S'}^-, x_{S'}^+]$. It follows from $t \notin \{\omega(a_S)\}_{S \in C_+ \cup I}$ that $u \notin [y_S^-, y_S^+]$ and $u' \notin [y_{S'}^-, y_{S'}^+]$. (In fact, assume that $u \in [y_S^-, y_S^+]$. Then, by $(\Xi 3)$, $p = \Xi_S(u) \in W_{\omega(a_S)}$. Thus, $t = \omega(p) = \omega(a_S)$, a contradiction to $t \notin \{\omega(a_S)\}_{S \in C_+ \cup I}$.) Thus, we have shown that if $S \in C_+$ then $u \in [x_S^-, y_S^-) \cup (y_S^+, x_S^+]$, and if $S \in I$, then $u \in [-1, y_S^-) \cup (y_S^+, 1]$. Analogously, if $S' \in C_+$ then $u' \in [x_{S'}^-, y_{S'}^-) \cup (y_{S'}^+, x_{S'}^+]$, and if $S' \in I$, then $u' \in [-1, y_{S'}^-) \cup (y_{S'}^+, 1]$.

If $u \in [x_S^-, y_S^-) \cup (y_S^+, x_S^+]$ and $u' \in [x_{S'}^-, y_{S'}^-) \cup (y_{S'}^+, x_{S'}^+]$, then $q = \alpha_S(u) \in B_S^+ \cup B_S^-$ and $q' = \alpha_{S'}(u') \in B_{S'}^+ \cup B_{S'}^-$. As f_1 is injective on $\bigcup_{S'' \in C_+ \cup I}(B_{S''}^- \cup B_{S''}^+)$ and $q \neq q'$, we obtain $f_1(q) \neq f_1(q')$. Thus, we may assume that $S \in I$ and $u \in [-1, x_S^-] \cup [x_S^+, 1]$. Then, it follows from $(\Xi 2)$ that $p = \Xi_S(u) \notin Q$. Thus, by $(\Xi 1)$, we have $\Xi(p) = p = \Xi(q)$ and $p = q$. Now we obtain from $q = \alpha_S(u) \in \alpha_S([-1, t_S^-] \cup [t_S^+, 1]) = S \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$ that $t = \omega(p') = \omega(p) = \omega(q) \in [0,\varepsilon] \cup [1-\varepsilon, 1]$. Thus, $p' \in \Xi(S') \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. By $(\Xi 4)$ we can conclude that $q' = \Xi^{-1}(p') \in W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$. Since $q, q' \in S(F) \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$, there exist $T, T' \in I_{[0,\varepsilon]} \cup I_{[1-\varepsilon,1]}$ such that $q \in T$ and $q' \in T'$. As $q' \in T'$ and $\Xi(q') = p' \in W_{[0,\varepsilon]\cup[1-\varepsilon,1]}$, it follows from $(\Xi 5)$ that $p' \in \Xi(T') \cap W_{[0,\varepsilon]\cup[1-\varepsilon,1]} \subset T'$. It follows from $\omega(T), \omega(T') \in \{[0,\varepsilon], [1-\varepsilon, 1]\}$ and $t = \omega(q) = \omega(p) = \omega(p') \in \omega(T) \cap \omega(T')$ that $\omega(T) = \omega(T')$. Moreover, we have $T \cap W_t = \{q\} = \{p\} \neq \{p'\} = T' \cap W_t$. Finally, by (4), it follows from $\omega(T) = \omega(T')$ and $T \neq T'$ that $f_1(T) \cap f_1(T') = \emptyset$. In particular, $f_1(q) \neq f_1(q')$.

After the theorem has been proven for all simple cobordisms W, the proof for arbitrary W ($\neq \emptyset$) is as follows. By Definition 3.4.3, the restrictions of F to the simple cobordisms $W(k)$, $F(k)\colon W(k) \to \mathbb{C}$, $k \in \mathbb{N}$, are fold pre-fields on $W(k)$. Application of the theorem separately to $F(k) \in \mathcal{F}^{\mathrm{pre}}(W(k))$ yields for every $k \in \mathbb{N}$ a fold field $G(k)\colon W(k) \to \mathbb{C}$ such that $F(k) \sim_W G(k)$. The fold fields $G(k) \in \mathcal{F}(W(k))$ give rise to a fold field $G := \bigsqcup_{k\in\mathbb{N}} G(k) \in \mathcal{F}(W)$ by Definition 3.4.2. Finally, it follows from $F(k) \sim_W G(k)$ for every $k \in \mathbb{N}$ that $F \sim_W G$. (In fact, since $F(k)$ and $G(k)$ agree on an neighbourhood $U(k)$ of $\partial(W(k)) = (\partial W) \cap W(k)$ in $W(k)$ for every $k \in \mathbb{N}$, it follows that F and G agree on the neighbourhood $U := \bigsqcup_{k\in\mathbb{N}} U(k)$ of $\partial W = \bigsqcup_{k\in\mathbb{N}} \partial(W(k))$ in $W = \bigsqcup_{k\in\mathbb{N}} W(k)$. Moreover, we obtain from $\mathbb{S}(F(k)) = \mathbb{S}(G(k))$ for every $k \in \mathbb{N}$ that $\mathbb{S}(F) = \bigotimes_{k\in\mathbb{N}} \mathbb{S}(F(k)) = \bigotimes_{k\in\mathbb{N}} \mathbb{S}(G(k)) = \mathbb{S}(G)$.)

□

Definition 3.4.10. For a nonempty, closed, smooth $(n-1)$-dimensional manifold $P \subset \mathbb{R}^D$, set

$$\mathcal{F}^{\mathrm{pre}}(P) := \{f \in \mathcal{F}^{\mathrm{pre}}([0,1] \times P);\ \mathbb{S}(f) = 1 \in \mathrm{Mor}(\mathbf{Br})\}.$$

Let W be a cobordism from M to N and let $(f_M, f_N) \in \mathcal{F}^{\mathrm{pre}}(M) \times \mathcal{F}^{\mathrm{pre}}(N)$ be a boundary condition. If W is simple, set

$$\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) := \{F\colon W \to \mathbb{C} \text{ fold map};\ \exists \varepsilon, \varepsilon' \in (0, \varepsilon_W)\colon F|_{[0,\varepsilon]\times M} \approx f_M, F|_{[1-\varepsilon',1]\times N} \approx f_N\}.$$

For arbitrary W, we set

$$\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) := \{F\colon W \to \mathbb{C};\ F(k) \in \mathcal{F}^{\mathrm{pre}}(W(k); f_M(k), f_N(k)) \text{ for all } k \in \mathbb{N}\}.$$

Note that $\mathcal{F}(P) = \mathcal{F}^{\mathrm{pre}}(P) \cap \mathcal{F}([0,1] \times P)$. Moreover, if $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, then $\mathcal{F}(W; f_M, f_N) = \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) \cap \mathcal{F}(W)$.

Lemma 3.4.11. *If $(f_M, f_N) \in \mathcal{F}^{\mathrm{pre}}(M) \times \mathcal{F}^{\mathrm{pre}}(N)$, then $\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) \subset \mathcal{F}^{\mathrm{pre}}(W)$.*

Proof. By Definition 3.4.10 elements in $\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ are maps $F\colon W \to \mathbb{R}^2$ that restrict for every $k \in \mathbb{N}$ to a fold map $F(k)\colon W(k) \to \mathbb{R}^2$ such that there exist $\varepsilon(k), \varepsilon'(k) \in (0, \varepsilon_W)$ with

$$F(k)|_{([0,\varepsilon(k)]\times M)\cap W(k)} \cong f_M(k),$$
$$F(k)|_{([1-\varepsilon'(k),1]\times N)\cap W(k)} \cong f_N(k).$$

Here, $f_M(k)\colon ([0,1] \times M)(k) \to \mathbb{R}^2$ and $f_N(k)\colon ([0,1] \times N)(k) \to \mathbb{R}^2$ are fold maps satisfying property $(ff1)$ of Definition 3.4.2 by Definition 3.4.10 and Definition 3.4.3. Therefore, Lemma 3.4.5 implies that, if $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$, then $F(k)\colon W(k) \to \mathbb{C}$ is a fold map satisfying (ff1) for all $k \in \mathbb{N}$. The claim now follows from Definition 3.4.3. □

Proposition 3.4.12. *If $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ and $(g_M, g_N) \in \mathcal{F}^{\mathrm{pre}}(M) \times \mathcal{F}^{\mathrm{pre}}(N)$ are boundary conditions such that $f_M \sim_{[0,1]\times M} g_M$ and $f_N \sim_{[0,1]\times N} g_N$, then*

$$Z_W(f_M, f_N) = \sum_{G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)} Y(\mathbb{S}(G)) \otimes 1.$$

Proof. Let us first assume that W is a simple cobordism. Define the sets

$$\mathfrak{S}(W) := \{\mathbb{S}(F);\ F \in \mathcal{F}(W; f_M, f_N)\}, \qquad \mathfrak{S}^{\mathrm{pre}}(W) := \{\mathbb{S}(G);\ G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)\}.$$

In the following, we will show that $\mathfrak{S}(W) = \mathfrak{S}^{\mathrm{pre}}(W)$. Consequently, applying the functor $Y\colon \mathbf{Br} \to \mathbf{Vect}$, one obtains $Y(\mathfrak{S}(W)) = Y(\mathfrak{S}^{\mathrm{pre}}(W))$. By [4, Proposition 6.16] the semiring $(Q, +, \times)$ is continuous. Thus, the claim of the proposition follows from [4, Proposition 4.3]:

$$\begin{aligned} Z_W(f_M, f_N) &= \sum_{F \in \mathcal{F}(W; f_M, f_N)} Y(\mathbb{S}(F)) \otimes 1 = \sum_{\phi \in Y(\mathfrak{S}(W))} \phi \otimes 1 \\ &= \sum_{\phi \in Y(\mathfrak{S}^{\mathrm{pre}}(W))} \phi \otimes 1 = \sum_{G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)} Y(\mathbb{S}(G)) \otimes 1. \end{aligned}$$

Let us show $\mathfrak{S}(W) \subset \mathfrak{S}^{\mathrm{pre}}(W)$. Given $F \in \mathcal{F}(W; f_M, f_N)$, we have to produce $G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)$ such that $\mathbb{S}(F) = \mathbb{S}(G)$. Since W is a simple cobordism, there exist $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$ such that $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$. By [4, Definition 7.16], there exist diffeomorphisms $\xi_M\colon [0,\varepsilon] \to [0,1]$ with $\xi_M(0) = 0$ and $\xi_N\colon [1-\varepsilon',1] \to [0,1]$ with $\xi_N(1) = 1$, such that $F|_{[0,\varepsilon]\times M} = f_M \circ (\xi_M \times \mathrm{id}_M)$ and $F|_{[1-\varepsilon',1]\times N} = f_N \circ (\xi_N \times \mathrm{id}_N)$. Define $\widetilde{g}_M := g_M \circ (\xi_M \times \mathrm{id}_M)\colon W_{[0,\varepsilon]} = [0,\varepsilon] \times M \to \mathbb{C}$ and $\widetilde{g}_N := g_N \circ (\xi_N \times \mathrm{id}_N)\colon W_{[1-\varepsilon',1]} = [1-\varepsilon',1] \times N \to \mathbb{C}$. As $\widetilde{g}_M \approx g_M$ and $\widetilde{g}_N \approx g_N$, we obtain from $(g_M, g_N) \in \mathcal{F}^{\mathrm{pre}}(M) \times \mathcal{F}^{\mathrm{pre}}(N)$ and Lemma 3.4.5 that $\widetilde{g}_M \in \mathcal{F}^{\mathrm{pre}}([0,\varepsilon] \times M)$, $\mathbb{S}(\widetilde{g}_M) = \mathbb{S}(g_M) = 1$, and $\widetilde{g}_N \in \mathcal{F}^{\mathrm{pre}}([1-\varepsilon',1] \times N)$, $\mathbb{S}(\widetilde{g}_N) = \mathbb{S}(g_N) = 1$. It follows from $f_M \sim_{[0,1]\times M} g_M$ and $f_N \sim_{[0,1]\times N} g_N$ that there exist compact subsets $K_M \subset W_{(0,\varepsilon)}$ and $K_N \subset W_{(1-\varepsilon',1)}$ such that $F|_{W_{[0,\varepsilon]}\setminus K_M} = \widetilde{g}_M|_{W_{[0,\varepsilon]}\setminus K_M}$ and $F|_{W_{[1-\varepsilon',1]}\setminus K_N} = \widetilde{g}_N|_{W_{[1-\varepsilon',1]}\setminus K_N}$. Consider the open covering $W = (W \setminus (K_M \sqcup K_N)) \cup (W_{(0,\varepsilon)} \sqcup W_{(1-\varepsilon',1)})$. The fold maps $F|_{W\setminus(K_M\sqcup K_N)}$ on $W \setminus (K_M \sqcup K_N)$ and $\widetilde{g}_M|_{W_{(0,\varepsilon)}} \sqcup \widetilde{g}_N|_{W_{(1-\varepsilon',1)}}$ on $W_{(0,\varepsilon)} \sqcup W_{(1-\varepsilon',1)}$ agree on the intersection $(W\setminus(K_M\sqcup K_N))\cap(W_{(0,\varepsilon)}\sqcup W_{(1-\varepsilon',1)}) = (W_{(0,\varepsilon)}\setminus K_M)\sqcup(W_{(1-\varepsilon',1)}\setminus K_N)$. Thus, we obtain a fold map

$$G\colon W \to \mathbb{C}, \qquad G(p) = \begin{cases} (\widetilde{g}_M|_{W_{(0,\varepsilon)}} \sqcup \widetilde{g}_N|_{W_{(1-\varepsilon',1)}})(p), & \text{if } p \in K_M \sqcup K_N, \\ F(p), & \text{if } p \in W \setminus (K_M \sqcup K_N). \end{cases}$$

It follows from $F|_{W_{[0,\varepsilon]}\setminus K_M} = \widetilde{g}_M|_{W_{[0,\varepsilon]}\setminus K_M}$ and $F|_{W_{[1-\varepsilon',1]}\setminus K_N} = \widetilde{g}_N|_{W_{[1-\varepsilon',1]}\setminus K_N}$ that $G|_{W_{[0,\varepsilon]}} = \widetilde{g}_M \approx g_M$ and $G|_{W_{[1-\varepsilon',1]}} = \widetilde{g}_N \approx g_N$. Consequently, $G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)$. By Lemma 3.4.11, we have $G \in \mathcal{F}^{\mathrm{pre}}(W)$. It remains to show that $\mathbb{S}(F) = \mathbb{S}(G)$. It follows from $f_M \sim_{[0,1]\times M} g_M$ that $\mathbb{S}(g_M) = \mathbb{S}(f_M)$. By Lemma 3.4.5, it follows from $F|_{[0,\varepsilon]\times M} \approx f_M$ that $\mathbb{S}(f_M) = \mathbb{S}(F|_{W_{[0,\varepsilon]}})$. All in all, $\mathbb{S}(\widetilde{g}_M) = \mathbb{S}(g_M) = \mathbb{S}(F|_{W_{[0,\varepsilon]}})$. Analagously, $\mathbb{S}(\widetilde{g}_N) = \mathbb{S}(F|_{W_{[1-\varepsilon',1]}})$. It follows from $G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)$ and $(g_M, g_N) \in \mathcal{F}^{\mathrm{pre}}(M) \times \mathcal{F}^{\mathrm{pre}}(N)$ that $\varepsilon, 1-\varepsilon' \in (0,1) \cap \pitchfork(G) \cap \mathrm{GenIm}(G)$. Thus, by Lemma 3.4.4, we have $\mathbb{S}(G) = \mathbb{S}(\widetilde{g}_N) \circ \mathbb{S}(F|_{W_{[\varepsilon,1-\varepsilon']}}) \circ \mathbb{S}(\widetilde{g}_M) = \mathbb{S}(F|_{W_{[1-\varepsilon',1]}}) \circ \mathbb{S}(F|_{W_{[\varepsilon,1-\varepsilon']}}) \circ \mathbb{S}(F|_{W_{[0,\varepsilon]}})$. It follows from $F \in \mathcal{F}(W; f_M, f_N)$ and $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ that $\varepsilon, 1-\varepsilon' \in (0,1) \cap \pitchfork(F) \cap \mathrm{GenIm}(F)$. Thus, by Lemma 3.4.4, we can conclude $\mathbb{S}(G) = \mathbb{S}(F)$.

Conversely, let us show that $\mathfrak{S}(W) \supset \mathfrak{S}^{\mathrm{pre}}(W)$. Given $G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)$, we have to produce $F \in \mathcal{F}(W; f_M, f_N)$ such that $\mathbb{S}(G) = \mathbb{S}(F)$. Since W is a simple cobordism, there exist $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$ such that $G|_{[0,\varepsilon]\times M} \approx g_M$ and $G|_{[1-\varepsilon',1]\times N} \approx g_N$. By [4, Definition 7.16], there exist diffeomorphisms $\xi_M\colon [0,\varepsilon] \to [0,1]$ with $\xi_M(0) = 0$ and $\xi_N\colon [1-\varepsilon',1] \to [0,1]$ with $\xi_N(1) = 1$, such that $G|_{[0,\varepsilon]\times M} = g_M \circ (\xi_M \times \mathrm{id}_M)$ and $G|_{[1-\varepsilon',1]\times N} = g_N \circ (\xi_N \times \mathrm{id}_N)$. Define $\widetilde{f}_M := f_M \circ (\xi_M \times \mathrm{id}_M)\colon W_{[0,\varepsilon]} = [0,\varepsilon] \times M \to \mathbb{C}$ and $\widetilde{f}_N := f_N \circ (\xi_N \times \mathrm{id}_N)\colon W_{[1-\varepsilon',1]} = [1-\varepsilon',1] \times N \to \mathbb{C}$. As $\widetilde{f}_M \approx f_M$ and $\widetilde{f}_N \approx f_N$, we obtain from $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ and [4, Lemma 7.17] that $\widetilde{f}_M \in \mathcal{F}([0,\varepsilon] \times M)$, $\mathbb{S}(\widetilde{f}_M) = \mathbb{S}(f_M) = 1$, and $\widetilde{f}_N \in \mathcal{F}([1-\varepsilon',1] \times N)$, $\mathbb{S}(\widetilde{f}_N) = \mathbb{S}(f_N) = 1$. We have $G|_{W_{[\varepsilon,1-\varepsilon']}} \in \mathcal{F}^{\mathrm{pre}}(W_{[\varepsilon,1-\varepsilon']})$. Thus, by Theorem 3.4.9, there exists

$\overline{F} \in \mathcal{F}(W_{[\varepsilon,1-\varepsilon']})$ such that $G|_{W_{[\varepsilon,1-\varepsilon']}} \sim_{W_{[\varepsilon,1-\varepsilon']}} \overline{F}$. Therefore, there exists a compact subset $K \subset W_{(\varepsilon,1-\varepsilon')}$ such that $G|_{W_{[\varepsilon,1-\varepsilon']}\setminus K} = \overline{F}|_{W_{[\varepsilon,1-\varepsilon']}\setminus K}$. It follows from $f_M \sim_{[0,1]\times M} g_M$ and $f_N \sim_{[0,1]\times N} g_N$ that there exist compact subsets $K_M \subset W_{(0,\varepsilon)}$ and $K_N \subset W_{(1-\varepsilon',1)}$ such that $G|_{W_{[0,\varepsilon]}\setminus K_M} = \widetilde{f}_M|_{W_{[0,\varepsilon]}\setminus K_M}$ and $G|_{W_{[1-\varepsilon',1]}\setminus K_N} = \widetilde{f}_N|_{W_{[1-\varepsilon',1]}\setminus K_N}$. Consider the open covering $W = (W \setminus (K_M \sqcup K \sqcup K_N)) \cup (W_{(0,\varepsilon)} \sqcup W_{(\varepsilon,1-\varepsilon')} \sqcup W_{(1-\varepsilon',1)})$. The fold maps $G|_{W\setminus(K_M\sqcup K\sqcup K_N)}$ on $W\setminus(K_M\sqcup K\sqcup K_N)$ and $\widetilde{f}_M|_{W_{(0,\varepsilon)}} \sqcup \overline{F}|_{W_{(\varepsilon,1-\varepsilon')}} \sqcup \widetilde{f}_N|_{W_{(1-\varepsilon',1)}}$ on $W_{(0,\varepsilon)} \sqcup W_{(\varepsilon,1-\varepsilon')} \sqcup W_{(1-\varepsilon',1)}$ agree on the intersection $(W \setminus (K_M \sqcup K \sqcup K_N)) \cap (W_{(0,\varepsilon)} \sqcup W_{(\varepsilon,1-\varepsilon')} \sqcup W_{(1-\varepsilon',1)}) = (W_{(0,\varepsilon)} \setminus K_M) \sqcup (W_{(\varepsilon,1-\varepsilon')} \setminus K) \sqcup (W_{(1-\varepsilon',1)} \setminus K_N)$. Thus, we obtain a fold map

$$F\colon W \to \mathbb{C}, \qquad F(p) = \begin{cases} (\widetilde{f}_M|_{W_{(0,\varepsilon)}} \sqcup \overline{F}|_{W_{(\varepsilon,1-\varepsilon')}} \sqcup \widetilde{f}_N|_{W_{(1-\varepsilon',1)}})(p), & \text{if } p \in K_M \sqcup K \sqcup K_N, \\ G(p), & \text{if } p \in W \setminus (K_M \sqcup K \sqcup K_N). \end{cases}$$

It follows from $G|_{W_{[0,\varepsilon]}\setminus K_M} = \widetilde{f}_M|_{W_{[0,\varepsilon]}\setminus K_M}$ and $G|_{W_{[1-\varepsilon',1]}\setminus K_N} = \widetilde{f}_N|_{W_{[1-\varepsilon',1]}\setminus K_N}$ that $F|_{W_{[0,\varepsilon]}} = \widetilde{f}_M \approx f_M$ and $F|_{W_{[1-\varepsilon',1]}} = \widetilde{f}_N \approx f_N$. Consequently, $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$. By Lemma 3.4.11, we have $F \in \mathcal{F}^{\mathrm{pre}}(W)$. Let us check that F satisfies (ff2). In fact, it follows from $F|_{W_{[0,\varepsilon]}} = \widetilde{f}_M \in \mathcal{F}([0,\varepsilon] \times M)$ and $F|_{W_{[1-\varepsilon',1]}} = \widetilde{f}_N \in \mathcal{F}([1-\varepsilon',1] \times N)$ that $[0,\varepsilon] \cap \pitchfork(F) \cap \mathrm{GenIm}(F)$ is residual in $[0,\varepsilon]$ and $[1-\varepsilon',1] \cap \pitchfork(F) \cap \mathrm{GenIm}(F)$ is residual in $[1-\varepsilon',1]$. Moreover, it follows from $F|_{W_{[\varepsilon,1-\varepsilon']}} = \overline{F} \in \mathcal{F}(W_{[\varepsilon,1-\varepsilon']})$ that $[\varepsilon,1-\varepsilon'] \cap \pitchfork(F) \cap \mathrm{GenIm}(F)$ is residual in $[\varepsilon,1-\varepsilon']$. Thus, it follows from [4, Lemmas 7.4, 7.5] that $\pitchfork(F) \cap \mathrm{GenIm}(F)$ is residual in $[0,1]$. Hence, F satisfies (ff2), and we can conclude that $F \in \mathcal{F}(W)$. Thus, $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) \cap \mathcal{F}(W) = \mathcal{F}(W; f_M, f_N)$. It remains to show that $\mathbb{S}(F) = \mathbb{S}(G)$. It follows from $F \in \mathcal{F}(W; f_M, f_N)$ and $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ that $\varepsilon, 1-\varepsilon' \in (0,1) \cap \pitchfork(F) \cap \mathrm{GenIm}(F)$. Thus, by [4, Lemma 7.11], we have $\mathbb{S}(F) = \mathbb{S}(F|_{W_{[1-\varepsilon',1]}}) \circ \mathbb{S}(F|_{W_{[\varepsilon,1-\varepsilon']}}) \circ \mathbb{S}(F|_{W_{[0,\varepsilon]}}) = \mathbb{S}(\widetilde{f}_N) \circ \mathbb{S}(\overline{F}) \circ \mathbb{S}(\widetilde{f}_M)$. It follows from $f_M \sim_{[0,1]\times M} g_M$ that $\mathbb{S}(f_M) = \mathbb{S}(g_M)$. By Lemma 3.4.5, it follows from $G|_{[0,\varepsilon]\times M} \approx g_M$ that $\mathbb{S}(g_M) = \mathbb{S}(G|_{W_{[0,\varepsilon]}})$. All in all, $\mathbb{S}(\widetilde{f}_M) = \mathbb{S}(f_M) = \mathbb{S}(G|_{W_{[0,\varepsilon]}})$. Analagously, $\mathbb{S}(\widetilde{f}_N) = \mathbb{S}(G|_{W_{[1-\varepsilon',1]}})$. Moreover, $\mathbb{S}(\overline{F}) = \mathbb{S}(G|_{W_{[\varepsilon,1-\varepsilon']}})$, because $G|_{W_{[\varepsilon,1-\varepsilon']}} \sim_{W_{[\varepsilon,1-\varepsilon']}} \overline{F}$. Therefore, we obtain $\mathbb{S}(F) = \mathbb{S}(G|_{W_{[1-\varepsilon',1]}}) \circ \mathbb{S}(G|_{W_{[\varepsilon,1-\varepsilon']}}) \circ \mathbb{S}(G|_{W_{[0,\varepsilon]}})$. Finally, it follows from $G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)$ and $(g_M, g_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ that $\varepsilon, 1-\varepsilon' \in (0,1) \cap \pitchfork(G) \cap \mathrm{GenIm}(G)$. Thus, by Lemma 3.4.4, we can conclude $\mathbb{S}(F) = \mathbb{S}(G)$.

Let W be an arbitrary cobordism. By [4, Theorem 7.20],

$$\begin{aligned} Z_W(f_M, f_N) &= Z_{\bigsqcup_{k\in\mathbb{N}} W(k)}(\sqcup_k f_M(k), \sqcup_k f_N(k)) \\ &= Z_{W(0)}(f_M(0), f_N(0)) \times \cdots \times Z_{W(k)}(f_M(k), f_N(k)) \times \cdots, \end{aligned}$$

where the product in Q is finite, since $W(k)$ is empty for all but finitely many $k \in \mathbb{N}$. As the theorem is already shown for simple cobordisms, we obtain for every $k \in \mathbb{N}$ that

$$Z_{W(k)}(f_M(k), f_N(k)) = \sum_{G_k \in \mathcal{F}^{\mathrm{pre}}(W(k); g_M(k), g_N(k))} Y(\mathbb{S}(G_k)) \otimes 1.$$

Hence, using the law [4, Equation (9)] in the complete semiring $(Q, +, \times)$,

$$Z_W(f_M, f_N) = \sum_{G \in \mathcal{F}^{\mathrm{pre}}(W; g_M, g_N)} ((Y(\mathbb{S}(G(k))) \otimes 1) \times \cdots \times (Y(\mathbb{S}(G(k))) \otimes 1) \times \cdots).$$

(Here, we have used the identification $\mathcal{F}^{\mathrm{pre}}(W; g_M, g_N) = \prod_{k\in\mathbb{N}} \mathcal{F}^{\mathrm{pre}}(W(k); g_M(k), g_N(k))$.) By

[4, Lemma 6.14], and since Y is a monoidal functor, we obtain for every $k \in \mathbb{N}$ that

$$(Y(\mathbb{S}(G(k))) \otimes 1) \times \cdots \times (Y(\mathbb{S}(G(k))) \otimes 1) \times \cdots = Y(\bigotimes_{k \in \mathbb{N}} \mathbb{S}(G(k))) \otimes 1 = Y(\mathbb{S}(F)) \otimes 1.$$

$\square$

3.4.2 Construction of Stable Fold Maps from Fold Maps

Definition 3.4.13. Let $\mathcal{F}^{\mathrm{pre}}_{\mathrm{s}\partial}(W) \subset \mathcal{F}^{\mathrm{pre}}(W)$ denote the subspace of all fold maps $F \colon W \to \mathbb{C}$ in $\mathcal{F}^{\mathrm{pre}}(W)$ such that $F(k) \colon W(k) \to \mathbb{C}$ is stable in a suitable open neighbourhood of $\partial W(k)$ in $W(k)$ for every $k \in \mathbb{N}$. Let $\mathcal{F}^{\mathrm{pre}}_{\mathrm{s}}(W) \subset \mathcal{F}^{\mathrm{pre}}(W)$ denote the subspace of all fold maps $F \colon W \to \mathbb{C}$ in $\mathcal{F}^{\mathrm{pre}}(W)$ such that $F(k) \colon W(k) \to \mathbb{C}$ is stable for every $k \in \mathbb{N}$. Moreover, set $\mathcal{F}_{\mathrm{s}}(W) := \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}}(W) \cap \mathcal{F}(W)$.

Thus, we have the inclusions $\mathcal{F}_s(W) \subset \mathcal{F}^{\mathrm{pre}}_s(W) \subset \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}\partial}(W)$.

Note that if $F \in \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}\partial}(W)$ and $G \in \mathcal{F}^{\mathrm{pre}}(W)$ with $F \sim_W G$, then $G \in \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}\partial}(W)$.

Theorem 3.4.14. *If $F \in \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}\partial}(W)$, then there exists $G \in \mathcal{F}^{\mathrm{pre}}_{\mathrm{s}}(W)$ such that $F \sim_W G$.*

Proof. Let $F \in \mathcal{F}^{\mathrm{pre}}_{\partial s}(W)$. Let us first assume that W is a simple cobordism. Then, by Definition 3.4.13, $F \in \mathcal{F}^{\mathrm{pre}}(W)$ is already stable on an open neighbourhood of ∂W in W. In the following, we will use Lemma 3.3.6 to perform a finite sequence of perturbations of F on compact subsets of $W_{(0,1)}$, such that the resulting fold map G is stable. Note that G will agree with F in an open neighbourhood of ∂W in W, because F was only perturbed on a compact subset of $W_{(0,1)}$. In particular, $G \in \mathcal{F}^{\mathrm{pre}}_{s}(W)$. Moreover, since the perturbations of Lemma 3.3.6 do not affect the fold locus, we will have $S(G) = S(F)$ and thus $\mathbb{S}(G) = \mathbb{S}(F)$. Hence, $F \sim_W G$.

For every $S \in C$, we fix an embedding $[0,1] \to S$. Let B_S denote the image of $[0,1]$ under this embedding, and set $B := \bigsqcup_{S \in C} B_s$. Since F restricts to an immersion $S(F) \to \mathbb{C}$, we may assume that F restricts to an embedding $B \to \mathbb{C}$. Moreover, since $S(F) \cap W_{\{0,1\}}$ is a finite set because of $0, 1 \in \pitchfork(F)$, we may in addition assume that $F(B) \cap F(S(F) \cap W_{\{0,1\}}) = \emptyset$.

Since $0, 1 \in \pitchfork(F)$, it follows from Lemma 3.4.8 that there exists $\varepsilon_0 \in (0, \varepsilon_W)$ such that $(\varepsilon 1)$ and $(\varepsilon 2)$ hold for all $\varepsilon \in (0, \varepsilon_0)$. In particular, if $\varepsilon \in (0, \varepsilon_0)$, then $T_{[0,\varepsilon]} := \bigsqcup_{T \in I_{[0,\varepsilon]}} T = S(F) \cap W_{[0,\varepsilon]}$ and $T_{[1-\varepsilon,1]} := \bigsqcup_{T \in I_{[1-\varepsilon,1]}} T = S(F) \cap W_{[1-\varepsilon,1]}$ by $(\varepsilon 2)$. We choose $\varepsilon \in (0, \varepsilon_0)$ with the following properties:

(T1) F restricts to an immersion with normal crossings $F_\varepsilon \colon T_{[0,\varepsilon]} \sqcup T_{[1-\varepsilon,1]} \to \mathbb{C}$.
(T2) $F_\varepsilon(p)$ is not a double point of F_ε for every $p \in (\partial T_{[0,\varepsilon]} \sqcup \partial T_{[1-\varepsilon,1]}) \cap W_{(0,1)}$.
(T3) $F(B) \cap F(T_{[0,\varepsilon]} \sqcup T_{[1-\varepsilon,1]}) = \emptyset$.

(In fact, by assumption, there exists an open neighbourhood U of ∂W in W, such that F restricts to an immersion with normal crossings $U \cap S(F) \to \mathbb{C}$. As $\partial W = M \sqcup N$ is compact, one can choose $\varepsilon_0' \in (0, \varepsilon_0)$ such that $W_{[0,\varepsilon] \cup [1-\varepsilon,1]} = (M \times [0,\varepsilon]) \sqcup (N \times [1-\varepsilon, 1]) \subset U$ for all $\varepsilon \in (0, \varepsilon_0')$. This implies (T1), since $T_{[0,\varepsilon]} \sqcup T_{[1-\varepsilon,1]} = S(F) \cap W_{[0,\varepsilon] \cup [1-\varepsilon,1]} \subset S(F) \cap U$. Property (T2) can be deduced from $(\varepsilon 1)$ and (T1), since the restriction of F to the immersion with normal crossings $T_{[0,\varepsilon]} \sqcup T_{[1-\varepsilon,1]} \to \mathbb{C}$ has only finitely many double points. Finally, property (T3) can be obtained by using $(\varepsilon 1)$, (T1) and $F(B) \cap F(S(F) \cap W_{\{0,1\}}) = \emptyset$.)

Let $J_0, ..., J_{\mu-1}$ be an enumeration of the intervals in $\{S \setminus \operatorname{int} B_S\}_{S\in C} \cup I_{[\varepsilon,1-\varepsilon]}$. We define $\Sigma_0 := T_{[0,\varepsilon]} \sqcup T_{[1-\varepsilon,1]} \sqcup B$. Moreover, for $m \in \{1, ..., \mu\}$, we define Σ_m inductively by $\Sigma_m := \Sigma_{m-1} \cup J_{m-1}$. Note that for every $m \in \{0, ..., \mu\}$, Σ_m is a 1-dimensional compact manifold with boundary $\partial\Sigma_m = (S(F) \cap W_{\{0,1\}}) \cup \bigcup_{i=m}^{\mu-1} \partial J_i$. Note that $\Sigma_\mu = S(F)$.

Set $F_0 := F$. The restriction $F_0|_{\Sigma_0} \colon \Sigma_0 \to \mathbb{C}$ is an immersion with normal crossings. (This follows from (T1) and (T3), and since F_0 restricts to an embedding $B \to \mathbb{C}$.) Moreover, using (T2), we can conclude that F_0 has the following property for $m = 0$:

(F_m) $F_m|_{\Sigma_m}$ is an immersion with normal crossings, and $F_m(p)$ is not a double point of $F_m|_{\Sigma_m}$ for every $p \in \partial\Sigma_m \cap W_{(0,1)} = \bigcup_{i=m}^{\mu-1} \partial J_i$. Moreover, $F_m \sim_W F$.

Assume that we are given $F_{m-1} \in \mathcal{F}^{\mathrm{pre}}(W)$ for some $m \in \{1, ..., \mu\}$, such that (F_{m-1}) holds. In the following, we will use Lemma 3.3.6 to perform a finite sequence of perturbations of F_{m-1} on compact subsets of $W_{(0,1)}$, such that the resulting fold map F_m satisfies (F_m). Eventually, the fold map $G := F_\mu$ will satisfy (F_μ). In particular, $F \sim_W G$, and G restricts to an immersion with normal crossings on $\Sigma_\mu = S(F)$ and is hence stable.

For simplicity, we will write $E := F_{m-1}$, $\Sigma := \Sigma_{m-1}$ and $J := J_{m-1}$. Thus, $\Sigma_m = \Sigma \cup J$.

Fix a diffeomorphism $[0,1] \cong J$. This diffeomorphism will be used in the following to identify points in J with points in $[0,1]$. For every point $p \in J$, Proposition 3.3.5 enables us to choose charts $\phi_p \colon X_p \to X'_p \subset \mathbb{R}^n$ around $p \in W_{(0,1)}$ and $\psi_p \colon Y_p \to Y'_p \subset \mathbb{C}$ around $E(p) \in \mathbb{C}$, such that $E(X_p) \subset Y_p$ and $\psi_p(E(\phi_p^{-1}(t,x))) = \Lambda_{i(p)}(t,x)$ for all $(t,x) \in X'_p \subset \mathbb{R} \times \mathbb{R}^{n-1}$ and some integer $0 \le i(p) \le n-1$. Application of the Lebesgue lemma to the open covering $J \subset \bigcup_{p\in J} X_p$ yields an integer $D > 0$ such that each of the intervals $[\frac{d}{D}, \frac{d+1}{D}] \subset [0,1] = J$, $d \in \{0, ..., D-1\}$, is completely contained in $X_d := X_{p(d)}$ for some $p(d) \in J$. In particular, we have $\{\frac{d}{D}\} = [\frac{d-1}{D}, \frac{d}{D}] \cap [\frac{d}{D}, \frac{d+1}{D}] \subset X_{d-1} \cap X_d$ for all $d \in \{1, ..., D-1\}$. For every $d \in \{1, ..., D-1\}$ we choose $x_d \in (\frac{d-1}{D}, \frac{d}{D})$ such that $[x_d, \frac{d}{D}] \subset X_{d-1} \cap X_d$.

Construction I. There exists a fold map $\widetilde{E} \colon W \to \mathbb{C}$ with the following properties:

(1) There is a compact subset $K \subset W_{(0,1)}$ such that $S(E) \cap K \subset \operatorname{int}\Sigma$ and $\widetilde{E}|_{W\setminus K} = E|_{W\setminus K}$.
(2) $S(\widetilde{E}) = S(E)$.
(3) For every $d \in \{1, ..., D-1\}$, there is a point $y_d \in [x_d, \frac{d}{D}]$ such that $E(y_d) \notin \widetilde{E}(\Sigma)$.
(4) $E(p) \notin \widetilde{E}(\Sigma \setminus \{p\})$ for all $p \in \partial\Sigma \cap W_{(0,1)}$.
(5) The restriction of $\widetilde{E}$ to Σ is an immersion with normal crossings.

Since E restricts to an immersion with normal crossings on Σ (with finitely many double points, since Σ is compact!) and to an immersion on $[x_d, \frac{d}{D}]$ for every $d \in \{1, ..., D-1\}$, we can choose for every $d \in \{1, ..., D-1\}$ a point $y_d \in [x_d, \frac{d}{D}]$ such that $E(y_d) \notin E(\partial\Sigma)$, and such that there exists at most one point $p_d \in \operatorname{int}\Sigma$ with $E(p_d) = E(y_d)$. Let $\Omega \subset \{1, ..., D-1\}$ be the subset of all $d \in \{1, ..., D-1\}$ such that the point p_d exists. In addition, we may assume that $E(p_d) \neq E(p_e)$ for all $d, e \in \Omega$ with $d \neq e$. For all $d \in \Omega$, we have

$(*)$ $E(p_d) = E(y_d) \notin E(\partial\Sigma) \cup E(\operatorname{int}\Sigma \setminus \{p_d\}) = E(\Sigma \setminus \{p_d\})$.

Proposition 3.3.5 enables us to choose for every $d \in \Omega$ charts $\phi_d \colon U_d \to U'_d \subset \mathbb{R}^n$ around p_d in $W_{(0,1)}$ and $\psi_d \colon V_d \to V'_d \subset \mathbb{C}$ around $E(p_d) \in \mathbb{C}$, such that $E(U_d) \subset V_d$ and $\psi_d(E(\phi_d^{-1}(t,x))) = \Lambda_{i_d}(t,x)$ for all $(t,x) \in U'_d \subset \mathbb{R} \times \mathbb{R}^{n-1}$ and some integer $0 \le i_d \le n-1$. Without loss of generality, we may assume that

$(\Omega 1)$ $V_d \cap V_e = \emptyset$ for all $d, e \in \Omega$ with $d \neq e$.

(Ω2) $U_d \cap S(E) \subset \operatorname{int}\Sigma$ for all $d \in \Omega$.
(Ω3) $E(\Sigma \setminus U_d) \cap V_d = \emptyset$ for all $d \in \Omega$.

(In fact, (Ω2) can be achieved by avoiding the compact subset $\Sigma \setminus \operatorname{int}\Sigma \subset W$, using $p_d \in \operatorname{int}\Sigma$ for all $d \in \Omega$. In order to obtain (Ω1), one uses that $E(p_d) \neq E(p_e)$ for all $d, e \in \Omega$ with $d \neq e$. In order to obtain (Ω3), one can restrict the charts to the open subsets $\widetilde{V}_d := V_d \setminus E(\Sigma \setminus U_d) \subset V_d$ and $\widetilde{U}_d := U_d \cap E^{-1}(\widetilde{V}_d) \subset U_d$. Then, ($\Omega$1) and ($\Omega$2) will still be valid, and $E(\widetilde{U}_d) \subset \widetilde{V}_d$. Note that it follows from $p_d \in U_d$ and $E(p_d) \notin E(\Sigma \setminus U_d)$ (by $(*)$) that $p_d \in U_d \cap E^{-1}(\widetilde{V}_d) = \widetilde{U}_d$. Moreover, $E(\Sigma \setminus \widetilde{U}_d) \cap \widetilde{V}_d = \emptyset$. (In fact, if $\sigma \in \Sigma$ such that $E(\sigma) \in \widetilde{V}_d = V_d \setminus E(\Sigma \setminus U_d)$, then $\sigma \in U_d \cap E^{-1}(\widetilde{V}_d) = \widetilde{U}_d$.))

It follows from $E(U_d) \subset V_d$ that E restricts to a fold map $E_d \colon U_d \to V_d$ with fold locus $S(E_d) = S(E|_{U_d}) = U_d \cap S(E)$. Moreover, $E_d(p_d) \neq E_d(p)$ for all $p \in S(E_d) \setminus \{p_d\}$ by $(*)$. Following the proof of Proposition 3.3.7, there exist fold maps $\widetilde{E}_d \colon U_d \to V_d$ and compact subsets $K_d \subset U_d$ for every $d \in \Omega$, such that

(i) $\widetilde{E}_d|_{U_d \setminus K_d} = E_d|_{U_d \setminus K_d}$ for every $d \in \Omega$.
(ii) $S(\widetilde{E}_d) = S(E_d)$ for every $d \in \Omega$.
(iii) $E_d(p_d) \notin \widetilde{E}_d(S(F_d))$ for every $d \in \Omega$.
(iv) $\widetilde{E}_d$ restricts to an embedding $S(E_d) \to \mathbb{C}$ for every $d \in \Omega$.

Define $K := \bigsqcup_{d\in\Omega} K_d$, $U := \bigsqcup_{d\in\Omega} U_d$ and $V := \bigsqcup_{d\in\Omega} V_d$. (Note that the pairwise disjointness of the U_d follows from (Ω1).) $K \subset W$ is a compact subset, and $K \subset U \subset W_{(0,1)}$. Consider the open covering $W = U \cup (W \setminus K)$. The fold maps $\widetilde{E}_U := \bigsqcup_{d\in\Omega} \widetilde{E}_d$ on U and $E|_{W\setminus K}$ on $W \setminus K$ agree by (i) on the open subset $U \cap (W \setminus K) = U \setminus K = \bigsqcup_{d\in\Omega} U_d \setminus K_d$ of W. Thus, we can assemble these maps to obtain the fold map

$$\widetilde{E} \colon W \to \mathbb{C}, \qquad \widetilde{E}(p) = \begin{cases} \widetilde{E}_U(p), & \text{for } p \in U, \\ E(p), & \text{for } p \notin U. \end{cases}$$

Let us check that $\widetilde{E}$ satisfies the claimed properties:

(1). It follows from (Ω2) that $S(E) \cap K \subset S(E) \cap U = \bigsqcup_{d\in\Omega} S(E) \cap U_d \subset \operatorname{int}\Sigma$. We have $W \setminus K = (W \setminus U) \cup (U \setminus K)$, where $U \setminus K = \bigsqcup_{d\in\Omega} U_d \setminus K_d$. By definition, $\widetilde{E}$ and E agree on $W \setminus U$. Moreover, for every $d \in \Omega$, we have $\widetilde{E}|_{U_d\setminus K_d} = \widetilde{E}_d|_{U_d\setminus K_d} = E_d|_{U_d\setminus K_d}$ by (i). Thus, $\widetilde{E}$ and E also agree on $U \setminus K$. Hence, $\widetilde{E}|_{W\setminus K} = E|_{W\setminus K}$.

(2). As $W = U \cup (W \setminus K)$, we have $S(\widetilde{E}) = \bigsqcup_{d\in\Omega} S(\widetilde{E}|_{U_d}) \cup S(\widetilde{E}|_{W\setminus K})$. For every $d \in \Omega$ we have $S(\widetilde{E}|_{U_d}) = S(\widetilde{E}_d) = S(E_d) = S(E|_{U_d})$ by (ii). Moreover, it follows from (1) that $S(\widetilde{E}|_{W\setminus K}) = S(E|_{W\setminus K})$. Thus, $S(\widetilde{E}) = \bigsqcup_{d\in\Omega} S(E|_{U_d}) \cup S(E|_{W\setminus K}) = S(E)$.

(3). Let $d \in \{1, ..., D-1\}$. Write $\Sigma = (\Sigma \setminus U) \cup (\Sigma \cap U)$. Since $p_d \in U_d \subset U$, it follows from $(*)$ that $E(y_d) \notin E(\Sigma \setminus U) = \widetilde{E}(\Sigma \setminus U)$. Moreover, $\widetilde{E}(\Sigma \cap U) = \bigsqcup_{e\in\Omega} \widetilde{E}_e(\Sigma \cap U_e) \subset \bigsqcup_{e\in\Omega} \widetilde{E}_e(S(\widetilde{E}_e))$, which uses $\Sigma \cap U_e \subset S(E) \cap U_e = S(E_e) = S(\widetilde{E}_e)$ (by (ii)) for all $e \in \Omega$. By (Ω1), we have $E_d(y_d) \notin \widetilde{E}_e(S(\widetilde{E}_e))$ for all $e \in \Omega$ with $d \neq e$, since $E_d(y_d) \in V_d$ and $\widetilde{E}_e(S(\widetilde{E}_e)) \subset V_e$. By (ii) and (iii), we have $E_d(y_d) \notin \widetilde{E}_d(S(\widetilde{E}_d))$. Thus, $E_d(y_d) \notin \widetilde{E}(\Sigma \cap U)$. All in all, $E_d(y_d) \notin \widetilde{E}(\Sigma)$.

(4). Let $p \in \partial\Sigma \cap W_{(0,1)}$. Write $\Sigma \setminus \{p\} = ((\Sigma \setminus \{p\}) \cap U) \cup ((\Sigma \setminus \{p\}) \setminus U)$. We have $\widetilde{E}((\Sigma \setminus \{p\}) \cap U) = \widetilde{E}_U((\Sigma \setminus \{p\}) \cap U) \subset V$. By ($\Omega$2), we have $p \notin U$. Hence, by (Ω3), $E(p) \notin \widetilde{E}((\Sigma \setminus \{p\}) \cap U)$. Moreover, $\widetilde{E}((\Sigma \setminus \{p\}) \setminus U) = E((\Sigma \setminus \{p\}) \setminus U) \subset E(\Sigma \setminus \{p\})$. Thus, by

(F_{m-1}), we have $E(p) \notin \widetilde{E}((\Sigma \setminus \{p\}) \setminus U)$. All in all, $E(p) \notin \widetilde{E}((\Sigma \setminus \{p\}) \cap U) \cup \widetilde{E}((\Sigma \setminus \{p\}) \setminus U) = \widetilde{E}(\Sigma \setminus \{p\})$.

(5). We have the open covering $\Sigma = (\Sigma \cap U) \cup (\Sigma \setminus K)$. It follows from (1) that $\widetilde{E}|_{\Sigma \setminus K} = E|_{\Sigma \setminus K}$, which is by assumption an immersion with normal crossings. Moreover, $\widetilde{E}|_{\Sigma \cap U} = \bigsqcup_{d \in \Omega} \widetilde{E}_d|_{\Sigma \cap U_d}$ is an immersion with normal crossings as well. (In fact, by (iv), $\widetilde{E}_d$ is an embedding on $S(E_d) = S(E) \cap U_d$ for every $d \in \Omega$, and $\widetilde{E}_d(\Sigma \cap U_d) \cap \widetilde{E}_e(\Sigma \cap U_e) \subset V_d \cap V_e = \emptyset$ by $(\Omega 1)$ for all $d, e \in \Omega$ with $d \neq e$.) Thus, it suffices to show that $\widetilde{E}(\Sigma \cap U) \cap \widetilde{E}((\Sigma \setminus K) \setminus (\Sigma \cap U)) = \emptyset$. In fact, we have $\widetilde{E}((\Sigma \setminus K) \setminus (\Sigma \cap U)) = \widetilde{E}(\Sigma \setminus U) = E(\Sigma \setminus U)$, and hence, by $(\Omega 3)$,

$$\widetilde{E}(\Sigma \cap U) \cap \widetilde{E}((\Sigma \setminus K) \setminus (\Sigma \cap U)) = \bigsqcup_{d \in \Omega} \widetilde{E}_d(\Sigma \cap U_d) \cap E(\Sigma \setminus U) \subset \bigsqcup_{d \in \Omega} V_d \cap E(\Sigma \setminus U_d) = \emptyset.$$

This completes construction I.

For every $d \in \{1, ..., D-1\}$ we choose an interval $A_d := [w_d, z_d] \subset [x_d, \frac{d}{D}]$ with $w_d < y_d < z_d$ and define $A := \bigsqcup_{d=1}^{D-1} A_d$. Note that $\widetilde{E}(y_d) \notin \widetilde{E}(\Sigma)$ for all $d \in \{1, ..., D-1\}$. (In fact, let $d \in \{1, ..., D-1\}$. As $y_d \in J$, it follows from (1) that $E(y_d) = \widetilde{E}(y_d)$. Hence, by (3), $\widetilde{E}(y_d) \notin \widetilde{E}(\Sigma)$.) As $\widetilde{E}$ restricts to an immersion $[x_d, \frac{d}{D}] \to \mathbb{C}$ for every $d \in \{1, ..., D-1\}$, we may assume that $\widetilde{E}$ restricts to an embedding $A \to \mathbb{C}$. Moreover, as Σ is compact, we may in addition assume that $\widetilde{E}(A) \cap \widetilde{E}(\Sigma) = \emptyset$. Hence, it follows from (5) that $\widetilde{E}$ restricts to an immersion with normal crossings $\Sigma \sqcup A \to \mathbb{C}$.

Set $x_0 = z_0 = 0$ and $x_D = w_D = 1$. In particular, $x_d \leq z_d \leq \frac{d}{D} < x_{d+1} \leq w_{d+1} \leq \frac{d+1}{D}$ for all $d \in \{0, ..., D-1\}$. For all $d \in \{0, ..., D-1\}$, define $Z_d := [z_d, w_{d+1}]$. Define $\Pi_0 := \Sigma \sqcup A$. Moreover, for $d \in \{1, ..., D\}$, we define Π_d inductively by $\Pi_d := \Pi_{d-1} \cup Z_{d-1}$. Note that for every $d \in \{0, ..., D\}$, Π_d is a 1-dimensional compact manifold with boundary $\partial \Pi_d = (S(E) \cap W_{\{0,1\}}) \cup \bigcup_{i=m+1}^{\mu-1} \partial J_i \cup \bigcup_{j=d}^{D-1} \partial Z_j$. Note that $\Pi_D = \Sigma \cup J = \Sigma_m$.

Construction II. For every $d \in \{0, ..., D\}$, there exists a fold map $G_d \colon W \to \mathbb{C}$ such that

(1') There is a compact subset $L_d \subset W_{(0,1)}$ with $L_d \cap S(E) \subset [0, w_d] \cup \operatorname{int} \Sigma$ and $G_d|_{W \setminus L_d} = E|_{W \setminus L_d}$. (As w_0 is not defined, we understand $[0, w_0] = \emptyset$.)

(2') $S(G_d) = S(E)$.

(3') $G_d(p) \notin G_d(\Pi_d \setminus \{p\})$ for all $p \in \partial \Pi_d \cap W_{(0,1)}$.

(4') $G_d|_{\Pi_d}$ is an immersion with normal crossings.

The construction will be done by induction on d. For the induction basis, set $G_0 := \widetilde{E}$ and $L_0 := K$. We have to check the properties (1') to (4') for $d = 0$. The properties (1') and (2') follow from (1) and (2). Property (3') says for $d = 0$ that $\widetilde{E}(p) \notin \widetilde{E}((\Sigma \sqcup A) \setminus \{p\})$ for all $p \in (\partial \Sigma \cap W_{(0,1)}) \cup \partial A$. If $p \in \partial \Sigma \cap W_{(0,1)}$, then it follows from (1), (4) and $\widetilde{E}(\Sigma) \cap \widetilde{E}(A) = \emptyset$ that $\widetilde{E}(p) = E(p) \notin \widetilde{E}(\Sigma \setminus \{p\}) \sqcup \widetilde{E}(A) = \widetilde{E}((\Sigma \sqcup A) \setminus \{p\})$. If $p \in \partial A$, then $\widetilde{E}(p) \notin \widetilde{E}(\Sigma) \sqcup \widetilde{E}(A \setminus \{p\}) = \widetilde{E}((\Sigma \sqcup A) \setminus \{p\})$, since $\widetilde{E}(\Sigma) \cap \widetilde{E}(A) = \emptyset$, and $\widetilde{E}$ restricts to an embedding $A \to \mathbb{C}$. Moreover, by choice of A, $\widetilde{E}$ restricts to an immersion with normal crossings on $\Pi_0 = \Sigma \sqcup A$, which is (4').

Assume that for some $d \in \{0, ..., D-1\}$, we are given a fold map $G_d \colon W \to \mathbb{C}$ with the properties (1') to (4'). Let us construct a fold map $G_{d+1} \colon W \to \mathbb{C}$ such that (1') to (4') hold.

By construction, there exist charts $\phi \colon X \to X' \subset \mathbb{R}^n$ on $W_{(0,1)}$ and $\psi \colon Y \to Y' \subset \mathbb{C}$ on $\mathbb{C}$ such that $[x_d, \frac{d+1}{D}] = [x_d, \frac{d}{D}] \cup [\frac{d}{D}, \frac{d+1}{D}] \subset X$, $E(X) \subset Y$ and $\psi(E(\phi^{-1}(t, x))) = \Lambda_i(t, x)$ for all

$(t,x) \in X' \subset \mathbb{R} \times \mathbb{R}^{n-1}$ and for some integer $i \in \{0,...,n-1\}$. In particular, $\Lambda_i(X') \subset Y'$.

Note that it follows from $x_d \le z_d$ and $w_{d+1} \le \frac{d+1}{D}$ that $Z_d = [z_d, w_{d+1}] \subset [x_d, \frac{d+1}{D}] \subset X$. By (1'), we have $Z_d \cap L_d = \emptyset$. Thus, after restricting ϕ to a chart $X \setminus L_d \to \phi(X \setminus L_d)$, we may assume without loss of generality that $X \cap L_d = \emptyset$ and we still have $Z_d \subset X$. Again by (1'), we obtain $G_d|_X = E|_X$. Thus, $G_d(X) \subset Y$ and $\psi(G_d(\phi^{-1}(t,x))) = \Lambda_i(t,x)$ for all $(t,x) \in X' \subset \mathbb{R} \times \mathbb{R}^{n-1}$.

Since $Z_d \subset S(E) \cap X$, the chart $\phi\colon X \to X' \subset \mathbb{R} \times \mathbb{R}^{n-1}$ restricts to an embedding $Z_d \to \mathbb{R} \times \{0\} \cap X'$. In particular, there are real numbers $s < s'$ such that Z_d can be identified with $\phi(Z_d) = [s,s'] \times \{0\} \subset X'$. Choose $\rho > 0$ such that $[s,s'] \times \{x \in \mathbb{R}^{n-1};\ \|x\|^2 \le \rho\} \subset X'$. Define the compact subset $P := \phi^{-1}([s,s'] \times \{x \in \mathbb{R}^{n-1};\ \|x\|^2 \le \rho\}) \subset X$.

Choose a smooth function $\beta\colon \mathbb{R} \to \mathbb{R}$ such that $\beta(0) = 1$ and $\beta(r) = 0$ for $r \ge \rho$. Choose $R > 0$ such that $|\beta(r)| \le R$ and $|\beta'(r)| \le R$ for all $r \in \mathbb{R}$. (In particular, $R \ge |\beta(0)| = 1$.)

Choose $\delta \in (0,1]$ such that $\|y - z\| \ge \delta$ for all $y \in \Lambda_i(\phi(P))$ and $z \in \mathbb{C} \setminus Y'$. (This is possible, since $\Lambda_i(\phi(P))$ and $\mathbb{C} \setminus Y'$ are disjoint subsets of the metric space $(\mathbb{C}, \|\cdot\|)$, where $\Lambda_i(\phi(P))$ is compact and $\mathbb{C} \setminus Y'$ is a closed subset.) Note that $[s,s'] \times (-\frac{\delta}{R}, \frac{\delta}{R}) \subset Y'$. (In fact, suppose that $z := (a,b) \in [s,s'] \times (-\frac{\delta}{R}, \frac{\delta}{R})$ satisfies $z \in \mathbb{C} \setminus Y'$. We have $(a,0) \in [s,s'] \times \{0\} \subset \phi(P)$. Setting $y := (a,0) = \Lambda_i(a,0,...,0) \in \Lambda_i(\phi(P))$, we obtain $\frac{\delta}{R} \le \delta \le ||y - z|| = |b|$.)

Choose $u \in (0, \frac{s+s'}{2})$ and $v \in (0, \frac{\delta}{R})$ so small that $[s, s+u] \times [-v,v] \cap \psi(G_d(\Pi_d) \cap Y) = (s,0)$ and $[s'-u, s'] \times [-v,v] \cap \psi(G_d(\Pi_d) \cap Y) = (s',0)$. (This is possible by (3'), using that $\psi(G_d(\Pi_d) \cap Y)$ is a closed subset of Y' and that $G_d(\Pi_d)$ has only finitely many double points.)

Define $Y_0' := (s,s') \times (-v,v)$, $Y_0 := \psi^{-1}(Y_0')$ and $X_0 := (G_d)^{-1}(Y_0)$. $\Pi_d \cap X_0$ is a (not necessarily compact) 1-dimensional manifold with boundary. The map $H_d\colon \Pi_d \cap X_0 \to Y_0'$, $H_d(p) = \psi(G_d(p))$, is an immersion with normal crossings by (4'). Let $\Delta_d \subset Y_0'$ be the (finite!) set of double points of H_d. By construction, $H_d(\Pi_d \cap X_0) \subset Y_0' \cap \psi(G_d(\Pi_d) \cap Y) \subset (s+u, s'-u) \times (-v,v)$.

Set $h_d := \operatorname{Im} \circ H_d$. By Brown's theorem, $\operatorname{Reg}(h_d)$ is residual in $\mathbb{R}$. Hence, there exists a point $v_0 \in (-v,v) \cap \operatorname{Reg}(h_d)$ such that $\operatorname{Im}^{-1}(v_0) \cap H_d(\partial\Pi_d \cap X_0) = \emptyset$ and $\operatorname{Im}^{-1}(v_0) \cap \Delta_d = \emptyset$.

Choose a smooth function $\alpha\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) = 0$ for $t \in \mathbb{R} \setminus (s,s')$, $\alpha(t) = v_0$ for $t \in [s+u, s'-u]$ and $|\alpha(t)| < v$ for all $t \in \mathbb{R}$.

By construction, the graph $\Gamma_\alpha := \{(t, \alpha(t));\ t \in (s,s')\} \subset Y_0'$ of α on (s,s') satisfies

(α1) $\Gamma_\alpha \cap H_d(\partial\Pi_d \cap X_0) = \emptyset$. (In fact, it follows from $H_d(\partial\Pi_d \cap X_0) \subset (s+u, s'-u) \times (-v,v)$ and $\Gamma_\alpha \cap (s+u, s'-u) \times (-v,v) = (s+u, s'-u) \times \{v_0\}$ that $\Gamma_\alpha \cap H_d(\partial\Pi_d \cap X_0) = (s+u, s'-u) \times \{v_0\} \cap H_d(\partial\Pi_d \cap X_0) \subset \operatorname{Im}^{-1}(v_0) \cap H_d(\partial\Pi_d \cap X_0) = \emptyset$.)

(α2) $\Gamma_\alpha \cap \Delta_d = \emptyset$. (In fact, it follows from $\Delta_d \subset H_d(\Pi_d \cap X_0)$ that $\Gamma_\alpha \cap \Delta_d = (s+u, s'-u) \times \{v_0\} \cap \Delta_d \subset \operatorname{Im}^{-1}(v_0) \cap \Delta_d = \emptyset$.)

(α3) $H_d \pitchfork \Gamma_\alpha$. (In fact, assume that there exists $p \in \Pi_d \cap X_0$ such that H_d is not transversal to Γ_α at p. In particular, $H_d(p) \in \Gamma_\alpha \cap (s+u, s'-u) \times (-v,v) = (s+u, s'-u) \times \{v_0\}$. This implies that $p \in \Pi_d \cap X_0$ is a singular point of $h_d = \operatorname{Im} \circ H_d$ with $h_d(p) = v_0$, which is a contradiction to $v_0 \in \operatorname{Reg}(h_d)$. Therefore, $H_d \pitchfork \Gamma_\alpha$.)

Since $|\alpha(t)\beta'(r)| \le |\alpha(t)|R < vR < \delta \le 1$ for all $(t,r) \in \mathbb{R}^2$, it follows from Lemma 3.3.6 that the perturbation

$$\widetilde{\Lambda}_i \colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^2, \qquad \widetilde{\Lambda}_i(t,x) = (t, \lambda_i(x) + \alpha(t)\beta(\|x\|^2)),$$

of Λ_i is a fold map such that $S(\widetilde{\Lambda}_i) = S(\Lambda_i)$. Note that $\widetilde{\Lambda}_i(X') \subset Y'$. (In fact, assume that there exists a point $(t,x) \in X'$ such that $\widetilde{\Lambda}_i(t,x) \in \mathbb{C} \setminus Y'$. Then it follows from $\widetilde{\Lambda}_i(X' \setminus \phi(P)) = \Lambda_i(X' \setminus \phi(P)) \subset Y'$ that $(t,x) \in \phi(P)$. But then, by choice of d, we obtain the contradiction $\delta \leq \|\Lambda_i(t,x) - \widetilde{\Lambda}_i(t,x)\| = \|(0, \alpha(t)\beta(\|x\|^2))\| = |\alpha(t)\beta(\|x\|^2)| \leq |\alpha(t)|R < vR < \delta$.) Thus, $\widetilde{\Lambda}_i$ induces a fold map

$$\widetilde{G}_d \colon X \to Y, \qquad \widetilde{G}_d(p) = \psi^{-1}(\widetilde{\Lambda}_i(\phi(p))).$$

The compact subset $P \subset X$ gives rise to the open covering $W = X \cup (W \setminus P)$. Since $\widetilde{\Lambda}_i|_{X' \setminus \phi(P)} = \Lambda_i|_{X' \setminus \phi(P)}$ by construction, we have $\widetilde{G}_d|_{X \setminus P} = G_d|_{X \setminus P}$. This shows that the fold maps $\widetilde{G}_d$ and $G_d|_{W \setminus P}$ agree on the intersection $X \cap (W \setminus P) = X \setminus P$ and can thus be assembled to a fold map

$$G_{d+1} \colon W \to \mathbb{C}, \qquad G_{d+1}(p) = \begin{cases} \widetilde{G}_d(p), & \text{for } p \in X, \\ G_d(p), & \text{for } p \notin X. \end{cases}$$

Let us check that G_{d+1} satisfies the properties (1') to (4'):

(1'). Define the compact subset $L_{d+1} := L_d \cup P \subset W_{(0,1)}$. Let us first show that $L_{d+1} \cap S(E) \subset [0, w_{d+1}] \cup \operatorname{int} \Sigma$. It suffices to show that $L_d \cap S(E) \subset [0, w_d] \cup \operatorname{int} \Sigma$ and $P \cap S(E) \subset [z_d, w_{d+1}]$. The first statement follows from (1'). Moreover, since $P \subset X$, we have $P \cap S(E) = P \cap \phi^{-1}(S(\Lambda_i) \cap X') = \phi^{-1}(\phi(P) \cap S(\Lambda_i)) = \phi^{-1}([s,s'] \times \{0\}) = Z_d = [z_d, w_{d+1}]$. It remains to show that $G_{d+1}|_{W \setminus L_{d+1}} = E|_{W \setminus L_{d+1}}$. Since $P \subset L_{d+1}$, we have by construction that $G_{d+1}|_{W \setminus L_{d+1}} = G_d|_{W \setminus L_{d+1}}$. Since $L_d \subset L_{d+1}$, it follows from (1') that $G_d|_{W \setminus L_{d+1}} = E|_{W \setminus L_{d+1}}$.

(2'). Using the open covering $W = X \cup (W \setminus P)$, we have $S(G_{d+1}) = S(G_{d+1}|_X) \cup S(G_{d+1}|_{W \setminus P})$. We have $S(G_{d+1}|_X) = S(\widetilde{G}_d) = \phi^{-1}(S(\widetilde{\Lambda}_i) \cap X') = \phi^{-1}(S(\Lambda_i) \cap X') = S(G_d) \cap X = S(G_d|_X)$. Moreover, by construction, $S(G_{d+1}|_{W \setminus P}) = S(G_d|_{W \setminus P})$. Thus, $S(G_{d+1}) = S(G_d)$. Furthermore, by (2'), $S(G_d) = S(E)$. Hence, $S(G_{d+1}) = S(E)$.

(3'). Let $p \in \partial \Pi_{d+1} \cap W_{(0,1)}$. It follows from $\Pi_{d+1} = \Pi_d \cup Z_d$ and $p \notin Z_d$ that $\Pi_{d+1} \setminus \{p\} = (\Pi_d \setminus \{p\}) \cup Z_d$. Hence, in order to show that $G_{d+1}(p) \notin G_{d+1}(\Pi_{d+1} \setminus \{p\})$, it suffices to show that $G_{d+1}(p) \notin G_{d+1}(\Pi_d \setminus \{p\})$ and $G_{d+1}(p) \notin G_{d+1}(\operatorname{int} Z_d)$. (Note that $\Pi_d \cap Z_d = \partial Z_d$.) As in the proof of (1'), we have $P \cap S(E) = Z_d$. Thus, it follows from $p \in S(E) \setminus Z_d$ and $Z_d \subset P$ that $G_{d+1}(p) = G_d(p)$ and $G_{d+1}(Z_d) = \widetilde{G}_d(Z_d)$. Moreover, $G_{d+1}(\Pi_d \setminus \{p\}) = G_d(\Pi_d \setminus \{p\})$. (Use $P \cap \Pi_d = \partial Z_d$ and check explicitly that G_d and G_{d+1} agree on ∂Z_d.) Hence, $G_d(p) \notin G_d(\Pi_d \setminus \{p\})$ by (3'). It remains to show that $G_d(p) \notin \widetilde{G}_d(\operatorname{int} Z_d)$. In fact, $\widetilde{G}_d(\operatorname{int} Z_d) = \psi^{-1}(\widetilde{\Lambda}_i(\phi(\operatorname{int} Z_d))) = \psi^{-1}(\widetilde{\Lambda}_i((s,s') \times \{0\})) = \psi^{-1}(\Gamma_\alpha)$. By $(\alpha 1)$, we obtain $\psi^{-1}(\Gamma_\alpha) \cap G_d(\partial \Pi_d) = \emptyset$. Hence, $G_d(p) \notin \psi^{-1}(\Gamma_\alpha) = \widetilde{G}_d(\operatorname{int} Z_d)$.

(4'). Using that $P \cap S(E) = Z_d$, $\Pi_{d+1} = \Pi_d \cup Z_d$ and $\Pi_d \cap Z_d = \partial Z_d$, we obtain $\Pi_{d+1} \setminus P = \Pi_{d+1} \setminus Z_d = \Pi_d \setminus \partial Z_d = \Pi_d \setminus P$ and $\Pi_{d+1} \cap P = Z_d$. It follows from $G_{d+1}|_{\Pi_{d+1} \setminus P} = G_{d+1}|_{\Pi_d \setminus P} = G_d|_{\Pi_d \setminus P}$ and $G_{d+1}|_{\partial Z_d} = G_d|_{\partial Z_d}$ that $G_{d+1}|_{\Pi_d} = G_d|_{\Pi_d}$. Thus, by (4'), $G_{d+1}|_{\Pi_d}$ is an immersion with normal crossings. In order to show that $G_{d+1}|_{\Pi_{d+1}}$ is an immersion with normal crossings, it suffices to note the following:

- G_{d+1} restricts to an embedding $\operatorname{int} Z_d \to \mathbb{C}$. (In fact, since $\operatorname{int} Z_d \subset X$ and $\phi(\operatorname{int} Z_d) =$

$(s, s') \times \{0\} \subset X'$, it suffices to show that $(s, s') \times \{0\} \to \mathbb{C}$, $(t, 0) \mapsto G_{d+1}(\phi^{-1}(t, 0))$, is an embedding. Indeed, we have $G_{d+1}(\phi^{-1}(t, 0)) = \widetilde{G}_d(\phi^{-1}(t, 0)) = \psi^{-1}(\widetilde{\Lambda}_i(t, 0)) = \psi^{-1}(t, \alpha(t))$ for all $(t, 0) \in (s, s') \times \{0\}$.)

- $G_{d+1}(\operatorname{int} Z_d)$ does not contain any double points of $G_{d+1}|_{\Pi_d}$. (Since $G_{d+1}(\operatorname{int} Z_d) = \psi^{-1}(\Gamma_\alpha) \subset Y_0$, it suffices to show that $\psi^{-1}(\Gamma_\alpha)$ does not contain any double points of $G_{d+1}|_{\Pi_d \cap X_0} = G_d|_{\Pi_d \cap X_0}$. This is equivalent to $(\alpha 2)$.)
- $G_{d+1}|_{\Pi_d} \pitchfork G_{d+1}(\operatorname{int} Z_d)$. (Since $G_{d+1}(\operatorname{int} Z_d) = \psi^{-1}(\Gamma_\alpha) \subset Y_0$ and $G_{d+1}|_{\Pi_d} = G_d|_{\Pi_d}$, it suffices to show that $G_d|_{\Pi_d \cap X_0} \pitchfork \psi^{-1}(\Gamma_\alpha)$. This is equivalent to $(\alpha 3)$.)

This completes construction II. Eventually, the fold map $F_m := G_D$ satisfies condition (F_m). (In fact, $F_m|_{\Sigma_m} = G_D|_{\Pi_D}$ is an immersion with normal crossings by (4'). Moreover, $F_m(p)$ is not a double point of $F_m|_{\Sigma_m}$ for all $p \in \partial\Sigma_m \cap W_{(0,1)}$ by (3'). Furthermore, it follows from (1') and (2') that $F_m \sim_W F$.)

After the theorem has been proven for all simple cobordisms W, the proof for arbitrary W is as follows. By Definition 3.4.13, the restrictions of F to the simple cobordisms $W(k)$, $F(k) \colon W(k) \to \mathbb{C}$, $k \in \mathbb{N}$, are fold pre-fields such that $F(k)$ is stable in a suitable open neighbourhood of $\partial W(k)$ in $W(k)$ for every $k \in \mathbb{N}$. Application of the theorem to $F(k) \in \mathcal{F}^{\text{pre}}_{\text{s}\partial}(W(k))$ yields for every $k \in \mathbb{N}$ a fold map $G(k) \in \mathcal{F}^{\text{pre}}_{\text{s}}(W(k))$ such that $G(k) \sim_{W(k)} F(k)$. The fold maps $G(k) \in \mathcal{F}^{\text{pre}}_{\text{s}}(W(k))$ give rise to a fold map $G := \bigsqcup_{k \in \mathbb{N}} G(k) \in \mathcal{F}^{\text{pre}}_{\text{s}}(W)$ by Definition 3.4.13. Finally, it follows from $F(k) \sim_W G(k)$ for every $k \in \mathbb{N}$ that $F \sim_W G$. (See the end of the proof of Theorem 3.4.9.)

□

Theorem 3.4.15. *If $F \in \mathcal{F}_{\mathrm{s}}^{\mathrm{pre}}(W)$, then there exists $G \in \mathcal{F}_{\mathrm{s}}(W)$ such that $F \sim_W G$.*

Proof. We return to the proof of Theorem 3.4.9 and assume that F is stable. By Definition 3.4.13, we may assume that W is a simple cobordism. Then it suffices to show that construction I can be performed in such a way that the resulting fold map F_1 is stable. (In this case, $G := F_1 \circ \Xi$ will also be stable, being the precomposition of a stable map with a diffeomorphism. Hence, $G \in \mathcal{F}_{\mathrm{s}}^{\mathrm{pre}}(W) \cap \mathcal{F}(W) = \mathcal{F}_{\mathrm{s}}(W)$ and $F \sim_W G$.)

Since F is stable, the restriction of F to $S(F)$ is an immersion with normal crossings. The set of normal crossings (double points) is given by $D(F) := \{z \in \mathbb{C};\ |S(F) \cap F^{-1}(z)| > 1\}$. Since $S(F)$ is compact, $D(F)$ is a finite set. As F is stable, we have

(i) F restricts to an embedding $S(F) \setminus F^{-1}(D(F)) \to \mathbb{C}$.

We may assume that $F(p_T) \notin D(F)$ for every $T \in \Omega$. Thus, for the choice of the open neighbourhoods V_T of $F(p_T)$ in $\mathbb{C}$, we may in addition to $(V1)$, $(V2)$, $(V3)$ and $(V4)$ assume that

$(V5)$ $V_T \cap D(F) = \emptyset$ for every $T \in \Omega$.

The modification of F in construction I is based on the application of Proposition 3.3.7. Following the proof of Proposition 3.3.7, we choose for every $T \in \Omega$ a chart $\phi_T \colon U_{T0} \to U_{T1} \subset \mathbb{R}^n$ around p_T in U_T and a chart $\psi_T \colon V_{T0} \to V_{T1} \subset \mathbb{C}$ around $F(p_T)$ in V_T, such that $\phi_T(p_T) = 0 \in U_{T1}$, $F(U_{T0}) \subset V_{T0}$ and $\psi(F(p_T)) = 0 \in V_{T1}$, and there exists an integer $0 \leq i(T) \leq n-1$, such that for all $(t,x) \in U_{T1} \subset \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ we have $\psi_T(F(\phi_T^{-1}(t,x))) = \Lambda_{i(T)}(t,x)$ $(= (t, \lambda_{i(T)}(x)))$. Without loss of generality, we can assume that

(ii) $F(S(F) \setminus U_{T0}) \cap V_{T0} = \emptyset$ for all $T \in \Omega$.

(In fact, fix $T \in \Omega$. Define the open subsets $V'_{T0} := \mathbb{C} \setminus F(S(F) \setminus U_{T0}) \subset \mathbb{C}$ and $U'_{T0} := U_{T0} \cap F^{-1}(V'_{T0}) \subset W_{(0,\varepsilon)\cup(1-\varepsilon,1)}$. It follows from (i) and $(V5)$ that $F(p_T) \notin F(S(F) \setminus U_{T0})$. Therefore, $p_T \in U_{T0} \cap F^{-1}(V'_{T0}) = U'_{T0}$. Moreover, $F(S(F) \setminus U'_{T0}) \cap V'_{T0} = \emptyset$. (In fact, if $\sigma \in S(F)$ such that $F(\sigma) \in V'_{T0} = \mathbb{C} \setminus F(S(F) \setminus U_{T0})$, then $\sigma \in U_{T0} \cap F^{-1}(V'_{T0}) = U'_{T0}$.))

For every $T \in \Omega$ we follow the proof of Proposition 3.3.7 and introduce the perturbed fold map

$$\widetilde{F}_{T0} \colon U_{T0} \to V_{T0}, \qquad \widetilde{F}_{T0}(p) = \psi_T^{-1}(\widetilde{\Lambda}_{i(T)}(\phi_T(p))).$$

Define $U_0 := \bigsqcup_{T\in\Omega} U_{T0}$ and $V_0 := \bigsqcup_{T\in\Omega} V_{T0}$. Note that $K_1 \subset U_0 \subset U_1$, since $K_T \subset U_{T0} \subset U_T$ for all $T \in \Omega$.

Consider the open covering $W = (W \setminus K_1) \cup U_0$. F_1 restricts to a stable fold map on $W \setminus K_1$, since, by (1), $F_1|_{W\setminus K_1} = F|_{W\setminus K_1}$, which is stable. Moreover, F_1 restricts to a stable fold map on on U_0. (In fact, $F_1|_{U_0} = (F_1|_{U_1})|_{U_0} = \widetilde{F}_1|_{U_0} = \bigsqcup_{T\in\Omega} \widetilde{F}_T|_{U_{T0}} = \bigsqcup_{T\in\Omega} \widetilde{F}_{T0}$. This is stable, since $\widetilde{F}_{T0}$ is stable for all $T \in \Omega$ and the $V_{T0} \subset V_T$ are pairwise disjoint by $(V3)$.) We have $(W\setminus K_1)\setminus U_0 = W \setminus U_0$ and $U_0 \setminus (W \setminus K_1) = K_1$. It suffices to show that $F_1(S(F_1)\cap(W\setminus U_0))\cap F_1(S(F_1)\cap K_1) = \emptyset$. Note that $F_1(S(F_1)\cap(W\setminus U_0)) = F_1(S(F)\cap(W\setminus U_0)) = F_1(S(F)\setminus U_0) = F(S(F)\setminus U_0)$ and $F_1(S(F_1)\cap K_1) = F_1(S(F)\cap K_1) \subset F_1(U_0) = \bigsqcup_{T\in\Omega} \widetilde{F}_{T0}(U_{T0}) \subset \bigsqcup_{T\in\Omega} V_{T0} = V_0$. Finally, it follows from (ii) that $F(S(F)\setminus U_0)\cap V_0 = \bigsqcup_{T\in\Omega} F(S(F)\setminus U_0)\cap V_{T0} \subset \bigsqcup_{T\in\Omega} F(S(F)\setminus U_{T0})\cap V_{T0} = \emptyset$. ☐

Part II

Fold Maps from Cobordisms into the Plane

Chapter 4

Generic Smooth Maps into the Plane

Let X denote a smooth manifold of dimension $n \geq 2$ (without boundary).

Generically, the singular locus $S(F)$ of a smooth map $F\colon X \to \mathbb{R}^2$ will not only consist of fold singularities (as studied in Section 3.3), but will also contain so-called *cusps*. It turns out that the set of cusps of F forms a discrete subset of the 1-dimensional submanifold $S(F)$ of X. Historically, the equidimensional case $n = 2$ has been studied by Whitney in [60]. Based on [17], the theory of fold and cusp singularities is systematically introduced in Section 4.1 and Section 4.3. Furthermore, using the concept of *intrinsic derivative*, Section 4.5 presents a method to determine cusps and the absolute index of fold points in practice.

The determination of the state sets defined in Section 3.1.6 requires to extend given boundary conditions to fold maps from a cobordism into the plane while controlling the number of loops. The following two-step program indicates that the study of fold maps from the perspective of generic smooth maps is a promising approach to this construction problem:

1. Extend the given boundary conditions to an *arbitrary* generic smooth map from the cobordism into the plane.
2. Use suitable local modifications for generic smooth maps to produce a fold map with the desired properties.

Step 1. is solved in Section 4.4 by employing a relative version of the Thom transversality theorem (see Proposition A.3.2). As far as step 2. is concerned, the present chapter will present two local modifications for generic smooth maps that will be combined in Chapter 5 (when $n = 2$) and Chapter 6 (when $n > 2$) to control to some extent the number of components of the singular locus of a generic smooth map into the plane:

- Elimination of cusps (see Section 4.6) due to Levine [32]. In order to make this process applicable as a local modification, [32, Lemma (4.9), p. 293] modifies the homotopy of the local normal form that describes the elimination of cusps carefully to maintain the identity map outside a compact subset.
- Creation of cusps (see Section 4.7). By lack of a detailed reference, we give an ad hoc construction that makes the homotopy of the local normal form that describes the creation of cusps into the identity map outside a compact subset.

Note that the cases $n = 2$ and $n > 2$ are different to handle since in dimension 2 it is not always possible to choose a suitable path between two cusps.

With regard to further investigations of the state sets defined in Section 3.1.6 one has to focus on the problem of modifying in a desired way the singular set $S(F)$ of a given generic smooth map $F\colon X \to \mathbb{R}^2$ defined on a smooth n-dimensional manifold X^n without boundary. Assuming that $S(F)$ consists of a finite number of components, the task is to modify F in such a way that the number of components changes in a controlled way.

All modifications considered in the present chapter are local in the sense that F is modified in small neighbourhoods of certain critical points. Therefore, the results can be applied to study generic smooth maps on cobordisms. (The modifications will not affect the behaviour near the boundary.) However, no topological phenomena are investigated.

4.1 One-generic Maps into the Plane

In this section, we collect general facts about *one-generic* smooth maps from X^n $(n \geq 2)$ into the plane. Recall from Section 3.3 that for $r \in \{0,1,2\}$ a submanifold $S_r(X,\mathbb{R}^2)$ of $J^1(X,\mathbb{R}^2)$ is defined by

$$S_r(X,\mathbb{R}^2) := \{\sigma \in J^1(X,\mathbb{R}^2);\ \operatorname{corank}\sigma = r\}.$$

In order to simplify the notation, we set $S_r := S_r(X,\mathbb{R}^2)$. The codimension of S_r in $J^1(X,\mathbb{R}^2)$ is given by $r(n-2+r)$. The jet manifold has the decomposition $J^1(X,\mathbb{R}^2) = S_0 \cup S_1 \cup S_2$.

Lemma 4.1.1. *(a) The submanifold $S_0 \subset J^1(X,\mathbb{R}^2)$ is an open subset of $J^1(X,\mathbb{R}^2)$.*
(b) The submanifold $S_2 \subset J^1(X,\mathbb{R}^2)$ is a closed subset of $J^1(X,\mathbb{R}^2)$.

Proof. (a). The codimension of the submanifold S_0 in $J^1(X,\mathbb{R}^2)$ is given by $r(n-2+r) = 0$ since $r = 0$. Hence, S_0 is an open subset of $J^1(X,\mathbb{R}^2)$.

(b). Let $\alpha\colon J^1(X,\mathbb{R}^2) \to X$ denote the source map and let $\beta\colon J^1(X,\mathbb{R}^2) \to \mathbb{R}^2$ denote the target map. It is well-known that $\alpha\times\beta\colon J^1(X,\mathbb{R}^2) \to X\times\mathbb{R}^2$ can be considered as a vector bundle over $X\times\mathbb{R}^2$ with fiber $\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^2)$. Moreover, the restriction $(\alpha\times\beta)|\colon S_2 \to X\times\mathbb{R}^2$ is a subfiberbundle of this vector bundle with fiber

$$L^2(\mathbb{R}^n,\mathbb{R}^2) = \{A \in \operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^2);\ \operatorname{corank}A = 2\} = \{0\}.$$

Thus, the total space S_2 of the fiber bundle $(\alpha\times\beta)|\colon S_2 \to X\times\mathbb{R}^2$ can be identified with the image in $J^1(X,\mathbb{R}^2)$ of the zero section of the vector bundle $\alpha\times\beta\colon J^1(X,\mathbb{R}^2) \to X\times\mathbb{R}^2$. Hence, S_2 is a closed subset of $J^1(X,\mathbb{R}^2)$. □

Lemma 4.1.2. *Let $f\colon X \to \mathbb{R}^2$ be smooth and let $x \in X$ be a point.*

(a) $j^1(f) \pitchfork S_2$ at x if and only if $j^1(f)(x) \notin S_2$.
(b) If $j^1(f) \pitchfork S_2$ at x, then there exists an open neighbourhood $U \subset X$ of x such that $j^1(f) \pitchfork S_2$ on U.
(c) If $j^1(f) \pitchfork S_2$ at x and $j^1(f)(x) \in \overline{S_1}^{J^1(X,\mathbb{R}^2)}$, then $j^1(f)(x) \in S_1$.

Proof. (a). If $j^1(f)(x) \notin S_2$, then $j^1(f) \pitchfork S_2$ at x by Definition A.1.1. Conversely, assume that $j^1(f) \pitchfork S_2$ at x. The codimension of the submanifold S_2 in $J^1(X,\mathbb{R}^2)$ is given by $\operatorname{codim}S_2 = r(n-2+r) = 2n$ since $r = 2$. As $\dim X = n < 2n = \operatorname{codim}S_2$, we obtain from the proof of [17, Proposition II.4.2, page 51] that $j^1(f)(x) \notin S_2$.

(b). Let $j^1(f) \pitchfork S_2$ at x. By part (a) we have $j^1(f)(x) \notin S_2$. Hence, x is an element of

$$U := j^1(f)^{-1}(J^1(X,\mathbb{R}^2) \setminus S_2).$$

It follows from Lemma 4.1.1(b) that U is an open subset of X. By construction, we have $j^1(f)(x') \notin S_2$ for all $x' \in U$. Therefore, part (a) implies that $j^1(f) \pitchfork S_2$ on U.

(c). We have $j^1(f)(x) \in J^1(X,\mathbb{R}^2) = S_0 \cup S_1 \cup S_2$. It follows from $j^1(f) \pitchfork S_2$ at x and part (a) that $j^1(f)(x) \notin S_2$. Therefore, it suffices to show that $j^1(f)(x) \notin S_0$. By assumption, we have $j^1(f)(x) \in \overline{S_1}^{J^1(X,\mathbb{R}^2)}$. Since $S_1 \cap S_0 = \emptyset$ and S_0 is an open subset of $J^1(X,\mathbb{R}^2)$ by Lemma 4.1.1(a), we obtain $\overline{S_1}^{J^1(X,\mathbb{R}^2)} \cap S_0 = \emptyset$. In particular, $j^1(f)(x) \notin S_0$. □

Definition 4.1.3. Let $f\colon X \to \mathbb{R}^2$ be smooth.

(a) Given a point $x \in X$, we say that f is *one-generic at* x if $j^1(f) \pitchfork S_1$ at x and $j^1(f) \pitchfork S_2$ at x.
(b) Given a subset $A \subset X$, we say that f is *one-generic on* A if f is one-generic at x for all $x \in A$.
(c) Finally, we say that f is *one-generic* if f is one-generic on X.

The following Lemma shows that the property of being one-generic is compatible with restriction to open subsets:

Lemma 4.1.4. *Let $f\colon X \to \mathbb{R}^2$ be smooth and let $X' \subset X$ be an open subset. Then the restriction $f'\colon X' \to \mathbb{R}^2$ of f to X' is one-generic if and only if f is one-generic on X'. In particular, if f is one-generic, then f' is one-generic.*

Proof. The restriction $f'\colon X' \to \mathbb{R}^2$ of f to X' is one-generic if and only if $j^1(f') \pitchfork S_r(X', \mathbb{R}^2)$ at x for all $x \in X'$ and $r \in \{1, 2\}$. This holds if and only if $j^1(f)|_{X'} \pitchfork S_r \cap \alpha^{-1}(X')$ at x for all $x \in X'$. By Lemma A.1.2(b) this is equivalent to $j^1(f) \pitchfork S_r \cap \alpha^{-1}(X')$ at x for all $x \in X'$. By Lemma A.1.2(b) this is equivalent to $j^1(f) \pitchfork S_r$ at x for all $x \in X'$. (In fact, note that $S_r \cap \alpha^{-1}(X')$ is an open subset of S_r. Moreover, $j^1(f)(x') \notin S_r \cap \alpha^{-1}(X')$ implies $j^1(f)(x') \notin S_r$ for all $x' \in X'$ since $j^1(f)(x') \in \alpha^{-1}(X')$ for all $x' \in X'$.) Equivalently, f is one-generic on X'. □

Lemma 4.1.5. *Assume that $f\colon X \to \mathbb{R}^2$ is a smooth map which is one-generic at a point $x \in X$. Then there exists an open neighbourhood $U \subset X$ of x such that f is one-generic on U.*

Proof. By assumption, f is one-generic at x. Hence, by Definition 4.1.3(a), $j^1(f) \pitchfork S_1$ at x and $j^1(f) \pitchfork S_2$ at x. Set $y := j^1(f)(x)$. By Lemma 4.1.2(a), $j^1(f) \pitchfork S_2$ at x implies that $y \notin S_2$. Since $y \in J^1(X, \mathbb{R}^2) = S_0 \cup S_1 \cup S_2$, we can distinguish the following two cases:

- $y \in S_0$. Note that S_0 is an open subset of $J^1(X, \mathbb{R}^2)$ by Lemma 4.1.1(a). Thus, $U_0 := j^1(f)^{-1}(S_0)$ is an open neighbourhood of x in X such that $j^1(f)(x_0) \notin S_1 \cup S_2$ for all $x_0 \in U_0$. Hence, for every $x_0 \in U_0$ we have $j^1(f) \pitchfork S_1$ at x_0 and $j^1(f) \pitchfork S_2$ at x_0. Therefore, f is one-generic on $U := U_0$.
- $y \in S_1$. Since $j^1(f) \pitchfork S_1$ at x, Lemma A.1.3 implies that there exists an open neighbourhood $U_1 \subset X$ of x such that $j^1(f) \pitchfork S_1$ on U_1. Since $j^1(f) \pitchfork S_2$ at x, Lemma 4.1.2(b) implies that there exists an open neighbourhood U_2 of x in X such that $j^1(f) \pitchfork S_2$ on U_2. Hence, f is one-generic on $U := U_1 \cap U_2$.

□

4.2 The Intrinsic Derivative

Following [17, Section VI.3, p. 149 ff], we present briefly the concept of *intrinsic derivative*, which will be employed in Section 4.5 to define the index of fold lines and to determine cusps.

The intrinsic derivative allows to differentiate smooth vector bundle homomorphisms in a way that is related to the vector bundle structure. First consider trivial vector bundles $E := X \times \mathbb{R}^n$ and $F := X \times \mathbb{R}^q$ over X. Let $\rho\colon E \to F$ be a smooth vector bundle homomorphism (covering the identity map on X). Note that ρ is nothing but a smooth map $\rho\colon X \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q)$. Evaluation at a given point $p \in X$ yields a linear map $\sigma := \rho(p)\colon \mathbb{R}^n \to \mathbb{R}^q$. Let $\iota_\sigma\colon K_\sigma \to \mathbb{R}^n$ denote the inclusion of the kernel $K_\sigma := \ker \sigma \subset \mathbb{R}^n$ and let $\pi_\sigma\colon \mathbb{R}^q \to L_\sigma$ denote the canonical projection to the cokernel $L_\sigma := \operatorname{coker} \sigma = \frac{\mathbb{R}^q}{\operatorname{Im} \sigma}$. The differential of ρ at $p \in X$ is a linear map

$$D_p\rho\colon T_pX \to T_\sigma \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q) = \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q).$$

Definition 4.2.1. In this situation, the *intrinsic derivative* of ρ at p is the linear map

$$\mathcal{D}_p\rho\colon T_pX \to \operatorname{Hom}(K_\sigma, L_\sigma), \qquad \mathcal{D}_p\rho(v) = \pi_\sigma \circ D_p\rho(v) \circ \iota_\sigma.$$

It is shown in [17, Section VI.3] that the intrinsic derivative of ρ at p transforms as $\overline{B(p)} \circ \mathcal{D}_p\rho(v) \circ A(p)^{-1}$ under changes of trivializations $A\colon X \to \mathrm{GL}(\mathbb{R}^n)$ of E and $B\colon X \to \mathrm{GL}(\mathbb{R}^q)$ of F. (The proof exploits the reduction $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q) \to \operatorname{Hom}(K_\sigma, L_\sigma)$.) Hence, the above definition of a linear map $\mathcal{D}_p\rho\colon T_pX \to \operatorname{Hom}(K_\sigma, L_\sigma)$ carries over to homomorphisms $\rho\colon E \to F$ between arbitrary smooth vector bundles over X (covering the identity map on X), where now $K_\sigma := \ker \sigma \subset E_p$ and $L_\sigma := \operatorname{coker} \sigma = \frac{F_p}{\operatorname{Im} \sigma}$.

Remark 4.2.2. There also exists the following description of the intrinsic derivative whose formulation does not rely on the choice of local trivializations of vector bundles. Consider $\rho\colon E \to F$ as a section $\rho\colon X \to \operatorname{Hom}(E, F)$ of the vector bundle $\operatorname{Hom}(E, F)$ over X. Let $p \in X$ and $\sigma := \rho(p)$. In contrast to the above construction on trivial bundles, the tangent map $D_p\rho\colon T_pX \to T_\sigma \operatorname{Hom}(E, F)$ does *not* induce a map $T_pX \to \operatorname{Hom}(E_p, F_p)$ because there is in general no canonical projection of the form $T_\sigma \operatorname{Hom}(E, F) \to T_\sigma(\operatorname{Hom}(E, F)_p)$ $(= \operatorname{Hom}(E_p, F_p))$. Nevertheless, if r denotes the corank of $\sigma \in \operatorname{Hom}(E, F)$, then $\sigma \in L^r(E, F)$, and one can compose $D_p\rho$ with the projection at σ to the normal bundle N of $L^r(E, F)$ in $\operatorname{Hom}(E, F)$,

$$T_\sigma \operatorname{Hom}(E, F) \to T_\sigma \operatorname{Hom}(E, F)/T_\sigma L^r(E, F) = N_\sigma.$$

Now the trick is to note that N_σ is also the normal space at σ of $L^r(E_p, F_p)$ in $\operatorname{Hom}(E_p, F_p)$. This normal bundle can naturally be described as $N_\sigma \cong \operatorname{Hom}(K_\sigma, L_\sigma)$ because the tangent space $T_\sigma L^r(E_p, F_p)$ can be shown to be the kernel of the natural surjection

$$T_\sigma \operatorname{Hom}(E_p, F_p) = \operatorname{Hom}(E_p, F_p) \to \operatorname{Hom}(K_\sigma, L_\sigma), \qquad A \mapsto \pi_\sigma \circ A \circ \iota_\sigma.$$

Finally, the resulting linear map $T_pX \to \operatorname{Hom}(K_\sigma, L_\sigma)$ can be shown to be the intrinsic derivative $\mathcal{D}_p\rho$ defined above. It is surjective if and only if $\rho\colon X \to \operatorname{Hom}(E, F)$ is transverse to $L^r(E, F)$ at p (see [17, Proposition VI.3.7, p. 151]).

Of particular interest is the following application to smooth maps $f\colon X \to Y$. Setting $E :=$

TX and $F := f^*TY$, the tangent map of f can be considered as a smooth vector bundle homomorphism $\rho := Df\colon E \to F$ (covering the identity map on X). Let $p \in X$ and $\sigma := \rho(p)$. In this concrete situation, we have $K_\sigma = \ker \sigma \subset E_p = T_pX$. Therefore, with the canonical identification $\mathrm{Hom}(T_pX, \mathrm{Hom}(K_\sigma, L_\sigma)) \cong \mathrm{Hom}(T_pX \otimes K_\sigma, L_\sigma)$ the intrinsic derivative $\mathcal{D}_p\rho\colon T_pX \to \mathrm{Hom}(K_\sigma, L_\sigma)$ restricts to a linear map $K_\sigma \otimes K_\sigma \to L_\sigma$. It can be shown (see [17, Exercise VI.3(3), p. 152]) that this linear map is symmetric, that is, it takes the same value on $v \otimes w$ and $w \otimes v$ for all $v, w \in K_\sigma$. Hence, it induces an element

$$\delta_p^2 f \in \mathrm{Hom}(K_\sigma \circ K_\sigma, L_\sigma),$$

where $V \circ V := V \otimes V / \langle v_1 \otimes v_2 - v_2 \otimes v_1;\ v_1, v_2 \in V \rangle$ denotes the symmetric product of a vector space V with itself.

Remark 4.2.3. In the special case that $Y = \mathbb{R}$ and that p is a critical point of f, $\delta_p^2 f$ can be identified with the Hessian of f at p, whose symmetry is a consequence of Schwarz's theorem (see [17, Exercise VI.3(4), p. 152].)

Let $\pi\colon J^2(X,Y) \to J^1(X,Y)$ denote the canonical projection described in [17, Exercise II.2(1), page 42]). By [17, Exercise VI.3(2), p. 152], there exists for every $\sigma \in J^1(X,Y) = \mathrm{Hom}(TX, TY)$ with source $p := \alpha(\sigma)$ a map

$$\Gamma_\sigma\colon \pi^{-1}(\sigma) \to \mathrm{Hom}(K_\sigma \circ K_\sigma, L_\sigma)$$

such that $\Gamma_\sigma(j^2(f)(p)) = \delta_p^2 f$ for all smooth maps $f\colon X \to Y$ with $j^1(f)(p) = \sigma$. Note that $\pi^{-1}(\sigma)$ is diffeomorphic to a Euclidean space, but does not possess a canonical linear structure (compare [17, Remark II.2(1), p. 41]). However, choosing local coordinates on X and Y, $\pi^{-1}(\sigma)$ inherits a linear structure, and Γ_σ turns out to be linear in these coordinates. It follows from [17, Exercise VI.3(2), p. 152] that Γ_σ is surjective (compare the argument after [17, Formular VI.3(4.1), p. 153]). This shows that Γ_σ is a submersion.

Finally, varying over $\sigma \in S_r$ for fixed corank r, K_σ and L_σ form vector bundles K and L over S_r, and the maps Γ_σ fit together to a map

$$\Gamma\colon \pi^{-1}(S_r) \to \mathrm{Hom}(K \circ K, L)$$

of fiber bundles over S_r such that Γ is a submersion. Define a smooth map

$$\Sigma\colon\ \mathrm{Hom}(K \circ K, L) \to \mathrm{Hom}(K, \mathrm{Hom}(K, L))$$

by composing a vector bundle homomorphism $K \circ K \to L$ with the canonical projection $K \otimes K \to K \circ K$, and then viewing the composition as a map $K \to \mathrm{Hom}(K, L)$. Note that the two-jet extension $j^2(f)\colon X \to J^2(X,Y)$ restricts to a map $j^2(f)|\colon S_r(f) \to S_r^{(2)}$ because

$$j^2(f)^{-1}(S_r^{(2)}) = j^2(f)^{-1}(\pi^{-1}(S_r)) = (\pi \circ j^2(f))^{-1}(S_r) = j^1(f)^{-1}(S_r) = S_r(f).$$

By construction, $\Sigma \circ \Gamma \circ j^2(f)|_{S_r(f)} = \mathcal{D}(Df)|_K$ for all smooth maps $f\colon X \to Y$.

4.3 (Two-)generic Maps into the Plane

Generalizing Definition 4.1.3, a smooth map $f\colon X^n \to Y^m$ is called *one-generic* if $j^1(f) \pitchfork S_r$ for all r. (The definition of $S_r := S_r(X,Y)$ is as in Section 3.3. Note that S_r is a submanifold of $J^1(X,Y)$ of codimension $(n-q+r)(m-q+r)$, where $q = \min(n,m)$.) If $S_r(f)$ denotes the set of points $x \in X$ where $D_x f\colon T_x X \to T_{f(x)} Y$ drops rank by r, then by construction $S_r(f) = j^1(f)^{-1}(S_r)$. Assuming $f\colon X \to Y$ to be one-generic, $S_r(f)$ is a submanifold, and we define $S_{r,s}(f)$ as the set of points $x \in S_r(f)$ where $D_x f|_{T_x S_r(f)}\colon T_x S_r(f) \to T_{f(x)} Y$ drops rank by s. The following theorem (see [17, Section VI.4, p. 152ff]) states that there exist universal submanifolds $S_{r,s} \subset J^2(X,Y)$ such that $S_{r,s}(f) = j^2(f)^{-1}(S_{r,s})$ for any one-generic map $f\colon X \to Y$.

Concerning notation, let $\alpha\colon J^2(X,Y) \to X$ denote the source map and let $\beta\colon J^2(X,Y) \to Y$ denote the target map. Furthermore, let $S_r^{(2)} := \pi^{-1}(S_r)$, where $\pi\colon J^2(X,Y) \to J^1(X,Y)$ denotes the canonical projection described in [17, Exercise II.2(1), page 42]). Finally, we will frequently use (for fixed r) the smooth vector bundles K and L over S_r whose fibers at $\sigma \in S_r \cong L^r(TX,TY)$ are given by $K_\sigma = \ker \sigma$ and $L_\sigma = \operatorname{coker} \sigma$.

Theorem 4.3.1. *For all pairs (r,s) of non-negative integers there exist fiber subbundles*

$$S_{r,s} := S_{r,s}(X,Y) \to X \times Y$$

of the smooth fiber bundle $\alpha \times \beta\colon J^2(X,Y) \to X \times Y$ such that

$$x \in S_{r,s}(f) \Leftrightarrow j^2(f)(x) \in S_{r,s}$$

for all one-generic maps $f\colon X \to Y$ (see [17, Theorem VI.4.7, p. 154]).

Moreover, $S_{r,s}$ is a submanifold of $S_r^{(2)}$ of codimension (see [17, Formula VI(4.4), p. 153])

$$\frac{l}{2}k(k+1) - \frac{l}{2}(k-s)(k-s+1) - s(k-s),$$

where $k := \operatorname{rank} K = n-q+r$ and $l := \operatorname{rank} L = m-q+r$.

Furthermore, $S_{r,s}$ is natural with respect to restriction to open subsets of X. (More precisely, if $X' \subset X$ is an open subset, then $\iota(S_{r,s}(X',Y)) = S_{r,s}(X,Y) \cap \alpha^{-1}(X')$, where $\iota\colon J^2(X',Y) \to J^2(X,Y)$ denotes the canonical inclusion.)

Proof. We briefly indicate the construction of $S_{r,s}$ given in [17, Chapter VI.4, p. 152 ff], which is based on the concept of intrinsic derivative of Section 4.2. Recall that the intrinsic derivative induces a map of fiber bundles over S_r,

$$\Gamma\colon S_r^{(2)} \to \operatorname{Hom}(K \circ K, L).$$

It is shown in [17, Proposition VI.4.3, p. 153] that a submanifold of $\operatorname{Hom}(K \circ K, L)$ is given by

$$\operatorname{Hom}(K \circ K, L)_s := \Sigma^{-1}(L^s(\operatorname{Hom}(K, \operatorname{Hom}(K,L))),$$

where the smooth map

$$\Sigma\colon \operatorname{Hom}(K \circ K, L) \to \operatorname{Hom}(K, \operatorname{Hom}(K, L))$$

composes a vector bundle homomorphism $K \circ K \to L$ with the canonical projection $K \otimes K \to K \circ K$, and then views the composition as a map $K \to \operatorname{Hom}(K, L)$. Note that Σ is in general *not* transverse to $L^s(\operatorname{Hom}(K, \operatorname{Hom}(K, L))$, which makes the proof that $\operatorname{Hom}(K \circ K, L)_s$ is in fact a manifold more involved (it uses the "Grassmannian trick"). In fact, the codimension of $L^s(\operatorname{Hom}(K, \operatorname{Hom}(K, L))$ in $\operatorname{Hom}(K, \operatorname{Hom}(K, L))$ is given by a different formular than the codimension of $\operatorname{Hom}(K \circ K, L)_s$ in $\operatorname{Hom}(K \circ K, L)$ (see [17, Formula VI(4.4), p. 153]).

Finally, the desired submanifold $S_{r,s} \subset J^2(X, Y)$ is defined via the submersion Γ as

$$S_{r,s} := \Gamma^{-1}(\operatorname{Hom}(K \circ K, L)_s).$$

□

Following the theory of Thom-Boardman singularities (see [17, Chapter VI.5, p. 165 ff]), the notion of two-generic maps can now be defined in terms of the $S_{r,s}$.

Definition 4.3.2. A one-generic map $f\colon X \to Y$ is called *two-generic* if $j^2(f) \pitchfork S_{r,s}$ for all r, s. If $Y = \mathbb{R}^2$, then two-generic maps $X \to \mathbb{R}^2$ are called *generic*.

A consequence of the proof of the previous theorem is the following sufficient criterion for checking that the two-jet extension of a one-generic map is transverse to $S_{r,s}$. It will be helpful for identifying cusp singularities in Section 4.5. (Indeed, given smooth vector bundles E and F over X, the condition that a section $X \to \operatorname{Hom}(E, F)$ of the vector bundle $\operatorname{Hom}(E, F) \to X$ is transverse to $L^s(E, F)$ can be checked by means of [17, Proposition VI.3.7, p. 151].)

Proposition 4.3.3. *Let $f\colon X \to Y$ be one-generic. If the intrinsic derivative*

$$\mathcal{D}(Df)|_K\colon S_r(f) \to \operatorname{Hom}(K, \operatorname{Hom}(K, L))$$

is transverse to $L^s(K, \operatorname{Hom}(K, L))$, then $j^2(f)\colon X \to J^2(X, Y)$ is transverse to $S_{r,s}$.

Proof. The following lemma will be employed in the proof.

Lemma 4.3.4. *Let $f\colon X \to Y$ and $g\colon W \to X$ be smooth maps of smooth manifolds. Suppose that Z is a submanifold of Y such that $f^{-1}(Z)$ is a submanifold of X. If the composition $f \circ g$ is transverse to Z, then $g \pitchfork f^{-1}(Z)$.*

Proof. Let $w \in W$ such that $x := g(w) \in f^{-1}(Z)$. It suffices to show that $g \pitchfork f^{-1}(Z)$ at w. The proof makes use of [17, Lemma II.4.3, p. 52].

As Z is a submanifold of Y of, say, codimension a, there exist an open neighbourhood U of $f(x)$ in Y and a submersion $\phi\colon U \to \mathbb{R}^a$ such that $\phi^{-1}(0) = Z \cap U$.

Since the composition $f \circ g$ is transverse to Z and $(f \circ g)(w) = f(x) \in Z$, [17, Lemma II.4.3, p. 52] implies that the composition $\phi \circ f \circ g$ is a submersion at w. Hence, $\phi \circ f$ is a submersion at $x = g(w)$. (Note that, again by [17, Lemma II.4.3, p. 52], f is transverse to Z at x.) Choose an open neighbourhood $V \subset f^{-1}(U)$ of x in X such that $\phi \circ f$ restricts to a submersion $\psi\colon V \to \mathbb{R}^a$. Note that the submanifold $f^{-1}(Z)$ of X satisfies

$$f^{-1}(Z) \cap V = (f|_V)^{-1}(Z) = (f|_V)^{-1}(Z \cap U) = (f|_V)^{-1}(\phi^{-1}(0)) = (\phi \circ f|_V)^{-1}(0) = \phi^{-1}(0).$$

Finally, since $g(w) \in f^{-1}(Z)$ and it was shown above that $\psi \circ g = \phi \circ f \circ g$ is a submersion at w, [17, Lemma II.4.3, p. 52] implies that $g \pitchfork f^{-1}(Z)$. □

Recall from Section 4.2 that the intrinsic derivative $\mathcal{D}(Df)|_K$ factorizes as

$$S_r(f) \xrightarrow{j^2(f)|} S_r^{(2)} \xrightarrow{\Gamma} \mathrm{Hom}(K \circ K, L) \xrightarrow{\Sigma} \mathrm{Hom}(K, \mathrm{Hom}(K, L)).$$

Since the intrinsic derivative $\mathcal{D}(Df)|_K$ is by assumption transverse to $L^s(K, \mathrm{Hom}(K, L))$, Lemma 4.3.4 implies that the restriction $j^2(f)|\colon S_r(f) \to S_r^{(2)}$ is transverse to $S_{r,s} = (\Sigma \circ \mathcal{D})^{-1}(L^s(K, \mathrm{Hom}(K, L)))$.

Furthermore, $j^2(f)\colon X \to J^2(X, Y)$ is transverse to $S_r^{(2)}$. (Indeed, the one-jet extension $j^1(f)\colon X \to J^2(X, Y)$ is by assumption transverse to S_r, and factorizes as

$$X \xrightarrow{j^2(f)} J^2(X, Y) \xrightarrow{\pi} J^1(X, Y).$$

Hence, Lemma 4.3.4 implies that $j^2(f)$ is transverse to $\pi^{-1}(S_r) = \pi^{-1}(\pi(S_r^{(2)})) = S_r^{(2)}$.)

All in all, $j^2(f)\colon X \to J^2(X, Y)$ is transverse to $S_{r,s}$. (In fact, let $p \in X$ such that $q := j^2(f)(p) \in S_{r,s}$. It suffices to show that any vector $v \in T_q J^2(X, Y)$ can be written as the sum of a vector in $D_p j^2(f)(T_p X)$ and a vector in $T_q S_{r,s}$. Since $j^2(f)\colon X \to J^2(X, Y)$ is transverse to $S_r^{(2)}$ and $y \in S_r^{(2)}$, there exist vectors $u_1 \in T_p X$ and $v' \in T_q S_r^{(2)}$ such that $v = D_p j^2(f)(u_1) + v'$. Moreover, since $j^2(f)|\colon S_r(f) \to S_r^{(2)}$ is transverse to $S_{r,s}$, there exist vectors $u_2 \in T_p S_r(f)$ and $w \in T_q S_{r,s}$ such that $v' = D_p j^2(f)(u_2) + w$. Finally, $v = D_p j^2(f)(u_1 + u_2) + w$ is the desired decomposition.) □

In the following, we focus on the case $Y = \mathbb{R}^2$ and $n = \dim X \geq 2$. A smooth map $f\colon X \to \mathbb{R}^2$ is one-generic if and only if $j^1(f) \pitchfork S_1$ and $j^1(f)(X) \cap S_2 = \emptyset$. Let $f\colon X \to \mathbb{R}^2$ be one-generic. Consequently, $S_0(f)$ is an open subset of X, $S_1(f)$ is a submanifold of X of codimension $r(n-2+r) = n-1$, and $S_2(f) = \emptyset$. In particular, $X = S_0(f) \cup S_1(f)$, and $S_1(f)$ is a 1-dimensional submanifold of X which is closed as a subset. Note that $X = S_{0,0}(f) \cup S_{1,0}(f) \cup S_{1,1}(f)$ by definition of the sets $S_{r,s}(f)$. Therefore, $j^2(f)(X) \subset S_{0,0} \cup S_{1,0} \cup S_{1,1}$.

Lemma 4.3.5. *(a) The submanifold $S_{0,0} \subset S_0^{(2)}$ is an open subset of $J^2(X, \mathbb{R}^2)$.*
(b) The submanifold $S_{1,0} \subset S_1^{(2)}$ is an open subset of $S_1^{(2)}$.

Proof. Setting $s = 0$, we obtain that the codimension of $S_{r,0}$ in $S_r^{(2)}$ is zero. This shows that $S_{r,0}$ is an open subset of $S_r^{(2)}$. In particular, part (b) follows. Note that $S_0^{(2)} = \pi^{-1}(S_0)$ is an open subset of $J^2(X, \mathbb{R}^2)$ by Lemma 4.1.1(a). This implies part (a). □

Lemma 4.3.6. *Assume that the smooth map $f\colon X \to \mathbb{R}^2$ is one-generic at a point $x \in X$. Then $j^2(f)(x) \in S_{0,0} \cup S_{1,0} \cup S_{1,1}$.*

Proof. Since f is one-generic at x, Lemma 4.1.5 implies that there exists an open neighbourhood $X' \subset X$ of x such that f is one-generic on X'. By Lemma 4.1.4, the restriction $f'\colon X' \to \mathbb{R}^2$ of f to X' is one-generic. Hence, $x \in X' = S_{0,0}(f') \cup S_{1,0}(f') \cup S_{1,1}(f')$. Consequently, [17, Theorem VI.4.7, page 154] implies that $j^2(f')(x) \in S'_{0,0} \cup S'_{1,0} \cup S'_{1,1}$. Hence, $j^2(f)|_{X'}(x) \in (S_{0,0} \cup S_{1,0} \cup S_{1,1}) \cap \alpha^{-1}(X')$. □

Lemma 4.3.7. *Assume that the smooth map $f\colon X \to \mathbb{R}^2$ is one-generic at $x \in X$. If $j^2(f)(x) \in \overline{S_{1,1}}^{J^2(X,\mathbb{R}^2)}$, then $j^2(f)(x) \in S_{1,1}$.*

Proof. We assume that f is one-generic at x and that the point $y := j^2(f)(x)$ satisfies $y \in \overline{S_{1,1}}^{J^2(X,\mathbb{R}^2)}$. Since f is one-generic at x, it follows from Lemma 4.3.6 that $y \in S_{0,0} \cup S_{1,0} \cup S_{1,1}$. Hence, in order to show that $y \in S_{1,1}$, it suffices to show that $y \notin S_{0,0}$ and $y \notin S_{1,0}$:

- By Lemma 4.3.5 (a), $S_{0,0}$ is an open subset of $J^2(X,\mathbb{R}^2)$. Hence, we obtain from $S_{1,1} \subset S_1^{(2)} \subset J^2(X,\mathbb{R}^2) \setminus S_0^{(2)} \subset J^2(X,\mathbb{R}^2) \setminus S_{0,0}$ that $y \in \overline{S_{1,1}}^{J^2(X,\mathbb{R}^2)} \subset J^2(X,\mathbb{R}^2) \setminus S_{0,0}$. Consequently, $y \notin S_{0,0}$.
- By Lemma 4.3.5(b), $S_{1,0}$ is an open subset of $S_1^{(2)}$. Thus, there exists an open subset $U \subset J^2(X,\mathbb{R}^2)$ such that $S_{1,0} = U \cap S_1^{(2)}$. Note that $S_{1,1} \cap U = S_{1,1} \cap S_1^{(2)} \cap U = S_{1,1} \cap S_{1,0} = \emptyset$ because $S_{1,1} \subset S_1^{(2)}$. Hence, we obtain from $S_{1,1} \subset J^2(X,\mathbb{R}^2) \setminus U$ that $y \in \overline{S_{1,1}}^{J^2(X,\mathbb{R}^2)} \subset J^2(X,\mathbb{R}^2) \setminus U \subset J^2(X,\mathbb{R}^2) \setminus S_{1,0}$. Consequently, $y \notin S_{1,0}$.

□

By Lemma 4.3.6, a one-generic map $f\colon X \to \mathbb{R}^2$ is (two-)generic (compare Definition 4.3.2) if and only if $j^2(f) \pitchfork S_{0,0}$, $j^2(f) \pitchfork S_{1,0}$ and $j^2(f) \pitchfork S_{1,1}$. Note that $j^2(f) \pitchfork S_{0,0}$ holds automatically since $S_{0,0}$ is an open subset of $J^2(X,\mathbb{R}^2)$ by Lemma 4.3.5 (a). Furthermore, it follows from $j^1(f) \pitchfork S_1$ and Lemma 4.3.5 (b) that $j^2(f) \pitchfork S_{1,0}$. Hence, a one-generic map $f\colon X \to \mathbb{R}^2$ is generic if and only if $j^2(f) \pitchfork S_{1,1}$. One can consider this property pointwise:

Definition 4.3.8. Let $f\colon X \to \mathbb{R}^2$ be a smooth map.

(a) Given a point $x \in X$, we say that f is *generic at* x if f is one-generic at x and $j^2(f) \pitchfork S_{1,1}$ at x.

(b) Given a subset $A \subset X$, we say that f is *generic on* A if f is generic at x for all $x \in A$.

In particular, a smooth map $f\colon X \to \mathbb{R}^2$ is generic if and only if f is generic on X.

The following lemma states that genericity is an open condition.

Lemma 4.3.9. *Assume that $f\colon X \to \mathbb{R}^2$ is a smooth map which is generic at a point $x \in X$. Then there exists an open neighbourhood $U \subset X$ of x such that f is generic on U.*

Proof. By assumption, f is generic at x. Hence, by Definition 4.3.8(a), f is one-generic at x and $j^2(f) \pitchfork S_{1,1}$ at x. Since f is one-generic at x, Lemma 4.1.5 implies that there exists an open neighbourhood $U_1 \subset X$ of x such that f is one-generic on U_1. We distinguish the following two cases for the point $y := j^2(f)(x)$:

- $y \in S_{1,1}$. Since $j^2(f) \pitchfork S_{1,1}$ at x, Lemma A.1.3 implies that there exists a neighbourhood $U_2 \subset X$ of x such that $j^2(f) \pitchfork S_{1,1}$ on U_2. Hence, f is generic on $U := U_1 \cap U_2$.
- $y \notin S_{1,1}$. Since f is one-generic at x, Lemma 4.3.7 implies that $V := J^2(X,\mathbb{R}^2) \setminus \overline{S_{1,1}}^{J^2(X,\mathbb{R}^2)}$ is an open neighbourhood of y in $J^2(X,\mathbb{R}^2)$. Hence, $U_3 := j^2(f)^{-1}(V)$ is an

open neighbourhood of x in X such that $j^2(f)(U_3) \cap S_{1,1} = \emptyset$. Thus, $j^2(f) \pitchfork S_{1,1}$ on U_3. Hence, f is generic on $U := U_1 \cap U_3$.

□

The following lemma shows that genericity is compatible with restriction to open subsets.

Lemma 4.3.10. *Let $f \colon X \to \mathbb{R}^2$ be a smooth map and let $X' \subset X$ be an open subset. Then the restriction $f' \colon X' \to \mathbb{R}^2$ of f to X' is generic if and only if f is generic on X'. In particular, if f is generic, then f' is generic.*

Proof. By Lemma 4.1.4, f' is one-generic if and only if f is one-generic on X'. Hence, by Definition 4.3.8, it suffices to show that $j^2(f') \pitchfork S'_{1,1}$ if and only if $j^2(f) \pitchfork S_{1,1}$ on X'. In fact, we have $j^2(f') \pitchfork S'_{1,1}$ if and only if $j^2(f)|_{X'} \pitchfork S_{1,1} \cap \alpha^{-1}(X')$ at x for all $x \in X'$. By Lemma A.1.2(b) this is equivalent to $j^2(f) \pitchfork S_{1,1} \cap \alpha^{-1}(X')$ at x for all $x \in X'$. By Lemma A.1.2(b) this is equivalent to $j^2(f) \pitchfork S_{1,1}$ at x for all $x \in X'$. (In fact, note that $S_{1,1} \cap \alpha^{-1}(X')$ is an open subset of $S_{1,1}$. Moreover, $j^2(f)(x') \notin S_{1,1} \cap \alpha^{-1}(X')$ implies $j^2(f)(x') \notin S_{1,1}$ for all $x' \in X'$ since $j^2(f)(x') \in \alpha^{-1}(X')$ for all $x' \in X'$.) Equivalently, $j^2(f) \pitchfork S_{1,1}$ on X'. □

We end this section with the definition of fold and cusp singularities of a generic smooth map $X \to \mathbb{R}^2$ (compare [17, Exercise VI.4(7), page 156]).

Definition 4.3.11. Let $f \colon X \to \mathbb{R}^2$ be a generic smooth map. Since f is one-generic, the singular locus of f is given by $S(f) = S_1(f) = S_{1,0}(f) \cup S_{1,1}(f)$ of X. The points of $S_{1,0}(f)$ are called *fold points* of f, and the points of $S_{1,1}(f)$ are called *cusps* of f.

Theorem 4.3.1 implies that $S_{1,1}(f) = j^2(f)^{-1}(S_{1,1})$ is a 0-dimensional submanifold of X. Hence, cusps are isolated points on the 1-dimensional submanifold $S_1(f)$ of X.

Remark 4.3.12. One observes directly that a smooth map $f \colon X \to \mathbb{R}^2$ is a fold map in the sense of Definition 3.3.1 if and only if f is a generic map whose singular points are all fold points. (Indeed, f satisfies conditions $(fm1)$ and $(fm2)$ if and only if f is one-generic. In this case, condition $(fm3)$ is by Remark 3.3.2 (iii) equivalent to saying that f restricts to an immersion $S_1(f) \to \mathbb{R}^2$, which means $S_1(f) = S_{1,0}(f)$ or $S_{1,1}(f) = j^2(f)^{-1}(S_{1,1}) = \emptyset$.)

4.4 Extension of Generic Smooth Maps

Proposition 4.4.1. *Let $C \subset U \subset X$, where C is compact and U is an open subset of X. Given a generic map $f\colon X \setminus C \to \mathbb{R}^2$, there exists a generic map $F\colon X \to \mathbb{R}^2$ such that $F|_{X\setminus U} = f|_{X\setminus U}$.*

Proof. By Lemma A.3.1(b) there exist open subsets $U_i \subset X$ for $i \in \{0,1,2,3,4\}$ such that $C_i := \overline{U_i}^X$ is compact for $i \in \{0,1,2,3,4\}$ and $C \subset U_0$, $C_i \subset U_{i+1}$ for $i \in \{0,1,2,3\}$ and $C_4 \subset U$.

We choose a smooth map $F_1\colon X \to \mathbb{R}^2$ such that $F_1|_{X\setminus U_1} = f|_{X\setminus U_1}$. F_1 can be constructed in the following way. By [22, Chapter 2, Theorem 2.1, page 43] the open cover $X = U_1 \cup (X\setminus C_0)$ has a subordinate smooth partition of unity. In other words, there exist smooth maps $\mu, \nu\colon X \to \mathbb{R}$ such that $\mu(X), \nu(X) \subset [0,1]$, $\operatorname{supp}\mu \subset U_1$, $\operatorname{supp}\nu \subset X \setminus C_0$ and $\mu(x) + \nu(x) = 1$ for all $x \in X$. In particular, we have $\nu(x) = 0$ for all $x \in C_0$ and $\nu(x) = 1$ for all $x \in X \setminus U_1$. We define the map

$$F_1\colon X \to \mathbb{R}^2, \qquad F_1(x) = \begin{cases} \nu(x)\cdot f(x), & \text{for } x \in X \setminus C, \\ 0, & \text{for } x \in C. \end{cases}$$

In order to show that F_1 is smooth, we consider the open covering $X = U_0 \cup (X \setminus C)$. The restriction of F_1 to $X \setminus C$ is a smooth map, since $\nu|_{X\setminus C}$ and f are smooth. Moreover, the restriction of F_1 to U_0 is identically zero, since $F_1(x) = 0$ for all $x \in C (\subset U_0)$ by definition of F_1 and $F_1(x) = \nu(x)\cdot f(x) = 0\cdot f(x) = 0$ for all $x \in U_0 \setminus C (\subset C_0)$. Hence, F_1 is smooth. Finally, $F_1(x) = \nu(x)\cdot f(x) = 1\cdot f(x) = f(x)$ for all $x \in X \setminus U_1$. This completes the construction of F_1. Note that $F_1|_{X\setminus C_1} = f|_{X\setminus C_1}$ is generic by Lemma 4.3.10, since f is generic by assumption. Hence, Lemma 4.3.10 implies that F_1 is generic on $X \setminus C_1$, that is, $j^1(F_1) \pitchfork S_1$ on $X \setminus C_1$, $j^1(F_1) \pitchfork S_2$ on $X \setminus C_1$ and $j^2(F_1) \pitchfork S_{1,1}$ on $X \setminus C_1$.

We apply Proposition A.3.2 to the smooth map $F_1\colon X \to \mathbb{R}^2$, the subsets $C_1 \subset U_2 \subset X$ (where C_1 is compact and U_2 is an open subset of X with compact closure C_2 in X) and the submanifold $S_2 \subset J^1(X,\mathbb{R}^2)$. (Indeed, condition (1) is satisfied since $j^1(F_1) \pitchfork S_2$ on $X \setminus C_1$. Moreover, note that condition (2) is satisfied since S_2 is a closed subset of $J^1(X,\mathbb{R}^2)$ by Lemma 4.1.1(b).) Hence, there exists a smooth map $F_2\colon X \to \mathbb{R}^2$ such that $F_2|_{X\setminus U_2} = F_1|_{X\setminus U_2}$ and $j^1(F_2) \pitchfork S_2$.

Application of Corollary A.2.3(b) to the submanifold $S_2 \subset J^1(X,\mathbb{R}^2)$ (which is a closed subset of $J^1(X,\mathbb{R}^2)$ by Lemma 4.1.1(b)) yields the open subset

$$V_2 := \{h \in C^\infty(X,\mathbb{R}^2);\ j^1(h) \pitchfork S_2 \text{ on } S_2\} = \{h \in C^\infty(X,\mathbb{R}^2);\ j^1(h) \pitchfork S_2\}$$

of $C^\infty(X,\mathbb{R}^2)$ in the C^∞ topology. By construction of F_2, V_2 is an open neighbourhood of $F_2 \in C^\infty(X,\mathbb{R}^2)$. We apply Proposition A.3.2 to the smooth map $F_2 \in V_2$, the subsets $C_2 \subset U_3 \subset X$ (where C_2 is compact and U_3 is an open subset of X with compact closure C_3 in X) and the submanifold $S_1 \subset J^1(X,\mathbb{R}^2)$. (Indeed, condition (1) follows from Lemma A.1.2(b) since $F_2|_{X\setminus C_2} = F_1|_{X\setminus C_2}$ and $j^1(F_1) \pitchfork S_1$ on $X \setminus C_2$. Moreover, note that condition (2) follows from Lemma 4.1.2(c) since $j^1(F_2) \pitchfork S_2$.) Hence, there exists a smooth map $F_3 \in V_2$ such that $F_3|_{X\setminus U_3} = F_2|_{X\setminus U_3}$ and $j^1(F_3) \pitchfork S_1$.

$K := j^1(F_3)(C_4)$ is compact. Note that $L := K \cap S_1$ is also compact. (In fact, it follows from $F_3 \in V_2$ and Lemma 4.1.2(c) that $\overline{S_1}^{J^1(X,\mathbb{R}^2)} \cap K = S_1 \cap K = L$. Hence, L is compact, being the intersection of the closed subset $\overline{S_1}^{J^1(X,\mathbb{R}^2)} \subset J^1(X, \mathbb{R}^2)$ and the compact space K.)

As L is a compact subspace of the manifold S_1, Lemma A.3.1(b) implies that there exists an open subset $Z \subset S_1$ such that $L \subset Z$ and $\overline{Z}^{S_1}$ is compact.

Let $\alpha\colon J^1(X, \mathbb{R}^2) \to X$ denote the source map. Note that $A := (\overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z) \cap \alpha^{-1}(C_4)$ is a closed subset of $J^1(X, \mathbb{R}^2)$. (In fact, since Z is an open subset of the submanifold $S_1 \subset J^1(X, \mathbb{R}^2)$, Lemma A.3.1(a) implies that $A' := \overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z$ is a closed subset of $J^1(X, \mathbb{R}^2)$. Note that $\alpha^{-1}(C_4)$ is also a closed subset of $J^1(X, \mathbb{R}^2)$. Consequently, $A = A' \cap \alpha^{-1}(C_4)$ is a closed subset of $J^1(X, \mathbb{R}^2)$.) Hence, it follows from Corollary A.2.3(a) that

$$V_A := \{h \in C^\infty(X, \mathbb{R}^2);\ j^1(h)(X) \cap A = \emptyset\}$$

is an open subset of $C^\infty(X, \mathbb{R}^2)$ in the C^∞ topology. Note that $F_3 \in V_A$. (In fact, we have to show that $j^1(F_3)(x) \notin A$ for all $x \in X$. Assume that $j^1(F_3)(x) \in A$ for some $x \in X$. Then $x = \alpha(j^1(F_3)(x)) \in \alpha(A) \subset C_4$. Therefore, $j^1(F_3)(x) \in A \cap K \subset \overline{S_1}^{J^1(X,\mathbb{R}^2)} \cap K = S_1 \cap K = L \subset Z$. Hence, we obtain the contradiction $j^1(F_3)(x) \in A \cap Z \subset (\overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z) \cap Z = \emptyset$.)

As $\overline{Z}^{S_1}$ is compact, it is in particular a closed subset of $J^1(X, \mathbb{R}^2)$ which is contained in the submanifold $S_1 \subset J^1(X, \mathbb{R}^2)$. Hence, Corollary A.2.3(b) implies that

$$T := \{h \in C^\infty(X, \mathbb{R}^2);\ j^1(h) \pitchfork S_1 \text{ on } \overline{Z}^{S_1}\}$$

is an open subset of $C^\infty(X, \mathbb{R}^2)$ in the C^∞ topology. Note that $F_3 \in T$ since $j^1(F_3) \pitchfork S_1$.

We apply Proposition A.3.2 to the smooth map $F_3 \in V_2 \cap V_A \cap T$, the subsets $C_3 \subset U_4 \subset X$ (where C_3 is compact and U_4 is an open subset of X with compact closure C_4 in X) and the submanifold $S_{1,1} \subset J^2(X, \mathbb{R}^2)$. (Indeed, condition (1) follows from Lemma A.1.2(b) since $F_3|_{X\setminus C_3} = F_2|_{X\setminus C_3} = F_1|_{X\setminus C_3}$ and $j^2(F_1) \pitchfork S_{1,1}$ on $X \setminus C_3$. Moreover, note that condition (2) follows from Lemma 4.3.7 since $j^1(F_3) \pitchfork S_1$ and $j^1(F_3) \pitchfork S_2$.) Hence, there exists a smooth map $F \in V_2 \cap V_A \cap T$ such that $F|_{X\setminus U_4} = F_3|_{X\setminus U_4}$ and $j^2(F) \pitchfork S_{1,1}$. In particular, $F|_{X\setminus U} = F_3|_{X\setminus U} = F_2|_{X\setminus U} = F_1|_{X\setminus U} = f|_{X\setminus U}$. It follows from $F \in V_2$ that $j^1(F) \pitchfork S_2$. It remains to show that $j^1(F) \pitchfork S_1$. We fix a point $x \in X$ and have to show that $j^1(F) \pitchfork S_1$ at x. By Definition A.1.1(a), we may assume that $j^1(F)(x) \in S_1$. Note that $F|_{X\setminus C_4} = F_3|_{X\setminus C_4}$ and $j^1(F_3) \pitchfork S_1$. Hence, if $x \in X \setminus C_4$, then we can apply Lemma A.1.2(b) to obtain $j^1(F) \pitchfork S_1$ at x. Next, we assume that $x \in C_4$. Thus, setting $y := j^1(F)(x)$, we have $y \in \alpha^{-1}(C_4)$. It follows from $F \in V_A$ that $y \notin A = (\overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z) \cap \alpha^{-1}(C_4)$. Hence, $y \in \alpha^{-1}(C_4)$ implies that $y \notin \overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z$. On the other hand, we have $y \in S_1$ by assumption. Therefore, $y \in Z$. (In fact, if $y \notin Z$, then we obtain the contradiction $y \in S_1 \setminus Z \subset \overline{S_1}^{J^1(X,\mathbb{R}^2)} \setminus Z$.) Finally, we obtain from $F \in T$ and $y \in Z \subset \overline{Z}^{S_1}$ that $j^1(F) \pitchfork S_1$ at x. □

4.5 Practical Determination of Fold Points and Cusps

Let $n \geq q \geq 1$ be integers. Suppose that $F\colon X^n \to Y^q$ is a fold map, and that $p \in S(F)$ is a fold point of F. As usual, set $\sigma := D_pF \in \mathrm{Hom}(T_pX, T_{F(p)}Y)$, and let $K_\sigma := \ker\sigma$ and $L_\sigma := \operatorname{coker}\sigma = \frac{T_{F(p)}Y}{\operatorname{Im}\sigma}$. Since $S(F)$ is a $(q-1)$-dimensional submanifold of X, and $T_pX = T_pS(F) \oplus K_\sigma$ by the end of Remark 3.3.2 (iii), we conclude that $\dim K_\sigma = n-q+1$. Hence, $\dim \operatorname{Im}\sigma = n-(n-q+1) = q-1$, and $\dim L_\sigma = q-(q-1) = 1$. Therefore, L_σ is linearly isomorphic to $\mathbb{R}$. As explained in Section 4.2, the intrinsic derivative $\mathcal{D}_p(DF)$ induces a linear map

$$\delta_p^2 F\colon K_\sigma \circ K_\sigma \to L_\sigma.$$

Up to the choice of a linear isomorphism $L_\sigma \cong \mathbb{R}$, the map δ_p^2F is nothing but a symmetric bilinear form on the real vector space K_σ. We claim that this bilinear form is non-degenerate for any fold point p of F. Before we check this, we use this bilinear form to define the non-reduced and the absolute index of a fold point:

Definition 4.5.1. Given an orientation of $L_\sigma = \frac{T_{F(p)}Y}{\operatorname{Im}\sigma} (\cong \mathbb{R})$, the index $\lambda(p)$ of the resulting symmetric bilinear form δ_p^2F, i.e. the number of negative diagonal entries of δ_p^2F after diagonalization, is called *non-reduced index* of F at p. (For instance, L_σ inherits an orientation by choice of an orientation of $T_{F(p)}Y$ and of $T_pS(F) \cong \operatorname{Im}\sigma$, where the isomorphism is the restriction of $\sigma\colon T_pX \to T_{F(p)}Y$.) If no orientations are given, then the number $\tau(p) := \max\{\lambda(p), n-q+1-\lambda(p)\}$ is still well-defined for any choice of orientation on L_p, and is called the *absolute index* of F at p.

Example 4.5.2. We illustrate the calculation of the non-reduced index for the fold points of the stable Whitney cusp (compare Proposition 4.5.4, whose proof gives the steps of the following calculation in full detail)

$$F\colon \mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2} \to \mathbb{R}^2, \qquad F(u,v,z_1,\dots,z_{n-2}) = (u, uv+v^3+\sum_{i=1}^{n-2}\varepsilon_i z_i^2),$$

where $n \geq 2$ is an integer and $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{n-2}) \in \{-1\}^{n-2}$ is an $(n-2)$-tuple of signs. The singular locus $S(F)$ can be parametrized via

$$\alpha\colon \mathbb{R} \to S(F), \qquad \alpha(v) = (-3v^2, v, 0),$$

and $\alpha(0) = 0$ is the unique cusp of F. What is the non-reduced index of F at the fold point $p := \alpha(v)$ for fixed $v \neq 0$? Concerning the orientations required by Definition 4.5.1, we assume that the target space $\mathbb{R}^2$ has the canonical orientation, and that $S(F)$ is oriented via α. It can be shown that, for $v \neq 0$,

$$\sigma := D_pF = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ v & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$D_p(DF)\colon \mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^2), \qquad (a,b,c_1,\dots,c_{n-2}) \mapsto \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ b & a+6vb & 2\varepsilon_1 c_1 & \dots & 2\varepsilon_{n-2}c_{n-2} \end{pmatrix}.$$

Note that

$$K_\sigma := \ker\sigma = 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^n,$$
$$L_\sigma := \operatorname{coker}\sigma = \frac{\mathbb{R}^2}{\mathbb{R}\cdot(1,v)}.$$

What is in our situation the correct orientation of $L_\sigma \cong \mathbb{R}$? The orientation of $S(F)$ at p is determined by the tangent vector $w := (-6v, 1, 0)$. As remarked in Definition 4.5.1, $\sigma(w) = (-6v, -6v^2)$ defines an orientation of $\operatorname{im}\sigma$. Since $\mathbb{R}^2$ is equipped with the standard orientation, the isomorphism $L_\sigma \cong \mathbb{R}$ induced by the inner product of vectors in $\mathbb{R}^2$ with the rotation $(6v^2, -6v)$ of $\sigma(w)$ by $\pi/2$ will define the correct orientation on L_σ. All in all, the symmetric bilinear form $\delta_p^2 F\colon \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}$ is then given by the diagonal matrix $\operatorname{diag}(-36v^2, -12\varepsilon_1 v, \ldots, -12\varepsilon_{n-2}v)$. Consequently, if ν is the number of negative signs in $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-2})$, then the non-reduced index of F at $\alpha(v)$ is $n-1-\nu$ for $v<0$ and $\nu+1$ for $v>0$. In particular, the non-reduced index remains unchanged after passage through the cusp if there is an equal number of positive and negative signs in ε. This is for instance case when $n=2$.

To check that the above bilinear form $\delta_p^2 F$ is non-degenerate it suffices by definition of the intrinsic derivative in terms of local trivializations to work in convenient local charts around p and $F(p)$. For instance, one could use charts in which F has the normal form of Proposition 3.3.5, but we work a bit more general in the setting of Proposition 3.3.4 in order to have a practical method for computing indices of fold points.

Proposition 4.5.3. *Following the notation of Proposition 3.3.4, let $X \subset \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1}$ be an open subset, let $f\colon X \to \mathbb{R}$ be a smooth function, and consider the smooth map*

$$F\colon X \to \mathbb{R}^q, \qquad p = (t,x) \mapsto F(t,x) = (t, f(p)).$$

Then, for every $p = (t,x) \in (D^x f)^{-1}(0)$, there exist bases of K_σ and L_σ in which $\delta_p^2 F$ is the Hessian $H_x(f_t)$. In particular, by Proposition 3.3.4 (d), F is a fold map if and only if $\delta_p^2 F$ is non-degenerate for all $p \in (D^x f)^{-1}(0)$. Furthermore, if F is a fold map, then the orientation of $L_\sigma = \frac{T_{F(p)}\mathbb{R}^q}{\operatorname{Im}\sigma}$ determined by any basis of L_σ as above is induced by the standard orientations of $T_{F(p)}\mathbb{R}^{n-q+1} = \mathbb{R}^{n-q+1}$ and $\operatorname{Im}\sigma = \sigma(\mathbb{R}^{q-1} \times 0)$.

Proof. The Jacobian of $F\colon X \to \mathbb{R}^q$ can be considered as a smooth map

$$\rho\colon X \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q), \qquad \rho(p) = D_p F = \begin{pmatrix} I_{q-1} & 0 \\ D_p^t f & D_p^x f \end{pmatrix}.$$

In particular, evaluation of ρ at $p \in S(F) = (D^x f)^{-1}(0)$ (see Proposition 3.3.4 (a)) yields

$$\sigma := \rho(p) = \begin{pmatrix} I_{q-1} & 0 \\ D_p^t f & 0 \end{pmatrix} \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q).$$

We define

$$K_\sigma := \ker \sigma = 0 \times \mathbb{R}^{n-q+1} \subset \mathbb{R}^{q-1} \times \mathbb{R}^{n-q+1} = \mathbb{R}^n,$$
$$L_\sigma := \operatorname{coker} \sigma = \frac{\mathbb{R}^q}{\operatorname{Im} \sigma}.$$

Note that $\{\partial_{x_k}\}$ is a basis of K_σ. Let $\iota\colon K_\sigma \to \mathbb{R}^n$ denote the inclusion, and let $\pi\colon \mathbb{R}^q \to L_\sigma$ denote the canonical projection.

As $\operatorname{Im} \sigma = \{\begin{pmatrix} v \\ (D_p^t f)v \end{pmatrix};\ v \in \mathbb{R}^{(q-1)\times 1}\}$ is the kernel of the linear map

$$A\colon \mathbb{R}^q \to \mathbb{R}, \qquad A(w) = w_q - (D_p^t f) \begin{pmatrix} w_1 \\ \vdots \\ w_{q-1} \end{pmatrix},$$

we obtain an induced linear map $\overline{A}\colon L_\sigma \to \mathbb{R}$, $\overline{A}(\pi(w)) = A(w)$. Note that A is surjective since $\dim \operatorname{Im} \sigma = q-1$. Hence, $\overline{A}$ is an isomorphism.

The Jacobian of ρ at p is a linear map

$$D_p\rho\colon T_pX \to T_\sigma \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q) \cong \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^q),$$

which is explicitly given on the basis $\{\partial_{t_i}\} \cup \{\partial_{x_j}\}$ of T_pX by

$$D_p\rho(\partial_{t_i}) = \partial_{t_i}\rho(p) = \begin{pmatrix} 0 & 0 \\ \partial_{t_i} D^t f(p) & \partial_{t_i} D^x f(p) \end{pmatrix},$$
$$D_p\rho(\partial_{x_j}) = \partial_{x_j}\rho(p) = \begin{pmatrix} 0 & 0 \\ \partial_{x_j} D^t f(p) & \partial_{x_j} D^x f(p) \end{pmatrix}.$$

The intrinsic derivative of ρ at p is given by

$$\mathcal{D}_p\rho\colon T_pX \to \operatorname{Hom}(K_\sigma, L_\sigma), \qquad v \mapsto \pi \circ D_p\rho(v) \circ \iota.$$

In particular, we have

$$\mathcal{D}_p\rho(\partial_{t_i}) = \pi \begin{pmatrix} 0 \\ \partial_{t_i} D^x f(p) \end{pmatrix}, \qquad \mathcal{D}_p\rho(\partial_{x_j}) = \pi \begin{pmatrix} 0 \\ \partial_{x_j} D^x f(p) \end{pmatrix}.$$

The intrinsic derivative of ρ at p can also be considered as a map

$$\mathcal{D}_p\rho\colon T_pX \otimes K_\sigma \to L_\sigma,$$
$$\partial_{t_i} \otimes \partial_{x_k} \mapsto \pi \begin{pmatrix} 0 \\ \partial_{t_i}\partial_{x_k} f(p) \end{pmatrix},$$
$$\partial_{x_j} \otimes \partial_{x_k} \mapsto \pi \begin{pmatrix} 0 \\ \partial_{x_j}\partial_{x_k} f(p) \end{pmatrix},$$

Restriction to $K_\sigma \otimes K_\sigma \subset T_pX \otimes K_\sigma$ yields the map

$$K_\sigma \otimes K_\sigma \to L_\sigma, \qquad \partial_{x_j} \otimes \partial_{x_k} \mapsto \pi \begin{pmatrix} 0 \\ \partial_{x_j}\partial_{x_k} f(p) \end{pmatrix}.$$

(By construction, this map corresponds to $\delta_p^2 F\colon K_\sigma \circ K_\sigma \to L_\sigma$.) Composition with the isomorphism $\overline{A}\colon L_\sigma \to \mathbb{R}$ yields the bilinear form $K_\sigma \otimes K_\sigma \to \mathbb{R}$ which is given by the Hessian $H_x(f_t)$ in the basis $\{\partial_{x_k}\}$ of K_σ. □

Next, we turn to the practical determination of cusps. Let $n \geq 2$ be an integer. Points $p \in \mathbb{R}^n$ will be written as triples $p = (u, x, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$. Given an $(n-2)$-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-2}) \in \{-1\}^{n-2}$ of signs, define the quadratic form $Q_\varepsilon(z) := \sum_{i=1}^{n-2} \varepsilon_i z_i^2$.

Proposition 4.5.4. *Given an $(n-2)$-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-2}) \in \{-1\}^{n-2}$ of signs and a smooth function $h\colon \mathbb{R} \to \mathbb{R}$, consider the smooth map*

$$F\colon \mathbb{R}^n \to \mathbb{R}^2, \qquad F(u, x, z) = (u, h(x) + ux + Q_\varepsilon(z)).$$

The singular locus of F is given by the image of the embedding $\varphi\colon \mathbb{R} \to \mathbb{R}^n$, $\varphi(x) = (-h'(x), x, 0)$. Fix $x \in \mathbb{R}$. The point $\varphi(x) \in S(F)$ is a fold point of F if and only if $h''(x) \neq 0$. Furthermore, if $h''(x) = 0$ and $h'''(x) \neq 0$, then $\varphi(x) \in S(F)$ is a cusp of F.

Proof. The tangent map $DF\colon \mathbb{R}^n \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^2)$ is given at $p = (u, x, z) \in \mathbb{R}^n$ by

$$D_pF = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ x & h'(x) + u & 2\varepsilon_1 z_1 & \ldots & 2\varepsilon_{n-2} z_{n-2} \end{pmatrix}.$$

This matrix has not maximal rank 2 if and only if $z_1 = \cdots = z_{n-2} = 0$ and $h'(x) + u = 0$. Hence, $S(F)$ is just the image of the embedding φ. The characterization of fold points $\varphi(x)$ of F by the condition $h''(x) \neq 0$ follows from Proposition 3.3.4 (d).

Finally, suppose that $h''(x) = 0$ and $h'''(x) \neq 0$ for some fixed $x \in \mathbb{R}$. Set $p := (u, x, z) := \varphi(x) = (-h'(x), x, 0) \in S(F)$ and $\sigma := D_pF = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ x & 0 & 0 & \ldots & 0 \end{pmatrix}$. Then,

$$K_\sigma := \ker \sigma = 0 \times \mathbb{R}^{n-1} \subset \mathbb{R}^n,$$
$$L_\sigma := \operatorname{coker} \sigma = \frac{\mathbb{R}^2}{\mathbb{R} \cdot (1, x)}.$$

We will use the linear isomorphism $\lambda\colon L_\sigma \xrightarrow{\cong} \mathbb{R}$, $(a, b) + \mathbb{R} \cdot (1, x) \mapsto b - xa$.

The tangent map $D(DF)\colon \mathbb{R}^n \to \operatorname{Hom}(\mathbb{R}^n, \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^2))$ is given at $p' = (u', x', z') \in \mathbb{R}^n$ by

$$\mathbb{R}^n \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^2),$$
$$(v_1, \ldots, v_n) \mapsto D_p(DF)(v_1, \ldots, v_n) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ v_2 & v_1 + h''(x')v_2 & 2\varepsilon_1 v_3 & \ldots & 2\varepsilon_{n-2} v_n \end{pmatrix}.$$

Thus, the intrinsic derivative $\mathcal{D}(DF)\colon S(F) \to \operatorname{Hom}(\mathbb{R}^n, \operatorname{Hom}(K, L))$ is given at $p' = (u', x', z') \in$

$S(F)$ by

$$\begin{aligned}\mathcal{D}_{p'}(DF)\colon \mathbb{R}^n &\to \operatorname{Hom}(\mathbb{R}^{n-1}, \mathbb{R}),\\ (v_1, \ldots, v_n) &\mapsto \Big(v_1 + h''(x')v_2 \quad 2\varepsilon_1 v_3 \quad \ldots \quad 2\varepsilon_{n-2} v_n\Big),\end{aligned}$$

where $\sigma' := D_{p'}F$, and we have identified $K_{\sigma'} = 0 \times \mathbb{R}^{n-1} = \mathbb{R}^{n-1}$ and $L_{\sigma'} = \mathbb{R}$ via λ. In particular, F is one-generic because the linear map $\mathcal{D}_{p'}(DF)$ is surjective for all $p' \in S(F)$. By Proposition 4.3.3 it suffices to show that the map

$$\Delta\colon S(F) \to \operatorname{Hom}(K, \operatorname{Hom}(K, L)), \qquad \Delta(p') = \mathcal{D}_{p'}(DF)|_{K_{\sigma'}},$$

is transverse to $L^1(K, \operatorname{Hom}(K, L))$ at $p = \varphi(x)$. Using the identifications $K_{\sigma'} = 0 \times \mathbb{R}^{n-1} = \mathbb{R}^{n-1}$, $L_{\sigma'} = \mathbb{R}$ via λ, and the diffeomorphism $\varphi\colon \mathbb{R} \xrightarrow{\cong} S(F)$, one has to show by [17, Proposition VI.3.7, p. 151] that the intrinsic derivative

$$\mathcal{D}_x\tau\colon T_x\mathbb{R} \to \operatorname{Hom}(\ker \tau(x), \operatorname{coker} \tau(x))$$

of the vector bundle homomorphism

$$\begin{aligned}\tau\colon \mathbb{R} &\to \operatorname{Hom}(\mathbb{R}^{n-1}, \operatorname{Hom}(\mathbb{R}^{n-1}, \mathbb{R})),\\ x' &\mapsto [(v_2, \ldots, v_n) \mapsto \Big(h''(x')v_2 \quad 2\varepsilon_1 v_3 \quad \ldots \quad 2\varepsilon_{n-2} v_n\Big)],\end{aligned}$$

is surjective. Under the canonical identification $\operatorname{Hom}(\mathbb{R}^{n-1}, \mathbb{R}) = \mathbb{R}^{n-1}$ one has

$$\tau\colon \mathbb{R} \to \operatorname{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}), \qquad \tau(x') = \operatorname{diag}\Big(h''(x') \quad 2\varepsilon_1 \quad \ldots \quad 2\varepsilon_{n-2}\Big).$$

The tangent map of τ at $x' \in \mathbb{R}$ is given by

$$D_{x'}\tau\colon T_{x'}\mathbb{R} \to T_{\tau(x')} \operatorname{Hom}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}), \qquad D_{x'}\tau(v) = \operatorname{diag}\Big(h'''(x') \quad 0 \quad \ldots \quad 0\Big) \cdot v.$$

Since $h''(x) = 0$ by assumption, it follows that

$$\begin{aligned}\ker \tau(x) &= \mathbb{R} = \mathbb{R} \times 0 \subset \mathbb{R}^{n-1},\\ \operatorname{coker} \tau(x) &= \frac{\mathbb{R}^{n-1}}{\operatorname{im} \tau(x)} = \frac{\mathbb{R}^{n-1}}{0 \times \mathbb{R}^{n-2}} \cong \mathbb{R}.\end{aligned}$$

Therefore, the intrinsic derivative of τ at x is given by

$$\mathcal{D}_x\tau\colon T_x\mathbb{R} \to \operatorname{Hom}(\ker \tau(x), \operatorname{coker} \tau(x)) \cong \operatorname{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}, \qquad \mathcal{D}_x\tau(v) = h'''(x) \cdot v.$$

Since $h'''(x) \neq 0$ by assumption, it follows that the linear map $\mathcal{D}_x\tau$ is indeed surjective.

□

4.6 Elimination of Cusps

Let X^n be a smooth manifold (without boundary) of dimension $n \geq 2$, and let $f\colon X \to \mathbb{R}^2$ be a generic smooth map as defined in Definition 4.3.2. Recall that the singular set $S(f)$ of f is a 1-dimensional submanifold of X and consists of fold points and cusps of f. The set of cusps is a 0-dimensional submanifold (i.e., a discrete subset) of $S(f)$.

Based on the choice of an orientation of $\mathbb{R}^2$, Levine [32] defines an *index*

$$I(p) \in \{0, \ldots, n-2\}$$

for every cusp p of f. In the following, we assume that $\mathbb{R}^2$ is equipped with the standard orientation. Levine also gives a description of the cusp index via local coordinates. This description can be explained as follows. It is a fact that if p is a cusp of f, then there are charts $\phi\colon U \to U' \subset \mathbb{R}^n$ around p and $\psi\colon V \to V' \subset \mathbb{R}^2$ around $f(p)$ such that $f(U) \subset V$, and there is some $k \in \{0, \ldots, n-2\}$ such that the composition $\psi \circ f \circ \phi^{-1}$ has the normal form

$$(\psi \circ f \circ \phi^{-1})(u, v, z_1, \ldots, z_{n-2}) = (u, v^3 + uv - \sum_{i=1}^{k} z_i^2 + \sum_{i=k+1}^{n-2} z_i^2).$$

(Note that this normal form does indeed describe a cusp at the origin by Proposition 4.5.4.)

If we require in addition that ψ is *orientation preserving*, then the index of p is given by k:

Lemma 4.6.1. *Let p be a cusp of f. Then there exist charts $\phi\colon U \to U' \subset \mathbb{R}^n$ around p and $\psi\colon V \to V' \subset \mathbb{R}^2$ around $f(p)$, where $f(U) \subset V$ and ψ is an orientation preserving diffeomorphism, such that the composition $\psi \circ f \circ \phi^{-1}\colon U' \to V'$ has the form*

$$(\psi \circ f \circ \phi^{-1})(u, v, z_1, \ldots, z_{n-2}) = (u, v^3 + uv - \sum_{i=1}^{k} z_i^2 + \sum_{i=k+1}^{n-2} z_i^2)$$

for some $k \in \{0, \ldots, n-2\}$. In this situation, $I(p) = k$. If, instead, ψ is orientation reversing, then $I(p) = n-2-k$.

Proof. As p is a cusp of f, there are charts $\check{\phi}\colon U \to \check{U}' \subset \mathbb{R}^n$ around p and $\check{\psi}\colon V \to \check{V}' \subset \mathbb{R}^2$ around $f(p)$ such that $f(U) \subset V$ and there is some $l \in \{0, \ldots, n-2\}$ such that the composition $\check{\psi} \circ f \circ \check{\phi}^{-1}\colon \check{U}' \to \check{V}'$ has the normal form

$$(\check{\psi} \circ f \circ \check{\phi}^{-1})(u, v, z_1, \ldots, z_{n-2}) = (u, v^3 + uv - \sum_{i=1}^{l} z_i^2 + \sum_{i=l+1}^{n-2} z_i^2).$$

If $\check{\psi}$ is orientation preserving, then we may take $U' := \check{U}'$, $\phi := \check{\phi}$, $V' := \check{V}'$, $\psi := \check{\psi}$, and $k := l$.

If $\check{\psi}$ is orientation reversing, then we define the following diffeomorphisms:

$$\alpha\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad \alpha(u, v, z_1, \ldots, z_{n-2}) = (u, -v, z_{l+1}, \ldots, z_{n-2}, z_1, \ldots, z_l),$$
$$\beta\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad \beta(a, b) = (a, -b).$$

Setting $U' := \alpha(\check{U}')$ and $V' := \beta(\check{V}')$, we define the compositions

$$\phi := \alpha| \circ \check{\phi} \colon U \to U',$$
$$\psi := \beta| \circ \check{\psi} \colon V \to V'.$$

Hence, ψ is orientation preserving, and we obtain

$$\begin{aligned}(\psi \circ f \circ \phi^{-1})(u, v, z_1, \dots, z_{n-2}) &= \beta((\check{\psi} \circ f \circ \check{\phi}^{-1})(\alpha^{-1}(u, v, z_1, \dots, z_{n-2}))) \\ &= \beta((\check{\psi} \circ f \circ \check{\phi}^{-1})(u, -v, z_{n-1-l}, \dots, z_{n-2}, z_1, \dots, z_{n-2-l})) \\ &= \beta(u, -v^3 - uv - \sum_{i=n-1-l}^{n-2} z_i^2 + \sum_{i=1}^{n-2-l} z_i^2) \\ &= (u, v^3 + uv + \sum_{i=n-1-l}^{n-2} z_i^2 - \sum_{i=1}^{n-2-l} z_i^2).\end{aligned}$$

Therefore, we may take U', ϕ and V', ψ as defined above and $k := n - 2 - l$.

Finally, as ψ is orientation preserving, we can deduce from [32, §4.3, page 284] that $I(p) = k$. □

The following result is [32, Lemma (3.2)(2), p. 274].

Lemma 4.6.2. *Let p be a cusp of f and let C denote the component of $S(F)$ which contains p. If n is even and $I(p) = \frac{n}{2} - 1$, then the two arcs abutting p on C have absolute index $\frac{n}{2}$. If $I(p) \neq \frac{n}{2} - 1$, then the two arcs abutting p on C have absolute indices τ and $\tau + 1$, where $\tau := \max\{I(p), n - 2 - I(p)\}$.*

Proof. Set $k := I(p)$. Then there exist charts $\phi \colon U \to U' \subset \mathbb{R}^n$ around p and $\psi \colon V \to V' \subset \mathbb{R}^2$ around $f(p)$, where $f(U) \subset V$ and ψ is an orientation preserving diffeomorphism, such that the composition $\overline{f} := \psi \circ f \circ \phi^{-1} \colon U' \to V'$ has the form

$$\overline{f}(u, v, z_1, \dots, z_{n-2}) = (u, v^3 + uv - \sum_{i=1}^{k} z_i^2 + \sum_{i=k+1}^{n-2} z_i^2).$$

The singular set of $\overline{f}$ is given by

$$S(\overline{f}) = U' \cap \{(-3v^2, v, 0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2};\ v \in \mathbb{R}\}.$$

Set $h_u(v, z_1, \dots, z_{n-2}) = v^3 + uv - \sum_{i=1}^{k} z_i^2 + \sum_{i=k+1}^{n-2} z_i^2$. Let $p' \in S(\overline{f}) \setminus \{p\}$, so $p' = (-3v^2, v, 0)$ for some $v \neq 0$. By Definition 4.5.1 and Proposition 4.5.3 the absolute index of p' is given by $\max\{\lambda, n - 1 - \lambda\}$, where λ denotes the index of the Hessian of h_{-3v^2} at $(v, 0)$:

$$H - (v, 0)(h_{-3v^2}) = \operatorname{diag}(6v, 2a_1, \dots, 2a_{n-2}),$$

where $a_i := -1$ for $i \in \{1, \dots, k\}$ and $a_i := 1$ for $i \in \{k+1, \dots, n-2\}$. Therefore,

$$\lambda = \begin{cases} k + 1, & \text{if } v < 0, \\ k, & \text{if } v > 0. \end{cases}$$

Set $\tau := \max\{I(p), n-2-I(p)\} = \max\{k, n-2-k\}$. We distinguish the following three cases:

- $k < \frac{n}{2} - 1$. In this case, $n-2-k > \frac{n}{2} - 1$. Therefore, $\tau = n-2-k$. If $v < 0$, then $\lambda = k+1 < \frac{n}{2}$ and $n-1-\lambda > \frac{n}{2} - 1$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = n-1-\lambda = \tau$. (In fact, if n is even, then $\lambda \le \frac{n}{2}-1$ and $n-1-\lambda \ge \frac{n}{2}$, so $\max\{\lambda, n-1-\lambda\} = n-1-\lambda$. If n is odd, then $\lambda \le \frac{n-1}{2}$ and $n-1-\lambda \ge \frac{n+1}{2}-1 = \frac{n-1}{2}$, so again $\max\{\lambda, n-1-\lambda\} = n-1-\lambda$.) If $v > 0$, then $\lambda = k < \frac{n}{2}-1$ and $n-1-\lambda > \frac{n}{2}$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = n-1-\lambda = \tau+1$.
- n is even and $k = \frac{n}{2} - 1$. If $v < 0$, then $\lambda = k+1 = \frac{n}{2}$ and $n-1-\lambda = \frac{n}{2} - 1$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = \frac{n}{2}$. If $v > 0$, then $\lambda = k = \frac{n}{2} - 1$ and $n-1-\lambda = \frac{n}{2}$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = \frac{n}{2}$.
- $k > \frac{n}{2}-1$. In this case, $n-2-k < \frac{n}{2}-1$. Therefore, $\tau = k$. If $v < 0$, then $\lambda = k+1 > \frac{n}{2}$ and $n-1-\lambda < \frac{n}{2}-1$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = \lambda = \tau+1$. If $v > 0$, then $\lambda = k > \frac{n}{2} - 1$ and $n-1-\lambda < \frac{n}{2}$, so the absolute index of p' is given by $\max\{\lambda, n-1-\lambda\} = \lambda = \tau$. (In fact, if n is even, then $\lambda \ge \frac{n}{2}$ and $n-1-\lambda \le \frac{n}{2} - 1$, so $\max\{\lambda, n-1-\lambda\} = \lambda$. If n is odd, then $\lambda \ge \frac{n+1}{2} - 1 = \frac{n-1}{2}$ and $n-1-\lambda \le \frac{n-1}{2}$, so again $\max\{\lambda, n-1-\lambda\} = \lambda$.)

□

Definition 4.6.3. A pair of cusps (p, p') of f is called a *matching pair* if

$$I(p) + I(p') = n-2.$$

A pair of cusps (p, p') of f is called a *removable pair* (see [32, Definition (4.5), p. 285]) if there exists a *joining curve for p and p'* which is a suitable embedding $\lambda\colon [0,1] \to X$ such that $\lambda(0) = p$, $\lambda(1) = p'$, and $\lambda^{-1}(S(f)) = \{p, p'\}$ (for the other required properties see [32, Section (4.4), p. 285]).

The following is Levine's main theorem on elimination of cusps (see [32, p. 286ff.]).

Theorem 4.6.4. *Every matching pair (p, p') of f that is also removable can be eliminated. More precisely, if $\lambda\colon [0,1] \to X$ is a joining curve for p and p', then (after a local homotopy of f in a neighbourhood of $\lambda([0,1]) \subset X$ that makes $f \circ \lambda$ an embedding) there exist local coordinates around $\lambda([0,1]) \subset X$ and $f(\lambda([0,1])) \subset \mathbb{R}^2$ on which f has the local form shown in [32, Lemma (4.9), p. 293 ff.]. Therefore, during the elimination of p and p', f needs only to be modified on an arbitrarily small neighbourhood of $\lambda([0,1]) \subset X$ (see property* (1) *of [32, Lemma (4.9), p. 293 ff.]).*

The case $n = 2$ of the cusp elimination theorem is exploited in Chapter 5. If $n \ge 3$ and X is connected, then any matching pair of cusps of f is automatically a removable pair by [32, Lemma (4.3)(a), p. 284] and [32, Lemma (4.4)(1), p. 285], and can thus be eliminated.

Remark 4.6.5. An examination of the local form shown in [32, Lemma (4.9), p. 293 ff.] reveals that the image $f(S(f)) \subset \mathbb{R}^2$ in the plane before and after the elimination of the pair (p, p') of cusps looks as in Figure 4.1. This gives control over the way the arcs that abut the cusps are connected to each other after the elimination.

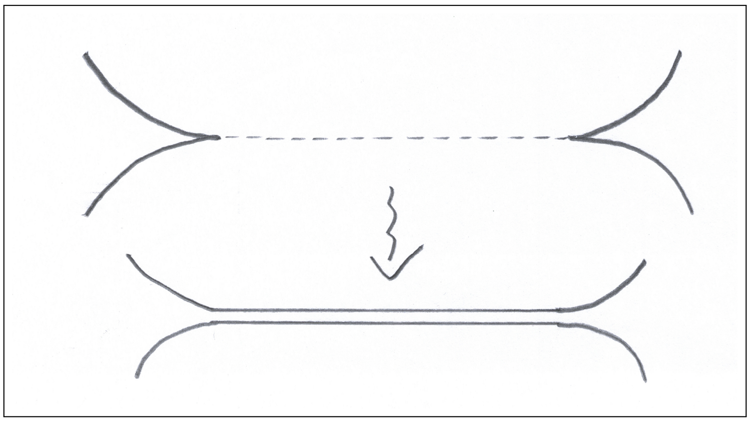

Figure 4.1: Effect of cusp elimination in the plane

4.7 Creation of Cusps

Let X^n be a smooth manifold (without boundary) of dimension $n \geq 2$. Having presented Levine's cusp elimination theorem in the previous section, we now discuss the complementary process of creating a pair of cusps on a given fold line of a generic smooth map $X \to \mathbb{R}^2$. For this purpose, points $p \in \mathbb{R}^m$ will be written as triples $p = (u, x, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{m-2}$. (If $n = 2$, then there are no z-coordinates.)

Let $i \in \{0, \ldots, n-2\}$. Setting $a_j := -1$ for $j \in \{1, \ldots, i\}$ and $a_j := 1$ for $j \in \{i+1, \ldots, n-2\}$, the standard quadratic form of index i is given by

$$Q \colon \mathbb{R}^{n-2} \to \mathbb{R}, \qquad Q(z) = \sum_{j=1}^{n-2} a_j z_j^2.$$

We consider the homotopy $F \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^2$ which is given at $t \in \mathbb{R}$ by

$$F_t \colon \mathbb{R}^n \to \mathbb{R}^2, \qquad F_t(p) = (u, \frac{x^4}{12} - t\frac{x^2}{2} + ux + Q(z)) = (u, f_t(p)),$$

where $f_t \colon \mathbb{R}^n \to \mathbb{R}$ is given at $p \in \mathbb{R}^n$ by

$$f_t(p) = \frac{x^4}{12} - t\frac{x^2}{2} + ux + Q(z).$$

The following lemma is actually a direct corollary of Proposition 4.5.4.

Lemma 4.7.1. *Given $t \in \mathbb{R}$, the singular set of F_t is given by the image of the embedding*

$$\varphi_t \colon \mathbb{R} \to \mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}, \qquad \varphi_t(x) = (-\frac{x^3}{3} + tx, x, 0).$$

If $t < 0$, then F_t is a fold map and the fold locus $S(F_t) = \varphi_t(\mathbb{R})$ has absolute index $\max\{i, n-1-i\}$. *If $t > 0$, then $(\varphi_t(-\sqrt{t}), \varphi_t(\sqrt{t}))$ is a matching pair of cusps of F_t, the points $\varphi_t(x)$ for $|x| > \sqrt{t}$ are fold points of F_t of absolute index* $\max\{i, n-1-i\}$, *and the points $\varphi_t(x)$ for $|x| < \sqrt{t}$ are fold points of F_t of absolute index* $\max\{i+1, n-2-i\}$.

Remark 4.7.2. The homotopy F is part of the *flipping move* discussed as "Deformation 3" in [31, p. 24]. For $n = 2$, the map $\mathbb{R}^3 \to \mathbb{R}^3$, $(t, u, x) \mapsto (t, u, f_t(u, x))$ is discussed in [17, Exercise VII.3(1), page 176]. The image of its singular set is called the *swallow's tail.*

Proposition 4.7.3. *If $U \subset \mathbb{R}^n$ is an open neighbourhood of the origin $0 \in \mathbb{R}^n$, then there exist a smooth map $G\colon \mathbb{R}^n \to \mathbb{R}^2$, a compact subset $K \subset U$ and an embedding $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ with the following properties:*

(i) $G(p) = F_{-1}(p)$ *for all* $p \in \mathbb{R}^n \setminus K$.
(ii) *The singular set of* G *is given by the image of* φ, $S(G) = \varphi(\mathbb{R})$.
(iii) *If* $|s| < 1$, *then* $\varphi(s)$ *is a fold point of* G *of absolute index* $\max\{i+1, n-2-i\}$.
(iv) $(\varphi(-1), \varphi(1))$ *is a matching pair of cusps of* G.
(v) *If* $|s| > 1$, *then* $\varphi(s)$ *is a fold point of* G *of absolute index* $\max\{i, n-1-i\}$.

Proof. Choose $\delta > 0$ such that $\{(0,0,z) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2};\ ||z|| \leq \delta\} \subset U$. Choose a smooth map $\gamma\colon \mathbb{R} \to [0,1]$ such that $\gamma(r) = 1$ for $r \leq \frac{\delta^2}{4}$ and $\gamma(r) = 0$ for $r \geq \delta^2$. Choose $\varepsilon > 0$ so small that

(ε1) $8\varepsilon(\varepsilon+1)\cdot\max\{|\gamma'(r)|;\ r \in \mathbb{R}\} < 1$.
(ε2) $K := \{(u,x,z);\ |u| \leq 2(\sqrt{\varepsilon})^3,\ |x| \leq 4\sqrt{\varepsilon},\ ||z|| \leq \delta\} \subset U$.

Choose a smooth map $\alpha\colon \mathbb{R} \to [0,1]$ such that $\alpha(r) = 1$ for $r \leq \varepsilon^3$ and $\alpha(r) = 0$ for $r \geq 4\varepsilon^3$. Choose a smooth map $\beta\colon \mathbb{R} \to [0,1]$ such that $\beta'(r) \leq 0$ for all $r \in \mathbb{R}$, $\beta(r) = 1$ for $r \leq 9\varepsilon$ and $\beta(r) = 0$ for $r \geq 16\varepsilon$. Define

$$\eta\colon \mathbb{R}^n \to [0,1], \qquad \eta(p) = \alpha(u^2)\beta(x^2)\gamma(||z||^2).$$

Finally, define the smooth map

$$G\colon \mathbb{R}^n \to \mathbb{R}^2, \qquad G(p) = (u, \eta(p)(f_\varepsilon(p) - f_{-1}(p)) + f_{-1}(p)) = (u, g(p)),$$

where $g\colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$g(p) = \eta(p)(f_\varepsilon(p) - f_{-1}(p)) + f_{-1}(p) = -\eta(p)(\varepsilon+1)\frac{x^2}{2} + f_{-1}(p).$$

Note that property (i) holds because we have $\eta(p) = 0$ for all $p \in \mathbb{R}^n \setminus K$ by construction.

As the Jacobian of G at $p \in \mathbb{R}^n$ is given by

$$D_pG = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \partial_u g(p) & \partial_x g(p) & \partial_{z_1} g(p) & \dots & \partial_{z_{n-2}} g(p) \end{pmatrix},$$

we conclude that $S(G) = \{p \in \mathbb{R}^n;\ \partial_x g(p) = \partial_{z_1} g(p) = \ldots = \partial_{z_{n-2}} g(p) = 0\}$.

In the following, let $E := \mathbb{R}\times\mathbb{R}\times 0$, which is the plane in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n-2} = \mathbb{R}^n$ defined by $z = 0$.

Let us show that $S(G) \subset E$. For $n = 2$ this follows from $E = \mathbb{R}^2 = \mathbb{R}^n$. If $n > 2$, then suppose that there exists $p = (u,x,z) \in S(G)$ with $z_j \neq 0$ for some $j \in \{1,\ldots,n-2\}$. It follows from $p \in S(G)$ that

$$\begin{aligned} 0 = \partial_{z_j} g(p) &= -2z_j\alpha(u^2)\beta(x^2)\gamma'(||z||^2)(\varepsilon+1)\frac{x^2}{2} + 2a_j z_j \\ &= -2z_j(\alpha(u^2)\beta(x^2)\gamma'(||z||^2)(\varepsilon+1)\frac{x^2}{2} - a_j). \end{aligned}$$

Since $z_j \neq 0$, we obtain $\alpha(u^2)\beta(x^2)\gamma'(||z||^2)(\varepsilon+1)\frac{x^2}{2} = a_j \in \{\pm 1\}$. We distinguish between the following two cases:

- $|x| \leq 4\sqrt{\varepsilon}$. In this case, we use $\alpha(u^2), \beta(x^2) \in [0,1]$ and $(\varepsilon 1)$ to deduce the following contradiction:

$$1 = |\alpha(u^2)\beta(x^2)\gamma'(||z||^2)(\varepsilon+1)\frac{x^2}{2}| \leq 8\varepsilon(\varepsilon+1)|\gamma'(||z||^2)| \overset{(\varepsilon 1)}{<} 1.$$

- $|x| > 4\sqrt{\varepsilon}$. In this case, $\beta(x^2) = 0$. Therefore,

$$0 = \partial_{z_j} g(p) = -2z_j(\alpha(u^2)\beta(x^2)\gamma'(||z||^2)(\varepsilon+1)\frac{x^2}{2} - a_j) = 2a_j z_j$$

  in contradiction to $a_j \in \{\pm 1\}$ and $z_j \neq 0$.

All in all, we have shown that $S(G) \subset E$.

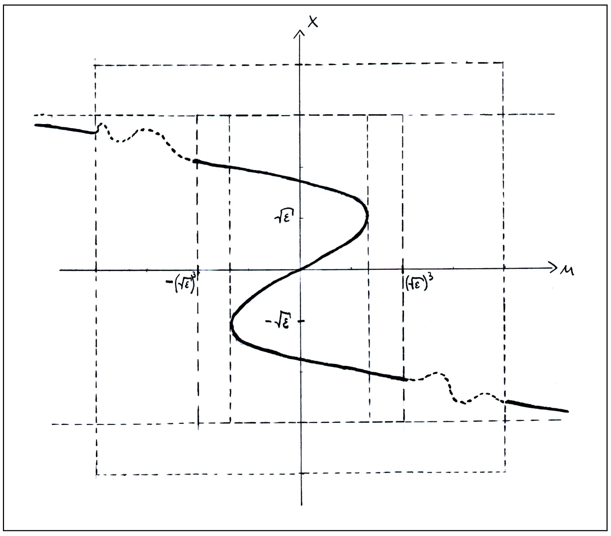

Figure 4.2: Open subsets U_0, U_1 and U_2 of the u-x-plane E.

Consider the following open subsets of E (see Figure 4.2):

$$\begin{aligned}
U_0 &:= E \cap (\mathbb{R}^n \setminus K), \\
U_1 &:= \{(u,x,0);\ |u| < (\sqrt{\varepsilon})^3,\ |x| < 3\sqrt{\varepsilon}\}, \\
U_2 &:= \{(u,x,0);\ |u| > \frac{2}{3}(\sqrt{\varepsilon})^3,\ |x| < 3\sqrt{\varepsilon}\}.
\end{aligned}$$

Let us show that $S(G) \subset U_0 \cup U_1 \cup U_2$. Suppose that $p = (u,x,0) \in S(G)$ $(\subset E)$. First assume that $|x| < 3\sqrt{\varepsilon}$. If $|u| < (\sqrt{\varepsilon})^3$, then $p \in U_1$. Otherwise, if $|u| \geq (\sqrt{\varepsilon})^3$ $(> \frac{2}{3}(\sqrt{\varepsilon})^3)$, then $p \in U_2$. Therefore, we may assume that $|x| \geq 3\sqrt{\varepsilon}$ in the following. Moreover, we may assume in the following that $p \in K$. (Indeed, if $p \notin K$, then $p \in U_0$.) In particular, we have $|u| \leq 2(\sqrt{\varepsilon})^3$. Now it suffices to show that the assumptions $|x| \geq 3\sqrt{\varepsilon}$ and $|u| \leq 2(\sqrt{\varepsilon})^3$ lead to a contradiction. The point $p = (u,x,0) \in S(G)$ satisfies

$$\begin{aligned}
0 = \partial_x g(p) &= \partial_x(\eta(p)f_\varepsilon(p) + (1-\eta(p))f_{-1}(p)) \\
&= (\partial_x \eta(p)) \cdot (f_\varepsilon(p) - f_{-1}(p)) + \eta(p)\partial_x f_\varepsilon(p) + (1-\eta(p))\partial_x f_{-1}(p) \\
&= \alpha(u^2)2x\beta'(x^2)\gamma(||0||^2)(f_\varepsilon(p) - f_{-1}(p)) + \eta(p)\partial_x f_\varepsilon(p) + (1-\eta(p))\partial_x f_{-1}(p).
\end{aligned}$$

Using $\gamma(||0||^2) = \gamma(0) = 1$ and $f_\varepsilon(p) - f_{-1}(p) = -(\varepsilon+1)\frac{x^2}{2}$, multiplication with x yields

$$0 = x^4(\varepsilon+1)\alpha(u^2)(-\beta'(x^2)) + \eta(p)x\partial_x f_\varepsilon(p) + (1-\eta(p))x\partial_x f_{-1}(p).$$

As $\alpha(r) \geq 0$ and $\beta'(r) \leq 0$ for all $r \in \mathbb{R}$, we obtain

$$0 \geq \eta(p)x\partial_x f_\varepsilon(p) + (1-\eta(p))x\partial_x f_{-1}(p).$$

Observing that $x\partial_x f_{-1}(p) = \frac{x^4}{3} + x^2 + ux \geq \frac{x^4}{3} - \varepsilon x^2 + ux = x\partial_x f_\varepsilon(p)$ and $\eta(\mathbb{R}^n) \subset [0,1]$, we get

$$(*) \qquad 0 \geq x\partial_x f_\varepsilon(p) = x(\frac{x^3}{3} - \varepsilon x + u) = x(u - q_\varepsilon(x)),$$

where we have introduced the function $q_\varepsilon \colon \mathbb{R} \to \mathbb{R}$, $q_\varepsilon(y) = -\frac{y^3}{3} + \varepsilon y$. Note that q_ε is strictly decreasing on $(-\infty, -3\sqrt{\varepsilon}]$ and on $[3\sqrt{\varepsilon}, \infty)$ because $q'_\varepsilon(y) < 0$ for all $y \in \mathbb{R}$ with $|y| > \sqrt{\varepsilon}$. Since $|x| \geq 3\sqrt{\varepsilon}$, we distinguish between the following two cases:

- $x \leq -3\sqrt{\varepsilon}$. As $x < 0$, $(*)$ implies that $u \geq q_\varepsilon(x) \geq q_\varepsilon(-3\sqrt{\varepsilon}) = 6(\sqrt{\varepsilon})^3$.
- $x \geq 3\sqrt{\varepsilon}$. As $x > 0$, $(*)$ implies that $u \leq q_\varepsilon(x) \leq q_\varepsilon(3\sqrt{\varepsilon}) = -6(\sqrt{\varepsilon})^3$.

Both cases are in contradiction to $|u| \leq 2(\sqrt{\varepsilon})^3 < 6(\sqrt{\varepsilon})^3$.

All in all, it follows that $S(G) \subset U_0 \cup U_1 \cup U_2$. Let us consider the intersections $S(G) \cap U_\nu$ for $\nu \in \{0,1,2\}$ in more detail:

$S(G) \cap U_0$:
Set $V_0 := \mathbb{R}^n \setminus K$. As $\eta(p) = 0$ for all $p \in V_0$ by construction, we obtain $G|_{V_0} = F_{-1}|_{V_0}$. Hence, by Lemma 4.7.1, $G|_{V_0}$ is a fold map with fold locus

$$S(G|_{V_0}) = S(F_{-1}|_{V_0}) = S(F_{-1}) \cap V_0 = \varphi_{-1}(\mathbb{R}) \cap E \cap V_0 = \varphi_{-1}(\mathbb{R}) \cap U_0,$$

where $\varphi_{-1}(x) = (q_{-1}(x), x, 0) = (-\frac{x^3}{3} - x, x, 0)$. As $S(G|_{V_0}) = S(G) \cap V_0 = S(G) \cap E \cap V_0 = S(G) \cap U_0$, we obtain

$$S(G) \cap U_0 = \varphi_{-1}(\mathbb{R}) \cap U_0.$$

Since the function $q_{-1} \colon \mathbb{R} \to \mathbb{R}$, $q_{-1}(y) = -\frac{y^3}{3} - y$, is strictly decreasing and $|q_{-1}(\pm 4\sqrt{\varepsilon})| = 4\sqrt{\varepsilon}(\frac{16}{3}\varepsilon + 1) > 2(\sqrt{\varepsilon})^3$, we conclude that $S(G) \cap U_0$ has two components, namely $\varphi_{-1}((-\infty, -x_0))$ and $\varphi_{-1}((x_0, \infty))$, where x_0 is uniquely determined by the equation $q_{-1}(x_0) = -2(\sqrt{\varepsilon})^3$. By the above consideration, we have $|x_0| < 4\sqrt{\varepsilon}$.

$S(G) \cap U_1$:
Set $V_1 := \{(u, x, z);\ |u| < (\sqrt{\varepsilon})^3,\ |x| < 3\sqrt{\varepsilon},\ ||z|| < \frac{\delta}{2}\}$. Obviously, $U_1 = V_1 \cap E$. As $\eta(p) = 1$ for all $p \in V_1$ by construction, we obtain $G|_{V_1} = F_\varepsilon|_{V_1}$.

Hence, by Lemma 4.7.1, $G|_{V_1}$ is a generic smooth map with singular set

$$S(G|_{V_1}) = S(F_\varepsilon|_{V_1}) = S(F_\varepsilon) \cap V_1 = \varphi_\varepsilon(\mathbb{R}) \cap E \cap V_1 = \varphi_\varepsilon(\mathbb{R}) \cap U_1,$$

where $\varphi_\varepsilon(x) = (q_\varepsilon(x), x, 0) = (-\frac{x^3}{3} + \varepsilon x, x, 0)$. Recall from Lemma 4.7.1 that the cusps of F_ε are the two points $\varphi_\varepsilon(\mp\sqrt{\varepsilon})$. As $S(G|_{V_1}) = S(G) \cap V_1 = S(G) \cap E \cap V_1 = S(G) \cap U_1$, we obtain

$$S(G) \cap U_1 = \varphi_\varepsilon(\mathbb{R}) \cap U_1.$$

A curve sketching of φ_ε reveals that

$$\varphi_\varepsilon(\mathbb{R}) \cap U_1 = \{\varphi_\varepsilon(x);\ x \in \mathbb{R},\ |q_\varepsilon(x)| < (\sqrt{\varepsilon})^3\} = \varphi_\varepsilon((-x_1, x_1)),$$

where $x_1 \in \mathbb{R}$ is the point that is uniquely determined by the equation $q_\varepsilon(x_1) = -(\sqrt{\varepsilon})^3$. It follows from

$$q_\varepsilon(\mp\sqrt{\varepsilon}) = \pm\frac{(\sqrt{\varepsilon})^3}{3} \mp (\sqrt{\varepsilon})^3 = \mp\frac{2}{3}(\sqrt{\varepsilon})^3$$

that $\varphi_\varepsilon(\mp\sqrt{\varepsilon})$ are the two cusps of $G|_{V_1} = F_\varepsilon|_{V_1}$.

$S(G) \cap U_2$:
Set $V_2 := \{(u, x, z);\ |u| > \frac{2}{3}(\sqrt{\varepsilon})^3,\ |x| < 3\sqrt{\varepsilon},\ ||z|| < \frac{\delta}{2}\}$. Obviously, $U_2 = V_2 \cap E$. For all $p = (u, x, z) \in V_2$ we set $y := (x, z)$ and

$$g_u(y) := g(p) = -\alpha(u^2)(\varepsilon + 1)\frac{x^2}{2} + f_{-1}(p).$$

Next, we show that $G|_{V_2}$ is a fold map. By Proposition 3.3.4 (d), it suffices to show that the Hessian of g_u at $y = (x, z)$,

$$H_y(g_u) = \operatorname{diag}(\partial_x^2 g_u(y), 2a_1, \ldots, 2a_{n-2}),$$

is non-degenerate for all $p = (u, x, z) \in S(G|_{V_2}) = S(G) \cap V_2$. Since $0 \neq 2a_j \in \{\pm 2\}$ for all j, it suffices to show that $\partial_x^2 g_u(y) \neq 0$ for all such p. If $p \in S(G|_{V_2}) = S(G) \cap V_2$, then

$$0 = \partial_x g_u(y) = -\alpha(u^2)(\varepsilon + 1)x + \frac{x^3}{3} + x + u = \frac{x^3}{3} - x(\alpha(u^2)(\varepsilon + 1) - 1) + u.$$

Suppose that $0 = \partial_x^2 g_u(y) = x^2 - (\alpha(u^2)(\varepsilon + 1) - 1)$. Hence, $\alpha(u^2)(\varepsilon + 1) - 1 = x^2 \geq 0$ and $x = \pm\sqrt{\alpha(u^2)(\varepsilon + 1) - 1}$. Consequently, we obtain from $\partial_x g_u(y) = 0$ that

$$u = -\frac{x^3}{3} + x(\alpha(u^2)(\varepsilon + 1) - 1) = \pm\frac{2}{3}\left(\sqrt{\alpha(u^2)(\varepsilon + 1) - 1}\right)^3.$$

Hence, from $0 \leq \alpha(u^2)(\varepsilon + 1) - 1 \leq \varepsilon$ (which uses that $\alpha(r) \leq 1$ for all $r \in \mathbb{R}$) we obtain $|u| \leq \frac{2}{3}(\sqrt{\varepsilon})^3$. This is a contradiction to $p \in V_2$. This shows that $G|_{V_2}$ is a fold map with fold locus $S(G|_{V_2}) = S(G) \cap V_2 = S(G) \cap E \cap U_2 = S(G) \cap U_2$.

Next, we prove that if $p, p' \in S(G) \cap U_2$ and $u = u'$, then $p = p'$.
Since $p = (u, x, 0)$ and $p' = (u, x', 0)$, the claim is that $x = x'$. Since $p, p' \in S(G) \cap U_2$, we have $0 = \partial_x g(p)$ and $0 = \partial_x g(p')$, which yields

$$\frac{x^3}{3} - x(\alpha(u^2)(\varepsilon + 1) - 1) + u \quad = \quad 0 \quad = \quad \frac{x'^3}{3} - x'(\alpha(u^2)(\varepsilon + 1) - 1) + u.$$

In other words, the function $q\colon \mathbb{R} \to \mathbb{R}$, $q(y) = -\frac{y^3}{3} + y(\alpha(u^2)(\varepsilon + 1) - 1)$, satisfies

$$q(x) = u = q(x').$$

By Rolle's theorem, q must have at least one critical point ξ. Hence, $0 = q'(\xi) = -\xi^2 + (\alpha(u^2)(\varepsilon + 1) - 1)$ implies that $\alpha(u^2)(\varepsilon + 1) - 1 = \xi^2 \geq 0$. A curve sketching of q implies that

(*a*) $q(y) > q(y') > 0$ for all $y < y' < -\sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}$,
(*b*) $|q(y)| \leq \frac{2}{3}\left(\sqrt{\alpha(u^2)(\varepsilon+1)-1}\right)^3$ for all $y \in [-\sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}, \sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}]$,
(*c*) $q(y) < q(y') < 0$ for all $y > y' > \sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}$.

It follows from $p \in U_2$ that $|u| > \frac{2}{3}(\sqrt{\varepsilon})^3$. Since $\varepsilon \geq \alpha(u^2)(\varepsilon+1)-1 \geq 0$, we have

$$|q(x)| = |q(x')| = |u| > \frac{2}{3}(\sqrt{\varepsilon})^3 \geq \frac{2}{3}\left(\sqrt{\alpha(u^2)(\varepsilon+1)-1}\right)^3.$$

Therefore, it follows from (*b*) that $x, x' \notin [-\sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}, \sqrt{3(\alpha(u^2)(\varepsilon+1)-1)}]$. But then, (*a*) and (*c*) imply that $x = x'$.

As $S(G) \cap U_\nu$ is a 1-dimensional submanifold of U_ν for $\nu \in \{0,1,2\}$ by the previous results, we conclude that $S(G)$ is a 1-dimensional submanifold of E. Therefore, every component of $S(G)$ is either an embedded circle in E or an embedded real line in E. Being the singular set of G, $S(G)$ is a closed subset of $\mathbb{R}^n$ and hence a closed subset of E. Therefore, all components of $S(G)$ are closed subsets of E as well. (In fact, let C be a component of $S(G)$. As $S(G)$ is a submanifold of E, any point $p \in S(G) \setminus C$ has an open neighbourhood W_p in E such that $S(G) \cap W_p$ is contained in the component of p in $S(G)$. Therefore, C can be written as the intersection of $S(G)$ and $E \setminus \bigcup_{p \in S(G)\setminus C} W_p$, which are both closed subsets of E.) In particular, if a component of $S(G)$ is an embedded real line, then it is unbounded in both directions.

Recall that $S(G) \cap U_0$ consists of two components, namely $\varphi_{-1}((-\infty, -x_0))$ and $\varphi_{-1}((x_0, \infty))$, and these are both unbounded in one direction. Therefore, $S(G)$ has exactly one component which is an embedded real line, say C, and we have $S(G) \cap U_0 \subset C$. We choose an embedding $\psi\colon \mathbb{R} \to E$ such that $\psi(\mathbb{R}) = C$. As C is a closed subset of E, it follows from $\varphi_{-1}((-\infty, -x_0)) \subset C$ and $\varphi_{-1}((x_0, \infty)) \subset C$ that $\varphi_{-1}(-x_0) \in C$ and $\varphi_{-1}(x_0) \in C$. Hence, there exist $a_-, a_+ \in \mathbb{R}$ such that $\psi(a_-) = \varphi_{-1}(-x_0)$ and $\psi(a_+) = \varphi_{-1}(x_0)$. Then we have $S(G) \cap U_0 = \psi((-\infty, a_-)) \cup \psi((a_+, \infty))$. Moreover, note that $\psi([a_-, a_+]) \subset U_1 \cup U_2 = \{(u, x, 0);\ |x| < 3\sqrt{\varepsilon}\}$. (In fact, this follows from $\psi([a_-, a_+]) \cap U_0 = \emptyset$ and $\psi([a_-, a_+]) \subset C \subset S(G) \subset U_0 \cup U_1 \cup U_2$.)

Next, we show that $S(G) \cap U_1 \subset C$. As $S(G) \cap U_1 = \varphi_\varepsilon((-x_1, x_1))$ is connected, it suffices to show that $S(G) \cap U_1 \cap C = C \cap U_1 \neq \emptyset$. Note that $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) \in S(G) \cap U_1 \cap U_2$. (In fact, we have $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) = (-\frac{(2.1\sqrt{\varepsilon})^3}{3} + \varepsilon 2.1\sqrt{\varepsilon}, 2.1\sqrt{\varepsilon}, 0) = (-0.987(\sqrt{\varepsilon})^3, 2.1\sqrt{\varepsilon}, 0) \in U_1 \cap U_2$. Thus, $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) \in \varphi_\varepsilon(\mathbb{R}) \cap U_1 = S(G) \cap U_1$.) As the u-component of $\psi(a_-) = \varphi_{-1}(-x_0)$ is given by $2(\sqrt{\varepsilon})^3$ and the u-component of $\psi(a_+) = \varphi_{-1}(x_0)$ is given by $-2(\sqrt{\varepsilon})^3$, there must be a point $a \in (a_-, b_+)$, such that the u-component of $\psi(a)$ is given by $-0.987(\sqrt{\varepsilon})^3 \in (-2(\sqrt{\varepsilon})^3, 2(\sqrt{\varepsilon})^3)$, which is the u-component of $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) \in S(G) \cap U_2$. Moreover, $\psi(a) \in S(G) \cap U_2$. (In fact, the u-component of $\psi(a)$ is given by $-0.987(\sqrt{\varepsilon})^3$. Since $\psi([a_-, a_+]) \subset U_1 \cup U_2 = \{(u, x, 0);\ |x| < 3\sqrt{\varepsilon}\}$, we have $|x_a| < 3\sqrt{\varepsilon}$, where x_a denotes the x-component of $\psi(a)$. By definition of U_2, we can conclude that $\psi(a) \in U_2$.) Hence, $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) = \psi(a) \in C$. Since $\varphi_\varepsilon(2.1\sqrt{\varepsilon}) \in U_1$, it follows that $C \cap U_1 \neq \emptyset$. Therefore, $S(G) \cap U_1 \subset C$.

Next, we show that $S(G) \cap U_2 \subset C$. Given $p = (u, x, 0) \in S(G) \cap U_2$, we may assume that $p \notin U_0$. (Indeed, if $p \in U_0$, then $p \in S(G) \cap U_0 \subset C$.) Hence, it follows from $p \in \mathbb{R}^n \setminus U_0 = K$ that $|u| \leq 2(\sqrt{\varepsilon})^3$. Therefore, there must be a point $a \in [a_-, a_+]$ such that $\psi(a) = (u_a, x_a, 0)$ satisfies $u_a = u$. Note that $\psi(a) \in S(G) \cap U_2$. (In fact, we have $|x_a| < 3\sqrt{\varepsilon}$ since $\psi(a) \in \psi([a_-, a_+]) \subset U_1 \cup U_2 = \{(u, x, 0);\ |x| < 3\sqrt{\varepsilon}\}$. Moreover, we have $|u_a| = |u| > \frac{2}{3}(\sqrt{\varepsilon})^3$ since

$p \in U_2$.) Thus, we conclude that $p = \psi(a) \in C$.

All in all, we have shown that $S(G) = S(G) \cap (U_0 \cap U_1 \cap U_2) = (S(G) \cap U_0) \cap (S(G) \cap U_1) \cap (S(G) \cap U_2) \subset C$. Hence, $S(G) = C$ is connected.

Finally, an embedding $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ with the desired properties (ii) to (v) can be defined as $\varphi := \psi \circ \sigma$, where σ is an automorphism of $\mathbb{R}$ with the property that $(\varphi \circ \sigma)(\mp 1)$ are the two cusps of G. $\square$

Chapter 5

Fold Maps on Two-dimensional Cobordisms

The purpose of the present chapter is to determine the state sum of any given 2-dimensional cobordism W within the fold map TFT studied in Chapter 3. Recall from Section 3.1.6 that the entire information of the state sum evaluated at a given boundary condition is precisely encoded in the family of *state sets*. Every such state set consists of those natural numbers that arise as the number of loops (i.e., closed components of the singular locus) of a fold map $W \to \mathbb{R}^2$ which extends the given boundary condition, and whose singular locus induces a Brauer morphism with prescribed open (i.e., loop-free) part. Creation and elimination of pairs of cusps as discussed in Chapter 4 will serve as the basic techniques to control the number of loops in the singular locus of a given generic smooth map. In Section 5.2 it is shown how these two fundamental local modifications can be combined in a careful way to define more complicated modifications.

Theorem 5.1.3 characterizes the non-emptiness of every state set in terms of the given boundary condition and properties of the prescribed open Brauer morphism φ. This characterization requires φ to be index-preserving (see Definition 5.1.2) and *admissible* (see Section 5.6). Furthermore, it involves the vanishing of the $\mathbb{Z}/2$-valued *cusp invariant* (see Section 5.3), which is defined on boundary conditions and measures the parity of the number of cusps of any generic smooth map that extends the given boundary condition. According to Theorem 5.1.1, the cusp invariant can in turn be calculated in terms of the newly introduced *boundary turning invariant* (see Section 5.4) which by construction measures the “turning” of boundary conditions in the plane.

Non-empty state sets are explicitly computed in Theorem 5.1.4 (and Corollary 5.1.5). If the underlying cobordism W is orientable, then the informational content of the state set is encoded in a certain integer Δ^σ, where σ denotes a suitable orientation of W. Intuitively, given a fold map $F\colon W \to \mathbb{R}^2$, Δ^σ measures the difference between the number of certain handles of W that are disjoint to $S(F)$, and certain loops of $S(F)$. As it turns out, Δ^σ is independent of F, and can be calculated from the data that describe the state set, i.e. the given boundary condition and the prescribed open Brauer morphism. For a non-orientable cobordism W the state sets turn out to be either $\mathbb{N}$ or $\mathbb{N} \setminus \{0\}$, a result which exploits the existence of Möbius bands.

5.1 Introduction and Statement of Results

Throughout the present chapter we fix a 2-dimensional connected cobordism W from M to N in the sense of Definition 3.1.1. (In particular, $W^2 \subset [0,1] \times \mathbb{R}^D$ is smoothly embedded with time function $\omega \colon W \to [0,1]$ and cylinder scale $\varepsilon_W > 0$.)

Let $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ be a pair of boundary conditions. If m_S and n_S denote the number of components of $S(f_M)$ and $S(f_N)$, respectively, then $\mathbb{S}(f_M) = \mathrm{id}_{[m_S]}$ and $\mathbb{S}(f_N) = \mathrm{id}_{[n_S]}$. Note that every Brauer morphism $[m_S] \to [n_S]$ contains the same number $k_S := (m_S + n_S)/2$ of intervals. Let $\varphi \in \mathrm{OP}_{m_S, n_S}$ be an open Brauer morphism.

The computation of $L_W(f_M, f_N; \varphi)$ is the content of Theorem 5.1.3 and Theorem 5.1.4 formulated below. (See Section 5.7 for the proof of Theorem 5.1.3 and Section 5.8 for the proof of Theorem 5.1.4.) The computation is completed by Corollary 5.1.5, which is shown in Section 5.9.

More precisely, Theorem 5.1.3 characterizes the non-emptiness of the state set $L_W(f_M, f_N; \varphi)$. In preparation of the formulation of Theorem 5.1.3 we introduce in Section 5.3 the *cusp invariant*

$$t_W \colon \mathcal{F}(M) \times \mathcal{F}(N) \to \mathbb{Z}/2$$

that is defined as the parity of the number of cusps of any generic smooth map that extends the given boundary condition (f_M, f_N). An explicit formula for $t_W(f_M, f_N)$ is deduced in Theorem 5.1.1 (see Section 5.5). This formula involves the new concept of *oriented boundary turning invariant* (see Definition 5.4.6)

$$\omega_\sigma \colon \mathcal{F}(P) \to \mathbb{Z},$$

which is defined for any closed smooth 1-dimensional manifold P equipped with an orientation σ. Composition with the quotient map $\mathbb{Z} \to \mathbb{Z}/2$, $m \mapsto \overline{m}$, yields the *reduced boundary turning invariant*

$$\varpi \colon \mathcal{F}(P) \to \mathbb{Z}/2,$$

which does not depend on an orientation of P any more.

Theorem 5.1.1. *For every connected 2-dimensional cobordism W from M to N, the value of the cusp invariant on $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ is given by*

$$t_W(f_M, f_N) = \overline{\chi(W)} + \overline{k_S} + \varpi(f_M) + \varpi(f_N) \in \mathbb{Z}/2.$$

The formulation of Theorem 5.1.3 involves further properties of the open Brauer morphism φ.

Recall the concept of *non-reduced index* (see Definition 4.5.1), which assigns to every oriented fold line of a fold map from a surface into the plane an element in $\{0,1\}$. Given the boundary condition (f_M, f_N), we use the convention that the orientation of every component of $S(f_M)$ points inwards and the orientation of every component of $S(f_N)$ points outwards of W. Hence, by means of these orientations, one can assign a non-reduced index to each component of $S(f_M) \sqcup S(f_N)$.

Definition 5.1.2. The open Brauer morphism $\varphi \in \mathrm{OP}_{m_S, n_S}$ is called *index-preserving* with respect to (f_M, f_N) if the non-reduced index is preserved in the obvious way for all pairs of components of $S(f_M) \sqcup S(f_N)$ that are connected by φ.

Furthermore, by Definition 5.6.5, the open Brauer morphism φ is called *admissible* if the points of $(S(f_M) \cap 0 \times M) \cup (S(f_N) \cap 1 \times N) \subset \partial W$ can be connected in pairs by disjoint arcs in W as dictated by φ. This condition is of course only relevant for 2-dimensional cobordisms.

Theorem 5.1.3. *Let W be a connected 2-dimensional cobordism from M to N. For all boundary conditions $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ and any $\varphi \in \mathrm{OP}_{m_S,n_S}$ the following statements are equivalent:*

(i) $L_W(f_M, f_N; \varphi) \neq \emptyset$.
(ii) $t_W(f_M, f_N) = 0$, *and φ is both index-preserving and admissible.*

If the state set $L_W(f_M, f_N; \varphi)$ is non-empty, then it can be computed explicitly according to Theorem 5.1.4. We distinguish between the two cases that W is either orientable or non-orientable. If W is orientable and σ denotes any orientation of W, then we put (see Definition 5.8.3)

$$\Delta^\sigma := (\chi(W) + k_S - \omega_{\sigma|_M}(f_M) - \omega_{\sigma|_N}(f_N))/2 - c_\sigma(f_M, f_N; \varphi).$$

Here, the *cycle number* $c_\sigma(f_M, f_N; \varphi) \in \mathbb{N}$ (see Definition 5.8.1) captures the combinatorial interplay between (f_M, f_N) and φ for given σ.

It can be shown that $\Delta^\sigma, \Delta^{-\sigma} \in \mathbb{Z}$, $\Delta^\sigma + \Delta^{-\sigma} \equiv 0 \bmod 2$.

Theorem 5.1.4. *Let W be a connected 2-dimensional cobordism from M to N. Suppose that $L_W(f_M, f_N; \varphi) \neq \emptyset$ (compare Theorem 5.1.3). Furthermore, suppose that $k_S > 0$. For the computation of $L_W(f_M, f_N; \varphi)$ we distinguish between the case that W is orientable and the case that W is non-orientable:*

- *Let W be orientable and fix an orientation σ of W. Then, $\Delta^\sigma + \Delta^{-\sigma} \leq 0$, and*

$$L_W(f_M, f_N; \varphi) = \begin{cases} \Delta^\sigma + 2\mathbb{N}, & \text{if } \Delta^\sigma > 0, \\ \Delta^{-\sigma} + 2\mathbb{N}, & \text{if } \Delta^{-\sigma} > 0, \\ \mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) = \mathbb{N} \cap (\Delta^{-\sigma} + 2\mathbb{N}), & \text{else.} \end{cases}$$

  Note that this can be summarized by the formula

$$L_W(f_M, f_N; \varphi) = \mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N}).$$

- *If W is non-orientable, then $L_W(f_M, f_N; \varphi) = \mathbb{N}$.*

Note that Theorem 5.1.4 does not cover the case that $k_S = 0$. Nevertheless, this case can be deduced as a corollary from Theorem 5.1.4. Note that $k_S = 0$ implies that $m_S = n_S = 0$ and $\varphi = 1_{[0]}$.

Corollary 5.1.5. *Let W be a connected 2-dimensional cobordism from M to N. Suppose that $L_W(f_M, f_N; \varphi) \neq \emptyset$ (compare Theorem 5.1.3). Furthermore, suppose that $k_S = 0$. For the computation of $L_W(f_M, f_N; \varphi)$ we distinguish between the case that W is orientable and the case that W is non-orientable:*

- *Let W be orientable and fix an orientation σ of W. Using [11] one can in principle decide whether (f_M, f_N) extends to an immersion $W \to \mathbb{R}^2$. If so, then $L_W(f_M, f_N; 1_{[0]}) = 2\mathbb{N}$. Otherwise, there must be at least one loop in the singular set of any fold map $W \to \mathbb{R}^2$*

that extends the boundary conditions (f_M, f_N)*, and we have*

$$L_W(f_M, f_N; 1_{[0]}) = 1 + \mathbb{N} \cap ((\Delta^{\sigma} - 1) + 2\mathbb{N}) \cap ((\Delta^{-\sigma} - 1) + 2\mathbb{N}).$$

- *If* W *is non-orientable, then* $L_W(f_M, f_N; 1_{[0]}) = 1 + \mathbb{N}$*.*

Remark 5.1.6. The special case of a closed cobordism W is covered by Corollary 5.1.5 because $\partial W = \emptyset$ implies that $k_S = 0$. As W is closed there cannot exist an immersion $W \to \mathbb{R}^2$, so

$$L_W(f_M, f_N; 1_{[0]}) = \begin{cases} 1 + \mathbb{N} \cap ((\Delta^{\sigma} - 1) + 2\mathbb{N}) \cap ((\Delta^{-\sigma} - 1) + 2\mathbb{N}), & \text{if } W \text{ is orientable,} \\ 1 + \mathbb{N}, & \text{else.} \end{cases}$$

If W is orientable and σ is an orientation of W, then $\Delta^{\pm\sigma} = \chi(W)/2 = 1 - g$, where g denotes the genus of W. Hence,

$$L_W(f_M, f_N; 1_{[0]}) = \begin{cases} 1 + 2\mathbb{N}, & \text{if } g \text{ is even,} \\ 2 + 2\mathbb{N}, & \text{else.} \end{cases}$$

These results for $\partial W = \emptyset$ are in accordance with [25, Theorem 2.5, p. 314].

5.2 Elimination and Creation of Cusps in Dimension Two

Throughout the present section let $F\colon W \to \mathbb{R}^2$ denote a generic smooth map that is the restriction to W of a generic smooth map $\tilde{F}\colon ((-\varepsilon, 0] \times M) \cup W \cup ([1, 1+\varepsilon) \times N) \to \mathbb{R}^2$ for suitable $\varepsilon > 0$ such that $S(\tilde{F}) \pitchfork \partial W$, and such that ∂W contains no cusps of $\tilde{F}$. Hence, the singular locus $S(F)$ is a 1-dimensional submanifold of W that consists of fold points and cusps, where the set of cusps is a 0-dimensional submanifold of $S(F) \setminus \partial W$.

Remark 5.2.1. The presentation will be supported by pictures of the singular locus $S(F)$ on open U of the cobordism W. The boundary of U will be indicated by a grey line. With $S(F)$ being a smooth 1-dimensional submanifold of W that is closed as a subset of W, $S(F) \cap U$ will be represented by massive lines, and $S(F) \cap (W \setminus U)$ will be indicated by spotted lines in the figures. The cusps of F form a discrete subset of $S(F)$ and can hence be represented by marked points on the massive lines. (The remaining open arcs on the massive lines consist of fold points.) Every cusp is furthermore equipped with a tangent vector that points *downward* in the sense of Levine [32, Definition on p. 284]. (A tangent vector points downward at a cusp if its image in the plane points in the direction of the cusp. Intuitively, the vector itself points into the half plane into which the cusp can propagate.) The direction of a downward pointing tangent vector will be referred to as the direction into which the cusp *points*. Finally, *joining curves* (see [32, section (4.4), p. 285]) between cusps will be symbolized by dashed lines.

The main technical tool to control the number of closed components of $S(F)$ (and in particular the number of loops of F) will be to modify F locally by creating and eliminating pairs of cusps on $S(F)$ in a careful way. Let us start with the presentation of these two fundamental local modifications since they form the building blocks for all other local modifications of F that will be introduced in the present section:

- (E) Elimination of cusps (see Section 4.6 and [32, Fig. 3, p. 286]):

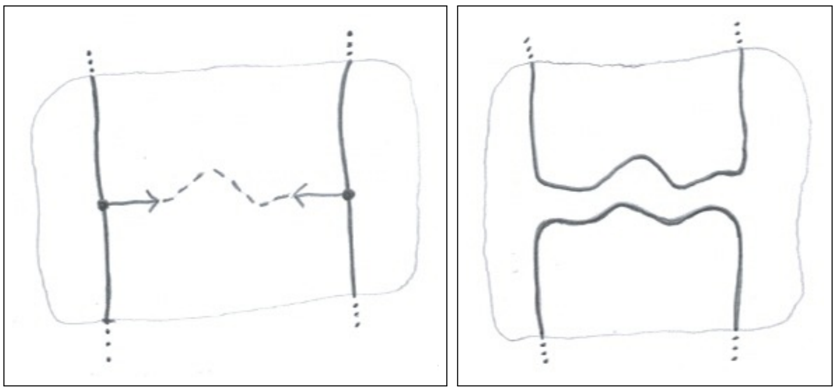

Figure 5.1: Elimination of cusps

A given pair of cusps of F can be eliminated if the cusps point into a common component of $W \setminus S(F)$. In fact, it follows from the proofs of [32, Lemma (1), section (4.4), p. 285] and [32, Lemma (2), section (4.4), p. 285] that such a pair of cusps can be connected by a joining curve (see the dashed line in Figure 5.1). As W is a 2-dimensional cobordism, such a pair of cusps of F is thus a matching pair in the sense of [32] and hence *removable* (see [32, Definition, section (4.5), p. 285]). As explained in [32, p. 286], the pair of cusps can therefore be eliminated by a homotopy of F that modifies F only in an arbitrarily small neighbourhood of the image of the joining curve. The singular locus $S(F)$ is changed by this process as pictured in Figure 5.1.

- (C) Creation of cusps (see Section 4.7 for a formal construction):

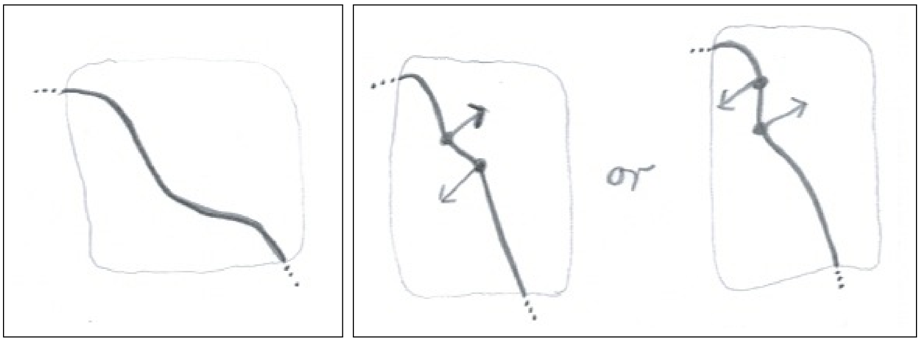

Figure 5.2: Creation of cusps

In an arbitrarily small open neighbourhood of a fold point of F a pair of two new cusps can be introduced on the fold line by a homotopy of F that modifies F only in this neighbourhood in such a way that the cusps point into opposite directions of the fold line.

Definition 5.2.2. A *loop* of F is a component of $S(F)$ that is diffeomorphic to the circle. (In general, loops may contain cusps.) A loop of F which does not contain any cusps will be called a *fold loop*. A fold loop C of F is called *trivial* if $W \setminus C$ has two components, and at least one of them is contractible and contains no singular points of F. A fold loop C is called *Möbius loop* if the normal bundle of C in W is the Möbius bundle.

It is helpful to have a loop contained only in parts in the neighbourhood shown in a figure since the the normal bundle of the loop might be nontrivial. (The parts of the loop that lie outside the neighbourhood will as usual be indicated by spotted lines.)

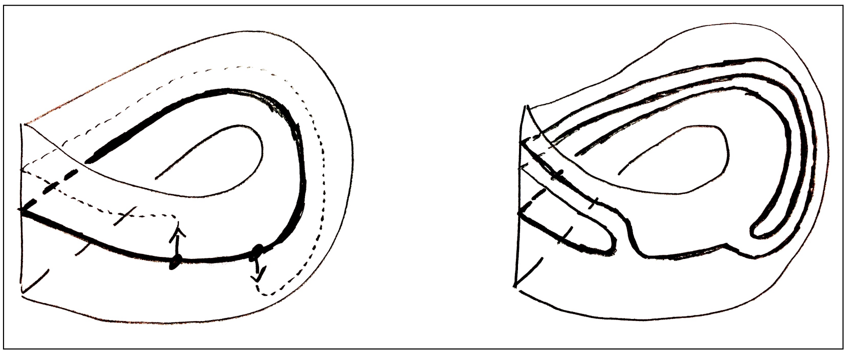

Figure 5.3: Creation and elimination of cusps near a Möbius loop.

Remark 5.2.3. Note that a pair of cusps that is created in a given neighbourhood U of a fold point p of F via (C) cannot be eliminated in general via (E). In fact, the two new cusps do not point into the same component of $U \cap (W \setminus S(F))$, and different components of $U \cap (W \setminus S(F))$ cannot be expected to be contained in the same component of $W \setminus S(F)$. However, if the component in $S(F)$ that contains p happens to be a Möbius loop, then the new cusps can indeed be eliminated via (E). Note that this modification increases the total number of fold loops of F by one by producing a new trivial fold loop (see Figure 5.3). This is essentially the reason why any higher number of loops can be realized on a non-orientable cobordism W (see Proposition 5.2.9), whereas it will be shown that the number of loops of any fold map that extends prescribed boundary conditions has the same parity when W is orientable. See Figure 5.17 for a local modification that increases the number of fold loops by two in the presence of a fold line.

As a first application, we combine the two fundamental modifications (E) and (C) to obtain

Proposition 5.2.4. *Any pair of cusps of the generic map $F\colon W \to \mathbb{R}^2$ can be eliminated.*

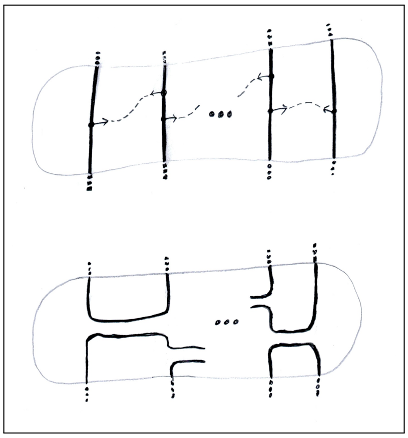

Figure 5.4: Global elimination of cusps in dimension 2.

Proof. As W is connected, the given pair of cusps can be connected by a smoothly embedded curve which is a joining curve (in the sense of [32, section (4.4), p. 285]) except for the fact that it intersects $S(F)$ transversely in a finite sequence $p_1, \dots, p_r$ of fold points. A tubular neighbourhood of this curve looks like in Figure 5.4. In a small open neighbourhood of every point p_i we create a pair of cusps via (C). Now one can find $r+1$ joining curves to eliminate the $2r+2$ cusps in pairs via (E) as shown in Figure 5.4. □

In the following we present five modifications of F that arise from convenient combinations of the fundamental modifications (E) and (C). (The proofs are given by illustration.) These modifications of F happen on compact subsets of $\operatorname{int} W$ and do not change the combinatorics of how the fold points in $S(F) \cap \partial W$ are connected by the 1-dimensional submanifold $S(F) \subset W$. However, these modifications will affect the number and other properties of the loops of $S(F)$.

Proposition 5.2.5. *(*Trivialization.*) If C is a fold loop of F and $S(F) \setminus C \neq \emptyset$, then C can be turned into a trivial fold loop.*

Proposition 5.2.6. *(*Absorbtion.*) Two fold loops of F, say C_0 and C_1, can be "absorbed" (see Figure 5.6) by a component L of $S(F) \setminus (C_0 \cup C_1)$ if there exists a smoothly embedded path $\gamma\colon [0,1] \to \operatorname{int} W$ such that $\gamma(i) \in C_i$ for $i \in \{0,1\}$, and $\gamma|_{(0,1)}$ intersects $S(F)$ transversely and in a single point of L.*

Proposition 5.2.7. *(*Tunneling.*) A loop of F (possibly containing cusps) with trivial normal bundle can pass through an even number of fold lines.*

Proposition 5.2.8. *(*Balancing.*) Two trivial fold loops of F, say C_0 and C_1, can be "balanced" (see Figure 5.8) with an embedded handle $H\colon [0,1] \times S^1 \to \operatorname{int} W \setminus S(F)$ if there exists a smoothly embedded path $\gamma\colon [0,1] \to \operatorname{int} W$ such that $\gamma^{-1}(C_i) = \{i\}$ for $i \in \{0,1\}$, and there exist $0 < s_0 < t_0 < t_1 < s_1 < 1$ such that*

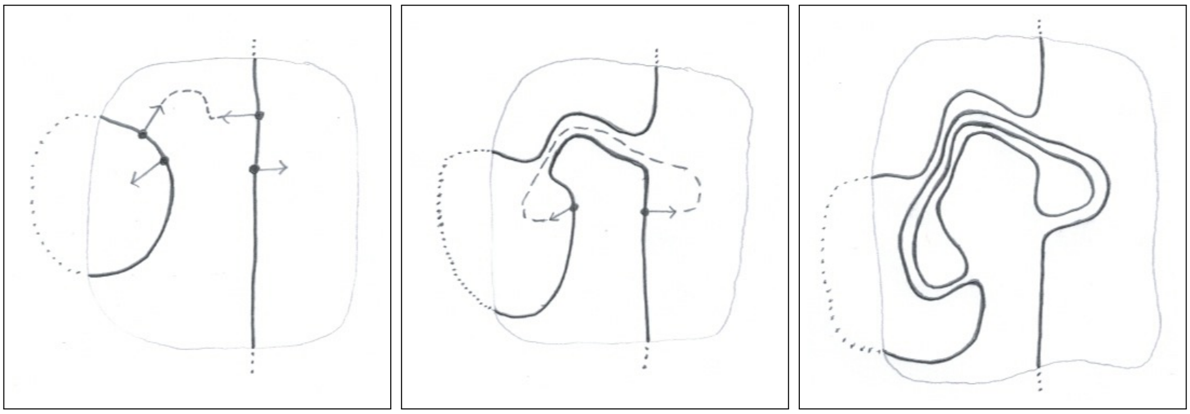

Figure 5.5: Trivialization. If $S(F) \setminus C \neq \emptyset$, then there always exists an embedded open 2-disc in $\operatorname{int} W$ which intersects C and another component of $S(F)$ as indicated.

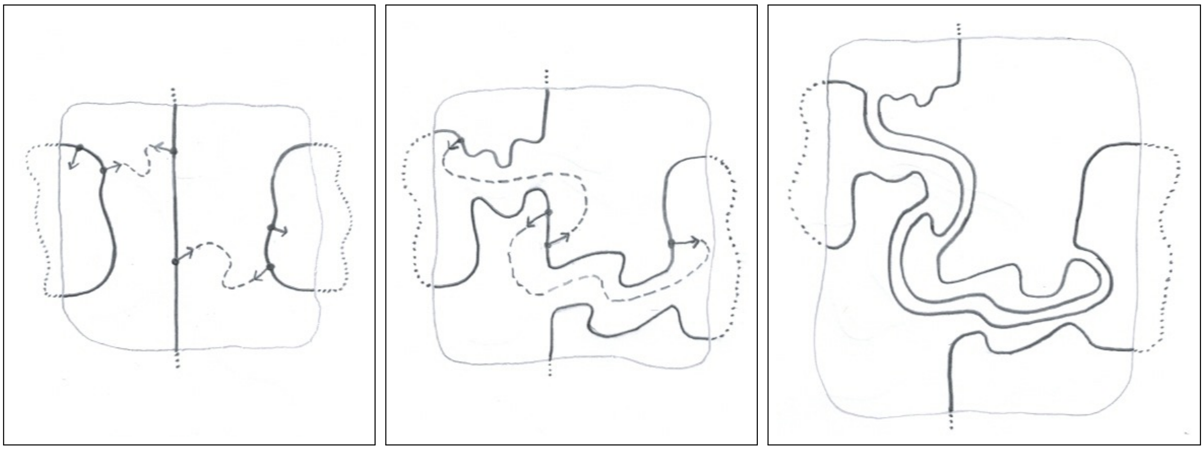

Figure 5.6: Absorbtion. Given γ, there always exists an embedded open 2-disc in $\operatorname{int} W$ which intersects $S(F)$ as indicated in C_0, C_1 and L.

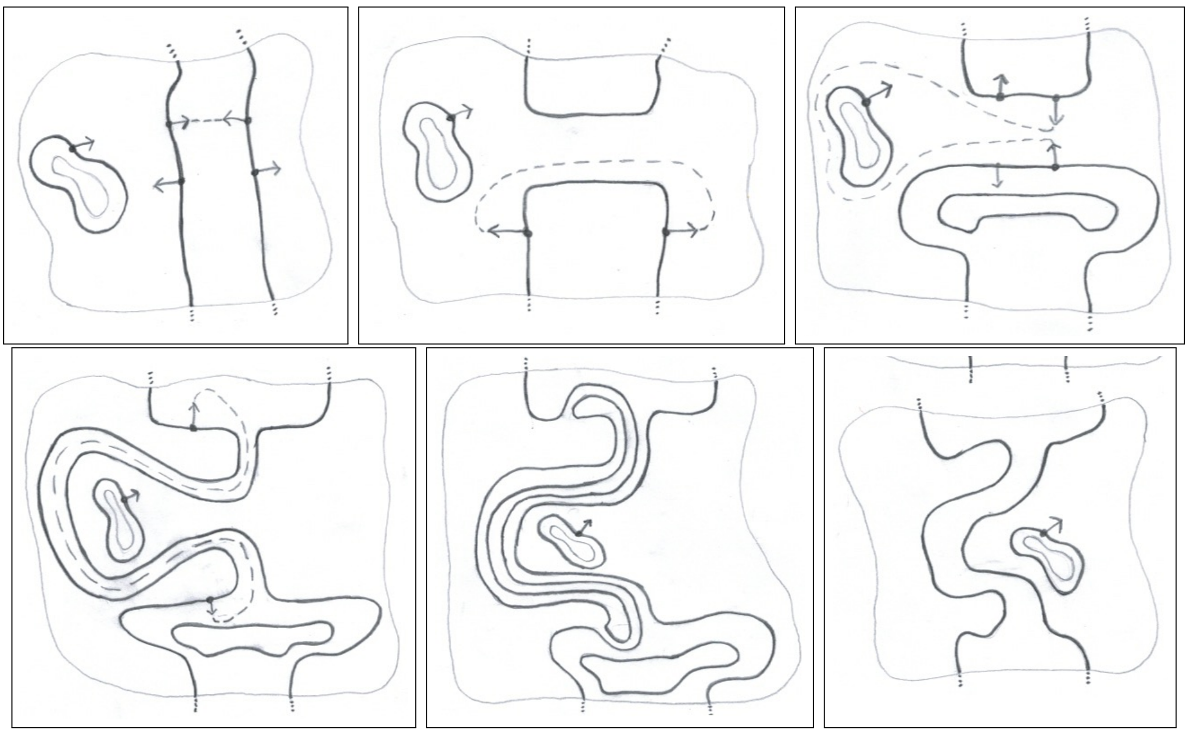

Figure 5.7: Tunneling. The modification takes place on an embedded open annulus in $\operatorname{int} W$. Note that in the last step the two additional trivial fold loops that were produced by earlier modifications are eliminated by absorbtion.

- $\gamma|_{(0,1)}$ *intersects* $S(F)$ *transversely, and only in the points* $\gamma(s_0)$ *and* $\gamma(s_1)$ *.*
- *for* $i \in \{0,1\}$ *,* $\gamma|_{(0,1)}$ *intersects* $H(i,S^1)$ *transversely, and only in the point* $\gamma(t_i)$ *.*

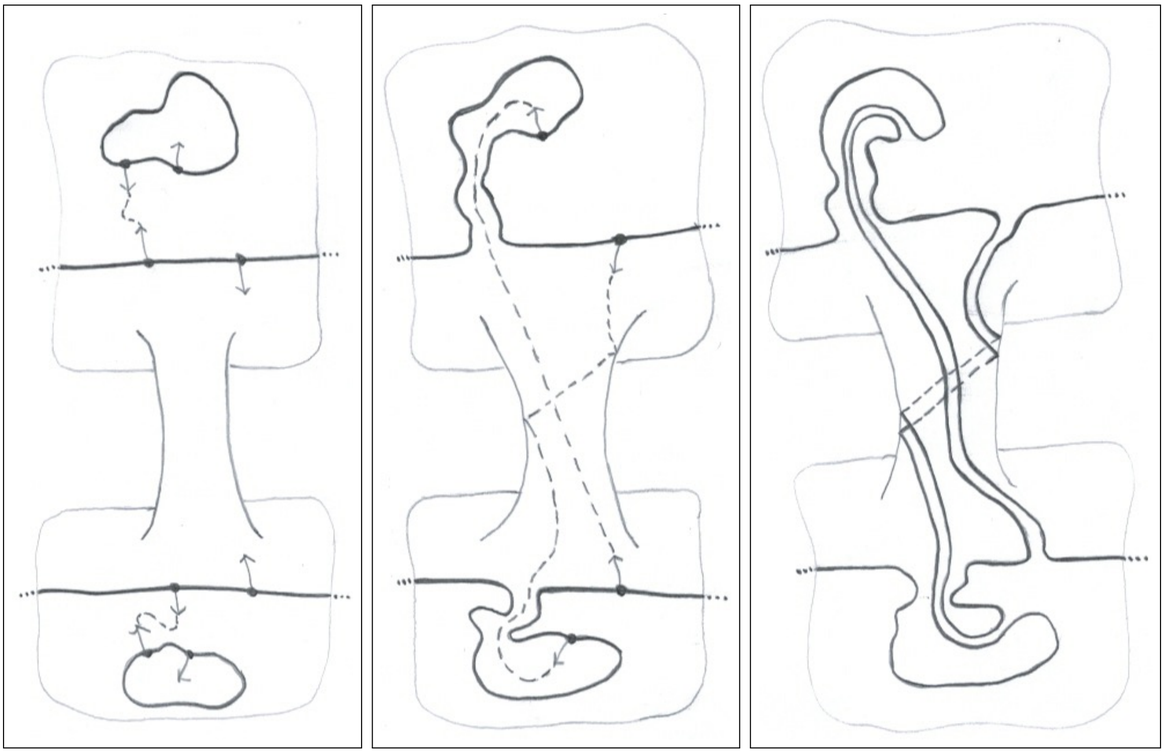

Figure 5.8: Balancing. The modification takes place on an embedded open annulus in $\operatorname{int} W$.

Proposition 5.2.9. *(*Möbius.*) Let* $C \subset \operatorname{int} W$ *be a smoothly embedded circle whose normal bundle is the Möbius bundle, and such that* $C \pitchfork S(F)$ *. Then,* F *can be modified on an arbitrarily small tubular neighbourhood of* C *to obtain a generic smooth map* $G\colon W \to \mathbb{R}^2$ *that has a Möbius loop. Furthermore,* G *can be modified on the same tubular neighbourhood of* C *to obtain a generic smooth map* $H\colon W \to \mathbb{R}^2$ *with the following properties:*

(i) $S(H)$ *contains exactly one more loop than* $S(F)$ *, namely a trivial fold loop.*
(ii) $S(H)$ *connects the same fold points of* $S(F) \cap \partial W$ *as* $S(F)$ *.*

Proof. It follows from $C \pitchfork S(F)$ that the finite set $C \cap S(F)$ has odd cardinality. (Indeed, an even cardinality would induce an orientation of a tubular neighbourhood of C because crossing a fold line in W alternates the property of $F|_{W\setminus S(F)}$ of being an orientation preserving or orientation reversing local diffeomorphism.) Therefore, as indicated in Figure 5.9, it is possible to modify F on a suitably small tubular neighbourhood of C to obtain a generic smooth map $G\colon W \to \mathbb{R}^2$ that has a Möbius loop. The generic map H with the desired properties can be constructed from G via the following three steps:

(1) Since G has a Möbius loop in the tubular neighbourhood of C, Remark 5.2.3 allows to modify G there in such a way that a new trivial fold loop arises in the singular locus of the resulting generic map G' .
(2) One can eliminate the cusps of G' that were produced during the construction of G from F in pairs as indicated in Figure 5.9. The resulting generic map G'' already satisfies property (ii) , but the elimination of cusps has produced an even number of fold loops in addition to the trivial fold loop constructed in step (2) .

(3) The even number of fold loops that were additionally produced in step (2) can be eliminated via tunneling and absorbtion. The resulting generic map H then satisfies both property (i) and property (ii).

□

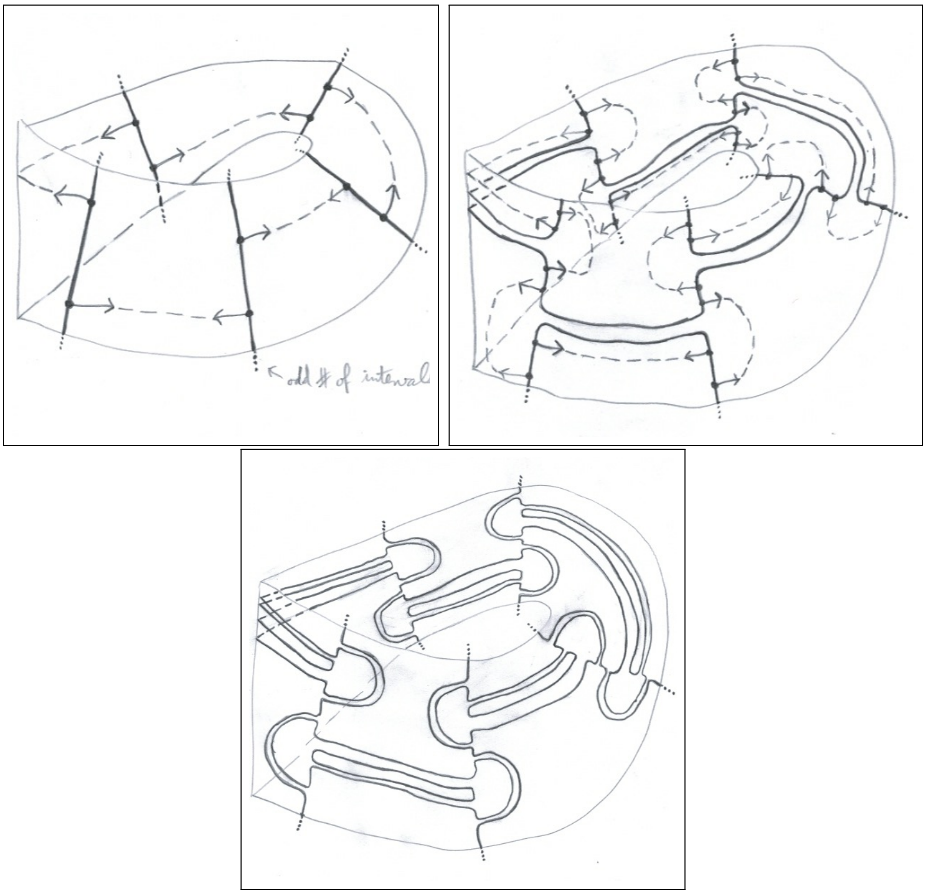

Figure 5.9: Möbius. The modification takes place on an embedded open Möbius band in $\operatorname{int} W$.

5.3 The Cusp Invariant

Proposition 5.3.1. *Given* $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, *there exists a generic smooth map* $F\colon W \to \mathbb{C}$ *such that* $F|_{[0,\varepsilon]\times M} \approx f_M$ *and* $F|_{[1-\varepsilon',1]\times N} \approx f_N$ *for some* $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$. *Moreover, any two such generic smooth maps have the same number of cusps mod* 2.

Proof. The first statement is a direct consequence of Lemma 4.3.9 and Proposition 4.4.1. To show the second statement, recall the fact from the theory of Thom polynomials that if V^n is a closed smooth n-dimensional manifold, then the number of cusps mod 2 of any generic smooth map $V \to \mathbb{C}$ equals the Stiefel-Whitney number $w_n[V]$ (see [32, Theorem (1.2), p. 264]). Taking V to be the double of W, which is a closed manifold that can be realized as the boundary of a compact smooth 3-dimensional manifold, we conclude that the number of cusps of any generic map $V \to \mathbb{C}$ is *even*. Given a generic map $F\colon W \to \mathbb{C}$ such that $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$, we choose a generic map $G\colon W \to \mathbb{C}$ such that $G|_{[0,\varepsilon]\times M} \approx \overline{f_M}$ and $G|_{[1-\varepsilon',1]\times N} \approx \overline{f_N}$. (Recall that $\overline{f}$ denotes the map $\overline{f}(t,s) = f(1-t,s)$ for any map $f\colon [0,1]\times S^1 \to \mathbb{C}$.) Gluing of F and G yields a generic map $H\colon V \to \mathbb{C}$ whose number of cusps is necessarily even. Hence, F and G have the same number of cusps mod 2. The claim follows because G was chosen independently of F. □

The previous result enables us to define a $\mathbb{Z}/2$-valued invariant

$$t_W(f_M, f_N) \in \mathbb{Z}/2$$

by taking the number of cusps mod 2 of any generic map $F\colon W \to \mathbb{C}$ such that $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$. It is clear by construction that $t_W(f_M, f_N)$ is an obstruction to the existence of an element in $\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$.

Proposition 5.3.2. *(Recall that* W *is connected.) The following statements are equivalent:*

(i) $t_W(f_M, f_N) = 0 \bmod 2$.
(ii) $\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N) \neq \emptyset$.
(iii) There exists $\varphi \in \mathrm{OP}_{m_S, n_S}$ *such that* $L_W(f_M, f_N; \varphi) \neq \emptyset$.

Proof. It is clear that $(iii) \Leftrightarrow (ii) \Rightarrow (i)$. Suppose that $t_W(f_M, f_N) = 0$. Then there exists a generic map $F\colon W \to \mathbb{C}$ with an *even* number of cusps and such that $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$. By Proposition 5.2.4 the cusps of F can now all be eliminated in pairs. □

Remark 5.3.3. (i). The arguments of Section 5.3 are still valid for higher-dimensional cobordisms as studied in Chapter 6. In paricular, Proposition 5.3.2 holds for cobordisms of any dimension $n \geq 2$.

(ii). In the spirit of the positive TQFTs of Part I one could define a family of action exponentials $\mathbb{T}$ that count the number of cusps of generic smooth maps $W \to \mathbb{R}^2$ which extend cusp-free boundary conditions. In this context $t_W(f_M, f_N)$ would just be the parity of $\mathbb{T}_W(f_M, f_N)$. Furthermore, it should be possible to relate generic smooth maps $F\colon W \to \mathbb{R}^2$ to fields within a stabilized version of the relative Stiefel-Whitney number TQFT discussed in Chapter 2 as follows. The differential $DF\colon TW \to \mathbb{R}^2$ induces a vector bundle homomorphism $\Phi\colon TW \oplus \mathbb{R} \to \mathbb{R}^2$ that is surjective everywhere except at the cusps of F. As $TW \oplus \mathbb{R}$ is trivial, the one-dimensional bundle $\ker \Phi|_{W \setminus \mathrm{cusps}(F)}$ can be trivialized by a section $W \setminus \mathrm{cusps}(F) \to TW \oplus \mathbb{R}$ of $TW \oplus \mathbb{R} \to W$. One would then have to show that this vector field can be modified in a small neighbourhood around every cusp of F to yield a generic section $W \to TW \oplus \mathbb{R}$ with singular locus $\mathrm{cusps}(F)$.

5.4 The Boundary Turning Invariant

In Section 5.4.1, the boundary turning invariant $\omega(f)$ of a boundary condition $f \in \mathcal{F}(S^1)$ is introduced in Definition 5.4.6 as the degree of the tangential Gauss map of the composition $f \circ \alpha\colon S^1 \to \mathbb{R}^2$ for a so-called f-adapted embedding $\alpha\colon S^1 \to [0,1] \times S^1$ (see Definition 5.4.4). Thus, the resulting map

$$\omega\colon \mathcal{F}(S^1) \to \mathbb{Z}$$

measures the "turning" in the plane of the regular part $([0,1]\times S^1)\setminus S(f)$ of a boundary condition $f\colon [0,1] \times S^1 \to \mathbb{R}^2$. In particular, Example 5.4.11 will show that ω vanishes on boundary conditions that are given by the suspension of a Morse function $S^1 \to \mathbb{R}$. Proposition 5.4.10 studies the transformation behaviour of ω under automorphisms of the cylinder $[0,1] \times S^1$ induced by time inversion (i.e., the automorphism $t \mapsto 1-t$ of $[0,1]$) or automorphisms of S^1.

As shown in Section 5.4.2, the boundary turning invariant can be extended for any closed smooth 1-dimensional manifold P to the *oriented boundary turning invariant*

$$\omega_\sigma\colon \mathcal{F}(P) \to \mathbb{Z},$$

where σ denotes a fixed orientation of P. Composition with the quotient map $\mathbb{Z} \to \mathbb{Z}/2$, $m \mapsto \overline{m}$, then yields the *reduced boundary turning invariant*

$$\overline{\omega}\colon \mathcal{F}(P) \to \mathbb{Z}/2,$$

which does not depend on an orientation of P any more.

Eventually, Section 5.4.3 reveals the importance of ω for Chapter 5, which is based the fact that Proposition 5.4.22 expresses certain seemingly global information of a pre-field $\mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ on the cobordism W purely in terms of the boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$.

Remark 5.4.1. We use the following convention for the induced orientation on the boundary of an oriented surface. If the first vector of an oriented frame points out of the surface at a boundary point, then the second vector defines the orientation of the boundary at that point.

5.4.1 Definition for Boundary Conditions on the Circle

In the following, the circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^D$ is equipped with the orientation determined by the canonical orientation of the unit disc D^2 as a submanifold of $\mathbb{R}^2$. According to Remark 5.4.1 this orientation is pointwise given by the vector field

$$u\colon S^1 \to TS^1 \subset T\mathbb{R}^2, \qquad u(x_1, x_2) = (-x_2, x_1).$$

Moreover, the cylinder $[0,1] \times S^1$ will always be oriented by the product orientation.

Definition 5.4.2. Let $F\colon X \to \mathbb{R}^2$ be a smooth map defined on a 2-dimensional smooth manifold X (possibly with boundary). A smooth curve $\alpha\colon S^1 \to X$ is called F*-regular* if $\alpha_*(u(s)) \notin \ker D_sF$ for all $s \in S^1$, where $\alpha_* := D_s\alpha$. (In particular, any F-regular curve is regular.) If α is F-regular, then the composition $F \circ \alpha$ is a regular closed curve in the plane, so its turning number $\gamma(F \circ \alpha)$ (see [59]) is defined and will be called F-*turning number* of α:

$$\gamma_F(\alpha) := \gamma(F \circ \alpha).$$

Two F-regular curves $\alpha, \beta\colon S^1 \to X$ are called F-*regularly homotopic* if there exists a C^1-homotopy $A\colon [0,1] \times S^1 \to X$ between $A_0 = \alpha$ and $A_1 = \beta$ such that $A_t := A(t, -)\colon S^1 \to X$ is F-regular for all $t \in [0,1]$.

Lemma 5.4.3. *If the F-regular curves $\alpha, \beta\colon S^1 \to X$ are F-regularly homotopic, then their F-turning numbers agree: $\gamma_F(\alpha) = \gamma_F(\beta)$.*

Proof. This is a direct consequence of the well-known fact that two regular closed curves in the plane that are C^1-homotopic through regular closed curves have the same turning number (see [59, Theorem 1, p. 279]). □

Given a boundary condition $f \in \mathcal{F}(S^1)$, the fold locus $S(f) \subset [0,1] \times S^1$ is a disjoint union of embedded intervals with one endpoint contained in $0 \times S^1$ and the other endpoint contained in $1 \times S^1$. We equip each of these intervals with the orientation that points from $0 \times S^1$ to $1 \times S^1$. Since f is a fold map, we have $TJ \oplus \ker Df|_J = T([0,1] \times S^1)|_J$ for any of these intervals J. Hence, the orientations of TJ and of $T([0,1] \times S^1)$ determine a unique orientation of the 1-dimensional vector bundle $\ker Df|_J$ over J. For every J we choose trivializations v_J of TJ and w_J of $\ker Df|_J$ that determine the chosen orientations of these trivial 1-dimensional vector bundles. (Note that v_J and w_J are determined up to multiplication with continuous functions $J \to \mathbb{R}_{>0}$.) By construction, the pair $(v_J(s), w_J(s))$ gives the orientation of the cylinder $[0,1] \times S^1$ at s for all points $s \in J$.

Given an embedding $\alpha\colon S^1 \to [0,1] \times S^1$, let $C_\alpha := \alpha(S^1)$ denote the image curve and let $u_\alpha := \alpha_*(u)$ denote the induced orientation on C_α.

Definition 5.4.4. Let $f \in \mathcal{F}(S^1)$ be a boundary condition. An embedding $\alpha\colon S^1 \to [0,1] \times S^1$ is called f-*adaped* if every interval J of $S(f)$ intersects C_α in a single point $s_J \in J$ (see Figure 5.10), and the pairs of vectors $(u_\alpha(s_J), v_J(s_J))$ and $(u_\alpha(s_J), w_J(s_J))$ both form oriented bases in the tangent space of the cylinder $[0,1] \times S^1$ at s_J.

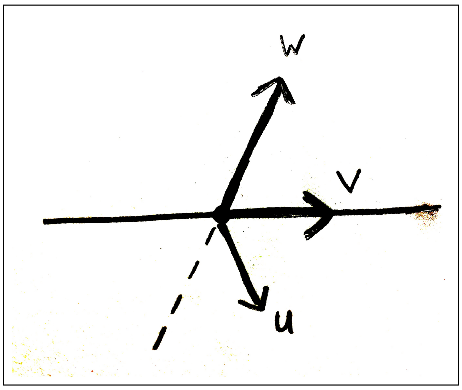

Figure 5.10: Definition of f-adapted embeddings.

Note that any f-adapted embedding α is f-regular because $u_\alpha(s_J)$ and $w_J(s_J)$ are linearly independent for every interval J of $S(f)$.

Proposition 5.4.5. *Let $f \in \mathcal{F}(S^1)$ be a boundary condition. There exists an f-adapted embedding $\alpha\colon S^1 \to [0,1]\times S^1$. Moreover, any two f-adapted embeddings are f-regularly homotopic.*

Proof. We construct an f-adapted embedding $\alpha\colon S^1 \to [0,1]\times S^1$. For every interval J in $S(f)$ we choose a point $s_J \in \operatorname{int} J$. Then, we choose for every J a tangent vector $u_J \in T_{s_J}([0,1]\times S^1)$ such that the pairs of vectors $(u_J, v_J(s_J))$ and $(u_J, w_J(s_J))$ both form oriented bases in $T_{s_J}([0,1]\times S^1)$. If $\delta > 0$ is sufficiently small, then we can choose for every J an embedding $\alpha_J\colon [-\delta,\delta] \to [0,1]\times S^1$ such that $\alpha_J^{-1}(S(f)) = \{0\}$, $\alpha_J(0) = s_J$ and $\alpha_{J*}(\partial_t|_0) = u_J$. Finally, it is not hard to extend the embeddings α_J to the desired embedding α by connecting $\alpha_J(\delta)$ and $\alpha_{J'}(-\delta)$ whenever J and J' are intervals of $S(f)$ that bound the same component of $([0,1]\times S^1)\setminus S(f)$, and u_J points into this component.

Consider two f-adapted embeddings $\alpha,\beta\colon S^1 \to [0,1]\times S^1$. We may assume that $\alpha^{-1}(J) = \beta^{-1}(J)$ for every interval J of $S(f)$. (In fact, this can be achieved by precomposition of β with a suitable orientation preserving diffeomorphism $S^1 \to S^1$. Note that this modification of β is f-regularly homotopic to β.) Now, β is f-regularly homotopic to an f-adapted embedding $\beta_1\colon S^1 \to [0,1]\times S^1$ that changes β only in small pairwise disjoint open ball neighbourhoods of the finitely many points in $\beta^{-1}(S(f))$ in such a way that $\beta_1^{-1}(S(f)) = \alpha^{-1}(S(f))$ and $u_{\beta_1}(C) = u_\alpha(C)$ for every $c \in S(f)\cap C_\alpha$. (Here, one exploits that α and β are both f-adapted. In fact, one may construct a homotopy of β that is at every time an f-adapted embedding.) Furthermore, β_1 is f-regularly homotopic to an embedding β_2 that coincides with α in small pairwise disjoint open ball neighbourhoods of the points in $\beta_2^{-1}(S(f)) = \alpha^{-1}(S(f))$. Finally, β_2 is f-regularly homotopic to α by a regular homotopy of β_2 supported in $S^1 \setminus \alpha^{-1}(S(f))$. $\square$

Lemma 5.4.3 and Proposition 5.4.5 enable us to give the following definition:

Definition 5.4.6. The *boundary turning invariant*

$$\omega\colon \mathcal{F}(S^1) \to \mathbb{Z}$$

is defined on $f \in \mathcal{F}(S^1)$ to be the f-turning number of α for any f-adapted embedding $\alpha\colon S^1 \to [0,1]\times S^1$:

$$\omega(f) := \gamma_f(\alpha).$$

Remark 5.4.7. The definition of $\omega(f)$ is also valid if $S(f) = \emptyset$. In this case an f-adapted curve is just an embedding of S^1 into the cylinder.

Example 5.4.8. Consider the generic smooth map $f\colon W \to \mathbb{R}^2$ (see Figure 5.11) given by the stable Whitney cusp $z = (t,x) \mapsto (t, tx + x^3)$ on the annulus $W = \{z \in \mathbb{R}^2;\ 1/2 \le ||z|| \le 4\}$ seen as a cobordism from $M = \{z \in \mathbb{R}^2;\ ||z|| = 4\}$ to $N = \{z \in \mathbb{R}^2;\ ||z|| = 1/2\}$. We equip W with the orientation that is opposite to the orientation induced by the inclusion $W \subset \mathbb{R}^2$. Hence, fixing a diffeomorphism $\rho\colon [0,1] \to [1/2, 4]$ such that $\rho(0) = 4$, we may identify W with the cylinder $[0,1]\times S^1$ via the orientation preserving diffeomorphism

$$[0,1]\times S^1 \to W, \qquad (t,s) \mapsto \rho(t)\cdot s.$$

Under this identification f can be considered as an element $f \in \mathcal{F}(S^1)$. (Note that $\operatorname{Im}\circ f|_{S(f)}$ is injective, where the imaginary part $\operatorname{Im}\colon \mathbb{C} \to \mathbb{R}$ is identified with the projection $\mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}$ to the second factor.) Let us show that $\omega(f) = \pm 1$. (The fact that $\omega(f)$ is odd will play a key

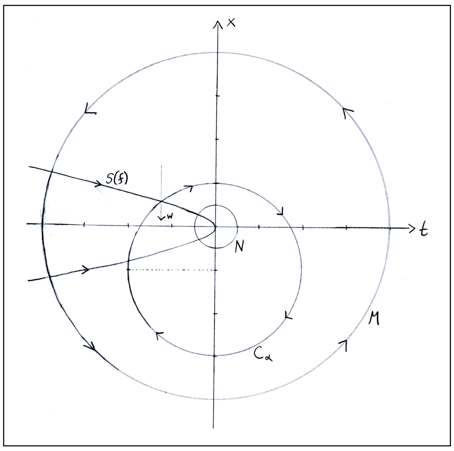

Figure 5.11: Boundary turning invariant around Whitney cusp.

role in the proof of Theorem 5.1.1.) For this purpose, we claim that an f-adapted embedding is given by

$$\alpha\colon S^1 \to W, \qquad \alpha(x_1, x_2) = (-2x_1, 2x_2 - 1).$$

(The image C_α of α is a circle of radius 2 around $(0,-1)$ as shown in Figure 5.11. Note that α induces the clockwise orientation on C_α.) The tangent vector field v on $S(F)$ points in the direction of the origin, and the vector field w that spans $\ker Df|_{S(f)} = 0 \times \mathbb{R}$ has to be chosen to point parallel to the negative x-axis. Thus, it is evident from Figure 5.11 that α is in fact f-adapted. In the following, we will show that the immersion $f \circ \alpha\colon S^1 \to \mathbb{R}^2$ is injective. (Hence, Hopf's Umlaufsatz will imply that $\omega(f) = \gamma(f \circ \alpha) = \pm 1$.) In the parametrization of S^1 given by $r \mapsto (\cos(r), \sin(r))$, $r \in [0, 2\pi)$, α takes the form $\alpha(r) = (-2\cos(r), 2\sin(r) - 1)$. To show that $f \circ \alpha$ is injective, suppose there are $r, r' \in [0, 2\pi)$ such that $(f \circ \alpha)(r) = (f \circ \alpha)(r')$ and $r < r'$. Writing $c^{(\prime)} = \cos(r^{(\prime)})$ and $s^{(\prime)} = \sin(r^{(\prime)})$, this means that

$$(-2c, -2c(2s-1) + (2s-1)^3) = (-2c', -2c'(2s'-1) + (2s'-1)^3).$$

In particular, it follows from $c = c'$ and $r, r' \in [0, 2\pi)$, $r \neq r'$, that $r + r' = 2\pi$. Therefore, $s' = -s$. Hence, by equality of the second components,

$$-2c(2s-1) + (2s-1)^3 = -2c(-2s-1) + (-2s-1)^3.$$

Consequently, $-2c(2s-1) + 2c(-2s-1) = -8cs$ implies that

$$\begin{aligned} -8cs &= (-2s-1)^3 - (2s-1)^3 = -4s((-2s-1)^2 + (-2s-1)(2s-1) + (2s-1)^2) \\ &= -4s(4s^2 + 4s + 1 - 4s^2 + 1 + 4s^2 - 4s + 1) = -4s(4s^2 + 3). \end{aligned}$$

Combining $r < r'$, $r, r' \in [0, 2\pi)$ and $r + r' = 2\pi$, we obtain $r \in (0, \pi)$ and thus, $s \neq 0$. Division by $-4s$ yields $2c = 4s^2 + 3$. This is impossible since $4s^2 + 3 \geq 3 > 2 \geq 2c$.

The boundary turning invariant of $f \in \mathcal{F}(S^1)$ can still be computed from embeddings $\alpha\colon S^1 \to [0,1] \times S^1$ that satisfy the definition of f-adapted embeddings (see Definition 5.4.4) up to a certain sign:

Lemma 5.4.9. *Let $f \in \mathcal{F}(S^1)$ be a boundary condition. Suppose that $\alpha\colon S^1 \to [0,1] \times S^1$ is an embedding such that every interval J of $S(f)$ intersects C_α in a single point $s_J \in J$, and the* *pairs of vectors* $(u_\alpha(s_J), v_J(s_J))$ *and* $(-u_\alpha(s_J), w_J(s_J))$ *both form oriented bases in the tangent space of the cylinder* $[0,1] \times S^1$ *at* s_J *. Then,* $\omega(f) = \gamma_f(\alpha)$ *.*

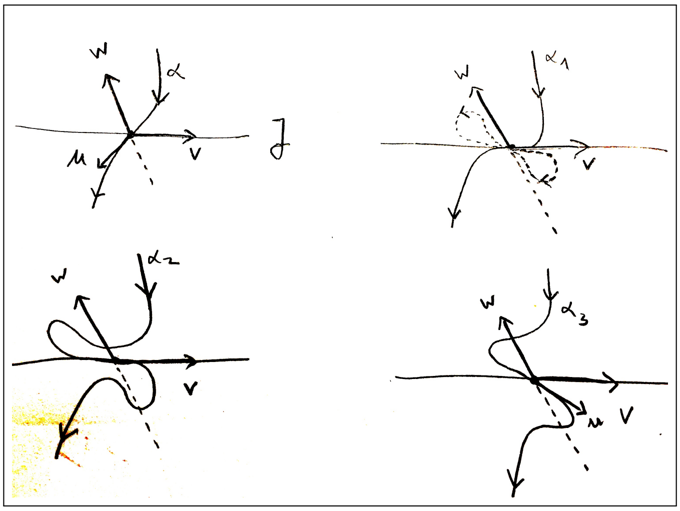

Figure 5.12: Alternative computation of boundary turning number.

Proof. Consider the sequence of local modifications of α near every point $x \in C_\alpha \cap S(f)$ shown in Figure 5.12. It is clear that α is f-regularly homotopic to α_1, $\overline{\alpha_1}$ is f-regularly homotopic to α_2, and α_2 is f-regularly homotopic to α_3. Therefore, it suffices to show that $\gamma_f(\alpha_1) = \gamma_f(\overline{\alpha_1})$. (Indeed, noting that α_3 is f-adapted, Lemma 5.4.3 will then imply that $\omega(f) = \gamma_f(\alpha_3) = \gamma_f(\overline{\alpha_1}) = \gamma_f(\alpha_1) = \gamma_f(\alpha)$.) Note that $\overline{\alpha_1}$ is obtained from α_1 by inserting two loops at every point $x \in C_{\alpha_1} \cap S(f)$. These are symbolized by dashed lines in Figure 5.12. The concatenation of the two loops at $x \in C_{\alpha_1} \cap S(f)$ is an immersion $\beta_x \colon S^1 \to [0,1] \times S^1$. How does this affect the f-turning number γ_f? Using the relative turning number of Chapter 1 one sees that

$$\gamma_f(\overline{\alpha_1}) = \gamma_f(\alpha_1) + \sum_{x \in C_{\alpha_1} \cap S(f)} \gamma_f(\beta_x).$$

Since f is a fold map, it follows that for every $x \in C_{\alpha_1} \cap S(f)$ the value of $\gamma_f(\beta_x)$ is either $+1$ or -1, and both values occur an equal amount of times. (In fact, f restricts on every component of $([0,1] \times S^1) \setminus S(f)$ either to an orientation preserving or an orientation reversing local diffeomorphism, and this orientation is different on any two such components with a common boundary part.) Hence, the sum vanishes. □

Recall from [4, Lemma 9.13, page 76] and [4, Lemma 9.14, page 77] that a diffeomorphism $\phi \colon S^1 \to S^1$ induces a homeomorphism $\phi_{\text{closed}} \colon \mathcal{F}(S^1) \to \mathcal{F}(S^1)$ in the following manner. A boundary condition $f \in \mathcal{F}(S^1)$ is mapped by ϕ_{closed} to $f \circ \overline{\phi}$, where $\overline{\phi}$ denotes the diffeomorphism $[0,1] \times S^1 \to [0,1] \times S^1$ given by $(t,x) \mapsto (t, \phi(x))$. The following lemma clarifies the behaviour of ω under composition with ϕ_{closed} and under composition with the *boundary inversion operator*

$$\iota \colon \mathcal{F}(S^1) \to \mathcal{F}(S^1), \qquad (\iota f)(t,x) = f(1-t, x).$$

Proposition 5.4.10. *The boundary turning invariant* $\omega \colon \mathcal{F}(S^1) \to \mathbb{Z}$ *has the following transformation properties:*

(*i*) *The boundary inversion operator fits into the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{F}(S^1) & \xrightarrow{\ \omega\ } & \mathbb{Z} \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{F}(S^1) & \xrightarrow{\ \omega\ } & \mathbb{Z} \end{array}$$

(ii) Every diffeomorphism $\phi\colon S^1 \to S^1$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(S^1) & \xrightarrow{\ \omega\ } & \mathbb{Z} \\ {\scriptstyle \phi_{\text{closed}}}\downarrow & & \downarrow{\scriptstyle \deg\phi} \\ \mathcal{F}(S^1) & \xrightarrow{\ \omega\ } & \mathbb{Z} \end{array}$$

Proof. Let $f \in \mathcal{F}(S^1)$. If $\alpha\colon S^1 \to [0,1]\times S^1$ is an f-adapted embedding, then $\omega(f) = \gamma_f(\alpha) = \gamma(f\circ\alpha)$. Recall from Definition 5.4.4 that every interval J of $S(f)$ intersects C_α in a single point $s_J \in J$, and when the vector fields v_J and w_J on J are chosen as required, then the pairs of vectors $(u_\alpha(s_J), v_J(s_J))$ and $(u_\alpha(s_J), w_J(s_J))$ both form oriented bases in the tangent space of the cylinder $[0,1]\times S^1$ at s_J.

(i). Let η denote the automorphism of $[0,1]\times S^1$ given by $(t,x)\mapsto(1-t,x)$. Let $g := \iota f = f\circ\eta \in \mathcal{F}(S^1)$, and define the embedding $\beta := \eta^{-1}\circ\alpha\colon S^1 \to [0,1]\times S^1$. It suffices to show that $\omega(g) = \gamma_g(\beta)$ because then,

$$\omega(g) = \gamma_g(\beta) = \gamma(g\circ\beta) = \gamma(f\circ\eta\circ\eta^{-1}\circ\alpha) = \gamma(f\circ\alpha) = \omega(f).$$

It follows from $S(g) = \eta^{-1}(S(f))$ that the components K of $S(g)$ correspond to the components J of $S(f)$ via $K = \eta^{-1}(J)$. Hence, every such component $K = \eta^{-1}(J)$ of $S(g)$ intersects $C_\beta = \eta^{-1}(C_\alpha)$ in the single point $s'_K := \eta^{-1}(s_J)$. Fix a component $K = \eta^{-1}(J)$ of $S(g)$. Set

$$v'_K := -\eta_*^{-1}(v_J), \qquad w'_K := \eta_*^{-1}(w_J).$$

Now v'_K is a trivialization of TK that defines the orientation of K that points from $0\times S^1$ to $1\times S^1$ because $\eta(0\times S^1) = 1\times S^1$ and $\eta(1\times S^1) = 0\times S^1$. Moreover, w'_K is a trivialization of $\ker Dg|_K$ such that $(v'_K, w'_K) = (-\eta_*^{-1}(v_J), \eta_*^{-1}(w_J))$ is an oriented frame of $[0,1]\times S^1$ along K because (v_J, w_J) is an oriented frame of $[0,1]\times S^1$ along J, and η^{-1} reverses orientation. Note that $u_\beta = \beta_*(u) = \eta_*^{-1}(u_\alpha)$. Since the pairs of vectors $(u_\alpha(s_J), v_J(s_J))$ and $(u_\alpha(s_J), w_J(s_J))$ both form oriented bases in the tangent space of the cylinder $[0,1]\times S^1$ at s_J, it follows that

$$(u_\beta(s'_K), v'_K(s'_K)) = (\eta_*^{-1}(u_\alpha(s_J)), -\eta_*^{-1}(v_J(s_J))),$$
$$(u_\beta(s'_K), -w'_K(s'_K)) = (\eta_*^{-1}(u_\alpha(s_J)), -\eta_*^{-1}(w_J(s_J))),$$

both form oriented bases in the tangent space of the cylinder $[0,1]\times S^1$ at s_J. Thus, Lemma 5.4.9 implies that $\omega(g) = \gamma_g(\beta)$.

(ii). Let $g := \phi_{\text{closed}}(f) = f\circ\overline{\phi} \in \mathcal{F}(S^1)$, and define the embedding $\beta := \overline{\phi}^{-1}\circ\alpha\circ\phi\colon S^1 \to [0,1]\times S^1$. It suffices to show that $\omega(g) = \gamma_g(\beta)$ because then,

$$\omega(g) = \gamma_g(\beta) = \gamma(g\circ\beta) = \gamma(f\circ\overline{\phi}\circ\overline{\phi}^{-1}\circ\alpha\circ\phi) = (\deg\phi)\cdot\gamma(f\circ\alpha) = \omega(f).$$

It follows from $S(g) = \overline{\phi}^{-1}(S(f))$ that the components K of $S(g)$ correspond to the components J of $S(f)$ via $K = \overline{\phi}^{-1}(J)$. Hence, every such component $K = \overline{\phi}^{-1}(J)$ of $S(g)$ intersects $C_\beta = \overline{\phi}^{-1}(C_\alpha)$ in the single point $s'_K := \overline{\phi}^{-1}(s_J)$. Fix a component $K = \overline{\phi}^{-1}(J)$ of $S(g)$. Set

$$v'_K := \overline{\phi}_*^{-1}(v_J), \qquad w'_K := (\deg \phi) \cdot \overline{\phi}_*^{-1}(w_J).$$

Now v'_K is a trivialization of TK that defines the orientation of K that points from $0 \times S^1$ to $1 \times S^1$ because $\overline{\phi}(0 \times S^1) = 0 \times S^1$ and $\overline{\phi}(1 \times S^1) = 1 \times S^1$. Moreover, w'_K is a trivialization of $\ker Dg|_K$ such that $(v'_K, w'_K) = (\overline{\phi}_*^{-1}(v_J), (\deg \phi) \cdot \overline{\phi}_*^{-1}(w_J))$ is an oriented frame of $[0,1] \times S^1$ along K because (v_J, w_J) is an oriented frame of $[0,1] \times S^1$ along J, and $\overline{\phi}^{-1}$ reverses orientation if and only if $(\deg \phi) = -1$.

Let $\lambda\colon S^1 \to \mathbb{R} \setminus \{0\}$ be the smooth function such that $\phi_*(u(x)) = \lambda(x) \cdot u(\phi(x))$ for all $x \in S^1$. Note that, for all $x \in S^1$,

$$u_\beta(\beta(x)) = \beta_*(u(x)) = \overline{\phi}_*^{-1} \alpha_*(\lambda(x) \cdot u(\phi(x))) = \lambda(x) \cdot \overline{\phi}_*^{-1}(u_\alpha(\alpha(\phi(x)))).$$

Since the pairs of vectors $(u_\alpha(s_J), v_J(s_J))$ and $(u_\alpha(s_J), w_J(s_J))$ both form oriented bases in the tangent space of the cylinder $[0,1] \times S^1$ at s_J, it follows from $(\deg \phi) \cdot \lambda > 0$ that, for $x := \beta^{-1}(s'_K)$,

$$\begin{aligned} (u_\beta(s'_K), v'_K(s'_K)) &= (\lambda(x) \cdot \overline{\phi}_*^{-1}(u_\alpha(s_J)), \overline{\phi}_*^{-1}(v_J(s_J))), \\ (u_\beta(s'_K), (\deg \phi) \cdot w'_K(s'_K)) &= (\lambda(x) \cdot \overline{\phi}_*^{-1}(u_\alpha(s_J)), \overline{\phi}_*^{-1}(w_J(s_J))), \end{aligned}$$

both form oriented bases in the tangent space of the cylinder $[0,1] \times S^1$ at s_J. Thus, Definition 5.4.4 (for $\deg \phi = 1$) and Lemma 5.4.9 (for $\deg \phi = -1$) imply that $\omega(g) = \gamma_g(\beta)$. □

Example 5.4.11. An important class of elements in $\mathcal{F}(S^1)$ are suspensions of Morse functions. We use the invariance of ω under time inversion from Proposition 5.4.10 (i) to show that the boundary turning invariant of the suspension of a Morse function on S^1 vanishes. In fact, let $\mu\colon S^1 \to \mathbb{R}$ be a Morse function with set of critical points $\Sigma \subset S^1$. Then the suspension $f := \mathrm{id}_{[0,1]} \times \mu\colon [0,1] \times S^1 \to \mathbb{R}^2$, $f(t,x) = (t, \mu(x))$ is a fold map with singular locus $S(f) = [0,1] \times \Sigma$. If μ is excellent (i.e. injective on Σ), then $f \in \mathcal{F}(S^1)$. Note that $\iota(f) = \varphi \circ f$, where the diffeomorphism $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\varphi(x,y) = (1-x, y)$. (Indeed, all points $(t,s) \in [0,1] \times S^1$ satisfy $(\iota(f))(t,s) = f(1-t,s) = (1-t, \mu(s)) = (\varphi \circ f)(t,s)$.) Therefore, if $\alpha\colon S^1 \to [0,1] \times S^1$ is an f-adapted embedding, then α is also an $(\varphi \circ f)$-adapted embedding. (Indeed, this is clear from Definition 5.4.4 because $\ker D(\varphi \circ f) = \ker Df$ and $S(\varphi \circ f) = S(f)$.) All in all, invariance of ω under time inversion implies that $\omega(f) = \omega(\iota f) = \gamma(\iota f \circ \alpha) = \gamma(\varphi \circ f \circ \alpha) = -\gamma(f \circ \alpha) = -\omega(f)$.

5.4.2 Oriented and Reduced Boundary Turning Invariant

Proposition 5.4.10 (ii) enables us to define the boundary turning invariant for boundary conditions on arbitrary closed 1-dimensional manifolds as follows.

Definition 5.4.12. Let P be a closed smooth 1-dimensional manifold and let σ be an orientation of P. For every component Q of P we choose an orientation preserving diffeomophism

$\psi^Q\colon S^1 \to Q$ (where the orientation of Q is given by $\sigma|_Q$) and obtain an induced homeomorphism

$$\psi^Q_{\text{closed}}\colon \mathcal{F}(Q) \to \mathcal{F}(S^1), \qquad \psi^Q_{\text{closed}}(f) = f \circ \overline{\psi^Q},$$

where $\overline{\psi^Q}$ denotes the diffeomorphism $[0,1] \times Q \to [0,1] \times Q$ given by $(t,q) \mapsto (t, \psi^Q(q))$. The *oriented boundary turning invariant*

$$\omega_\sigma\colon \mathcal{F}(P) \to \mathbb{Z}$$

is defined on $f \in \mathcal{F}(P)$ by

$$\omega_\sigma(f) = \sum_Q \omega((\psi^Q_{\text{closed}}(f|_{[0,1]\times Q})),$$

where the (finite) sum runs over all components Q of P. (Note that $f|_{[0,1]\times Q} \in \mathcal{F}(Q)$ for all Q.) Note that Proposition 5.4.10 (ii) ensures that the definition of ω_σ is independent of the choice of ψ^Q.

Remark 5.4.13. Note that if ρ denotes the standard orientation of S^1, then $\omega_\rho(f) = \omega(f)$ for all $f \in \mathcal{F}(S^1)$.

The following is a simple consequence of Definition 5.4.12:

Lemma 5.4.14. *Let P be a closed smooth 1-dimensional manifold and let σ be an orientation of P. Then for all $f \in \mathcal{F}(P)$ we have*

$$\omega_\sigma(f) = \sum_Q \omega_{\sigma|_Q}(f|_{[0,1]\times Q}),$$

where the (finite) sum runs over all components Q of P.

The following observation is a consequence of Definition 5.4.12:

Proposition 5.4.15. *Let P be a closed connected 1-dimensional smooth manifold and let σ be an orientation of P. Then, every diffeomorphism $\phi\colon P \to P$ induces a commutative diagram*

$$\begin{array}{ccc} \mathcal{F}(P) & \xrightarrow{\ \omega_\sigma\ } & \mathbb{Z} \\ {\scriptstyle \phi_{\text{closed}}}\downarrow & & \downarrow{\scriptstyle =} \\ \mathcal{F}(P) & \xrightarrow{\ \omega_{(\deg\phi)\cdot\sigma}\ } & \mathbb{Z} \end{array}$$

Proof. Let $\psi\colon S^1 \to P$ be an orientation preserving diffeomorphism, where P is equipped with the orientation σ. Then, $\phi^{-1} \circ \psi\colon S^1 \to P$ is an orientation preserving diffeomorphism when P is equipped with the orientation $\deg\phi$. Therefore, it follows from Definition 5.4.12 and from $(\phi^{-1} \circ \psi)_{\text{closed}} = \psi_{\text{closed}} \circ \phi^{-1}_{\text{closed}}$ that for every $f \in \mathcal{F}(P)$,

$$\omega_{(\deg\phi)\cdot\sigma}(f) = \omega((\phi^{-1} \circ \psi)_{\text{closed}}(\phi_{\text{closed}}(f))) = \omega(\psi_{\text{closed}}(f)) = \omega_\sigma(f).$$

∎

The following is a direct consequence of Proposition 5.4.10:

Proposition 5.4.16. *Let P be a closed smooth 1-dimensional manifold and let σ be an orientation of P. Then the following statements hold:*

(*i*) *The* boundary inversion operator

$$\iota_P \colon \mathcal{F}(P) \to \mathcal{F}(P), \qquad (\iota f)(t,x) = f(1-t,x),$$

fits into the following commutative diagram:

$$\begin{CD} \mathcal{F}(P) @>{\omega_\sigma}>> \mathbb{Z} \\ @V{\iota_P}VV @VV{=}V \\ \mathcal{F}(P) @>{\omega_\sigma}>> \mathbb{Z} \end{CD}$$

(*ii*) *If P is connected, then every diffeomorphism $\phi\colon P \to P$ induces a commutative diagram*

$$\begin{CD} \mathcal{F}(P) @>{\omega_\sigma}>> \mathbb{Z} \\ @V{\phi_{\text{closed}}}VV @VV{\deg \phi}V \\ \mathcal{F}(P) @>{\omega_\sigma}>> \mathbb{Z} \end{CD}$$

Corollary 5.4.17. *Let P be a closed smooth 1-dimensional manifold and let σ be an orientation of P. Then the following diagram commutes:*

$$\begin{CD} \mathcal{F}(P) @>{\omega_\sigma}>> \mathbb{Z} \\ @V{=}VV @VV{-1}V \\ \mathcal{F}(P) @>{\omega_{-\sigma}}>> \mathbb{Z} \end{CD}$$

Proof. By Lemma 5.4.14 we may assume that P is connected. Then, any diffeomorphism $\phi\colon P \to P$ satisfies

$$\omega_{(\deg\phi)\cdot\sigma} = \omega_{(\deg\phi)\cdot\sigma} \circ \phi_{\text{closed}} \circ \phi^{-1}_{\text{closed}} \overset{5.4.15}{=} \omega_\sigma \circ (\phi^{-1})_{\text{closed}} \overset{5.4.16}{=} (\deg\phi)\cdot\omega_\sigma.$$

The claim follows by choosing ϕ to be orientation reversing. □

Corollary 5.4.17 implies the existence of a $\operatorname{mod} 2$ reduced version of the boundary turning invariant, which will appear in Theorem 5.1.1.

Definition 5.4.18. For a closed 1-dimensional manifold P the *reduced boundary turning invariant*

$$\overline{\omega}\colon \mathcal{F}(P) \to \mathbb{Z}/2$$

is defined on $f \in \mathcal{F}(P)$ by

$$\overline{\omega}(f) := \omega_\sigma(f) \bmod 2,$$

where σ denotes a chosen orientation of P. Note that Corollary 5.4.17 ensures that the definition of $\overline{\omega}$ is independent of the choice of σ.

Reduction modulo 2 of Lemma 5.4.14 implies

Proposition 5.4.19. *Let P be a closed smooth 1-dimensional manifold. Then for all $f \in \mathcal{F}(P)$ we have*

$$\overline{\omega}(f) = \sum_Q \overline{\omega}(f|_{[0,1]\times Q}).$$

5.4.3 Boundary Turning Invariant and Euler Characteristic

The following lemma gives a method to compute the oriented boundary turning invariant on a connected boundary by means of the relative turning number of Chapter 1.

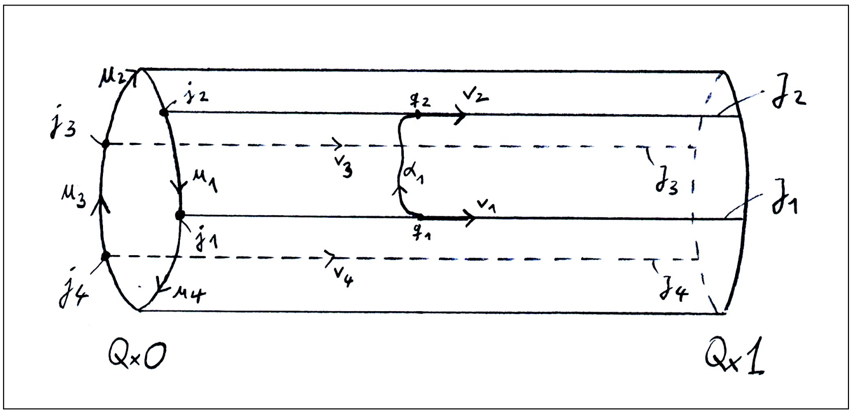

Figure 5.13: Setting of Lemma 5.4.20 for $r = 4$.

Lemma 5.4.20. *Let Q be a closed connected smooth 1-dimensional manifold, and let σ be an orientation of Q. Let $f \in \mathcal{F}(Q)$ be a boundary condition. Let $j_1, \ldots, j_r$ be a labeling of the finite set of points $(Q \times 0) \cap S(f)$ in $Q \times 0 = Q$, and let $u_1, \ldots, u_r$ be a labeling of the components of $Q \setminus \{j_1, \ldots, j_r\}$ such that, for all s (see Figure 5.13),*

- $\partial u_s = \{j_s, j_{s+1}\}$. *(Indices are understood* $\bmod r$*, so $j_{r+1} = j_1$.)*
- *the restriction of the orientation $-\sigma$ to u_s points from j_s to j_{s+1}. (By Remark 5.4.1, $-\sigma$ is the restriction of the product orientation of $[0,1] \times Q$ to $0 \times Q = Q$.)*

For all s, let J_s denote the component of $S(f)$ satisfying $J_s \cap (0 \times Q) = j_s$, and let U_s denote the component of $([0,1]\times Q)\setminus S(f)$ satisfying $U_s \cap (0\times Q) = u_s$. Finally, let v_s be a trivialization of TJ_s of the form $v_s = (\phi_s)_*(\partial_t)$ for some diffeomorphism $\phi_s\colon [0,1] \to J_s$ with $\phi_s(0) = j_s$.

Fix for every s a point $q_s \in \operatorname{int} J_s$, and let $\alpha_s\colon [0,1] \to [0,1] \times Q$ be an embedding such that $\alpha_s(0) = q_s$, $\alpha_s(1) = q_{s+1}$, $\alpha_s(0,1) \subset U_s$ and $(\alpha_s)_*(\partial_t|_0) = -v_s(q_s)$, $(\alpha_s)_*(\partial_t|_1) = v_{s+1}(q_{s+1})$. Then

$$\omega_\sigma(f) = \sum_{s=1}^{r} \gamma(\omega_{f\circ\alpha_s}).$$

Here, $\omega_{f\circ\alpha_s}\colon [0,1] \to S^1$ denotes the rotation map of $f\circ\alpha_s$ from Definition 1.1.2, and $\gamma(\omega) \in \mathbb{R}$ denotes the turning number of a map $\omega\colon [0,1] \to S^1$ from Definition 1.2.3. (Note that γ has another meaning than in the context of Definition 5.4.2.)

The conclusion of the lemma remains true if one replaces the requirement $-(\alpha_s)_*(\partial_t|_0) = v_s(q_s) = (\alpha_s)_*(\partial_t|_1)$ by $(\alpha_s)_*(\partial_t|_0) = v_s(q_s) = -(\alpha_s)_*(\partial_t|_1)$.

Proof. Let $\phi\colon S^1 \to Q$ be an orientation preserving diffeomorphism. Let $\overline{\phi}$ denote the orientation preserving diffeomorphism $[0,1] \times S^1 \to [0,1] \times Q$ given by $(t,x) \mapsto (t, \phi(x))$. By

Definition 5.4.12 we have $\omega_\sigma(f) = \omega(\phi_{\text{closed}}(f)) = \omega(f') \overset{5.4.13}{=} \omega_{\sigma'}(f')$, where we have defined $f' := f \circ \overline{\phi} = \phi_{\text{closed}}(f) \in \mathcal{F}(S^1)$, and σ' denotes the standard orientation of S^1. Moreover, setting $j'_s := \overline{\phi}^{-1}(j_s)$, $u'_s := \overline{\phi}^{-1}(u_s)$, $J'_s := \overline{\phi}^{-1}(J_s)$, $U'_s := \overline{\phi}^{-1}(U_s)$, $v'_s := \overline{\phi}_*^{-1}(v_s)$, $q'_s := \overline{\phi}^{-1}(q_s)$, $\alpha'_s := \overline{\phi}^{-1} \circ \alpha_s$, we have $q'_s \in \operatorname{int} J'_s$, and $\alpha'_s\colon [0,1] \to [0,1] \times S^1$ is an embedding such that $\alpha'_s(0) = q'_s$, $\alpha'_s(1) = q'_{s+1}$, $\alpha'_s(0,1) \subset U'_s$ and $(\alpha'_s)_*(\partial_t|_0) = -v'_s(q'_s)$, $(\alpha'_s)_*(\partial_t|_1) = v'_{s+1}(q'_{s+1})$. Hence, it suffices to assume that $Q = S^1$ and $\sigma = \sigma'$.

We use the embeddings $\alpha_s\colon [0,1] \to [0,1] \times S^1$ to construct an f-adapted embedding $\alpha\colon S^1 \to [0,1] \times S^1$ such that $(\omega_\sigma(f) =)\ \gamma_f(\alpha) = \sum_{s=1}^r \gamma(\omega_{f\circ\alpha_s})$.

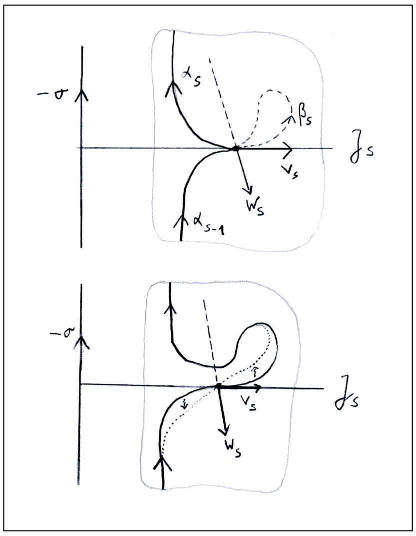

Figure 5.14: Local modification around q_s.

For every s choose an immersion $\beta_s\colon [0,1] \to (0,1) \times S^1$ as shown in Figure 5.14. In particular, $\beta_s(0) = \beta_s(1) = q_s$, $\beta_s(0,1) \subset U_s$ and $(\beta_s)_*(\partial_t|_0) = u_s(q_s) = -(\beta_s)_*(\partial_t|_1)$. Furthermore, by working in a sufficiently small neighbourhood of q_s, we may suppose that $f \circ \beta_s|_{(0,1)}$ is injective.

An application of axiom $(TGLUE)$ of Theorem 1.2.4 to the C^1 concatenation $\tilde{\alpha} := \beta_1 * \alpha_1 * \cdots * \beta_r * \alpha_r$ yields

$$\gamma_f(\tilde{\alpha}) = \gamma(\omega_{f\circ\tilde{\alpha}}) = \sum_{s=1}^r \gamma(\omega_{f\circ\alpha_s}) + \sum_{s=1}^r \gamma(\omega_{f\circ\beta_s}).$$

Note that $\gamma(f \circ \beta s) = \pm(-1)^s/2 = -\gamma(f \circ \beta_{s+1})$ for all s because $f \circ \beta_s|_{(0,1)}$ is injective by construction. Furthermore, r is even. (In fact, crossing a fold line of f in $[0,1] \times S^1$ alternates the property of $f|_{([0,1]\times S^1)\setminus S(f)}$ of being an orientation preserving or orientation reversing local diffeomorphism.) Consequently, $\sum_{s=1}^r \gamma(\omega_{f\circ\beta_s}) = 0$.

Note that $\tilde{\alpha}$ is f-regularly homotopic to an f-adapted embedding α (see the dashed line in Figure 5.14). Therefore, $\gamma_f(\tilde{\alpha}) = \gamma_f(\alpha)$ by Lemma 5.4.3.

Finally, if one replaces the requirement $-(\alpha_s)_*(\partial_t|_0) = v_s(q_s) = (\alpha_s)_*(\partial_t|_1)$ by $(\alpha_s)_*(\partial_t|_0) = v_s(q_s) = -(\alpha_s)_*(\partial_t|_1)$, then the proof is analogous, but one requires $-(\beta_s)_*(\partial_t|_0) = u_s(q_s) = (\beta_s)_*(\partial_t|_1)$ for the choice of β_s, such that the equality $\omega_\sigma(f) = \gamma_f(\alpha)$ will hold by Lemma 5.4.9. (The embedding α will not be f-adapted.) ☐

Definition 5.4.21. Suppose that W is orientable. Given an orientation σ of W and a fold pre-field $F \in \mathcal{F}^{\text{pre}}(W; f_M, f_N)$, let W_F^σ be the closure in W of the union of the components V of $W \setminus S(F)$ for which the local diffeomorphism $F|_V$ is orientation preserving ($\mathbb{R}^2$ is equipped with the standard orientation).

Note that W_F^σ is a 2-dimensional smooth manifold with corners. (The corner points are $S(F) \cap \partial W$.) Moreover, we have $W = W_F^\sigma \cup W_F^{-\sigma}$. In addition, $\partial W_F^\sigma \cap \partial W_F^{-\sigma} = S(F) = W_F^\sigma \cap W_F^{-\sigma}$. Furthermore, note that every component of $S(F)$ is contained in the boundary of exactly two different components of $W \setminus S(F)$ as W is orientable. One of these components belongs to W_F^σ and the other component belongs to $W_F^{-\sigma}$.

Proposition 5.4.22. *Assume that W is orientable, and let σ be an orientation of W. Let $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ be a fold pre-field for the boundary conditions $f_M \in \mathcal{F}(M)$ and $f_N \in \mathcal{F}(N)$. Then*

$$\chi(W_F^\sigma) - \chi(W_F^{-\sigma}) = -\omega_{\sigma|_M}(f_M) - \omega_{\sigma|_N}(f_N).$$

Proof. Since $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ and W is connected, there exist $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$ such that $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$. Hence, there exist diffeomorphisms $\xi\colon [0,1] \to [0,\varepsilon]$ and $\zeta\colon [0,1] \to [1-\varepsilon', 1]$ with $\xi(0) = 0$ and $\zeta(0) = 1-\varepsilon'$, and such that $f_M = F \circ (\xi \times \mathrm{id}_M)$ and $f_N = F \circ (\zeta \times \mathrm{id}_M)$.

Fix a component V of $W \setminus S(F)$, and let $\overline{V}$ denote the closure of V in W. If $V \cap \partial W = \emptyset$, then $\overline{V}$ is a submanifold of W with boundary, and we set $\tilde{V} := \overline{V}$. If $V \cap \partial W \neq \emptyset$, then $\overline{V}$ is a submanifold of W with corners. In this case, let $\tilde{V}$ denote a fixed smooth manifold with boundary obtained by cutting off the corners of $\overline{V}$ (which are all contained in $S(F) \cap \partial W$) along a smooth curve in $\overline{V}$ in such a way that $\overline{V} \setminus \tilde{V} \subset ([0,\varepsilon) \times M) \sqcup ((1-\varepsilon', 1] \times N)$ and $\overline{V} \simeq \tilde{V}$. In any case, we equip $\tilde{V}$ with the orientation $\sigma|_{\tilde{V}}$. Thus, $\partial\tilde{V}$ is a 1-dimensional closed manifold that inherits an orientation as the boundary of $\tilde{V}$ according to Remark 5.4.1. Note that the fold map F restricts to an immersion

$$\beta_V := F|_{\partial\tilde{V}}\colon \partial\tilde{V} \to \mathbb{R}^2,$$

which in turn induces a field $(\beta_V, \omega_{\beta_V})$ on $\tilde{V}$ (see Theorem 1.1.7) within the turning number TFT of Chapter 1. (Here, $\omega_{\beta_V}\colon \tilde{V} \to S^1$ denotes the rotation map of β_V from Definition 1.1.2.) Since $\overline{V} \simeq \tilde{V}$ and $F|_V$ is an immersion, a theorem by Haefliger [18] implies that

$$\chi(\overline{V}) = \chi(\tilde{V}) = \begin{cases} \gamma(\omega_{\beta_V}), & \text{if } \overline{V} \text{ is a component of } W_F^\sigma, \\ -\gamma(\omega_{\beta_V}), & \text{if } \overline{V} \text{ is a component of } W_F^{-\sigma}, \end{cases}$$

where $\gamma(\omega) \in \mathbb{R}$ denotes the turning number of a map $\omega\colon \tilde{V} \to S^1$ from Definition 1.2.3, which is by definition just the sum of the F-turning numbers $\gamma_F(\phi_Q)$ (see Definition 5.4.2) of orientation preserving parametrizations $\phi_Q\colon S^1 \to Q$ of the components Q of $\tilde{V}$. (To be technically proper, consider a submanifold with boundary $\hat{V}$ of $\tilde{V}$ such that $\tilde{V} \setminus \operatorname{int}\hat{V}$ is a closed collar neighbourhood of $\partial\tilde{V}$, and let $\hat{\beta}_V := F|_{\partial\hat{V}}$. Note that $\tilde{V} \simeq \hat{V}$ and $\partial\hat{V} \cong \partial\tilde{V}$. Since F is an immersion on $\hat{V}$, the claim follows for $\hat{\beta}_V$ instead of β_V. But $\gamma(\omega_{\hat{\beta}_V}) = \gamma(\omega_{\beta_V})$ by Lemma 5.4.3 because $\hat{\beta}_V$ is regularly homotopic to β_V.) Furthermore, application of axiom $(TGLUE)$ of Theorem 1.2.4 to the gluing

$$\partial\tilde{V} = (\partial\tilde{V} \cap W_{[0,\varepsilon]}) \cup_{\partial\tilde{V}\cap W_\varepsilon} (\partial\tilde{V} \cap W_{[\varepsilon,1-\varepsilon']}) \cup_{\partial\tilde{V}\cap W_{1-\varepsilon'}} (\partial\tilde{V} \cap W_{[1-\varepsilon',1]})$$

yields

$$\gamma(\omega_{\beta_V}) = \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[0,\varepsilon]}}}) + \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[\varepsilon,1-\varepsilon']}}}) + \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[1-\varepsilon',1]}}}).$$

Hence, varying over all components V of $W \setminus S(F)$, one obtains

$$\chi(W_F^{\sigma})-\chi(W_F^{-\sigma}) = \sum_V \gamma(\omega_{\beta_V}) = \sum_V \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[0,\varepsilon]}}})+\sum_V \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[\varepsilon,1-\varepsilon']}}})+\sum_V \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap W_{[1-\varepsilon',1]}}}).$$

The second sum vanishes. (Indeed, axiom $(TDISJ)$ of Theorem 1.2.4 allows to rewrite the sum as $\gamma(\omega_{\beta_V|_{\partial W_F^{\sigma}\cap W_{[\varepsilon,1-\varepsilon']}}}) + \gamma(\omega_{\beta_V|_{\partial W_F^{-\sigma}\cap W_{[\varepsilon,1-\varepsilon']}}})$ by noting that $\partial\tilde{V} \cap W_{[\varepsilon,1-\varepsilon']} = \partial\overline{V} \cap W_{[\varepsilon,1-\varepsilon']}$ and by splitting the original sum up into two sums according to whether $\overline{V}$ is a component of W_F^{σ} or of $W_F^{-\sigma}$. The resulting two summands cancel each other because $\partial W_F^{\sigma} \cap W_{[\varepsilon,1-\varepsilon']}$ and $\partial W_F^{-\sigma} \cap W_{[\varepsilon,1-\varepsilon']}$ are the same cobordism equipped with opposite orientations.)

The first sum is equal to $\omega_{-\sigma|_M}(f_M) \overset{5.4.17}{=} -\omega_{\sigma|_M}(f_M)$. (In fact, by Lemma 5.4.14 and by axiom $(TDISJ)$ of Theorem 1.2.4, it suffices to show that, for every component Q of M,

$$\sum_V \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap([0,\varepsilon]\times Q)}}) = \omega_{-\sigma|_Q}(f_M|_{[0,1]\times Q}).$$

Fix a component U of $([0,\varepsilon]\times Q)\setminus S(F)$ and let V_U denote the component of $W\setminus S(F)$ that contains U. By construction of $\tilde{V}_U$ there exists an embedding $\alpha_U\colon [0,1]\to[0,1]\times Q$ such that $\tilde{\alpha}_U := (\xi\times \mathrm{id}_Q)\circ\alpha_U$ is an orientation preserving parametrization of $\partial\tilde{V}_U\cap\overline{U}$. Then,

$$\gamma(\omega_{\beta_{V_U}|_{\partial\tilde{V}_U\cap\overline{U}}}) \overset{1.2.3}{=} \gamma(\omega_{\beta_{V_U}\circ\tilde{\alpha}_U}) = \gamma(\omega_{\beta_{V_U}\circ\tilde{\alpha}_U}) = \gamma(\omega_{f_M\circ\alpha_U})$$

Therefore, the claim follows from Lemma 5.4.20. (Note that the curves α_U have to be moved by an f_M-adapted homotopy to produce the desired curves α_s.))

The third sum is equal to $\omega_{-\sigma|_N}(\iota_N f_N) \overset{5.4.17,\ 5.4.16}{=} -\omega_{\sigma|_N}(f_N)$. (In fact, by Lemma 5.4.14 and by axiom $(TDISJ)$ of Theorem 1.2.4, it suffices to show that, for every component Q of N,

$$\sum_V \gamma(\omega_{\beta_V|_{\partial\tilde{V}\cap([1-\varepsilon',1]\times Q)}}) = \omega_{-\sigma|_Q}(\iota_Q f_N|_{[0,1]\times Q}).$$

Fix a component U of $([1-\varepsilon',1]\times Q)\setminus S(F)$ and let V_U denote the component of $W\setminus S(F)$ that contains U. By construction of $\tilde{V}_U$ there exists an embedding $\alpha_U\colon[0,1]\to[0,1]\times Q$ such that $\tilde{\alpha}_U := (\zeta\times\mathrm{id}_Q)\circ\eta\circ\alpha_U$ is an orientation preserving parametrization of $\partial\tilde{V}_U\cap\overline{U}$, where the automorphism $\eta\colon[0,1]\times Q\to[0,1]\times Q$ is given by $(t,x)\mapsto(1-t,x)$. Then,

$$\gamma(\omega_{\beta_{V_U}|_{\partial\tilde{V}_U\cap\overline{U}}}) \overset{1.2.3}{=} \gamma(\omega_{\beta_{V_U}\circ\tilde{\alpha}_U}) = \gamma(\omega_{\beta_{V_U}\circ\tilde{\alpha}_U}) = \gamma(\omega_{f_N\circ\alpha_U})$$

Therefore, the claim follows from Lemma 5.4.20. (Note that the curves α_U have to be moved by an f_M-adapted homotopy to produce the desired curves α_s.)) □

Note that $\chi(W_F^{\sigma})+\chi(W_F^{-\sigma}) = \chi(W)+k_S$ holds in the situation of Proposition 5.4.22. Combined with the difference $\chi(W_F^{\sigma})-\chi(W_F^{-\sigma}) = -\omega_{\sigma|_M}(f_M)-\omega_{\sigma|_N}(f_N)$, one obtains the following

Corollary 5.4.23. *Assume that W is orientable. Let σ be an orientation of W. Let $F\in\mathcal{F}^{\mathrm{pre}}(W;f_M,f_N)$ be a fold pre-field for the boundary condition $(f_M,f_N)\in\mathcal{F}(M)\times\mathcal{F}(N)$. Then,* *the integer* $\chi(W_F^{\pm\sigma})$ *is given by the formula*

$$\chi(W_F^{\pm\sigma}) = \chi^{\pm\sigma} := \frac{\chi(W) + k_S \mp \omega_{\sigma|_M}(f_M) \mp \omega_{\sigma|_N}(f_N)}{2},$$

and is in particular independent of F.

5.5 Proof of Theorem 5.1.1

The proof of Theorem 5.1.1 consists of showing the following two statements:

(*i*) If $\bar{t}_W(f_M, f_N) = \bar{0}$, then $\overline{\chi(W)} + \overline{k_S} + \overline{\omega}(f_M) + \overline{\omega}(f_N) = \bar{0}$.
(*ii*) If $\bar{t}_W(f_M, f_N) = \bar{1}$, then $\overline{\chi(W)} + \overline{k_S} + \overline{\omega}(f_M) + \overline{\omega}(f_N) = \bar{1}$.

We start with the proof of statement (i), which will then be used in the proof of statement (ii). Suppose that $t_W(f_M, f_N)$ is even. By Proposition 5.3.2 there exists a fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$. If W is orientable and σ denotes an orientation of W, then Corollary 5.4.23 implies that $\chi(W)+k_S-\omega_{\sigma|_M}(f_M)-\omega_{\sigma|_N}(f_N) = 2\chi^\sigma$ is even. If, however, W is not orientable, then there exists an integer $h \geq 1$ and embedded loops $C_1, \ldots, C_h \subset \operatorname{int} W$ whose normal bundles $\nu(C_i) \cong U_i \subset W$ are Möbius bundles, and such that $V := W \setminus \bigsqcup_{i=1}^h U_i$ is orientable. After some perturbation of C_i and some modification of F we may assume that $U_i \cap S(F) = C_i$ for every $i \in \{1, \ldots, h\}$. (In fact, C_i can be slightly moved in $\operatorname{int} W$ to achieve that $C_i \pitchfork S(F)$. Then one modifies F on U_i as in the first part of Figure 5.9.) For every i set $P_i := \partial U_i (\cong S^1)$. Furthermore, fix a suitably small collar neighbourhood $[0,1] \times P_i$ of $0 \times P_i = P_i \subset V$ such that $([0,1] \times P_i) \cap S(F) = \emptyset$. Thus, $f_{P_i} := F|_{[0,1]\times P_i} \in \mathcal{F}(P_i)$ with $\mathbb{S}(f_{P_i}) = \mathrm{id}_{[0]}$. If one considers V as a cobordism from $M \sqcup P_1 \sqcup \cdots \sqcup P_h$ to N, then F restricts on V to an element $G := F|_V \in \mathcal{F}^{\mathrm{pre}}(V; f_{M\sqcup P_1 \sqcup\cdots\sqcup P_h}, f_N)$, where $f_{M\sqcup P_1\sqcup\cdots\sqcup P_h} := f_M \sqcup f_{P_1} \sqcup \cdots \sqcup f_{P_h} \in \mathcal{F}(M \sqcup P_1 \sqcup \cdots \sqcup P_h)$. Since V is orientable, we have already shown that

$$\bar{0} = \bar{t}_V(f_{M\sqcup P_1\sqcup\cdots\sqcup P_h}, f_N) = \overline{\chi(V)} + \overline{k_S} + \overline{\omega}(f_{M\sqcup P_1\sqcup\cdots\sqcup P_h}) + \overline{\omega}(f_N).$$

(Note that $\mathbb{S}(f_{P_i}) = \mathrm{id}_{[0]}$ for all i implies that m_S and n_S are still the number of components of $S(f_{M\sqcup P_1\sqcup\cdots\sqcup P_h})$ and $S(f_N)$, so that $k_S = (m_S + n_S)/2$ is still the number of intervals of $S(G)$.)

Since for all i, $P_i \cong S^1$ and $\overline{U_i}^W \simeq S^1$ is a Möbius band, we obtain

$$\chi(W) = \chi(V) + \chi(\bigsqcup_{i=1}^h \overline{U_i}^W) - \chi(\bigsqcup_{i=1}^h P_i) = \chi(V).$$

Note that $\overline{\omega}(f_{M\sqcup P_1\sqcup\cdots\sqcup P_h}) = \overline{\omega}(f_M) + \overline{\omega}(f_{P_1}) + \cdots + \overline{\omega}(f_{P_h})$ by Proposition 5.4.19. Finally, note that $\overline{\omega}(f_{P_i}) = \bar{0}$ for all i. This completes the proof of statement (i).

In order to show statement (ii), we assume that $t_W(f_M, f_N)$ is odd. Hence, any generic smooth map $F\colon W \to \mathbb{R}^2$ with $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$ has at least one cusp, say $c \in S(F)$. Let $U \subset \operatorname{int} W$ be a small Euclidean open ball with origin c on which F looks in local coordinates like the stable Whitney cusp. Consider the cobordism $V := W \setminus U$. Set $P := \partial U (\cong S^1)$. Fix a suitably small collar neighbourhood $[0,1] \times P$ of $0 \times P = P \subset V$ such that $f_P := F|_{[0,1]\times P} \in \mathcal{F}(P)$ with $\mathbb{S}(f_P) = \mathrm{id}_{[2]}$. If one considers V as a cobordism from $M \sqcup P$ to N, then F restricts on V to an element $G := F|_V \in \mathcal{F}^{\mathrm{pre}}(V; f_{M\sqcup P}, f_N)$, where $f_{M\sqcup P} := f_M \sqcup f_P \in \mathcal{F}(M \sqcup P)$. As G has an even number of cusps, we have already shown in statement (i) that

$$\bar{0} = \bar{t}_V(f_{M\sqcup P}, f_N) = \overline{\chi(V)} + \overline{k_S + 1} + \overline{\omega}(f_{M\sqcup P}) + \overline{\omega}(f_N).$$

(Note that $\mathbb{S}(f_P) = \mathrm{id}_{[2]}$ implies that the number of intervals in $S(G)$ is by 1 bigger than the number k_S of intervals in $S(F)$.) Now, $\chi(W) = \chi(V) + \chi(\overline{U}^W) - \chi(P) = \chi(V) + 1$ implies that $\overline{\chi(V)} = \chi(W) + \overline{1}$. Furthermore, $\overline{\omega}(f_{M \sqcup P}) = \overline{\omega}(f_M) + \overline{\omega}(f_P) = \overline{\omega}(f_M) + \overline{1}$ by Proposition 5.4.19 and Example 5.4.8. All in all, statement (ii) follows.

This completes the proof of Theorem 5.1.1.

5.6 Admissible Open Brauer Morphisms

Recall that $M[r]$ ($r \in \mathbb{N}$) denotes the set $[r] := \{0, \dots, r\}$ seen as a 0-dimensional submanifold $\{0, \dots, r\} \subset \mathbb{R}$. Throughout this section we fix non-negative integers $p, q \in \mathbb{N}$ and an injective map

$$\alpha\colon M[p] \sqcup M[q] \to \partial W$$

such that $P := \alpha(M[p]) \subset M$ and $Q := \alpha(M[q]) \subset N$.

Remark 5.6.1. Any given pair of boundary conditions $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ gives rise to such a map α for $p := m_S$ and $q := n_S$. In fact, recall that f_M induces a canonical identification between the points of $S(f_M) \cap (0 \times M)$ and the points of $[m_S]$. Analogously, f_N induces a canonical identification between the points of $S(f_N) \cap (1 \times N)$ and the points of $[n_S]$. Identifying $0 \times M$ with $M \subset \partial W$ and $1 \times N$ with $N \subset \partial W$, we obtain canonical embeddings $M[m_S] \subset M$ and $M[n_S] \subset N$ which can be combined to the desired map $\alpha\colon M[m_S] \sqcup M[n_S] \to \partial W$.

Definition 5.6.2. An *arc* of an open Brauer morphism $\varphi\colon [r] \to [s]$ is a subset $\{x, y\} \subset M[r] \sqcup M[s]$, $x \neq y$, such that the corresponding points in $(0 \times M[r] \times 0 \times 0) \sqcup (1 \times M[s] \times 0 \times 0)$ are connected by an arc of some (and hence, any) representative $V \subset [0,1] \times \mathbb{R}^3$ of φ. Note that the arcs of φ form a partition of $M[r] \sqcup M[s]$ by $k(\varphi) := (r+s)/2$ subsets.

Definition 5.6.3. Given an open Brauer morphism $\varphi \in \mathrm{OP}_{p,q}$, a φ*-model* is a pair (X, ξ) consisting of a compact smooth 2-manifold X with boundary, and a diffeomorphism $\xi\colon \partial W \xrightarrow{\cong} \partial X$ with $(\xi \circ \alpha)^{-1}(\partial Y) \neq \emptyset$ for all components Y of X such that there exist smooth embeddings

$$\gamma_i\colon [0,1] \to X, \qquad i = 1, \dots, k(\varphi),$$

with the following properties:

(*i*) $\gamma_i([0,1]) \cap \gamma_j([0,1]) = \emptyset$ for all $i \neq j$.
(*ii*) $\gamma_i \pitchfork \partial X$ for all i. (In particular, $\gamma_i((0,1)) \cap \partial X = \emptyset$ for all i.)
(*iii*) For every arc $\{x, y\}$ of φ there exists an index i such that

$$\gamma_i(\{0,1\}) = \{\xi(\alpha(x)), \xi(\alpha(y))\}.$$

(In particular, $\gamma_i(\{0,1\}) \subset \xi(P \sqcup Q)$ for all i.)

Two φ-models (X, ξ) and (X', ξ') are *equivalent* if there exists a diffeomorphism $\Xi\colon X \xrightarrow{\cong} X'$ such that $\xi' = \Xi|_{\partial X} \circ \xi$. Let $\mathrm{Mod}(\varphi)$ denote the set of all equivalence classes of φ-models.

Remark 5.6.4. Let us check that $\mathrm{Mod}(\varphi)$ is indeed a set. Recall that the diffeomorphism classes of compact smooth 2-manifolds with boundary form a countable set. Let $\{X_i\}_{i \in \mathbb{N}}$ be a set of representatives. Now given any φ-model (X, ξ), there exists $i \in \mathbb{N}$ and a diffeomorphism $\Xi_i\colon X \xrightarrow{\cong} X_i$. Setting $\xi_i := \Xi_i|_{\partial X} \circ \xi$, it is now clear that (X_i, ξ_i) is also a φ-model which is by construction equivalent to (X, ξ). Thus, every equivalence class of φ-models has a representative in the set $\{(X', \xi') | X' \in \{X_i\}_{i \in \mathbb{N}},\ \xi'\colon \partial W \xrightarrow{\cong} \partial X' \text{ is a diffeomorphism}\}$.

Definition 5.6.5. An open Brauer morphism $\varphi \in \mathrm{OP}_{p,q}$ is called *admissible* if $(W, \mathrm{id}_{\partial W})$ is a φ-model. Note that $L_W(f_M, f_N; \varphi) \neq \emptyset$ implies that φ is admissible, where α is chosen as in Remark 5.6.1.

5.7 Proof of Theorem 5.1.3

If $L_W(f_M, f_N; \varphi) \neq \emptyset$ for some $\varphi \in \mathrm{OP}_{m_S,n_S}$, then obviously $t_W(f_M, f_N) = \overline{0}$, and φ is index-preserving with respect to (f_M, f_N) (see Definition 5.1.2) and admissible with respect to $\alpha(f_M, f_N)$ (see Remark 5.6.1). Refining Proposition 5.3.2, we prove the remaining implication $(ii) \Rightarrow (i)$ of Theorem 5.1.3. Suppose that statement (ii) holds. We have to construct a fold map $F\colon W \to \mathbb{R}^2$ such that

- $F|_{[0,\varepsilon]\times M} \approx f_M$ and $F|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$.
- $\mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l}$ for some $l \in \mathbb{N}$.

This can be achieved via the following three steps:

Step 1. Choose a generic map $F_0\colon W \to \mathbb{R}^2$ such that $F_0|_{[0,\varepsilon]\times M} \approx f_M$ and $F_0|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$.

Step 2. As φ is index-preserving and admissible, we can modify F_0 locally on $W_{(\varepsilon,1-\varepsilon')}$ via (E) and (C) to produce a generic map $F_1\colon W \to \mathbb{R}^2$ such that the components of $S(F_1)$ that are intervals do not contain any cusps, and $\mathbb{S}(F_1) = \varphi \otimes \lambda^{\otimes l'}$ for some $l' \in \mathbb{N}$.

Step 3. Since $t_W(f_M, f_N) = \overline{0}$, we can use (E) and (C) to modify F_1 locally on $W_{(\varepsilon,1-\varepsilon')}$ to produce the desired fold map $F\colon W \to \mathbb{R}^2$.

Step 1 is clear by Proposition 5.3.1. In order to produce the desired fold map F, we have to eliminate the cusps of F_0 in the remaining two steps more carefully than in the proof of Proposition 5.3.2 because the open Brauer morphism φ must be realized by the singular locus of F. In particular, the modification of F_1 in the third step must be careful enough to keep the properties of F_1. Also note that the resulting fold map F will satisfy the correct boundary conditions since F_0 does, and the local modifications (E) and (C) are supported on $W_{(\varepsilon,1-\varepsilon')}$.

We are left with the discussion of the remaining two steps:

Step 2. For every point $p \in S(F_0) \cap \partial W = M[m_S] \sqcup M[n_S]$ we choose a fold point p' in the same component of $S(F_0)$ as p such that $p' \in W \setminus (([0,\varepsilon] \times M) \sqcup ([1-\varepsilon', 1] \times N))$, and such that there are no cusps on the fold line segment between p and p'. Since φ is admissible for (W, f_M, f_N), there are embeddings $\gamma_i\colon [0,1] \to W_{(\varepsilon,1-\varepsilon')}$ for $i = 1, \ldots, k_S$ such that

- For every index i, we have $\{\gamma_i(0), \gamma_i(1)\} = \{p_i', q_i'\}$ for some points $p_i, q_i \in M[m_S] \sqcup M[n_S]$.
- Given two points $p, q \in M[m_S] \sqcup M[n_S]$, there exists an index i such that p' and q' are the endpoints of $\gamma_i([0,1])$ if and only if $\{p,q\}$ is an arc of φ (see Definition 5.6.2).
- Each γ_i is transverse to $S(F_0)$ and does not contain cusps of $S(F_0)$.
- $\gamma_i([0,1]) \cap \gamma_j([0,1]) = \emptyset$ for all $i \neq j$.

Choose a small open tubular neighbourhood $U_i \subset W_{(\varepsilon,1-\varepsilon')}$ around γ_i for every i such that $U_i \cap U_j = \emptyset$ for all $i \neq j$. We may also assume that $S(F_0) \cap U_i$ does not contain any cusp of F_0 for every i.

Fix an index i and set $\gamma := \gamma_i$, $(p,q) := (p_i, q_i)$ and $U := U_i$. The intersection $U \cap S(F_0)$ then looks as in Figure 5.15. As indicated in Figure 5.15, we insert via (C) on every component of $S(F_0) \cap U$ a pair of cusps in such a way that elimination of cusps on neighbouring components via (E) produces a fold line that connects p with q. This is possible since φ is index-preserving by assumption, and the non-reduced index of the fold points of an oriented component of $S(F_0)$

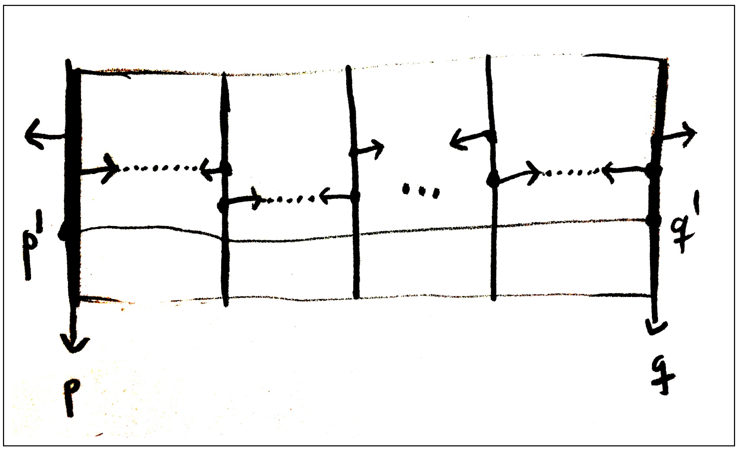

Figure 5.15: Realizing an arc between p and q.

is constant even in the presence of cusps by Example 4.5.2.

Repeating the above modifiactions of F_0 on U_i for all i, we obtain a generic map $F_1 \colon W \to \mathbb{C}$ such that for any two points $p, q \in M[m_S] \sqcup M[n_S]$ the following property is satisfied: p and q are connected by a component of $S(F_1)$ without cusps if and only if p and q are connected by φ. Consequently, all components of $S(F_1)$ that are intervals do not contain cusps. It remains to eliminate the remaining cusps of F_1 in the last step without losing this property.

STEP 3. Since $t_W(f_M, f_N) = \overline{0}$, the generic map $F_1 \colon W \to \mathbb{R}^2$ with $F_1|_{[0,\varepsilon]\times M} \approx f_M$ and $F_1|_{[1-\varepsilon',1]\times N} \approx f_N$ for some $\varepsilon, \varepsilon' \in (0, \varepsilon_W)$ has an *even* number of cusps. In consequence of the previous step, all cusps of F_1 lie on the closed components of $S(F_1)$. Let $T \subset S(F_1)$ denote the union of the components of $S(F_1)$ that are intervals. On each component of $W \setminus T$ one can eliminate the cusps up to at most one cusp by the same argument as in the proof of Proposition 5.3.2. As a result, we may assume that every cusp c_j is the unique cusp on a closed component C_j of $S(F_1)$. Moreover, by an adaption of the argument of Remark 5.2.3 one can achieve that the closed components C_j have trivial normal bundles in W. Fix an interval J of $S(F_1)$. (If $S(F_1)$ contains no intervals, then $W \setminus T$ is connected, so that all cusps have already been eliminated.) Since W is connected, it can be achieved by tunneling that the cusps c_j are as shown in Figure 5.16 collected in a tubular neighbourhood V of J such that $V \cap S(F_1) = J \cup \bigcup (V \cap C_j)$, where $c_j \in V \cap C_j$. By a single application of (C) and (E) we may also assume that all cusps point into the component of $W \setminus S(F_1)$ bounded by J as shown in Figure 5.16.

If each of the two components of $V \setminus J$ contains an even number of cusps, then one can eliminate all cusps in $V \setminus J$ by the same argument as in the proof of Proposition 5.3.2, which finishes the construction of F_2. Otherwise, the same method can be applied with the result that there is precisely one cusp left on each of the two components of $V \setminus J$. These last two cusps can finally be absorbed by J. (Indeed, insert a new pair of cusps on J via (C) and eliminate the four cusps in pairs via (E).)

This completes the proof of Theorem 5.1.3.

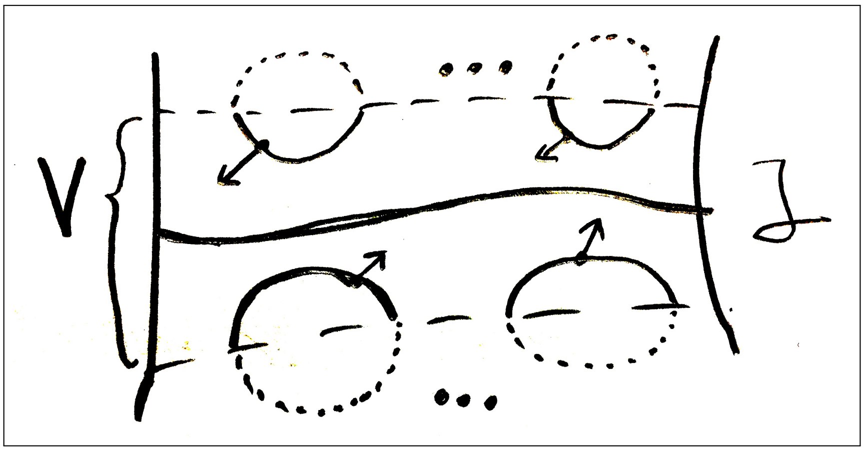

Figure 5.16: Elimination of cusps near J.

5.8 Proof of Theorem 5.1.4

Fix a boundary condition $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ and an open Brauer morphism $\varphi \in \mathrm{OP}_{m_S, n_S}$ such that $L_W(f_M, f_N; \varphi) \neq \emptyset$. Moreover, assume that $k_S > 0$. Given a fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$, let $\varphi_F \in \mathrm{OP}_{m_S, n_S}$ be the unique open Brauer morphism such that $\mathbb{S}(F) = \varphi_F \otimes \lambda^{\otimes l_F}$, where l_F denotes the number of loops of $S(F)$. The task is to to reduce l_F as much as possible by the local modifications of Section 5.2, and to determine this number.

First, suppose that W is non-orientable. Then it is claimed that $L_W(f_M, f_N, \varphi) = \mathbb{N}$. Since $l \in L_W(f_M, f_N, \varphi)$ implies $l + 1 \in L_W(f_M, f_N, \varphi)$ by Proposition 5.2.9, it suffices to show that $0 \in L_W(f_M, f_N, \varphi)$. For this purpose, choose a fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ satisfying $\mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l_F}$ for some *even* integer $l_F \geq 0$. (Such an F exists because $L_W(f_M, f_N, \varphi) \neq \emptyset$ by assumption, and $\mathbb{N} + L_W(f_M, f_N, \varphi) \subset L_W(f_M, f_N, \varphi)$.) Finally, since $k_S > 0$ and W is connected and non-orientable, it is possible to eliminate all loops of F in paris via tunneling and absorbtion. This completes the proof of Theorem 5.1.4 in the case that W is non-orientable.

From now on one may suppose that W is orientable. Fix an orientation σ of W. We start by introducing the *cycle number* $c_\sigma(f_M, f_N; \varphi) \geq 0$ in Definition 5.8.1.

Recall that f_M induces a canonical identification between the points of $S(f_M) \cap (0 \times M)$ and the points of $M[m_S]$. Analogously, f_N induces a canonical identification between the points of $S(f_N) \cap (1 \times N)$ and the points of $M[n_S]$. Identifying $0 \times M$ with $M \subset W$ and $1 \times N$ with $N \subset W$, we obtain canonical inclusions $M[m_S] \subset M$ and $M[n_S] \subset N$.

Fix collar neighbourhoods $[0,1] \times M$ of $0 \times M = M \subset W$ and $[0,1] \times N$ of $1 \times N = N \subset W$. Then σ induces orientations of the cylinders $[0,1] \times M$ and $[0,1] \times N$. Let $([0,1] \times M)_+$ and $([0,1] \times N)_+$ denote the closures of the unions of the components of $([0,1] \times M) \setminus S(f_M)$ and $([0,1] \times N) \setminus S(f_N)$ where f_M and f_N are orientation preserving. Set $M_+ = (0 \times M) \cap ([0,1] \times M)_+$ and $N_+ = (1 \times N) \cap ([0,1] \times N)_+$. Due to the identifications $0 \times M = M$ and $1 \times N = N$ we have inclusions $M_+ \subset M$ and $N_+ \subset N$. By construction, M_+ and N_+ are 1-dimensional cobordisms with boundaries $\partial M_+ = M[m_S]$ and $\partial N_+ = M[n_S]$. Note that M_+ and N_+ may have closed components since an element $f \in \mathcal{F}(P)$ might restrict to an immersions on components of $[0,1] \times P$. The given $\varphi \in \mathrm{OP}_{m_S, n_S}$ is represented by a 1-dimensional cobordism $V \subset [0,1] \times \mathbb{R}^3$ with boundary $(0 \times M[m_S] \times 0 \times 0) \sqcup (1 \times M[n_S] \times 0 \times 0)$.

Definition 5.8.1. The *cycle number* $c_\sigma(f_M, f_N; \varphi)$ is defined to be the number of components of the 1-dimensional closed manifold $M_+ \sqcup V \sqcup N_+$ which is obtained by gluing the boundaries of M_+, V and N_+ under the identifications $0 \times M[m_S] \times 0 \times 0 = M[m_S]$ and $1 \times M[n_S] \times 0 \times 0 = M[n_S]$.

The cycle number is constructed in such a way that the number of boundary components of W_F^σ (see Definition 5.4.21) is given by $l_F + c_\sigma(f_M, f_N; \varphi_F)$ for every fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$, where $\varphi_F \in \mathrm{OP}_{m_S,n_S}$ is the unique open Brauer morphism such that $\mathbb{S}(F) = \varphi_F \otimes \lambda^{\otimes l_F}$, and l_F denotes the number of loops of $S(F)$. Consequently, Corollary 5.4.23 implies the following

Proposition 5.8.2. *Let $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ be a fold pre-field, and let $\varphi_F \in \mathrm{OP}_{m_S,n_S}$ be the unique open Brauer morphism such that $\mathbb{S}(F) = \varphi_F \otimes \lambda^{\otimes l_F}$, where l_F denotes the number of loops of $S(F)$. Then, there exists an integer $h_F^\sigma \geq 0$ such that*

$$\chi^\sigma = l_F + c_\sigma(f_M, f_N; \varphi_F) - 2h_F^\sigma.$$

In fact, h_F^σ is the number of handles that have to be attached to the disjoint union of $l_F + c_\sigma(f_M, f_N; \varphi_F)$ 2-discs to obtain W_F^σ.

The following definition results from Proposition 5.8.2:

Definition 5.8.3. Let $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ be a fold pre-field, and let $\varphi_F \in \mathrm{OP}_{m_S,n_S}$ be the unique open Brauer morphism such that $\mathbb{S}(F) = \varphi_F \otimes \lambda^{\otimes l_F}$, where l_F denotes the number of loops of $S(F)$. Define the following difference, which is independent of F:

$$\Delta^\sigma := l_F - 2h_F^\sigma = \chi^\sigma - c_\sigma(f_M, f_N; \varphi_F).$$

By construction, $\Delta^\sigma, \Delta^{-\sigma} \in \mathbb{Z}$. Moreover, for any F, $\Delta^\sigma + \Delta^{-\sigma} = 2l_F - 2(h_F^\sigma + h_F^{-\sigma}) \equiv 0 \bmod 2$.

Next, we show that $\Delta^\sigma + \Delta^{-\sigma} \leq 0$. For this purpose, we start with any fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ and use trivialization (note that $k_S > 0$) in order to achieve that all loops in $S(F)$ are trivial (see Definition 5.2.2). Note that every trivial loop in $S(F)$ has exactly one contractible component because W is connected, and $k_S > 0$ implies that $\partial W \neq \emptyset$. Using tunneling and absorbtion (note that $k_S > 0$), one can then achieve that all (trivial) loops of $S(F)$ are boundary components of the same component of $W \setminus S(F)$, say a component V of W_F^ρ for suitable $\rho \in \{-\sigma, \sigma\}$, in such a way that for every (trivial) loop C of $S(F)$, the contractible component of $W \setminus C$ belongs to $W_F^{-\rho}$. In this situation, one has $l_F \leq h_F^\rho$. (In fact, recall from Proposition 5.8.2 that W_F^ρ can be obtained by attaching h_F^ρ handles to the disjoint union of $l_F + c_\rho(f_M, f_N; \varphi_F)$ 2-discs. In each step of this handle attaching process the number of components decreases by at most 1. Furthermore, every such component contains at least one of the $l_F + c_\rho(f_M, f_N; \varphi_F)$ boundary components. Now, note that the boundary of V contains by construction the l_F trivial loops of $S(F)$, and must have at least one more component because otherwise V would be part of a closed component of W. But then, W_F^ρ can have at most $c_\rho(f_M, f_N; \varphi_F)$ components. Consequently, at least l_F handle attachments are needed to produce W_F^ρ from the disjoint union of $l_F + c_\rho(f_M, f_N; \varphi_F)$ 2-discs. This shows that indeed $l_F \leq h_F^\rho$.) Hence, $h_F^{-\rho} \geq 0$ (see Proposition 5.8.2) implies that

$$\Delta^\sigma + \Delta^{-\sigma} = 2l_F - 2(h_F^\sigma + h_F^{-\sigma}) = 2(l_F - h_F^\sigma - h_F^{-\sigma}) \leq 2(l_F - h_F^\rho) \leq 0.$$

Therefore, we have either $\Delta^{\sigma} > 0$ or $\Delta^{-\sigma} > 0$ or $\Delta^{\sigma}, \Delta^{-\sigma} \leq 0$. We proceed to show the claim

$$L_W(f_M, f_N; \varphi) = \begin{cases} \Delta^{\sigma} + 2\mathbb{N}, & \text{if } \Delta^{\sigma} > 0, \\ \Delta^{-\sigma} + 2\mathbb{N}, & \text{if } \Delta^{-\sigma} > 0, \\ \mathbb{N} \cap (\Delta^{\sigma} + 2\mathbb{N}) = \mathbb{N} \cap (\Delta^{-\sigma} + 2\mathbb{N}), & \text{else.} \end{cases}$$

In order to prove that $L_W(f_M, f_N; \varphi)$ is included in the sets on the right-hand side, it suffices to note that every $l \in L_W(f_M, f_N; \varphi)$ satisfies $l \equiv \Delta^{\rho} \bmod 2$ and $l \geq \Delta^{\rho}$ for $\rho \in \{-\sigma, \sigma\}$. (In fact, given $l \in L_W(f_M, f_N; \varphi)$, there exists a fold pre-field $F \in \mathfrak{F}^{\text{pre}}(W; f_M, f_N)$ such that $\mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l}$. Then, Definition 5.8.3 implies that $\Delta^{\rho} = l - 2h_F^{\rho} \equiv l \bmod 2$ and $l \geq l - 2h_F^{\rho} = \Delta^{\rho}$ since $h_F^{\rho} \geq 0$.)

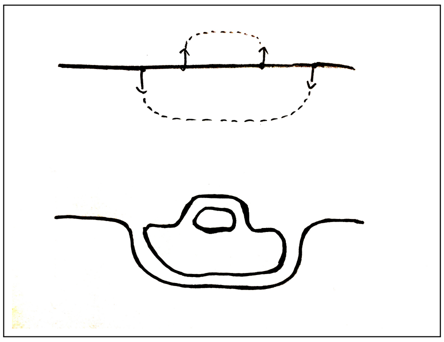

Figure 5.17: Creation of two additional fold loops.

Conversely, to show that the sets on the right-hand side are contained in $L_W(f_M, f_N; \varphi)$, it suffices to prove that $l + 2 \in L_W(f_M, f_N; \varphi)$ for every $l \in L_W(f_M, f_N; \varphi)$, and $\min(\mathbb{N} \cap (\Delta^{\sigma} + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})) \in L_W(f_M, f_N; \varphi)$. To show the first claim, let $l \in L_W(f_M, f_N; \varphi)$. Then, there exists a fold pre-field $F \in \mathfrak{F}^{\text{pre}}(W; f_M, f_N)$ such that $\mathbb{S}(F) = \varphi \otimes \lambda^{\otimes l}$. Since $S(F) \neq \emptyset$, Figure 5.17 shows how one can use (C) and (E) to modify F to a fold pre-field $F_1 \in \mathfrak{F}^{\text{pre}}(W; f_M, f_N)$ that satisfies $\mathbb{S}(F_1) = \varphi \otimes \lambda^{\otimes(l+2)}$. Hence, $l + 2 \in L_W(f_M, f_N; \varphi)$. Finally, let us show that $\min(\mathbb{N} \cap (\Delta^{\sigma} + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})) \in L_W(f_M, f_N; \varphi)$. As in the proof of $\Delta^{\sigma} + \Delta^{-\sigma} \leq 0$ above, one can construct a fold pre-field $F \in \mathfrak{F}^{\text{pre}}(W; f_M, f_N)$ satisfying $\mathbb{S}(F) = \varphi_F \otimes \lambda^{\otimes l_F}$ for some integer $l_F \geq 0$, and with the additional property that all loops of $S(F)$ are trivial and are contained in the boundary of the same component of $W \setminus S(F)$, say a component of W_F^{ρ} for suitable $\rho \in \{-\sigma, \sigma\}$, so that the contractible component of $W \backslash C$ belongs to $W_F^{-\rho}$ for every (trivial) loop C of $S(F)$. As shown above, $l_F \leq h_F^{\rho}$ in this situation. Then, note that $\Delta^{\rho} = l_F - 2h_F^{\rho} \leq l_F - h_F^{\rho} \leq 0$ because $h_F^{\rho} \geq 0$. Therefore, using $\Delta^{\sigma} \equiv \Delta^{-\sigma} \bmod 2$,

$$\min(\mathbb{N} \cap (\Delta^{\sigma} + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})) = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N})).$$

Finally, repeated balancing (see Proposition 5.2.8) of two loops of F for one handle of $W_F^{-\rho}$ (see Figure 5.8) yields a fold pre-field $G \in \mathfrak{F}^{\text{pre}}(W; f_M, f_N)$ satisfying $\mathbb{S}(G) = \varphi \otimes \lambda^{\otimes l_G}$ with $l_G = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N}))$. (Indeed, if $\Delta^{-\rho} = l_F - 2h_F^{-\rho} > 0$, then one repeats balancing until no handles are left in $W_F^{-\rho}$ to obtain $l_G = \Delta^{-\rho}$. If, however, $\Delta^{-\rho} = l_F - 2h_F^{-\rho} \leq 0$, then one repeats balancing until at most one loop is left in $S(F)$ to obtain $l_G \in \{0, 1\}$ such that $l_G \equiv \Delta^{-\rho} \bmod 2$. In both cases, $l_G = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N}))$.)

This completes the proof of Theorem 5.1.4.

5.9 Proof of Corollary 5.1.5

Our argument will reduce the case $k_S = 0$ to the case $k_S > 0$ of Theorem 5.1.4. We may suppose that (f_M, f_N) does not extend to an immersion $W \to \mathbb{R}^2$. (This is automatically the case if W is non-orientable. If W is orientable, then it can in principle be checked by using the results of [11].) Hence, $S(F) \neq \emptyset$ for all $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$.

Let V denote a cobordism from $M \sqcup P$ to N that is obtained by deleting a small open ball from $\operatorname{int} W$ whose boundary is $P \cong S^1$. (Strictly speaking, V is not a cobordism in the sense of Definition 3.1.1. Nevertheless, the arguments that are used in the proof of Theorem 5.1.4 do not make use of the special embedding requirements.)

Fix a collar neighbourhood $[0,1] \times P$ of $0 \times P = P \subset V$ and a diffeomorphism $P \cong S^1$, and let $f_P \colon [0,1] \times P \to \mathbb{R}^2$ denote the fold map $f_P = g \circ \alpha$ that is given by the composition of the embedding $\alpha \colon [0,1] \times P \to \mathbb{R}^2$, $\alpha(t,p) = (2-t)p$, (where $P = S^1 \subset \mathbb{R}^2$) with the fold map $g \colon \mathbb{R}^2 \to \mathbb{R}^2$, $g(x,y) = (x^2, y)$. It can be checked that $f_P \in \mathcal{F}(P)$ and $\mathbb{S}(f_P) = \mathrm{id}_{[2]}$. Hence, it follows from $S(f_M) = \emptyset$ that $f_M \sqcup f_P \in \mathcal{F}(M \sqcup P)$ and $\mathbb{S}(f_M \sqcup f_P) = \mathrm{id}_{[2]}$. Moreover, $S(f_N) = \emptyset$ implies that $\mathbb{S}(f_N) = \mathrm{id}_{[0]}$. Let φ denote the unique open Brauer morphism $[2] \to [0]$. Since any element $G \in \mathcal{F}^{\mathrm{pre}}(V; f_M \sqcup f_P, f_N)$ can by construction of V and f_P be extended to an element $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ such that $S(F)$ contains exactly one loop more than $S(G)$, it follows that

$$1 + L_V(f_M \sqcup f_P, f_N; \varphi) \subset L_W(f_M, f_N; 1_{[0]}).$$

If W is non-orientable, then so is V, and Theorem 5.1.4 implies that $L_V(f_M \sqcup f_P, f_N; \varphi) = \mathbb{N}$. Thus, $1 + \mathbb{N} \subset L_W(f_M, f_N; 1_{[0]})$. As $S(F) \neq \emptyset$ for all $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$, we also have $L_W(f_M, f_N; 1_{[0]}) \subset 1 + \mathbb{N}$. All in all, the claim $L_W(f_M, f_N; 1_{[0]}) = 1 + \mathbb{N}$ follows.

From now on, we may assume that W is orientable. Let σ denote an orientation of W. Since $\chi(W) = \chi(V) + 1$, $c_\sigma(f_M \sqcup f_P, f_N; \varphi) = c_\sigma(f_M, f_N; \varphi) + 1$ (the boundary component of V_F^σ that has points in P is disjoint to ∂W) and $\omega_{\sigma|_P}(f_P) = 0$, Theorem 5.1.4 implies

$$L_V(f_M \sqcup f_P, f_N; \varphi) = \mathbb{N} \cap ((\Delta^\sigma - 1) + 2\mathbb{N}) \cap ((\Delta^{-\sigma} - 1) + 2\mathbb{N}),$$

where Δ^σ and $\Delta^{-\sigma}$ are defined with respect to the cobordism W. Therefore, $1 + \mathbb{N} \cap ((\Delta^\sigma - 1) + 2\mathbb{N}) \cap ((\Delta^{-\sigma} - 1) + 2\mathbb{N}) \subset L_W(f_M, f_N; 1_{[0]})$. Finally, let us show that the converse inclusion is also valid. For every $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ one constructs a subcobordism V_F of W with boundary $\partial V_F = \partial W \sqcup P_F$, $P_F \cong S^1$, and $f_{P_F} \in \mathcal{F}(P_F)$ with $\mathbb{S}(f_{P_F}) = \mathrm{id}_{[2]}$, such that F restricts on V_F to an element $G \in \mathcal{F}^{\mathrm{pre}}(V_F; f_M \sqcup f_{P_F}, f_N)$ in such a way that $S(F)$ contains exactly one loop more than $S(G)$. (Indeed, recall that $S(F) \neq \emptyset$ for all $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$. Hence, V_F can be chosen by deleting from $\operatorname{int} W$ a small open ball with boundary $P_F \cong S^1$ around a point of $S(F)$. Moreover, f_{P_F} is obtained by restriction of F to a fixed suitably small collar neighbourhood $[0,1] \times P_F$ of $0 \times P_F = P_F \subset V_F$.) Hence, $1 + L_{V_F}(f_M \sqcup f_{P_F}, f_N; \varphi) \subset L_W(f_M, f_N; 1_{[0]})$. Note that, with the same arguments as for V above, Theorem 5.1.4 implies as desired that

$$L_{V_F}(f_M \sqcup f_{P_F}, f_N; \varphi) = \mathbb{N} \cap ((\Delta^\sigma - 1) + 2\mathbb{N}) \cap ((\Delta^{-\sigma} - 1) + 2\mathbb{N}).$$

This completes the proof of Corollary 5.1.5.

Chapter 6

Fold Maps on Higher-dimensional Cobordisms

6.1 Modifying Generic Smooth Maps to Fold Maps

Let X be a connected smooth manifold with boundary of dimension $m := \dim X \geq 3$. (The case that X is a surface is treated separately in Chapter 5. In the present section we exploit Levine's method of elimination of cusps [32] and the complementary procedure of creating a pair of cusps on a given fold line in order to construct a fold map with few closed fold lines starting from a given generic smooth map $X \to \mathbb{R}^2$.

Definition 6.1.1. A subset $A \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ is called *nice* if either $A = \{\frac{m}{2}\}$ (where m is necessarily even) or $A = \{a_1, a_1 + 1, \ldots, a_2\}$ for suitable integers $a_1, a_2 \in \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ such that $a_1 < a_2$. Given two subsets $A, A' \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$, we write $A \gg A'$ if there exists $i \in \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ such that $a > i > a'$ for all $(a, a') \in A \times A'$.

Consider a generic smooth map $G \colon X \to \mathbb{R}^2$ such that $S(G) \pitchfork \partial X$. It is always assumed that G has a finite number of cusps and that $S(G)$ consists of a finite number of components. (This is satisfied whenever X is compact.) Recall that $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ is the set of possible values for the absolute index of a fold point of G.

Definition 6.1.2. For every component J of $S(G)$ let $A_G(J) \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ denote the subset of integers that occur as the absolute index of a fold point of G on J. Moreover, let $A_G \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ denote the subset of integers that occur as the absolute index of a fold point of G. In other words, A_G is the union of the sets $A_G(J)$, where J runs through the components of $S(G)$.

Observe that if J is a component of $S(G)$ that contains at least one cusp of F, then $A_F(J)$ is a nice subset of $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$. (In fact, the map that assigns to every fold point of G on J its absolute index is locally constant. Furthermore, if two fold arcs in J abut on the same cusp, then their absolute indices are equal if the cusp has index $\frac{m}{2} - 1$ (this can only happen if m is even), and differ by 1 else.)

Now suppose that A_G is contained in the union $A_1 \cup \cdots \cup A_r$ of suitable nice subsets $A_1, \ldots, A_r \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$, $r \geq 1$, such that $A_1 \gg \cdots \gg A_r$. Then for every component J of $S(G)$

there exists a unique $s \in \{1, \ldots, r\}$ such that $A_G(J) \subset A_s$. For every $s \in \{1, \ldots, r\}$ let J_s denote the union of all components J of $S(G)$ that satisfy $A_G(J) \subset A_s$.

Assume for simplicity that all components of $S(G)$ are diffeomorphic to S^1, and that $S(G) = J_s$ for some $s \in \{1, \ldots, r\}$. The purpose of the present section is to modify G on a compact subset of $X \setminus \partial X$ to obtain a fold map $F\colon X \to \mathbb{R}^2$ whose fold lines (i.e. components of $S(F)$) correspond bijectively to the elements of A_s by assigning to every component of $S(F)$ its absolute index. Note that F has *few* closed components in the sense that it gets on with a single component per element of A_s. However, the more difficult problem of eliminating all fold points of G of a given absolute index during the construction of F seems to be inaccessible with the methods used here. The construction of F from G is realized in Proposition 6.1.3 in the general setting of a cobordism $X = W$, where boundary conditions are also taken into account. One should remark that the proof of Proposition 6.1.3 could also be adapted to the case of a non-compact manifold X.

Roughly speaking, Proposition 6.1.3 reduces the problem of constructing a fold map $F\colon W \to \mathbb{R}^2$ with desired pattern of fold lines to the construction of a suitable generic smooth map $G\colon W \to \mathbb{R}^2$. Apart from such a map G, Proposition 6.1.3 requires as input a set $\mathcal{R}$ of fold components of $S(G)$ that provides the relative character in the sense that G will not be modified in a neighbourhood of the components that are contained in $\mathcal{R}$, a certain set $\mathcal{P}$ that encodes the combinatorics of how the fold lines of F will end in the boundary of W, and certain nice subsets $A_1, \ldots, A_r \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ that cover the absolute indices of fold points of G on components not in $\mathcal{R}$ in an efficient way. (Note that there is in general no unique efficient choice of the sets A_s since all nice subsets apart from $\{\frac{m}{2}\}$ contain at least two adjacent integers according to Definition 6.1.1. For instance, there is in general more than one way to cover the subset of $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$ of odd integers efficiently by nice subsets.) Given these data, a fold map F with few closed fold lines can be produced that agrees with G near ∂W.

Proposition 6.1.3. *Let (W, W_0, W_1) be a smooth manifold triad of dimension $m = \dim W \geq 3$ such that W is connected.*

Suppose that $G\colon W \to \mathbb{R}^2$ is a generic smooth map such that

$$G|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}\colon \qquad W_0 \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$
$$G|_{W_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}\colon \qquad W_1 \times (1-\varepsilon, 1] \to \mathbb{R} \times (1-\varepsilon, 1],$$

where $f_0\colon W_0 \to \mathbb{R}$ and $f_1\colon W_1 \to \mathbb{R}$ are Morse functions, and $W_0 \times [0, \varepsilon)$ and $W_1 \times (1-\varepsilon, 1]$ are suitable collar neighbourhoods of $W_0 \times 0 = W_0 \subset W$ and $W_1 \times 1 = W_1 \subset W$, $\varepsilon > 0$.

If m is even, then suppose in addition that G has an even number of cusps.

Let $\mathcal{R}$ be a set of components of $S(G)$ that are fold lines. (The choice $\mathcal{R} = \emptyset$ is always possible.)

Let $\mathcal{P}$ be a partition of the set

$$\{c \in \partial W;\ c \text{ is a critical point of } f_0 \text{ or } f_1\} \setminus \{R \cap \partial W; R \in \mathcal{R}\}$$

such that every $P \in \mathcal{P}$ consists of two points, and

- *if $P \subset W_i$ for $i = 0$ or $i = 1$, then the indices of the two critical points of f_i in P add up to $m - 1$,*

- *if $P \cap W_i \neq \emptyset$ for $i = 0$ and $i = 1$, then the critical point $P \cap W_0$ of f_0 has the same index as the critical point $P \cap W_1$ of f_1.*

Let $A_1, \ldots, A_r \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$, $r \geq 1$, be nice subsets such that $A_1 \gg \cdots \gg A_r$, and every component J of $S(G)$ with $J \notin \mathcal{R}$ satisfies $A_G(J) \subset A_1 \cup \cdots \cup A_r$.

Then there exists a fold map $F \colon W \to \mathbb{R}^2$ with the following properties:

(*i*) *$F|_{W_0 \times [0,\varepsilon/2]} = G|_{W_0 \times [0,\varepsilon/2]}$ and $F|_{W_1 \times [1-\varepsilon/2,1]} = G|_{W_1 \times [1-\varepsilon/2,1]}$.*

(*ii*) *There exists an open subset $U \subset W$ that contains all $R \in \mathcal{R}$, and such that $F|_U = G|_U$. (In particular, $\mathcal{R}$ is also a set of components of $S(F)$.)*

(*iii*) *Every component $J \notin \mathcal{R}$ of $S(F)$ has absolute index in $A_1 \cup \cdots \cup A_r$, and*
  - *if $\partial J = \emptyset$, then $A_F(J) \neq A_F(J')$ for all components $J' \neq J$ of $S(F)$ with $J' \notin \mathcal{R}$.*
  - *if $\partial J \neq \emptyset$, then $J \cap \partial W \in \mathcal{P}$.*

Proof. Starting from the given generic smooth map $G^{(0)} := G$, we will inductively modify $G^{(s-1)}$ for $s \in \{1, \ldots, r\}$ by a finite sequence of creations and eliminations of cusps to obtain a generic smooth map $G^{(s)} \colon W \to \mathbb{R}^2$ with the following properties:

(1_s) There exists a compact subset $K_s \subset W$ such that
  - (a) $K_s \cap W_0 \times [0, \varepsilon/2] = \emptyset$ and $K_s \cap W_1 \times [1-\varepsilon/2, 1] = \emptyset$,
  - (b) $K_s \cap R = \emptyset$ for every $R \in \mathcal{R}$,
  - (c) if J is a component of $S(G^{(s)})$ such that $K_s \cap J \neq \emptyset$, then $A_{G^{(s)}}(J) \subset A_s$,
  - (d) $G^{(s)}|_{W \setminus K_s} = G^{(s-1)}|_{W \setminus K_s}$.

(2_s) Every component $J \notin \mathcal{R}$ of $S(G^{(s)})$ with $A_{G^{(s)}}(J) \subset A_s$ is a fold line of $G^{(s)}$, and
  - if $\partial J = \emptyset$, then $A_{G^{(s)}}(J) \neq A_{G^{(s)}}(J')$ for all components $J' \neq J$ of $S(G^{(s)})$ with $J' \notin \mathcal{R}$.
  - if $\partial J \neq \emptyset$, then $J \cap \partial W \in \mathcal{P}$.

Suppose that $G^{(s)}$ has been constructed for some $s \in \{1, \ldots, r\}$. By property $(1_t)(d)$, $t \in \{1, \ldots, s\}$, $G^{(s)}$ coincides with $G^{(0)} = G$ on the open subset $W \setminus \bigcup_{s=1}^{r} K_s$ of W. In particular, property $(1_t)(b)$, $t \in \{1, \ldots, s\}$, implies that the components of $S(G)$ in $\mathcal{R}$ are also components of $S(G^{(s)})$. Furthermore, the properties (1_t) and (2_t), $t \in \{1, \ldots s\}$, imply that

($2'_s$) Every component $J \notin \mathcal{R}$ of $S(G^{(s)})$ with $A_{G^{(s)}}(J) \subset A_1 \cup \cdots \cup A_s$ is a fold line of $G^{(s)}$, and
  - if $\partial J = \emptyset$, then $A_{G^{(s)}}(J) \neq A_{G^{(s)}}(J')$ for all components $J' \neq J$ of $S(G^{(s)})$ with $J' \notin \mathcal{R}$.
  - if $\partial J \neq \emptyset$, then $J \cap \partial W \in \mathcal{P}$.

(In fact, property $(2'_s)$ follows by induction on $t \in \{1, \ldots s\}$. The induction basis ($t = 1$) is provided by the fact that property (2_1) coincides with property $(2'_1)$. Furthermore, the inductive step for $t \geq 2$ is supplied by properties $(1_t)(c)$, $(1_t)(d)$, $(2'_{t-1})$ and (2_t) as follows. One has to check property $(2'_t)$. Let $J \notin \mathcal{R}$ be a component of $S(G^{(t)})$ such that $A_{G^{(t)}}(J) \subset A_1 \cup \cdots \cup A_t$. If $A_{G^{(t)}}(J) \subset A_t$, then the claims of property $(2'_t)$ hold by property (2_t). Therefore, we may assume in the following that $A_{G^{(t)}}(J) \subset A_1 \cup \cdots \cup A_{t-1}$. In this case, property $(1_t)(c)$ implies that J is contained in the open subset $W \setminus K_t$ of W. Hence, by property $(1_t)(d)$, $G^{(t-1)}$ and $G^{(t)}$ coincide on an open neighbourhood of J in W. Consequently, J is also a component of $S(G^{(t-1)})$ such that $A_{G^{(t-1)}}(J) \subset A_1 \cup \cdots \cup A_{t-1}$. Thus, using $J \notin \mathcal{R}$, property $(2'_{t-1})$ implies that J is a fold line of $G^{(t-1)}$, and

- if $\partial J = \emptyset$, then $A_{G^{(t-1)}}(J) \neq A_{G^{(t-1)}}(J')$ for all components $J' \neq J$ of $S(G^{(t-1)})$ with $J' \notin \mathcal{R}$.
- if $\partial J \neq \emptyset$, then $J \cap \partial W \in \mathcal{P}$. (This coincides with the second item of property $(2'_t)$.)

As $G^{(t-1)}$ and $G^{(t)}$ coincide on an open neighbourhood of J in W, J is also a fold line of $G^{(t)}$. It remains to show the first item of $(2'_t)$, i.e. if $\partial J = \emptyset$, then $A_{G^{(t)}}(J) \neq A_{G^{(t)}}(J')$ for all components $J' \neq J$ of $S(G^{(t)})$ with $J' \notin \mathcal{R}$. Indeed, assuming $\partial J = \emptyset$, let $J' \neq J$ be a component of $S(G^{(t)})$ with $J' \notin \mathcal{R}$. Suppose that $A_{G^{(t)}}(J) = A_{G^{(t)}}(J')$. Consequently, J' is a component of $S(G^{(t)})$ with $A_{G^{(t)}}(J') = A_{G^{(t)}}(J) \subset A_1 \cup \cdots \cup A_{t-1}$. Therefore, we obtain as above from property $(1_t)(c)$ and $(1_t)(d)$ that $G^{(t-1)}$ and $G^{(t)}$ coincide on an open neighbourhood of J' in W. In particular, $J' \neq J$ is a component of $S(G^{(t-1)})$ with $J' \notin \mathcal{R}$. However, by the first item of property $(2'_{t-1})$, this results in the contradiction $A_{G^{(t)}}(J) = A_{G^{(t-1)}}(J) \neq A_{G^{(t-1)}}(J') = A_{G^{(t)}}(J')$. This completes the proof of property $(2'_t)$.)

Next, one shows that $F := G^{(r)}$ is a fold map with the desired properties (i) to (iii). First of all, recall that $F = G^{(r)}$ coincides with $G = G^{(0)}$ on the open subset $U := W \setminus \bigcup_{s=1}^{r} K_r$ of W by property $(1_s)(d)$, $s \in \{1, \ldots, r\}$. This implies property (i) because $W_0 \times [0, \varepsilon/2], W_1 \times [1 - \varepsilon/2, 1] \subset U$ by property $(1_s)(a)$, $s \in \{1, \ldots, r\}$. Moreover, property (ii) follows from $F|_U = G|_U$ and $R \subset U$ for all $R \in \mathcal{R}$, which is a consequence of property $(1_s)(b)$, $s \in \{1, \ldots, r\}$. In order to check property (iii), it suffices by property $(2'_r)$ to show that every component $J \notin \mathcal{R}$ of $S(G^{(r)})$ satisfies $A_{G^{(r)}}(J) \subset A_1 \cup \cdots \cup A_r$. (Indeed, one shows by induction on $s \in \{0, \ldots, r\}$ that $A_{G^{(s)}}(J) \subset A_1 \cup \cdots \cup A_r$ for every component $J \notin \mathcal{R}$ of $S(G^{(s)})$. The induction basis ($s = 0$) holds by assumption on $G^{(0)} = G$. Furthermore, the inductive step for $s \geq 1$ is supplied by properties $(1_s)(c)$ and $(1_s)(d)$ as follows. If $J \notin \mathcal{R}$ is a component of $S(G^{(s)})$, then either $K_s \cap J \neq \emptyset$ or $K_s \cap J = \emptyset$. In the first case, property $(1_s)(c)$ yields $A_{G^{(s)}}(J) \subset A_s$. In the second case, property $(1_s)(d)$ implies that $G^{(s)}$ and $G^{(s-1)}$ coincide on the open neighbourhood $W \setminus K_s$ of J in W. Hence, J is also a component of $S(G^{(s-1)})$, and $A_{G^{(s)}}(J) = A_{G^{(s-1)}}(J) \subset A_1 \cup \cdots \cup A_r$, which completes the proof.) It remains to show that F is a fold map. Let J be a component of $S(F)$. If $J \in \mathcal{R}$, then $J \subset U$ is a fold line of $F|_U = G|_U$ by assumption on G. If $J \notin \mathcal{R}$, then it was already shown that $A_F(J) \subset A_1 \cup \cdots \cup A_r$, which implies by property $(2'_r)$ that J is a fold line of F.

Let us now turn to the recursive construction of the maps $G^{(s)}$. Fix $s \in \{1, \ldots, r\}$ and suppose inductively that $G^{(s-1)}$ has already been constructed. Let J_s denote the union of all components $J \notin \mathcal{R}$ of $S(G^{(s-1)})$ with the property that $A_{G^{(s-1)}}(J) \subset A_s$. As W is connected and of dimension $m \geq 3$, it follows that

$$U_s := W \setminus (S(G^{(s-1)}) \setminus J_s)$$

is a connected open subset of W such that $U_s \cap S(G^{(s-1)}) = J_s$. Writing $H := G^{(s-1)}|_{U_s}$, note that $S(H) = S(G^{(s-1)}|_{U_s}) = S(G^{(s-1)}) \cap U_s = J_s$. All local modifications of $G^{(s-1)}$ will take place on compact subsets of $U_s \setminus (W_0 \times [0, \varepsilon/2] \cup W_1 \times [1 - \varepsilon/2])$. By Definition 6.1.1 one can distinguish between the following two cases for the nice subset $A_s \subset \{\lfloor \frac{m}{2} \rfloor, \ldots, m-1\}$:

- $\frac{m}{2} \notin A_s$ (this always holds if m is odd). Note that in this case the absolute indices of any two fold arcs of H that abut on the same cusp differ by exactly 1. Moreover, Definition 6.1.1 implies that A_s has cardinality at least 2.

  In a first step we modify H on a compact subset $K \subset U_s \setminus (W_0 \times [0, \varepsilon/2] \cup W_1 \times [1 - \varepsilon/2])$

in such a way that the modified map $H'\colon U_s \to \mathbb{R}^2$ is a generic smooth map such that every component J' of $S(H')$ with $\partial J' \neq \emptyset$ is a fold line of H', and $J' \cap \partial W \in \mathcal{P}$. Secondly, we modify H' on a compact subset $K' \subset U_s \setminus (W_0 \times [0, \varepsilon/2] \cup W_1 \times [1-\varepsilon/2])$ in such a way that the modified map $H''\colon U_s \to \mathbb{R}^2$ is a fold map such that every component J'' of $S(H'')$ satisfies $A_{H''}(J'') \subset A_s$, and

- if $\partial J'' = \emptyset$, then $A_{H''}(J'') \neq A_{H''}(J''')$ for all components $J''' \neq J''$ of $S(H'')$,
- if $\partial J'' \neq \emptyset$, then $J'' \cap \partial W \in \mathcal{P}$.

Finally, using H'', it follows directly that

$$G^{(s)}\colon W \to \mathbb{R}^2, \qquad G^{(s)}(w) = \begin{cases} H''(w), & \text{if } w \in U_s, \\ G^{(s-1)}(w), & \text{else,} \end{cases}$$

is a generic smooth map with the desired properties (1_s) (set $K_s := K \cup K'$) and (2_s). In order to construct H', we consider the subset

$$\mathcal{P}_s := \{P \in \mathcal{P};\ P \subset J_s \cap \partial W\} \qquad \subset \mathcal{P}.$$

It follows from the assumptions on $\mathcal{P}$ that $\mathcal{P}_s$ is a partition of

$$J_s \cap \partial W = \{c \in S(G) \cap \partial W;\ c \text{ has absolute index contained in } A_s\}.$$

Given $P = \{c, d\} \in \mathcal{P}_s$, we proceed as follows. Let J^c denote the component of $S(H)$ that contains c and let J^d denote the component of $S(H)$ that contains d. Let ν denotes the absolute index of the fold points c and d of G. Choose $\mu \in A_s$ such that $|\mu - \nu| = 1$. (Recall that A_s has cardinality at least 2.) We use Proposition 4.7.3 to introduce a removable pair (c_1, c_2) of cusps of H on the fold arc of J^c that contains c in such a way that the fold arc between the cusps c_1 and c_2 has absolute index μ, and such that H is only modified on $W_0 \times (\varepsilon/2, \varepsilon) \sqcup W_1 \times (1-\varepsilon, 1-\varepsilon/2)$. Similarly, we introduce a removable pair (d_1, d_2) of cusps of H on the fold arc of J^d that contains d in such a way that the fold arc between the cusps d_1 and d_2 has absolute index μ, and such that H is only modified on $W_0 \times (\varepsilon/2, \varepsilon) \sqcup W_1 \times (1-\varepsilon, 1-\varepsilon/2)$. If we suppose that c_1 lies between c and c_2 on the component $J^c \cong [0,1]$, and d_2 lies between d_1 and c on the component $J^d \cong [0,1]$, then (c_1, d_2) will form a removable pair. (This fact follows from Lemma 4.6.1 using the assumptions on the indices of the critical points c and d of the Morse function $f_1 \sqcup f_2$.) Finally, elimination of the cusp pair (c_1, d_2) will produce a fold component J' of H such that $J' \cap \partial W = P$. Repetition of this argument for all elements of $\mathcal{P}_s$ yields the desired map H'.

Let us proceed to the construction of H'', which is an inductive construction over the elements $a \in A_s$. As long as a is not the smallest element of A_s, repeat the following procedure, starting with the greatest value a in A_s and decreasing the value of a by 1 after each step. Consider the union J'_a of those components of $S(H')$ that contain fold points of absolute index a. (If there are no fold points of index a, then modify H' on a compact subset of $U_s \setminus (W_0 \times [0, \varepsilon/2] \cup W_1 \times [1-\varepsilon/2])$ by introducing a pair of closed fold components of absolute indices $a-1$ and a.) We distinguish between the following two cases:

- None of the components of J'_a is diffeomorphic to $[0,1]$. If some of the components are fold lines, then introduce a removable pair of cusps of H' on each of these components such that the new fold arc between the created cusps has absolute index $a-1$. Then it is possible to eliminate all cusps that bound fold lines of absolute index a in a cycle, forming a single fold line of H' of absolute index a.

  – At least one of the components of J'_a is diffeomorphic to $[0,1]$. In this case, use an analogous procedure as in the previous case to "absorb" all fold arcs of H' of absolute index a that do not lie on a component diffeomorphic to $[0,1]$ in one of the components of J'_a that is diffeomorphic to $[0,1]$.

Finally, if a is the smallest element of A_s, then do the same as before, but introduce removable pairs of cusps that abut on a new fold arc of absolute index $a+1$. Then collect all the fold points of absolute index a in a single fold line of H' as before. In order to avoid the production of an additional fold line of absolute index $a+1$, use an existing fold line of H' of absolute index $a+1$ in order to "absorb" the fold arcs of absolute index $a+1$. This completes the construction of the desired fold map $H'' \colon U_s \to \mathbb{R}^2$.

- m is even, and $\frac{m}{2} \in A_s$. (In particular, it follows from $A_1 \gg \cdots \gg A_r$ that $s = r$.) Note that [32, Lemma (3.2)(2)(i), p. 274] implies that H may have cusps of index $\frac{m}{2}-1$, which have the property that the abutting fold arcs have absolute index $\frac{m}{2}$. It is important to note that any two cusps of index $\frac{m}{2}-1$ form a removable pair. The absolute indices of any two fold arcs of H that abut on the same cusp of index greater than $\frac{m}{2}-1$ differ by exactly 1 just as in the case $\frac{m}{2} \notin A_s$.

Firstly, if $\{\frac{m}{2}\} \subsetneq A_r$, then we proceed as in the case $\frac{m}{2} \notin A_r$ above, and ignore the cusps of index $\frac{m}{2}-1$ by treating them as fold points of absolute index $\frac{m}{2}$. Hence, we can apply the arguments of the case $\frac{m}{2} \notin A_s$ above to obtain a generic smooth map $\tilde{G}^{(r)} \colon W \to \mathbb{R}^2$ that has all properties of the desired map $G^{(r)}$ with the exception that the components of $S(\tilde{G}^{(r)})$ whose fold points are all of index $\frac{m}{2}$ might contain cusps of index $\frac{m}{2}-1$. The same situation is found for $G^{(r-1)}$ in the case $\{\frac{m}{2}\} = A_r$ (but possibly with more than one closed component whose fold points are all of index $\frac{m}{2}$), so we finally treat these two cases in one go by considering the case $\{\frac{m}{2}\} = A_r$.

In the case $\{\frac{m}{2}\} = A_r$ the purpose of the remaining argument is to show that H can be modified on a compact subset of $U_s \setminus (W_0 \times [0, \varepsilon/2] \cup W_1 \times [1-\varepsilon/2])$ via elimination and creation of pairs of cusps of index $\frac{m}{2}-1$ in such a way that the resulting map $H'' \colon U_r \to \mathbb{R}^2$ is a fold map such that every fold line J'' of $S(H'')$ satisfies $A_{H''}(J'') = \{\frac{m}{2}\}$, and

  – if $\partial J'' = \emptyset$, then J'' is the only component of $S(H'')$,
  – if $\partial J'' \neq \emptyset$, then $J'' \cap \partial W \in \mathcal{P}$.

As in the case $\frac{m}{2} \notin A_s$ above we need an intermediate construction of a generic smooth map $H' \colon U_s \to \mathbb{R}^2$ such that every component J' of $S(H')$ with $\partial J' \neq \emptyset$ is a fold line of H', and $J' \cap \partial W \in \mathcal{P}$. The only difference to the corresponding construction of H' in the case $\frac{m}{2} \notin A_s$ above is that we introduce here removable pairs of cusps of index $\frac{m}{2}-1$. In particular, the fold arc between two new cusps has still absolute index $\frac{m}{2}$. It can still be shown that one can analogously eliminate pairs of cusps in the correct way. (Again, this follows from Lemma 4.6.1 and the assumptions on the indices of the critical points c and d of the Morse function $f_1 \sqcup f_2$.)

The construction of H'' is performed in the following three steps:

Step 1. Using Levine's elimination of cusps as described above, one eliminates all cusps of H' in pairs. Note that H' has in fact an even number of cusps. (Indeed, G has an

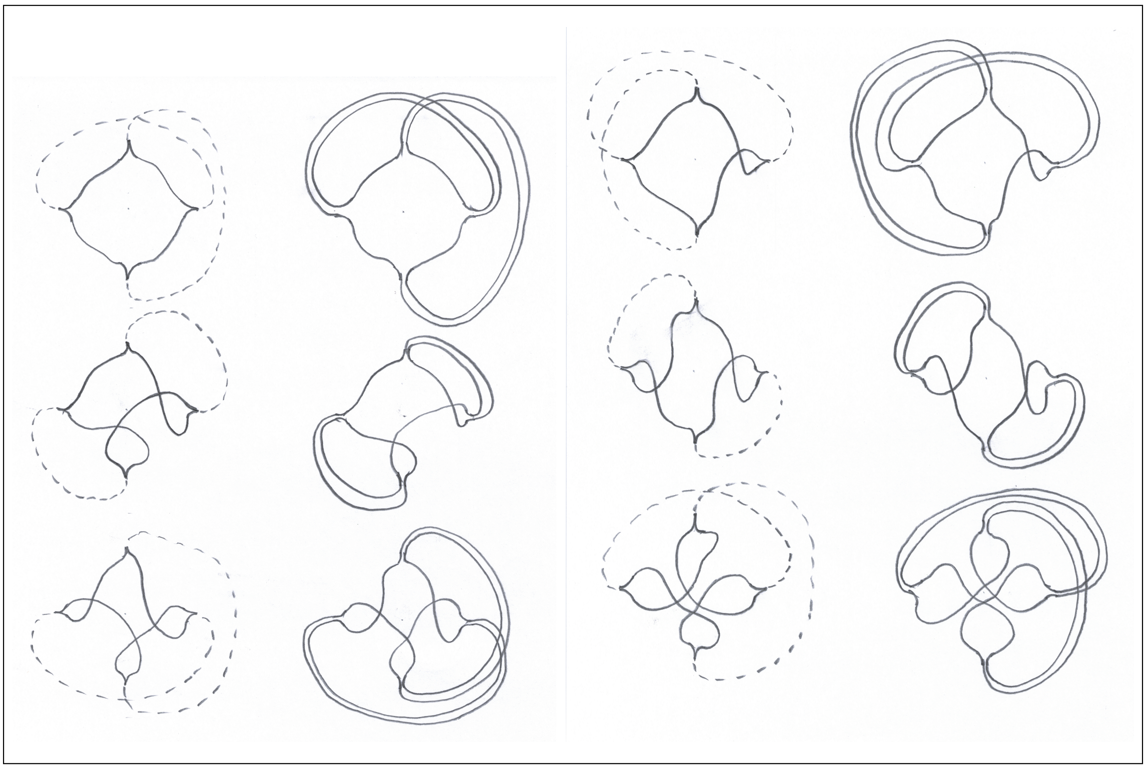

Figure 6.1: Combinatorics of how the four cusps of H_2' on C are connected to each other in the plane: for each of the six possible constellations (first and third column) the corresponding figure on the right shows the connected fold line resulting from the indicated eliminations

even number of cusps by assumption because m is even. Now in the preceding steps $s < r$ of the induction we always worked in the case $\frac{m}{2} \notin A_s$ above. In this case cusps were always eliminated in pairs, so that the parity of all cusps has not changed in the sequence $G = G^{(0)}, \ldots, G^{(r-1)}$. The same holds for the construction of H' from H, where H has the same number of cusps as $G^{(r-1)}$.) Thus, we end up with a fold map $H_1' \colon U_s \to \mathbb{R}^2$ whose fold lines have absolute index $\frac{m}{2}$.

Step 2. On each loop of H_1' one introduces a new pair of cusps of index $\frac{m}{2}-1$. Afterwards, one eliminates pairs of cusps in such a way that one ends up with a generic smooth map $U_s \to \mathbb{R}^2$ with at most one closed singular component (namely one if there are no singular components with nonempty boundary, and no closed singular component otherwise) and exactly two cusps (both of index $\frac{m}{2} - 1$) lying on the same singular component C (of absolute index $\frac{m}{2}$). Finally, we introduce a second pair of cusps of index $\frac{m}{2} - 1$ on C to obtain a generic smooth map $H_2' \colon U_s \to \mathbb{R}^2$ with at most one closed singular component (namely one if there are no singular components with nonempty boundary, and no closed singular component otherwise) and exactly four cusps (all of index $\frac{m}{2} - 1$) lying on C (of absolute index $\frac{m}{2}$).

Step 3. For the image of C under H_2' in the plane we can distinguish (up to symmetry) between six cases depending on how the four cusps are connected with each other by $H_2'(C)$ (see Figure 6.1). In each of these cases we are able to eliminate the cusps in two pairs in such a way that the resulting fold map $H'' \colon U_r \to \mathbb{R}^2$ has the desired properties.

□

6.2 Cusps and Euler Characteristic

Lemma 6.2.1. *Let* $f\colon (W, M_0, M_1) \to ([0,1], 0, 1)$ *be a Morse function on a smooth manifold triad. Then the number of critical points of* f *has the same parity as* $\chi(W) + \chi(M_0)$ *.*

Proof. By the alternate version of the rearrangement theorem [41, Theorem 4.8, p. 44] we may assume that f is a self-indexing Morse function (see [41, Definition 4.9, p. 44]). It is well-known that there exists a chain complex of free abelian groups (or $\mathbb{Z}$-modules)

$$C_{n-2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{r+1} \xrightarrow{\partial} C_r \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2$$

such that the rank of C_λ is given by the number of critical points of f of index λ (see [41, page 89] and [41, Section 3, page 36]), and $H_*(C_*) \cong H_*(W, M_0)$ (see [41, Theorem 7.4, page 90]). Hence, the proof of [20, Theorem 2.44, p. 146f] implies the formular

$$\sum_i (-1)^i \operatorname{rank} C_i = \sum_i (-1)^i \operatorname{rank} H_i(C_*).$$

The long exact homology sequence of the pair (W, M_0) yields

$$\sum_i (-1)^i \operatorname{rank} H_i(W, M_0) = \sum_i (-1)^i \operatorname{rank} H_i(W) - \sum_i (-1)^i \operatorname{rank} H_i(M_0) = \chi(W) - \chi(M_0).$$

(Indeed, consider a bounded exact sequence of finitely generated abelian groups of the form

$$\cdots \xrightarrow{\beta_{i+1}} C_{i+1} \xrightarrow{\gamma_{i+1}} A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \xrightarrow{\gamma_i} A_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

The three exact sequences

$$\begin{aligned} 0 &\to \operatorname{im}\gamma_{i+1} \to A_i \xrightarrow{\alpha_i|} \operatorname{im}\alpha_i \to 0, \\ 0 &\to \operatorname{im}\alpha_i \to B_i \xrightarrow{\beta_i|} \operatorname{im}\beta_i \to 0, \\ 0 &\to \operatorname{im}\beta_i \to C_i \xrightarrow{\gamma_i|} \operatorname{im}\gamma_i \to 0 \end{aligned}$$

imply that

$$\begin{aligned} \operatorname{rank} A_i &= \operatorname{rank}\operatorname{im}\gamma_{i+1} + \operatorname{rank}\operatorname{im}\alpha_i, \\ \operatorname{rank} B_i &= \operatorname{rank}\operatorname{im}\alpha_i + \operatorname{rank}\operatorname{im}\beta_i, \\ \operatorname{rank} C_i &= \operatorname{rank}\operatorname{im}\beta_i + \operatorname{rank}\operatorname{im}\gamma_i. \end{aligned}$$

Hence, by boundedness of the exact sequence,

$$\sum_i (-1)^i \operatorname{rank} B_i = \sum_i (-1)^i \operatorname{rank} A_i + \sum_i (-1)^i \operatorname{rank} C_i$$

Setting $A_i := H_i(M_0)$, $B_i := H_i(W)$ and $C_i := H_i(W, M_0)$ yields the claim.)

The claim now follows from the fact that $\operatorname{rank} C_i$ is the number of critical points of f of index i. □

For a unit vector $v \in S^1 \subset \mathbb{R}^2$ let $\pi_v \colon \mathbb{R}^2 \to \mathbb{R}$ denote the linear projection given by $\pi_v(w) = v \cdot w$ for all $w \in \mathbb{R}^2$.

Lemma 6.2.2. *Let (W, W_0, W_1) be a smooth manifold triad of dimension $m = \dim W \geq 2$. Suppose that $F \colon W \to \mathbb{R}^2$ is a generic smooth map such that*

$$F|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)} \colon \qquad W_0 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),$$
$$F|_{W_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]} \colon \qquad W_1 \times (1-\varepsilon,1] \to \mathbb{R} \times (1-\varepsilon,1],$$

where $f_0 \colon W_0 \to \mathbb{R}$ and $f_1 \colon W_1 \to \mathbb{R}$ are Morse functions, and $W_0 \times [0,\varepsilon)$ and $W_1 \times (1-\varepsilon,1]$ are suitable collar neighbourhoods of $W_0 \times 0 = W_0 \subset W$ and $W_1 \times 1 = W_1 \subset W$, $\varepsilon > 0$. Furthermore, suppose that $F^{-1}(\mathbb{R} \times [0,\varepsilon)) = W_0 \times [0,\varepsilon)$ and $F^{-1}(\mathbb{R} \times (1-\varepsilon,1]) = W_1 \times (1-\varepsilon,1]$. Then the following statements hold:

(a) There exists an open neighbourhood $V \subset S^1$ of $(0,1) \in S^1$ such that for every $v \in V$, the composition $\tau := \pi_v \circ F \colon W \to \mathbb{R}$ induces a smooth manifold triad

$$(W', W'_0, W'_1) := (\tau^{-1}([t_0, t_1]), \tau^{-1}(t_0), \tau^{-1}(t_1))$$

(for a suitable interval $[t_0, t_1] \subset \mathbb{R}$, $t_0 < t_1$) with the following properties:

(i) There exists a diffeomorphism $W \xrightarrow{\cong} W'$ that restricts to diffeomorphisms $W_i \xrightarrow{\cong} W'_i$ for $i = 0, 1$.

(ii) The composition $\tau = \pi_v \circ F \colon W \to \mathbb{R}$ restricts to a smooth map

$$\tau' \colon (W', W'_0, W'_1) \to ([t_0, t_1], t_0, t_1)$$

without critical points near the boundary, and such that $(\tau')^{-1}(t_i) = W'_i$ for $i = 0, 1$.

(iii) The restriction $F' := F|_{W'}$ gives rise to the smooth manifold triad

$$(S(F'), S(F') \cap W'_0, S(F') \cap W'_1).$$

Moreover, there exists a diffeomorphism $S(F') \xrightarrow{\cong} S(F)$ that restricts to diffeomorphisms $S(F') \cap W'_i \xrightarrow{\cong} S(F) \cap W_i$ for $i = 0, 1$. (Recall that $(S(F), S(F) \cap W_0, S(F) \cap W_1)$ is a smooth manifold triad.)

(b) There exists a dense subset $A \subset V$ such that for every $v \in A$, the map τ' of part (a)(ii) has the following properties:

(i) τ' is a Morse function such that every critical point of τ' of index $j \in \{0, \ldots, m\}$ is a fold point of F of absolute index

$$\begin{cases} \max\{j-1, m-j\} \text{ or } \max\{j, m-1-j\}, & \text{if } 0 < j < m, \\ m-1, & \text{if } j = 0 \text{ or } j = m. \end{cases}$$

(ii) τ' restricts to a Morse function

$$\tau'' \colon S(F) \cap (W', W'_0, W'_1) \to ([t_0, t_1], t_0, t_1)$$

whose set of critical points is the union of the cusps of F and the critical points of τ'.

Proof. (a). *Construction of* V. Choose $R > 0$ (see Figure 6.2) such that

$$F(W) \subset (-R, R) \times [0, 1].$$

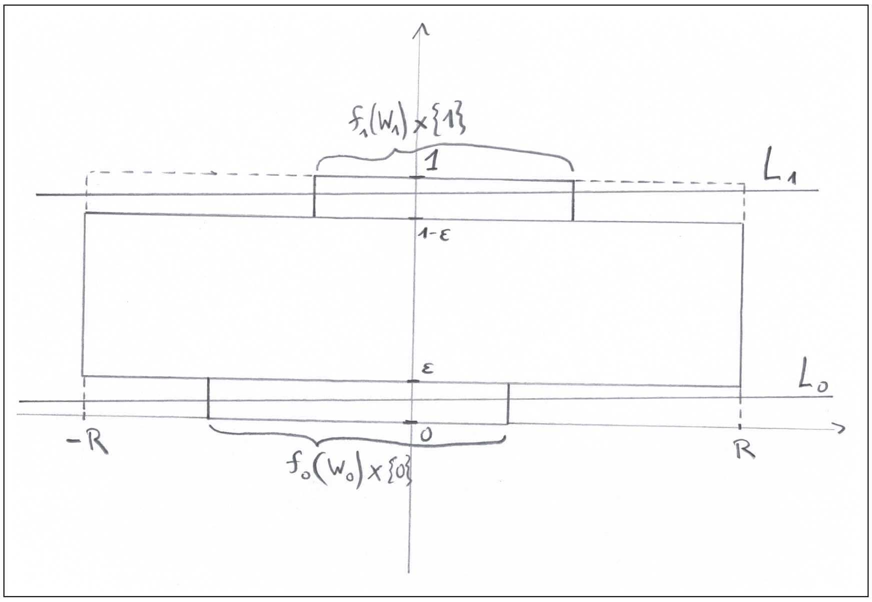

Figure 6.2: Existence of the lines L_0 and L_1

For suitable $u \in S^1 \setminus \{(0,1)\}$ (see Figure 6.2) there exist lines $L_0, L_1 \perp \mathbb{R}u$ such that

$$\begin{aligned} L_0 \cap [-R, R] \times [\varepsilon, 1-\varepsilon] = L_0 \cap f_1(W_1) \times (1-\varepsilon, 1] = \emptyset, \\ L_0 \cap f_0(W_0) \times [0, \varepsilon] \subset f_0(W_0) \times (0, \varepsilon), \\ L_1 \cap (f_0(W_0) \times [0, \varepsilon) = L_1 \cap [-R, R] \times [\varepsilon, 1-\varepsilon] = \emptyset, \\ L_1 \cap f_1(W_1) \times [1-\varepsilon, 1] \subset f_1(W_1) \times (1-\varepsilon, 1). \end{aligned}$$

If $t_i := \pi_u(L_i)$ for $i = 0, 1$, then $f_u^{-1}(t_i) = F^{-1}(\pi_u^{-1}(t_i)) = F^{-1}(L_i)$, where $f_u := \pi_u \circ F$. The above properties of the lines L_0 and L_1 imply that

$$f_u^{-1}(t_0) \subset W_0 \times (0, \varepsilon), \qquad f_u^{-1}(t_1) \subset W_1 \times (1-\varepsilon, 1).$$

(Indeed, if $w \in f_u^{-1}(t_0) = F^{-1}(L_0)$, then

$$\begin{aligned} F(w) &\in L_0 \cap F(W) \\ &\subset (L_0 \cap (f_0(W_0) \times [0, \varepsilon))) \cup (L_0 \cap ([R_-, R_+] \times [\varepsilon, 1-\varepsilon])) \cup (L_0 \cap (f_1(W_1) \times (1-\varepsilon, 1]))) \\ &= L_0 \cap (f_0(W_0) \times [0, \varepsilon]) \\ &\subset f_0(W_0) \times (0, \varepsilon). \end{aligned}$$

Hence, $w \in F^{-1}(\mathbb{R} \times [0, \varepsilon)) = W_0 \times [0, \varepsilon)$. Thus, it follows from $F|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ that $w \in W_0 \times (0, \varepsilon)$. Analogously, using $F^{-1}(\mathbb{R} \times (1-\varepsilon, 1]) = W_1 \times (1-\varepsilon, 1]$, one shows that $f_u^{-1}(t_1) \subset W_1 \times (1-\varepsilon, 1)$.) Set $\delta := ||(0,1) - u||$.

Define the desired open neighbourhood V of $(0,1)$ in S^1 by

$$V := \{v \in S^1;\ ||(0,1) - v|| < \delta\}.$$

Fix $v \in V$. We show first that (W', W_0', W_1') is a smooth manifold triad. For this purpose, observe that $\tau^{-1}([t_0, t_1]) = (\tau|_{W\setminus\partial W})^{-1}([t_0, t_1])$. (Indeed, it suffices to show for any $w \in W$ that $\tau(w) \in [t_0, t_1]$ implies $w \notin \partial W$. But this follows from $\pi_v(F(w)) = \tau(w) \in [t_0, t_1]$ and $\pi_v(F(\partial W)) = \pi_v(f_0(W_0) \times 0 \sqcup f_1(W_1) \times 1) \subset \mathbb{R} \setminus [t_0, t_1]$.) Next, note that t_0 and t_1 are regular values of $\tau|_{W\setminus\partial W}$. (Indeed, if $w \in (f_v|_{W\setminus\partial W})^{-1}(t_i) = f_v^{-1}(t_i)$, $i = 0, 1$, then w is either a regular point of F or a fold point of F such that $\mathrm{im}(dF|_{S(F)})_w$ is not perpendicular to v being parallel to the y-axis. In both cases it follows that w is a regular point of f_v.) Hence, [22, Exercise 5, p. 32] implies that W' is a smooth submanifold of W with boundary $W_0' \sqcup W_1'$. Moreover, $W' = \tau^{-1}([t_0, t_1])$ is compact being a closed subset of the compact space W. All in all, (W', W_0', W_1') is a smooth manifold triad. It remains to check properties (i) to (iii):

(i). Let c denote the cobordism from $(W/v)_0$ to $(W/v)_1$ given by $(W/v, (W/v)_0, (W/v)_1; \mathrm{id}, \mathrm{id})$ (see [41, Definition 1.5, p. 2]). Choose Morse functions with Morse number 0 on the two smooth manifold triads $(W_\leq, W_0, (W/v)_0)$ and $(W_\geq, (W/v)_1, W_1)$, where $W_\leq := (f/v)^{-1}((-\infty, t_0])$ and $W_\geq := (f/v)^{-1}([t_1, \infty))$. (Take the central projection to the line $\mathbb{R}(0,1)$ with centre $L_i \cap ((0,1) + \mathbb{R}(1,0))$.) Consequently, the tuples $(W_\leq, W_0, (W/v)_0; \phi_0, \mathrm{id})$ and $(W_\leq, (W/v)_1, W_1; \mathrm{id}, \phi_1)$ define the identity cobordism classes c_0 on $(W/v)_0$ and c_1 on $(W/v)_1$ for suitable diffeomorphisms $\phi_i \colon W_i \xrightarrow{\cong} (W/v)_i$, $i = 0, 1$. Therefore, the composition $c_0 c c_1$ of cobordism classes is on the one hand equal to c and on the other hand represented by the tuple $(W, W_0, W_1; \phi_0, \phi_1)$. Hence, there exists a diffeomorphism $W \cong W/v$ that restricts for $i = 0, 1$ to the diffeomorphisms $\phi_i \colon W_i \cong (W/v)_i$.

(ii). Note that the restriction $\tau' := \tau|_{W'} = (\tau|_{W\setminus\partial W})|_{W'}$ has no critical points near $\partial W'$ as t_0 and t_1 are regular values of $\tau|_{W\setminus\partial W}$. In addition, $W/v = f_v^{-1}([t_0, t_1])$ implies that $(f/v)^{-1}(t_i) = (f_v|_{W/v})^{-1}(t_i) = f_v^{-1}(t_i) = (W/v)_i$ for $i = 0, 1$.

(iii). This is clear from the behaviour of F near the boundary and the choice of v. As $S(F)$ is 1-dimensional, it is also not hard to construct the desired diffeomorphism.

(b). *Construction of A.* Recall that $S(F)$ is a 1-dimensional neat smooth submanifold of W. Let $S \subset W$ denote the set of fold points of F. In particular, $S(F) \setminus S$ is the (finite) set of cusps of F. Moreover, the restriction $\alpha \colon S \to \mathbb{R}^2$ of the fold map $F \colon W \to \mathbb{R}^2$ to S is an immersion. Since F is a fold map, we have:

(1) $T_sW = \ker(dF)_s \oplus T_sS$ for all $s \in S$.

If $G \colon S \to S^1$ denotes the Gauss map of the immersion $\alpha \colon S \to \mathbb{R}^2$, then we have:

(2) $G(s) \perp \mathrm{im}(d\alpha)_s$ for all $s \in S$.

Let $\iota \colon S^1 \to S^1$ be given by $\iota(x) = -x$ for all $x \in S^1$.

Recall that $\mathrm{im}(dF)_c$ is of dimension one for all cusps c of F. Define

$$U := \{v \in S^1;\ \mathrm{im}(dF)_c \cap \ker \pi_v = 0 \text{ for every cusp } c \text{ of } F\}$$
$$B := U \cap \{v \in S^1;\ v \text{ is a regular value of both } G \colon S \to S^1 \text{ and } \iota \circ G \colon S \to S^1\}$$

It follows from the Morse-Sard theorem that B is a dense subset of S^1. (In fact, by the Morse-Sard theorem, the sets of regular values of G and $\iota \circ G$ are both residual in S^1. Moreover, U is an open dense subset of S^1 being the complement of a finite subset of S^1. Hence, the intersection B is residual and in particular dense in S^1 containing a countable intersection of open dense subsets of S^1.) Consequently, $A := B \cap V$ is a dense subset of V (recall that V is an open subset of S^1). This completes the construction of A. Fix $v \in A$. For simplicity, we will also write $\pi := \pi_v$. It remains to check properties (i) and (ii):

(i). Let us show that τ' is a Morse function. By property $(a)(i)$ it suffices to show that all critical points of τ' on $W' \setminus \partial W'$ are non-degenerate. For this purpose, let $w \in W'$ such that $\tau(w) \in (t_0, t_1)$. By construction, $v \in A$ is a regular value of both $G\colon S \to S^1$ and $\iota \circ G\colon S \to S^1$. The subset

$$C := G^{-1}(v) \cup (\iota \circ G)^{-1}(v) = G^{-1}(\{-v, v\}) \qquad \subset S$$

has the following properties:

(C1) $(dG)_c \neq 0$ for all $c \in C$.
(C2) For $s \in S$ we have $\operatorname{im}(d\alpha)_s \subset \ker \pi$ if and only if $s \in C$.

Note that (C2) holds by construction of C. (Indeed, for $s \in S$, we have $s \in C$ if and only if $G(s) \in \mathbb{R}v$. By (2), the latter is equivalent to $\operatorname{im}(d\alpha)_s \perp \mathbb{R}v$.)

First, we show that C is the set of critical points of τ'. Since F restricts to a submersion $W \setminus S(F) \to \mathbb{R}^2$ and $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ is a submersion, it follows that the composition $f = \pi \circ F$ restricts to a submersion $W \setminus S(F) \to \mathbb{R}$. Moreover, if $s \in S \setminus C$, then $(df)_s\colon T_sW \to T_{f(s)}\mathbb{R}$ is surjective. (In fact, $(df)_s(T_sS) = (d\pi_v \circ dF)_s(T_sS) = (d\pi)_{\alpha(s)}((d\alpha)_s(T_sS)) = \pi((d\alpha)_s(T_sS)) \neq 0$, where the last inequality follows from (C2).) Furthermore, if c is a cusp of F, then $(df)_c\colon T_cW \to T_{f(c)}\mathbb{R}$ is surjective by construction. (Indeed, $(df)_c(T_cW) = (d\pi \circ dF)_c(T_cW) = (d\pi)_{F(c)}(\operatorname{im}(dF)_c) = \pi_v(\operatorname{im}(dF)_c) \neq 0$ because $v \in A \subset U$.) Finally, if $s \in C$, then $G(s) = \pm v$ and $df_v(T_sW) = (d\pi \circ dF)(T_sW) \overset{(1)}{=} (d\pi_v \circ d\alpha)(T_sS(F)) = \pm G(s) \cdot d\alpha(T_sS(F)) \overset{(2)}{=} 0$.

Next, we show that the critical points of τ' are nondegenerate. Let $c \in C$. Since $C \subset S(F)$, there exist charts $\varphi\colon U \to U' \subset \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ and $\psi\colon V \to V' \subset \mathbb{R}^2$, where $F(U) \subset V$, $c \in U$, $\varphi(c) = 0$, and such that, for some $i \in \{0, \dots, n-1\}$,

$$H = (H_1, H_2) := \psi \circ F \circ \varphi^{-1}\colon U' \to V', \qquad H(t, x) = (t, \lambda_i(x)).$$

In particular, the absolute index of F at c is given by $\max\{i, m-1-i\}$.

Define the compositions $p := \pi \circ \psi^{-1}\colon V' \to \mathbb{R}$ and $g := p \circ H = f \circ \varphi^{-1}\colon U' \to \mathbb{R}$. We have to consider the Hessian of g at $\varphi(c) = (0, 0) \in U' \subset \mathbb{R} \times \mathbb{R}^{n-1}$. For all $\mu \in \{1, ..., m-1\}$, we have by the chain rule

$$\begin{aligned}(\partial_{x_\mu} g)(t, x) &= (\partial_{x_\mu}(p \circ H))(t, x) \\ &= (\partial_1 p)(H(t, x)) \cdot \partial_{x_\mu}(H_1(t, x)) + (\partial_2 p)(H(t, x)) \cdot \partial_{x_\mu}(H_2(t, x)) \\ &= (\partial_1 p)(H(t, x)) \cdot \partial_{x_\mu}(t) + (\partial_2 p)(H(t, x)) \cdot \partial_{x_\mu}(\lambda_i(x)) \\ &= \sigma_i(\mu) 2x_\mu \cdot (\partial_2 p)(H(t, x)).\end{aligned}$$

Hence, for all $\mu, \nu \in \{1, \dots, m-1\}$, we obtain by the product rule

$$\begin{aligned}(\partial_{x_\nu}\partial_{x_\mu} g)(t, x) &= \partial_{x_\nu}(\sigma_i(\mu) 2x_\mu \cdot (\partial_2 p)(H(t, x))) \\ &= \sigma_i(\mu) 2\delta_{\mu\nu} \cdot (\partial_2 p)(H(t, x)) + \sigma_i(\mu) 2x_\mu \cdot (\partial_{x_\nu}((\partial_2 p) \circ H))(t, x).\end{aligned}$$

Moreover, the product rule also yields

$$\begin{aligned}(\partial_t\partial_{x_\mu} g)(t,x) &= \partial_t(\sigma_i(\mu)2x_\mu \cdot (\partial_2 p)(H(t,x)))\\ &= \sigma_i(\mu)2x_\mu \cdot (\partial_t((\partial_2 p)\circ H))(t,x).\end{aligned}$$

This shows that the Hessian $H_{(0,0)}(g)$ of g at $(0,0) \in U'$ has the form

$$H_{(0,0)}(g) = \operatorname{diag}\Big((\partial_t^2 g)(0,0) \quad 2\sigma_i(1)\cdot(\partial_2 p)(0,0) \quad \ldots \quad 2\sigma_i(n)\cdot(\partial_2 p)(0,0)\Big).$$

We have to show that all diagonal entries of $H_{(0,0)}(g)$ are nonzero. To see this for the last $(n-1)$ diagonal entries, it suffices to show that $(\partial_2 p)(0,0) \neq 0$. This can be shown in the following way. Since c is a critical point of f and $\varphi\colon U \to U'$ is a diffeomorphism, we conclude that $\varphi(c) = (0,0)$ is a critical point of $f \circ \varphi^{-1} = g = p \circ H\colon U' \to \mathbb{R}$. Hence, the chain rule implies $0 = (dg)_{(0,0)} = (dp)_{H(0,0)} \circ (dH)_{(0,0)} = J(p,(0,0))J(H,(0,0))$. This yields

$$0 = \Big((\partial_1 p)(0,0) \quad (\partial_2 p)(0,0)\Big)\begin{pmatrix}1 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0\end{pmatrix} = \Big((\partial_1 p)(0,0) \quad 0 \quad \ldots \quad 0\Big).$$

This shows that $(\partial_1 p)(0,0) = 0$. One concludes that $(\partial_2 p)(0,0) \neq 0$, since $p = \pi \circ \psi^{-1}$ is a submersion, being the composition of the diffeomorphism $\psi^{-1}\colon V' \to V$ and the linear projection $\pi\colon \mathbb{R}^2 \to \mathbb{R}$.

It remains to show that $(\partial_t^2 g)(0,0) = (\partial_t^2 g)(t,0)|_{t=0}$ does not vanish. Note that $\varphi(S(F) \cap U) = \varphi(S(F|_U)) = S(H) = (\mathbb{R} \times \{0\}) \cap U'$. Thus, there exists $\varepsilon' > 0$ and an embedding $\gamma\colon (-\varepsilon',\varepsilon') \to S(F)$ such that $\varphi^{-1}(t,0) = \gamma(t)$ for all $t \in (-\varepsilon',\varepsilon')$. In particular, $\gamma(0) = \varphi^{-1}(0,0) = c$. Choose a chart $\xi\colon S \to R \subset \mathbb{R}$ such that $c \in S$ and $\xi(c) = 0$. We may assume that $\gamma(-\varepsilon',\varepsilon') \subset S$ and set $\gamma^\xi := \xi \circ \gamma\colon (-\varepsilon',\varepsilon') \to \mathbb{R}$. For all $t \in (-\varepsilon',\varepsilon')$ we have

$$g(t,0) = (f \circ \varphi^{-1})(t,0) = (f \circ \gamma)(t) = (\pi \circ F \circ \gamma)(t) = \pi(\alpha(\gamma(t))) = \pi(\alpha_\xi(\gamma^\xi(t))).$$

Therefore, using the linearity of $\pi\colon \mathbb{R}^2 \to \mathbb{R}$,

$$\begin{aligned}(\partial_t^2 g)(t,0) &= \partial_t^2(\pi \circ \alpha_\xi \circ \gamma^\xi)(t)\\ &= \pi((\partial_t^2(\alpha_\xi \circ \gamma^\xi))(t))\\ &= \pi((\partial_t((\gamma^\xi)'(t)\alpha_\xi'(\gamma^\xi(t))))(t))\\ &= \pi((\gamma^\xi)''(t)\alpha_\xi'(\gamma^\xi(t)) + ((\gamma^\xi)'(t))^2\alpha_\xi''(\gamma^\xi(t)))\\ &= (\gamma^\xi)''(t)\pi(\alpha_\xi'(\gamma^\xi(t))) + ((\gamma^\xi)'(t))^2\pi(\alpha_\xi''(\gamma^\xi(t))).\end{aligned}$$

Using $\gamma^\xi(0) = \xi(\gamma(0)) = \xi(c) = 0$, we obtain

$$(\partial_t^2 g)(0,0) = (\gamma^\xi)''(0)\pi(\alpha_\xi'(0)) + ((\gamma^\xi)'(0))^2\pi(\alpha_\xi''(0)).$$

Since $\xi^{-1}(0) = c \in C$, it follows from (C2) that $\pi(\alpha_\xi'(0)) = 0$. Moreover, it follows from (C1) that $\pi(\alpha_\xi''(0)) \neq 0$. (In fact, it follows from $||G(s)|| = 1$ for all $s \in S(F)$ that $G(c) \perp G_\xi'(0)$. We have $\alpha_\xi'(0) \neq 0$, since α is an immersion. By (2), we have $G(c) \perp \alpha_\xi'(0)$. As $G_\xi'(0) \neq 0$ by (C1), there exists a scalar $0 \neq \kappa \in \mathbb{R}$, such that $\alpha_\xi'(0) = \kappa G_\xi'(0)$. By (2), we have $G_\xi(t) \cdot \alpha_\xi'(t) = 0$ for all $t \in (-\varepsilon',\varepsilon')$. This yields $G_\xi'(0) \cdot \alpha_\xi'(0) + G(c) \cdot \alpha_\xi''(0) = 0$. Hence,

$\pi(\alpha''_\xi(0)) = \pm G(c)\cdot\alpha''_\xi(0) = \mp G'_\xi(0)\cdot\alpha'_\xi(0) = \mp\kappa(G'_\xi(0))^2 \neq 0$.) In conclusion, $(\partial_t^2 g)(0,0)\neq 0$. The index j of f at c is the number of negative diagonal entries of $H_{(0,0)}(g)$. If $(\partial_2 p)(0,0) > 0$, then $j\in\{i,i+1\}$, depending on the sign of $(\partial_t^2 g)(0,0)$. If $(\partial_2 p)(0,0) < 0$, then $j\in\{n-1-i, n-i\}$, depending on the sign of $(\partial_t^2 g)(0,0)$.

(ii). First of all, $S(F)\cap(W', W'_0, W'_1)$ is a smooth manifold triad by part $(a)(iii)$. It is clear that τ' restricts to a smooth map $\tau''\colon S(F)\cap(W',W'_0,W'_1)\to([t_0,t_1],t_0,t_1)$ without critical points near the boundary and such that $(\tau'')^{-1}(t_i) = S(F)\cap W'_i$, $i=0,1$. Given a point $s\in S(F)\cap W'$, we distinguish between the following cases:

- s is a fold point of F and no critical point of τ', i.e. $s\in S\setminus C$. In this case, it was shown in the proof of part (b)(i) that $(d\tau')_s(T_sS)\neq 0$. This shows that s is a regular point of $\tau'' = \tau'|_{S(F)\cap W'}$.
- s is a critical point of τ', i.e. $s\in C$. Thus, $\ker(d\tau')_s = T_sW'$, which shows that s is also a singular point of $\tau''=\tau'|_{S(F)\cap W'}$. It remains to show that s is a non-degenerate critical point of τ''. For this purpose, recall from the proof of part (b)(i) that there exists an embedding $\gamma\colon(-\lambda,\lambda)\to S$ such that $\gamma(0)=s$ and $\hat g(t) := (\tau\circ\gamma)(t)$ satisfies $(\partial_t^2\hat g)(0)\neq 0$. This shows that s is a non-degenerate critical point of τ''.
- $s=c$ is a cusp of F. In this case, choose local coordinates $\varphi = (t, x_1, x_2, \dots, x_{m-1})$ around $(0,0,\dots,0)=\varphi(s)\in W$ and $\tilde\varphi=(p,q)$ around $(0,0)=\tilde\varphi(F(s))\in\mathbb{R}^2$ such that F takes the form

$$\hat F := \tilde\varphi\circ F\circ\varphi^{-1}\colon (t,x_1,\dots,x_{m-1})\mapsto (t, tx_1 + x_1^3 + Q(x_2,\dots,x_{m-1})).$$

The Jacobian at $(t,x_1,\dots,x_{m-1})$ of this local normal form of the cusp is given by

$$J(\hat F,(t,x_1,\dots,x_{m-1})) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ x_1 & t+3x_1^2 & 2\sigma_2x_2 & \dots & 2\sigma_{m-1}x_{m-1}\end{pmatrix}$$

The singular locus of $\hat F$ is parametrized by the embedding $\gamma\colon(-\lambda,\lambda)\to\mathbb{R}^m$, $u\mapsto(-3u^2,u,0,\dots,0)$, and $\Gamma(u) := (\hat F\circ\gamma)(u) = (-3u^2,-2u^3)$.
Hence, the embedding $\varphi^{-1}\circ\gamma\colon(-\lambda,\lambda)\to W$ parametrizes $S(F)$ around the cusp $c = (\varphi^{-1}\circ\gamma)(0)$.
It suffices to show that the composition $\tilde\tau\colon(-\lambda,\lambda)\to\mathbb{R}^2$ given by

$$\tilde\tau := \tau\circ(\varphi^{-1}\circ\gamma) = \pi_v\circ\tilde\varphi^{-1}\circ\hat F\circ\gamma = \pi_v\circ\tilde\varphi^{-1}\circ\Gamma$$

satisfies $\tilde\tau'(0)=0$ and $\tilde\tau''(0)\neq 0$. Writing $\tilde\varphi^{-1} = (\alpha,\beta)\colon\mathbb{R}^2\to\mathbb{R}^2$, one obtains

$$\tilde\tau'(0) = \frac{d}{du}\pi_v\begin{pmatrix}\alpha(\Gamma(u))\\ \beta(\Gamma(u))\end{pmatrix}|_{u=0} = -v\cdot\begin{pmatrix}\partial_1\alpha(\Gamma(u))\cdot 6u + \partial_2\alpha(\Gamma(u))\cdot 6u^2\\ \partial_1\beta(\Gamma(u))\cdot 6u + \partial_2\beta(\Gamma(u))\cdot 6u^2\end{pmatrix}|_{u=0} = 0.$$

Note that $v\cdot\operatorname{im}(dF)_c\neq 0$ implies

$$\begin{aligned}0 &\neq v\cdot\operatorname{im}J(\tilde\varphi^{-1}\circ\hat F,(0,0,\dots,0)) = v\cdot J((\alpha,\beta),(0,0))\cdot\operatorname{im}J(\hat F,(0,0,\dots,0))\\ &= \mathbb{R}\cdot v\cdot\begin{pmatrix}\partial_1\alpha((0,0)) & \partial_2\alpha((0,0))\\ \partial_1\beta((0,0)) & \partial_2\beta((0,0))\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix} = \mathbb{R}\cdot v\cdot\begin{pmatrix}\partial_1\alpha((0,0))\\ \partial_1\beta((0,0))\end{pmatrix}.\end{aligned}$$

Therefore,

$$\tilde{\tau}''(0) = -v \cdot \frac{d}{du}\left(u \cdot \begin{pmatrix} \partial_1\alpha(\Gamma(u)) \cdot 6 + \partial_2\alpha(\Gamma(u)) \cdot 6u \\ \partial_1\beta(\Gamma(u)) \cdot 6 + \partial_2\beta(\Gamma(u)) \cdot 6u \end{pmatrix}\right) |_{u=0} = -6v \cdot \begin{pmatrix} \partial_1\alpha((0,0)) \\ \partial_1\beta((0,0)) \end{pmatrix} \neq 0.$$

□

The following Proposition is implicitly shown in [24].

Proposition 6.2.3. *Let (W, W_0, W_1) be a smooth manifold triad. Suppose that $F\colon W \to \mathbb{R}^2$ is a generic smooth map such that*

$$F|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}\colon \qquad W_0 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),$$
$$F|_{W_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}\colon \qquad W_1 \times (1-\varepsilon,1] \to \mathbb{R} \times (1-\varepsilon,1],$$

where $f_0\colon W_0 \to \mathbb{R}$ and $f_1\colon W_1 \to \mathbb{R}$ are Morse functions, and $W_0 \times [0,\varepsilon)$ and $W_1 \times (1-\varepsilon, 1]$ are suitable collar neighbourhoods of $W_0 \times 0 = W_0 \subset W$ and $W_1 \times 1 = W_1 \subset W$, $\varepsilon > 0$.

Let k denote the number of components of the compact 1-dimensional smooth manifold $S(F)$ that are diffeomorphic to $[0,1]$. Moreover, let c denote the number of cusps of F. Then

$$\chi(W) \equiv c + k \pmod 2.$$

Proof. By a modification of F as in the proof of [4, Theorem 10.2, p. 81f.] we may assume that $F^{-1}(\mathbb{R} \times [0,\varepsilon)) = W_0 \times [0,\varepsilon)$ and $F^{-1}(\mathbb{R} \times (1-\varepsilon, 1]) = W_1 \times (1-\varepsilon, 1]$. Note that this does not change k and c.

By Lemma 6.2.2 there exists $v \in S^1$ such that all properties of the lemma hold.

We apply Lemma 6.2.1 thrice:

- Application to the Morse function

  $$f_0\colon W_0 \to \mathbb{R}$$

  on the closed manifold W_0 yields

  $$N_0 \equiv \chi(W_0) \qquad \pmod 2,$$

  where N_0 denotes the number of critical points of f_0.
- Application to the Morse function

  $$\tau'\colon (W', W'_0, W'_1) \to ([t_0, t_1], t_0, t_1)$$

  yields

  $$N' \equiv \chi(W') + \chi(W'_0) \qquad \pmod 2,$$

  where N' denotes the number of critical points of τ'.
- Application of Lemma 6.2.1 to the Morse function

  $$\tau''\colon S(F) \cap (W', W'_0, W'_1) \to ([t_0, t_1], t_0, t_1)$$

yields

$$N'' \equiv \chi(S(F) \cap W') + \chi(S(F) \cap W_0') \qquad (\mathrm{mod}\, 2),$$

where N'' denotes the number of critical points of τ''.

Addition of the three obtained equations yields

$$N_0 + N' + N'' + \chi(W_0) + \chi(W') + \chi(W_0') + \chi(S(F) \cap W') + \chi(S(F) \cap W_0') \equiv 0 \qquad (\mathrm{mod}\, 2).$$

This equation can now be simplified by using various properties of Lemma 6.2.2:

(1) By Property (b)(ii), $N'' = c + N'$.

(2) By Property (a)(iii), $\chi(S(F) \cap W') = \chi(S(F)) = k$. (For the second equality, note that $S(F)$ is a compact smooth 1-dimensional manifold. Each component J of $S(F)$ is either diffeomorphic to the circle S^1 or to the interval $[0,1]$. If $J \cong S^1$, then $\chi(J) = 0$. If $J \cong [0,1]$, then $\chi(J) = 1$. Hence, $\chi(S(F))$ is equal to the number k of components of $S(F)$ that are diffeomorphic to the interval.)

(3) By Property (a)(iii), $\chi(S(F) \cap W_0') = \chi(S(F) \cap W_0) = N_0$. (Note that the Euler characteristic of a finite discrete set is equal to its cardinality, and the cardinality of $S(F) \cap W_0$ is given by the number N_0 of critical points of f_0 because $F|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$.)

(4) By Property (a)(i), $\chi(W_0') = \chi(W_0)$.

(5) By Property (a)(i), $\chi(W') = \chi(W)$.

Finally,

$$\begin{aligned} 0 &\cong N_0 + N' + N'' + \chi(W_0) + \chi(W') + \chi(W_0') + \chi(S(F) \cap W') + \chi(S(F) \cap W_0') \\ &\overset{(1)}{\cong} N_0 + c + \chi(W_0) + \chi(W') + \chi(W_0') + \chi(S(F) \cap W') + \chi(S(F) \cap W_0') \\ &\overset{(2)}{\cong} N_0 + c + \chi(W_0) + \chi(W') + \chi(W_0') + k + \chi(S(F) \cap W_0') \\ &\overset{(3)}{\cong} c + \chi(W_0) + \chi(W') + \chi(W_0') + k \\ &\overset{(4)}{\cong} c + \chi(W') + k \\ &\overset{(5)}{\cong} c + \chi(W) + k. \end{aligned}$$

□

An important special case of the previous Proposition is the following

Corollary 6.2.4. *Let (W, W_0, W_1) be a smooth manifold triad and let $f_0 \colon W_0 \to \mathbb{R}$ and $f_1 \colon W_1 \to \mathbb{R}$ be Morse functions. Suppose that $F \colon W \to \mathbb{R}^2$ is a generic smooth map such that*

$$\begin{aligned} F|_{W_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)} &\colon \qquad W_0 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon), \\ F|_{W_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]} &\colon \qquad W_1 \times (1-\varepsilon,1] \to \mathbb{R} \times (1-\varepsilon,1], \end{aligned}$$

where $W_0 \times [0,\varepsilon)$ and $W_1 \times (1-\varepsilon,1]$ are suitable collar neighbourhoods of $W_0 \times 0 = W_0 \subset W$ and $W_1 \times 1 = W_1 \subset W$, $\varepsilon > 0$.

*If f_0 and f_1 have the same number of critical points and $(W, W_0, W_1) = (W_0 \times [0,1], W_0 \times$

$0, W_0 \times 1)$ *is a cylinder, then* F *has an even number of cusps.*

Proof. First of all, $\chi(W) = \chi(W_0)$ holds because $W = W_0 \times [0,1] \simeq W_0$. Let k denote the number of components of the compact 1-dimensional smooth manifold $S(F)$ that are diffeomorphic to the interval $[0,1]$. As f_0 and f_1 have the same number of critical points, this number is also equal to k. Application of Lemma 6.2.1 to the Morse function $f_0 \colon W_0 \to \mathbb{R}$ on the closed manifold W_0 yields $k \equiv \chi(W_0) \pmod 2$. Furthermore, if c denotes the number of cusps of F, then Proposition 6.2.3 implies that $\chi(W) \equiv c + k \pmod 2$. All in all, $c \equiv 0 \pmod 2$. □

6.3 Implications for Higher-Dimensional State Sets

Fix an integer $m \geq 3$. In the present section some direct consequences on the state sets of a connected m-dimensional cobordism W from M to N are formulated.

In an analogous manner to Theorem 5.1.3 there is the following criterion for the vanishing of state sets (compare also Remark 5.3.3 (i)).

Theorem 6.3.1. *For all boundary conditions $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ and any $\varphi \in \mathrm{OP}_{m_S,n_S}$ the following statements are equivalent:*

(i) $L_W(f_M, f_N; \varphi) \neq \emptyset$.
(ii) $t_W(f_M, f_N) = 0$, *and φ is index-preserving.*

As a reformulation of Proposition 6.2.3, one obtains the following higher-dimensional version of Theorem 5.1.1 in the special case that the boundary conditions are suspensions of Morse functions. The computation of the cusp invariant for general boundary conditions remains open.

Theorem 6.3.2. *Suppose that $f_M \colon M \to \mathbb{R}$ and $f_N \colon N \to \mathbb{R}$ are excellent Morse functions with m_S and n_S critical points. Let $k_S := (m_S + n_S)/2$. Then, the value of the cusp invariant on the suspensions $(\overline{f}_M, \overline{f}_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$ is given by*

$$t_W(\overline{f}_M, \overline{f}_N) = \overline{\chi(W)} + \overline{k_S} \in \mathbb{Z}/2.$$

One notable consequence of Proposition 6.1.3 is the following result, which implies that state sums are rational with *linear* denominator (compare [4, Theorem 8.3, p. 71]), and the degree of the polynomial in the nominator can be estimated in terms of the dimension m of W.

Theorem 6.3.3. *Let $(f_M, f_N) \in \mathcal{F}(M) \times \mathcal{F}(N)$, and let $\varphi \in \mathrm{OP}_{m_S,n_S}$ be an open Brauer morphism such that $L_W(f_M, f_N; \varphi) \neq \emptyset$ (see Theorem 6.3.1 (i)). Let r denote the cardinality of the set of absolute indices that occur among the fold lines of f_M and f_N. Then:*

(i) Given a fold pre-field $F \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ such that $S(F) \neq \emptyset$, there exists a fold pre-field $F^1 \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ such that $\mathbb{S}(F^1) = \mathbb{S}(F) \otimes \lambda$ (compare [5, Lemma 8.1, page 67]).
(ii) There exists a fold pre-field $G \in \mathcal{F}^{\mathrm{pre}}(W; f_M, f_N)$ such that $\mathbb{S}(G) = \varphi \otimes \lambda^{\lfloor (m-1)/2 \rfloor + 1 - r}$.

Consequently,

$$\lfloor (m-1)/2 \rfloor + 1 - r + \mathbb{N} \subset L_W(f_M, f_N; \varphi).$$

Remark 6.3.4. The complete computation of the state sets $L_W(f_M, f_N; \varphi)$ remains an open problem. Note that Eliashberg's method for the construction of fold maps [14] cannot be used for this purpose since it requires the presence of all absolute indices in the singular set.

Part III

Detecting Exotic Smooth Structures on Spheres via Indefinite Fold Singularities

Chapter 7

Constructing Fold Maps from Cobordisms into the Plane

The main result of the present chapter is the following

Theorem 7.0.1. *Fix integers* $m \geq 8$, $k \in \{4, \ldots, \lfloor \frac{m}{2} \rfloor\}$ *and* $\lambda \in \{k, \ldots, m-k\}$. *Suppose that* (W, W_0, W_1) *is a smooth manifold triad of dimension* $m = \dim W$ *(see Figure 7.1) such that* W_0 *and* W_1 *are* $(k-2)$*-connected. Furthermore, let*

$$\tau \colon (W, W_0, W_1) \to ([0,1], 0, 1)$$

be a Morse function with only critical points of index λ *that are all contained in* $\tau^{-1}(1/2)$. *Then there exists a smooth map*

$$\sigma \colon W \to \mathbb{R}$$

with the following properties:

(i) σ *restricts for every* $t \in [0,1] \setminus \{1/2\}$ *to an excellent Morse function* $\tau^{-1}(t) \to \mathbb{R}$.

(ii) σ *and* τ *form the components of a fold map*

$$F := (\sigma, \tau) \colon W \to \mathbb{R} \times [0,1],$$

and the absolute index of every fold line of F *is contained in* $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-k\} \cup \{m-1\}$.

The proof of Theorem 7.0.1 (which will be given in Section 7.3) makes use of techniques due to Gay and Kirby [28], notably *standard Morse functions* (see [28, p. 26]) and *forward handles* (see [28, Fig. 29, p. 44]). Their original intention is to navigate generically between *Morse* 2*-functions*, i.e. generic smooth maps from a cobordism into a 2-dimensional manifold. Motivated by the study of Lefschetz fibrations in the context of complex and symplectic geometry, they focus on Morse 2-functions without definite fold points and with connected fibers. However, according to [28, Remark 1.6, p. 8], they do not impose further constraints on the occuring indefinite absolute indices of fold points (as of interest in Theorem 7.0.1).

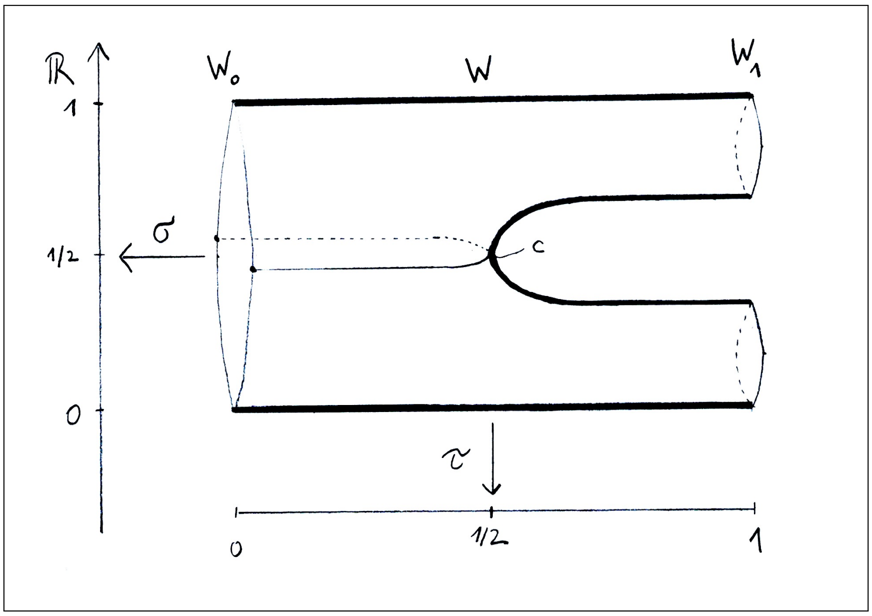

Figure 7.1: Construction of σ as the height function on a 2-dimensional cobordism W. The fold lines of (σ, τ) are marked as bold lines. Note that σ restricts to an excellent Morse function on W_0 that is standard with respect to the left-hand sphere of the critical point c of τ in W_0.

The main ideas that will be used for the construction of the desired smooth map σ can be illustrated on a 2-dimensional cobordism W by means of Figure 7.1, where σ could be taken to be the *height function*. Let c be a critical point of τ (of index λ). In some local chart centered at c in which τ has the usual normal form of a Morse critical point, consider the so-called *forward* λ*-handles* (see [28, Fig. 29, p. 44])

$$\mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda} \to \mathbb{R}, \qquad (x, y) \mapsto (y_1, -||x||^2 + ||y||^2 + 1/2),$$

which can be shown to be a fold map with a single fold line, namely the y_1-axis, of absolute index $\max\{\lambda, m-1-\lambda\}$. Still working in the local chart around c, the proof of Proposition 7.2.6 in Section 7.2 uses a bump function to modify this forward handle outside a compact neighbourhood of the origin in a way that is convenient for extending it to the desired function σ on all of W. Then the use of integral curves of a gradient-like vector field of τ reduces this extension problem for suitable $t_- \in (0, 1/2)$ to the construction of an excellent Morse function $\sigma_- \colon \tau^{-1}(t_-) \to \mathbb{R}$ with index constraints such that σ_- is in addition *standard* (see [28, p. 26] and Definition 7.1.2 of Section 7.1) with respect to the left-hand spheres of the critical points of τ. Requiring σ_- to be standard is needed to fit it together with the forward handles that have been constructed locally around every critical point of τ. Finally, as indicated in Figure 7.1, the fold lines of σ will correspond to the suspended fold points of σ_-, plus one new indefinite fold line of absolute index $\max\{\lambda, m-1-\lambda\}$ per critical point of τ coming from the forward handles.

7.1 Standard Morse Functions

Consider a Morse function

$$f\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

on a smooth manifold triad (Y, Y^0, Y^1) (see [41, Definition 1.3, p. 2]) of dimension $n = \dim Y$.

Definition 7.1.1. Following [28, Definition 2.11, page 17], we call the Morse function f *indefinite* if it has neither critical points of index 0 nor of index n. Furthermore, following [28, Definition 2.12, page 17], f is called *ordered* if $f(p) < f(q)$ whenever p and q are critical points of f such that the index of p is strictly smaller than the index of q. We call f *well-ordered* if f is ordered and, in addition, $f(p) = f(q)$ whenever p and q are critical points of f of the same index.

Suppose that (ϕ, z) is a pair consisting of an embedding

$$\phi\colon L \times \operatorname{int}(\varepsilon \cdot D^{n-d}) \to Y \setminus \partial Y, \qquad \varepsilon > 0,$$

where L denotes a closed smooth manifold of dimension d, and a real number $z \in (0,1)$.

Definition 7.1.2. Following [28, page 26], the Morse function f is called (ϕ, z)*-standard* if, for some $\varepsilon' \in (0, \varepsilon)$,

$$f(\phi(u,v)) = v_1 + z, \qquad (u,v) \in L \times \operatorname{int}(\varepsilon' \cdot D^{n-d}).$$

In particular, if f is (ϕ, z)-standard, then the submanifold $\phi(L \times 0) \subset Y$ lies in the fiber $f^{-1}(z)$ of f. Furthermore, the tubular neighbourhood of $\phi(L \times 0)$ in Y induced by ϕ is nicely compatible with f in such a way that f has no critical points on the image $\phi(L \times \operatorname{int}(\varepsilon \cdot D^{n-d}))$.

The notion of a standard Morse function will eventually be brought to bear in the case where $L = S^d$ and the embedding ϕ plays the part of an attaching map of a $(d+1)$-handle.

We need the following result on the existence of standard Morse functions with index constraints:

Lemma 7.1.3. *Let (Y, Y^0, Y^1) be a smooth manifold triad of dimension $n = \dim Y \geq 7$. Suppose that Y, Y^0 and Y^1 are nonempty and simply connected. Let $l \in \{3, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ be an integer such that*

$$(*) \qquad H_i(Y, Y^j) = 0, \qquad i = 0, \ldots, l-1, \quad j = 0, 1.$$

(All homology groups in the present statement are taken with integer coefficients.)

Let $\mathcal{C}$ be a finite set. Suppose that L_c, $c \in \mathcal{C}$, are closed smooth manifolds of the same dimension $d \in \{l, \ldots, \lceil \frac{n}{2} \rceil - 1\}$. For every $c \in \mathcal{C}$ let $z_c \in (0,1)$ be a real number such that $z_c \neq z_{c'}$ for $c \neq c'$. Furthermore, for some $\varepsilon > 0$, suppose that

$$\phi_c\colon L_c \times \operatorname{int}(\varepsilon \cdot D^{n-d}) \to Y \setminus \partial Y, \qquad c \in \mathcal{C},$$

are pairwise disjoint embeddings. Then, there exists an excellent Morse function

$$f\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

with the following properties:

(i) For every $c \in \mathcal{C}$, f *is* (ϕ_c, z_c)*-standard, i.e. there exists* $\varepsilon' \in (0, \varepsilon)$ *such that*

$$f(\phi_c(u,v)) = v_1 + z_c, \qquad (u,v) \in L_c \times \operatorname{int}(\varepsilon' \cdot D^{n-d}).$$

(ii) All indices of critical points of f *are contained in the set* $\{l, \ldots, n-l\}$.

Proof. Given $i \in \{0, \ldots, n\}$ and $0 < a < b < 1$, call a Morse function

$$g\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

i-$[a,b]$-*separated* if every critical point s of g of index $\leq i$ satisfies $g(s) < a$ and every critical point s of g of index $> i$ satisfies $g(s) > b$.

Set $z_- := \min\{z_c\}_{c \in \mathcal{C}}$ and $z_+ := \max\{z_c\}_{c \in \mathcal{C}}$.

By [28, Theorem 4.2, page 27], there exists an indefinite ordered Morse function

$$f_0\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

which is d-$[z_-, z_+]$-separated and (ϕ_c, z_c)-standard for all $c \in \mathcal{C}$. (In fact, note that (Y, Y^0, Y^1) fits the requirement (1), and the pairs (ϕ_c, z_c), $c \in \mathcal{C}$, fit the requirements (2) and (3) stated in [28, p. 26]. Moreover,

$$d = \dim L_c \leq \lceil \frac{n}{2} \rceil - 1 = \lfloor \frac{n+1}{2} \rfloor - 1 \leq \frac{n+1}{2} - 1 < \frac{n}{2}.$$

Moreover, note that Y^0 and Y^1 are assumed to be non-empty.)

Additionally, f_0 can be assumed to be well-ordered (see Definition 7.1.1) by [41, Theorem 4.4, p. 40] and [41, Theorem 4.2, p. 39].

In the following, assumption $(*)$ will be exploited in order to cancel critical points of f_0 in such a way that the resulting Morse function f satisfies (i) and (ii). Afterwards, one can in addition assume that f is excellent. (In fact, small perturbations of f around its critical points as described in the proof of [41, Lemma 2.8, p. 17] have no effect on the properties (i) and (ii).)

As Y and Y^0 are simply connected and $n = \dim Y \geq 7$, it follows from [41, Theorem 8.1, page 100] that all critical points of f_0 of index 1 can be traded for an equal number of critical points of index 3. (Note that f_0 has no critical points of index 0 being indefinite.) Thus, there exists a well-ordered Morse function

$$f_1\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

without critical points of index contained in the set $\{0,1\} \cup \{n\}$ that is still d-$[z_-, z_+]$-separated and (ϕ_c, z_c)-standard for all $c \in \mathcal{C}$. (Indeed, f_0 needs only be modified on a compact subset of $f_0^{-1}([0, z_-))$ because f_0 is ordered and d-$[z_-, z_+]$-separated, where $3 \leq l \leq d$.)

Next, we use Lemma C.0.1(b) and $H_i(Y, Y^0) = 0$ iteratively for $i = 2, \ldots, l-1$ to produce a well-ordered Morse function

$$f_i\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$$

without critical points of index contained in the set $\{0, 1, \ldots, l-1\} \cup \{n\}$ that is still d-$[z_-, z_+]$-separated and (ϕ_c, z_c)-standard for all $c \in \mathcal{C}$ by cancelling all critical points of f_{i-1} of index i against an equal number of critical points of index $i+1$. (Indeed, f_{i-1} has only to be modified on a compact subset of $f_{i-1}^{-1}([0, z_-))$ because f_{i-1} is ordered and d-$[z_-, z_+]$-separated, where $i+1 \leq l \leq d$.)

Turning the smooth manifold triad (Y, Y^0, Y^1) around yields a well-ordered Morse function

$$g_0 := 1 - f_{l-1} \colon (Y, Y^1, Y^0) \to ([0,1], 0, 1)$$

with no critical points of index in $\{0\} \cup \{n+1-l, \ldots, n\}$ that is $(n-1-d)$-$[1-z_+, 1-z_-]$-separated and $(\overline{\phi}_c, 1-z_c)$-standard for all $c \in \mathcal{C}$, where $\overline{\phi}_c(x,y) = \phi_c(x,-y)$ for all $(x,y) \in L_c \times \mathrm{int}(\varepsilon \cdot D^{n-d})$, $c \in \mathcal{C}$. By the same arguments as before (and noting that $l \leq n-1-d$ because $l, d \leq \lceil \frac{n}{2} \rceil - 1 \leq \frac{n-1}{2}$), we can now iteratively cancel the critical points of g_0 of index $i = 1, \ldots, l-1$ to obtain a well-ordered Morse function

$$g_{l-1} \colon (Y, Y^1, Y^0) \to ([0,1], 0, 1)$$

that has only critical points of index contained in the set $\{l, \ldots, n-l\}$ and is $(\overline{\phi}_c, 1-z_c)$-standard for all $c \in \mathcal{C}$. Hence, $f := 1 - g_{l-1}$ will be the desired Morse function. $\square$

7.2 Constructing Fold Maps from Local Handles into the Plane

Fix a pair (m, λ) consisting of integers $m \geq 2$ and $\lambda \in \{1, \ldots, m-1\}$.

Writing $|x|^2 := x_1^2 + \cdots + x_r^2$ for $x = (x_1, \ldots, x_r) \in \mathbb{R}^r$, the *standard Morse function* μ on $\mathbb{R}^m = \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$ with a single critical point of index λ at the origin is given by

$$\mu\colon \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \to \mathbb{R}, \qquad \mu(p, q) = -|p|^2 + |q|^2.$$

The *standard gradient-like vector field* v for μ is given by

$$v\colon \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \to \mathbb{R}^m, \qquad v(p, q) = (-p, q).$$

Note that the *flow* η of v is given by

$$\eta\colon \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \times \mathbb{R} \to \mathbb{R}^m, \qquad \eta(p, q, t) = (e^{-t}p, e^t q).$$

Indeed, for any point $x = (p, q) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$, the *integral curve*

$$\eta_x\colon \mathbb{R} \to \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}, \qquad \eta_x(t) = \eta(p, q, t) = (e^{-t}p, e^t q),$$

satisfies $\eta_x(0) = x$ and $\eta_x'(t) = (-e^{-t}p, e^t q) = v(\eta_x(t))$ for all $t \in \mathbb{R}$.

In the following, let

$$Z := (\mathbb{R}^\lambda \times 0) \cup (0 \times \mathbb{R}^{m-\lambda}).$$

Throughout the present section, let $\delta > 0$. (Note that δ will have to be chosen sufficiently small in Proposition 7.2.6.)

Note that the composition $\mu \circ \eta_x$ yields for every $x = (p, q) \in (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z$ a diffeomorphism

$$\mu \circ \eta_x\colon \mathbb{R} \xrightarrow{\cong} \mathbb{R}, \qquad t \mapsto -e^{-2t}|p|^2 + e^{2t}|q|^2.$$

(In fact, its first derivative is given by the positive function $t \mapsto 2e^{-2t}|p|^2 + 2e^{2t}|q|^2$, and $(\mu \circ \eta_x)(t) \to \pm\infty$ for $t \to \pm\infty$.) In particular, for any $x = (p, q) \in (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z$, there exists a unique $\theta(x) \in \mathbb{R}$ such that $(\mu \circ \eta_x)(\theta(x)) = -\delta^2$. (In other words, $\eta_x(\theta(x))$ is the unique intersection point of $\eta_x(\mathbb{R})$ with the $(n-1)$-dimensional submanifold $\mu^{-1}(-\delta^2) \setminus Z \subset \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$. Note that θ depends on $\delta > 0$ which has been fixed before.) Explicitly, if $\gamma\colon (0, \infty) \to (0, \infty)$ is given by $\gamma(r) = \frac{\delta^2}{2r^2}$, then

$$\theta\colon (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z \to \mathbb{R}, \qquad \theta(x) = 1/2 \cdot \log\left(-\gamma(|q|) + \sqrt{\gamma(|q|)^2 + |p|^2/|q|^2}\right).$$

(Indeed, writing $S := \sqrt{\gamma(|q|)^2 + |p|^2/|q|^2}$, one obtains

$$\begin{aligned}
(\mu \circ \eta_x)(\theta(x)) &= -|p|^2(-\gamma(|q|) + S)^{-1} + |q|^2(-\gamma(|q|) + S) \\
&= -|p|^2(-\gamma(|q|) - S)(\gamma(|q|)^2 - S^2)^{-1} + |q|^2(-\gamma(|q|) + S) \\
&= |q|^2(-\gamma(|q|) - S) + |q|^2(-\gamma(|q|) + S) = -\delta^2.)
\end{aligned}$$

Lemma 7.2.1. *Let $\delta > 0$ and $x = (p, q) \in (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z$. Then, $-\delta^2 \leq \mu(x)$ if and only* *if* $\theta(x) \leq 0$. *Moreover,* $\mu(x) \leq \delta^2$ *if and only if* $\theta(x) \leq \log(|p|/|q|)$. *The statements also hold with all inequality signs "*$\leq$*" replaced by equality signs "*$=$*".*

Proof. Suppose that $-\delta^2 \leq \mu(x) = -|p|^2 + |q|^2$. Equivalently, $|p|^2 \leq \delta^2 + |q|^2$, or $|p|^2/|q|^2 \leq \delta^2/|q|^2 + 1 = 2\gamma(|q|) + 1$. This is equivalent to $\gamma(|q|)^2 + |p|^2/|q|^2 \leq \gamma(|q|)^2 + 2\gamma(|q|) + 1 = (\gamma(|q|) + 1)^2$. This holds if and only if $\sqrt{\gamma(|q|)^2 + |p|^2/|q|^2} \leq \gamma(|q|) + 1$, or $\theta(x) \leq 0$.

Suppose that $-|p|^2 + |q|^2 = \mu(x) \leq \delta^2$. Equivalently, $|q|^2 \leq \delta^2 + |p|^2$, or $1 \leq 2\gamma(|q|) + |p|^2/|q|^2$. This is equivalent to $\gamma(|q|)^2 + |p|^2/|q|^2 \leq \gamma(|q|)^2 + |p|^2/|q|^2(2\gamma(|q|) + |p|^2/|q|^2) = (\gamma(|q|) + |p|^2/|q|^2)^2$. This holds if and only if $e^{2\theta(x)} = -\gamma(|q|) + \sqrt{\gamma(|q|)^2 + |p|^2/|q|^2} \leq |p|^2/|q|^2$, or $\theta(x) \leq \log(|p|/|q|)$. □

Lemma 7.2.2. *If* $\delta > 0$, *then* η *restricts to a diffeomorphism*

$$(\mu^{-1}(-\delta^2) \setminus Z) \times \mathbb{R} \xrightarrow{\cong} (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z,$$
$$(y,t) = (u,v,t) \mapsto \eta_y(t) = \eta(u,v,t) = (e^{-t}u, e^t v),$$

with inverse

$$(\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z \xrightarrow{\cong} (\mu^{-1}(-\delta^2) \setminus Z) \times \mathbb{R},$$
$$x = (p,q) \mapsto (\eta_x(\theta(x)), -\theta(x)) = (e^{-\theta(x)}p, e^{\theta(x)}q, -\theta(x)).$$

Proof. It is clear that both maps are smooth maps between smooth m-dimensional manifolds. It remains to check that the maps are inverse to each other. Given $(y,t) = (u,v,t) \in (\mu^{-1}(-\delta^2)\setminus Z) \times \mathbb{R}$, let $x = (p,q) = (e^{-t}u, e^t v) \in (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z$. Since $-\delta^2 = \mu(y) = -|u|^2 + |v|^2$ implies $|u|^2 = \delta^2 + |v|^2$, we have

$$\gamma(|q|)^2 + |p|^2/|q|^2 = \frac{\delta^4 + 4|p|^2|q|^2}{4|q|^4} = \frac{\delta^4 + 4|u|^2|v|^2}{4|q|^4} = \frac{\delta^4 + 4(\delta^2 + |v|^2)|v|^2}{4|q|^4} = \left(\frac{\delta^2 + 2|v|^2}{2|q|^2}\right)^2.$$

Hence,

$$\theta(x) = 1/2 \cdot \log\left(-\gamma(|q|) + \frac{\delta^2 + 2|v|^2}{2|q|^2}\right) = 1/2 \cdot \log\left(\frac{|v|^2}{|q|^2}\right) = -t.$$

All in all,

$$x = (p,q) \mapsto (e^{-\theta(x)}p, e^{\theta(x)}q, -\theta(x)) = (e^t e^{-t}u, e^{-t}e^t v, t) = (y,t).$$

Conversely, given $x = (p,q) \in (\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z$, let $(y,t) = (u,v,t) = (e^{-\theta(x)}p, e^{\theta(x)}q, -\theta(x)) \in (\mu^{-1}(-\delta^2) \setminus Z) \times \mathbb{R}$. Then

$$(y,t) = (u,v,t) \mapsto (e^{-t}u, e^t v) = (e^{\theta(x)}e^{-\theta(x)}p, e^{-\theta(x)}e^{\theta(x)}q) = x.$$

□

Definition 7.2.3. For any $\varepsilon, \delta > 0$ one defines the *local handle* (see Figure 7.2) by

$$H_\delta^\varepsilon = \{(p,q) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda};\ -\delta^2 \leq -|p|^2 + |q|^2 \leq \delta^2,\ |p| \cdot |q| < \varepsilon \cdot \sqrt{\varepsilon^2 + \delta^2}\}.$$

The following result is essentially observed in the proof of [41, Theorem 3.12, page 30].

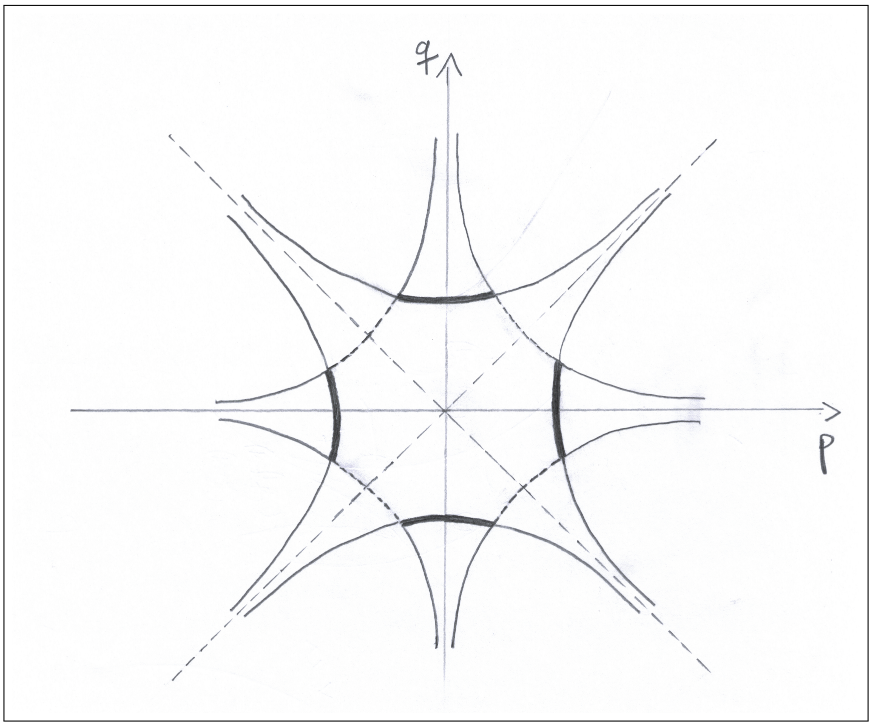

Figure 7.2: The local handle H^ε_δ for $(m, \lambda) = (2, 1)$

Proposition 7.2.4. *For any* $\varepsilon, \delta > 0$, H^ε_δ *is an* m*-dimensional submanifold of* $\mathbb{R}^m = \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$ *with boundary* $\partial H^\varepsilon_\delta = T^\varepsilon_- \sqcup T^\varepsilon_+$, *where* $T^\varepsilon_\pm := H^\varepsilon_\delta \cap \mu^{-1}(\pm\delta^2)$. *There are diffeomorphisms*

$$\phi^\varepsilon_- : S^{\lambda-1} \times \operatorname{int}(\varepsilon \cdot D^{n-\lambda}) \xrightarrow{\cong} T^\varepsilon_-, \qquad \phi^\varepsilon_-(u, v) = (\sqrt{|v|^2 + \delta^2} \cdot u, v),$$
$$\phi^\varepsilon_+ : \operatorname{int}(\varepsilon \cdot D^{\lambda}) \times S^{n-\lambda-1} \xrightarrow{\cong} T^\varepsilon_+, \qquad \phi^\varepsilon_+(u, v) = (u, \sqrt{|u|^2 + \delta^2} \cdot v).$$

Moreover, η *restricts to a diffeomorphism (with inverse given in Lemma 7.2.2)*

$$\{(y, t) = (u, v, t) \in T^\varepsilon_- \times \mathbb{R};\ v \neq 0,\ 0 \leq t \leq \log(|u|/|v|)\} \xrightarrow{\cong} H^\varepsilon_\delta \setminus Z,$$
$$(y, t) = (u, v, t) \mapsto \eta_y(t) = \eta(u, v, t) = (e^{-t}u, e^t v).$$

Proof. It is clear that $\mu^{-1}([-\delta^2, \delta^2])$ is an m-dimensional submanifold of $\mathbb{R}^m$ with boundary $\mu^{-1}(-\delta^2) \sqcup \mu^{-1}(\delta^2)$. By definition, H^ε_δ is the intersection of $\mu^{-1}([-\delta^2, \delta^2])$ with the open subset $\{(p, q) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda};\ |p| \cdot |q| < \varepsilon \cdot \sqrt{\varepsilon^2 + \delta^2}\} \subset \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$, and $\partial H^\varepsilon_\delta$ is the intersection of $\mu^{-1}(-\delta^2) \sqcup \mu^{-1}(\delta^2)$ with this open subset.

The map ϕ^ε_- is well-defined and smooth. (In fact, it suffices to note that the image of the smooth map $S^{\lambda-1} \times \operatorname{int}(\varepsilon \cdot D^{m-\lambda}) \to \mathbb{R}^m$, $(u, v) \mapsto (\sqrt{|v|^2 + \delta^2} \cdot u, v)$, lies in the submanifold $T^\varepsilon_- \subset \mathbb{R}^m$.) Moreover, the smooth map $\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \to \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$, $(p, q) \mapsto ((\sqrt{|q|^2 + \delta^2})^{-1} \cdot p, q)$, restricts to a smooth map $\psi^\varepsilon_- : T^\varepsilon_- \to S^{\lambda-1} \times \operatorname{int}(\varepsilon \cdot D^{m-\lambda})$. (In fact, if $(p, q) \in T^\varepsilon_-$, then $|p| = \sqrt{|q|^2 + \delta^2}$ and $|p| \cdot |q| < \varepsilon \cdot \sqrt{\varepsilon^2 + \delta^2}$. Hence, $|(\sqrt{|q|^2 + \delta^2})^{-1} \cdot p| = 1$ and $|q| < \varepsilon$.) Obviously, ϕ^ε_- and ψ^ε_- are mutually inverse to each other. Analogously, one can show that ϕ^ε_+ is a diffeomorphism.

It remains to show that the diffeomorphisms considered in Lemma 7.2.2 restrict in the desired way.

Suppose that $(y, t) = (u, v, t) \in T^\varepsilon_- \times \mathbb{R}$ such that $v \neq 0$ and $0 \leq t \leq \log(|u|/|v|)$. To show that $\eta_y(t) = \eta(u, v, t) = (e^{-t}u, e^t v) \in H^\varepsilon_\delta \setminus Z$, one checks the following:

- $(e^{-t}u, e^t v) \in H^\varepsilon_\delta$, i.e. $-\delta^2 \le -|e^{-t}u|^2 + |e^t v|^2 \le \delta^2$ and $|e^{-t}u| \cdot |e^t v| < \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$. The second statement holds since $|e^{-t}u| \cdot |e^t v| = |u| \cdot |v| < \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$. To show the first statement, use that $\gamma(|e^t v|) = e^{-2t}\gamma(|v|)$ and note that $\theta(u,v) = 0$ by Lemma 7.2.1:

$$\theta(e^{-t}u, e^t v) = 1/2 \cdot \log\left(-\gamma(|e^t v|) + \sqrt{\gamma(|e^t v|)^2 + |e^{-t}u|^2/|e^t v|^2}\right) = \theta(u,v) - t = -t.$$

  Using Lemma 7.2.1, we obtain from $\theta(e^{-t}u, e^t v) = -t \le 0$ that $-\delta^2 \le -|e^{-t}u|^2 + |e^t v|^2$. Moreover, by the same lemma, it follows from $\theta(e^{-t}u, e^t v) = -t \le \log(|u|/|v|) - 2t = \log(|e^{-t}u|/|e^t v|)$ that $-|e^{-t}u|^2 + |e^t v|^2 \le \delta^2$.
- $(e^{-t}u, e^t v) \notin Z$, i.e. $e^{-t}u \neq 0$ and $e^t v \neq 0$. Indeed, $-|u|^2 + |v|^2 = \mu(y) = -\delta^2 < 0$ implies that $u \neq 0$. Moreover, $v \neq 0$ holds by assumption.

Suppose that $x = (p,q) \in H^\varepsilon_\delta \setminus Z$. One has to check that $(e^{-\theta(x)}p, e^{\theta(x)}q, -\theta(x)) \in ((\mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}) \setminus Z) \times \mathbb{R}$ has the following properties:

- $(e^{-\theta(x)}p, e^{\theta(x)}q) \in T^\varepsilon_-$, i.e. $(e^{-\theta(x)}p, e^{\theta(x)}q) \in \mu^{-1}(-\delta^2)$ and $(e^{-\theta(x)}p, e^{\theta(x)}q) \in H^\varepsilon_\delta$. The first statement holds by Lemma 7.2.2. Hence, the second statement follows from $|e^{-\theta(x)}p| \cdot |e^{\theta(x)}q| = |p| \cdot |q| < \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$.
- $0 \le -\theta(x) \le \log(|e^{-\theta(x)}p|/|e^{\theta(x)}q|)$, i.e. $\theta(x) \le 0$ and $\theta(x) \le \log(|p|/|q|)$. This follows directly from Lemma 7.2.1 since $-\delta^2 \le \mu(x) \le \delta^2$.
- $e^{\theta(x)}q \neq 0$ is immediate.

□

Note that $\phi^\varepsilon_-(S^{\lambda-1} \times 0)$ is the left-hand sphere of the critical point 0 of μ in $\mu^{-1}(-\delta^2)$ with a tubular neighbourhood given by T^ε_-.

Lemma 7.2.5. *For all $\varepsilon, \delta > 0$, H^ε_δ has the following properties:*

(i) H^ε_δ is a bounded subset of $\mathbb{R}^m$.
(ii) If $c > 0$, then $c \cdot H^\varepsilon_\delta = H^{c\varepsilon}_{c\delta}$.
(iii) If $\varepsilon \ge \varepsilon' > 0$ and $\delta \ge \delta' > 0$, then $H^{\varepsilon'}_{\delta'} \subset H^\varepsilon_\delta$.

Proof. (i). Suppose that $x_1, x_2, \dots$ is a sequence of points in H^ε_δ such that $|x_i| \to \infty$ for $i \to \infty$. Writing $x_i = (p_i, q_i) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}$ for all i, one of the sequences $p_1, p_2, \dots$ and $q_1, q_2, \dots$ has a subsequence $r_1, r_2, \dots$ such that $|r_i| \to \infty$ for $i \to \infty$. (Indeed, if both $p_1, p_2, \dots$ and $q_1, q_2, \dots$ are bounded by some $C > 0$, then $|x_i| = \sqrt{|p_i|^2 + |q_i|^2} \le \sqrt{C^2 + C^2} = C\sqrt{2}$ for all i in contradiction to $|x_i| \to \infty$ for $i \to \infty$.) By passing to subsequences, we may assume that $|p_i| \to \infty$ for $i \to \infty$ or $|q_i| \to \infty$ for $i \to \infty$. If $|p_i| \to \infty$ for $i \to \infty$, then it follows from $|p_i| \cdot |q_i| < \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$ for all i that there exists i_0 such that $|q_i| < 1$ for all $i \ge i_0$. But then $-\delta^2 \le -|p_i|^2 + |q_i|^2$ for all i implies that $|p_i|^2 \le \delta^2 + |q_i|^2 < \delta^2 + 1$ for all $i \ge i_0$, a contradiction. Analogously, the assumption $|q_i| \to \infty$ for $i \to \infty$ leads to a contradiction.

(ii). Given $c > 0$, we have $(p,q) \in c \cdot H^\varepsilon_\delta$ if and only if there exists a point $(p', q') \in H^\varepsilon_\delta$ such that $p = cp'$ and $q = cq'$. Now it suffices to note that $-\delta^2 \le -|p'|^2 + |q'|^2 \le \delta^2$ is equivalent to $-(c\delta)^2 \le -|p|^2 + |q|^2 \le (c\delta)^2$ and $|p'| \cdot |q'| < \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$ is equivalent to $|p| \cdot |q| < (c\varepsilon) \cdot \sqrt{(c\varepsilon)^2 + (c\delta)^2}$.

(iii). If $\varepsilon \ge \varepsilon' > 0$, $\delta \ge \delta' > 0$ and $(p,q) \in H^{\varepsilon'}_{\delta'}$, then it follows from $-\delta^2 \le -\delta'^2 \le -|p|^2 + |q|^2 \le \delta'^2 \le \delta^2$ and $|p| \cdot |q| < \varepsilon' \cdot \sqrt{\varepsilon'^2 + \delta'^2} \le \varepsilon \cdot \sqrt{\varepsilon^2+\delta^2}$ that $(p,q) \in H^\varepsilon_\delta$. □

Recall that $\delta > 0$ is fixed. For $\delta' \in (0, \delta]$ define

$$R^{\varepsilon}_{\pm\delta'} := H^{\varepsilon}_{\delta} \cap \mu^{-1}(\pm\delta'^2).$$

Note that $R^{\varepsilon}_{\pm\delta} = T^{\varepsilon}_{\pm}$. By construction, $R^{\varepsilon}_{\pm\delta'}$ is an open subset of $\mu^{-1}(\pm\delta'^2)$.

The rest of the present section is concerned with the proof of the following result, which implements the method of forward handles (see [28, Fig. 29, p. 44]):

Proposition 7.2.6. *Let $\varepsilon > 0$. For any sufficiently small $\delta > 0$ there exists a smooth function $\nu\colon H^{\varepsilon}_{\delta} \to \mathbb{R}$ with the following properties:*

(i) The map $(\mu|_{H^{\varepsilon}_{\delta}}, \nu)\colon H^{\varepsilon}_{\delta} \to \mathbb{R}^2$ is a fold map whose singular locus is the fold line $0 \times [-\delta, \delta] \times 0 \subset \mathbb{R}^{\lambda} \times \mathbb{R} \times \mathbb{R}^{m-1-\lambda}$ of absolute index $\max\{\lambda, m-1-\lambda\}$.

(ii) If $\delta' \in (0, \delta]$, then $\nu|_{R^{\varepsilon}_{-\delta'}}\colon R^{\varepsilon}_{-\delta'} \to \mathbb{R}$ has no critical points. Moreover, the restriction $\nu|_{T^{\varepsilon}_{-}}\colon T^{\varepsilon}_{-} \to \mathbb{R}$ is the projection $(p, q) \mapsto q_1$.

(iii) If $\delta' \in (0, \delta]$, then the set of critical points of the restriction $\nu|_{R^{\varepsilon}_{\delta'}}\colon R^{\varepsilon}_{\delta'} \to \mathbb{R}$ is given by

$$R^{\varepsilon}_{\delta'} \cap (0 \times [-\delta, \delta] \times 0) = \{(0, \pm\delta', 0)\} =: \{x_{\pm}\}.$$

The critical point x_- is non-degenerate of index λ and the critical point x_+ is non-degenerate of index $m - \lambda - 1$. Moreover, $\nu(x_{\pm}) = \pm\delta'$.

(iv) There exists $\varepsilon' \in (0, \varepsilon)$ such that $\nu(\eta_y(t)) = \nu(\eta_y(0))$ for all $y = (u, v) \in \phi^{\varepsilon}_{-}(S^{\lambda-1} \times \{v \in \mathbb{R}^{m-\lambda};\ \varepsilon' < |v| < \varepsilon\}) = T^{\varepsilon}_{-} \setminus \overline{T^{\varepsilon'}_{-}}$ and all $t \in [0, \log(|u|/|v|)]$ (compare Proposition 7.2.4).

Proof. We begin with the construction of ν. Set $\varepsilon_0 := \varepsilon/3$. Given any $\delta > 0$, we construct a smooth map $\nu_{<}\colon H^{\varepsilon_0}_{\delta} \to \mathbb{R}$ on the open subset $H^{\varepsilon_0}_{\delta} \subset H^{\varepsilon}_{\delta}$ and a smooth map $\nu_{>}\colon H^{\varepsilon}_{\delta} \setminus Z \to \mathbb{R}$ on the open subset $H^{\varepsilon}_{\delta} \setminus Z \subset H^{\varepsilon}_{\delta}$ such that $\nu_{<}$ and $\nu_{>}$ agree on the intersection $H^{\varepsilon_0}_{\delta} \cap (H^{\varepsilon}_{\delta} \setminus Z) = H^{\varepsilon_0}_{\delta} \setminus Z$. Afterwards, ν will be defined to be the glued map on the union $H^{\varepsilon_0}_{\delta} \cup (H^{\varepsilon}_{\delta} \setminus Z) = H^{\varepsilon}_{\delta}$. To satisfy property (i), it will be necessary to choose $\delta > 0$ sufficiently small.

Let $\nu_{<}$ be the smooth map given by the projection to the $(\lambda+1)$-st coordinate:

$$\nu_{<}\colon H^{\varepsilon_0}_{\delta} \to \mathbb{R}, \qquad x = (p, q) \mapsto q_1.$$

For the construction of $\nu_{>}$, choose a smooth map $\xi\colon [0, \infty) \to \mathbb{R}$ (see Figure 7.3) such that $\xi([0, \infty)) \subset [0, 1]$, $\xi(t) = 1$ for $t < \varepsilon_0^2$ and $\xi(t) = 0$ for $t > (2\varepsilon_0)^2$.

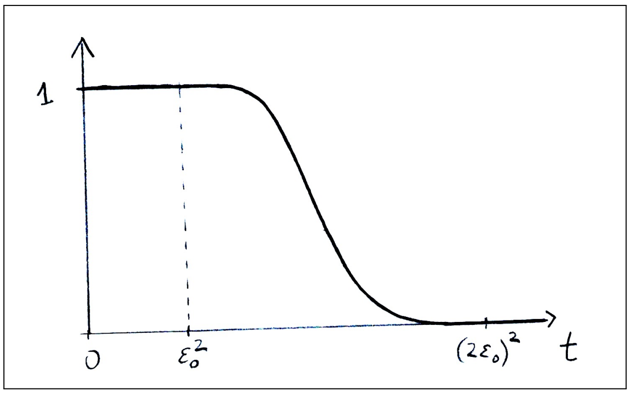

Figure 7.3: Graph of the bump function ξ

We first define the following smooth map:

$$\tilde{\nu}_>\colon \{(y,t)=(u,v,t)\in T_-^\varepsilon\times\mathbb{R};\ v\neq 0,\ 0\leq t\leq \log(|u|/|v|)\}\to\mathbb{R},$$
$$(y,t)=(u,v,t)\mapsto e^{t\cdot\xi(|v|^2)}v_1.$$

Recall from Proposition 7.2.4 that the inverse of the diffeomorphism $\eta\colon (\mu^{-1}(-\delta^2)\setminus Z)\times\mathbb{R}\xrightarrow{\cong}(\mathbb{R}^\lambda\times\mathbb{R}^{m-\lambda})\setminus Z$ from Lemma 7.2.2 restricts to a diffeomorphism

$$\Gamma\colon H_\delta^\varepsilon\setminus Z\xrightarrow{\cong}\eta^{-1}(H_\delta^\varepsilon\setminus Z)=\{(y,t)=(u,v,t)\in T_-^\varepsilon\times\mathbb{R};\ v\neq 0,\ 0\leq t\leq\log(|u|/|v|)\},$$
$$x=(p,q)\mapsto(\eta_x(\theta(x)),-\theta(x))=(e^{-\theta(x)}p,e^{\theta(x)}q,-\theta(x)).$$

Define $\nu_>\colon H_\delta^\varepsilon\setminus Z\to\mathbb{R}$ to be the composition $\nu_>:=\tilde{\nu}_>\circ\Gamma$.

To complete the construction of ν, we have to show that $\nu_<$ and $\nu_>$ agree on the intersection $H_\delta^{\varepsilon_0}\cap(H_\delta^\varepsilon\setminus Z)=H_\delta^{\varepsilon_0}\setminus Z$. In fact, given $x=(p,q)\in H_\delta^{\varepsilon_0}\setminus Z$, we set $(y,t):=(u,v,t):=(e^{-\theta(x)}p,e^{\theta(x)}q,-\theta(x))$. Using $\xi(|v|^2)=1$ (note that $|v|^2<\varepsilon_0^2$, which follows from $(u,v)\in T_-^{\varepsilon_0}$ and the definition of ϕ_-^ε in Proposition 7.2.4), we obtain

$$\nu_>(x)=\tilde{\nu}_>(u,v,t)=e^{t\cdot\xi(|v|^2)}v_1=e^{-\theta(x)\cdot\xi(|v|^2)}e^{\theta(x)}q_1=q_1=\nu_<(x).$$

Finally, define the well-defined smooth map

$$\nu\colon H_\delta^\varepsilon\to\mathbb{R},\qquad \nu(x)=\begin{cases}\nu_<(x), & \text{if}\quad x\in H_\delta^{\varepsilon_0},\\ \nu_>(x), & \text{if}\quad x\in H_\delta^\varepsilon\setminus Z.\end{cases}$$

It remains to show that ν has the desired properties for $\delta>0$ sufficiently small.

(i). Given $x=(p,q)\in H_\delta^{\varepsilon_0}$, we have $(\mu(x),\nu(x))=(-|p|^2+|q|^2,q_1)$. This is the *forward* λ-*handle* (see [28, Fig. 29, p.44]). Postcomposition with the diffeomorphism $\mathbb{R}^2\to\mathbb{R}^2$, $(a,b)\mapsto(b,a-b^2)$, yields the fold map

$$x=(p,q)\mapsto(q_1,-(p_1^2+\cdots+p_\lambda^2)+q_2^2+\cdots+q_{m-\lambda}^2),$$

whose singular locus on $H_\delta^{\varepsilon_0}$ is the fold line $0\times[-\delta,\delta]\times 0\subset Z\subset\mathbb{R}^\lambda\times\mathbb{R}\times\mathbb{R}^{m-1-\lambda}$ of absolute index $\max\{\lambda,m-1-\lambda\}$. As $H_\delta^\varepsilon=(H_\delta^\varepsilon\setminus Z)\cup H_\delta^{\varepsilon_0}$, it remains to show that $(\mu|_{H_\delta^\varepsilon},\nu)$ is a submersion on $H_\delta^\varepsilon\setminus Z$ (for a sufficiently small choice of $\delta>0$). For this purpose, we will show that the precomposition α of $(\mu|_{H_\delta^\varepsilon},\nu)$ with the diffeomorphism Γ^{-1} $(=\eta|)$ is a submersion:

$$\alpha\colon\eta^{-1}(H_\delta^\varepsilon\setminus Z)\to\mathbb{R}^2,$$
$$(y,t)=(u,v,t)\mapsto(\mu(\eta_y(t)),\nu_>(\eta_y(t)))=(-e^{-2t}|u|^2+e^{2t}|v|^2,e^{t\cdot\xi(|v|^2)}v_1).$$

By Proposition 7.2.4 we have a diffeomorphism $\phi_-^\varepsilon\colon S^{\lambda-1}\times\operatorname{int}(\varepsilon\cdot D^{m-\lambda})\xrightarrow{\cong}T_-^\varepsilon$ which is given by $\phi_-^\varepsilon(u,v)=(\sqrt{|v|^2+\delta^2}\cdot u,v)$. This gives rise to a diffeomorphism β from

$$S^{\lambda-1}\times\{(v,t)\in\operatorname{int}(\varepsilon\cdot D^{m-\lambda})\times\mathbb{R};\ v\neq 0,\ 0\leq t\leq\log(\sqrt{|v|^2+\delta^2}/|v|)\}$$

to $\eta^{-1}(H_\delta^\varepsilon \setminus Z) = \{(y,t) = (u,v,t) \in T_-^\varepsilon \times \mathbb{R};\ v \neq 0,\ 0 \leq t \leq \log(|u|/|v|)\}$ given by

$$(u,v,t) \mapsto \beta(u,v,t) = (\sqrt{|v|^2 + \delta^2} \cdot u, v, t).$$

As the composition $\alpha \circ \beta$ is constant in the variable $u \in S^{\lambda-1}$ (because $|u| = 1$), it suffices to show that the following map $(\overline{\mu}, \overline{\nu})$ is a submersion:

$$(\overline{\mu}, \overline{\nu})\colon \{(v,t) \in \mathbb{R}^{m-\lambda} \times \mathbb{R};\ 0 < |v| < \varepsilon,\ 0 \leq t \leq \log(\sqrt{|v|^2+\delta^2}/|v|)\} \to \mathbb{R}^2,$$
$$(v,t) \mapsto (-e^{-2t}(|v|^2+\delta^2) + e^{2t}|v|^2, e^{t\cdot\xi(|v|^2)}v_1) = (-\delta^2 e^{-2t} + |v|^2(e^{2t}-e^{-2t}), e^{t\cdot\xi(|v|^2)}v_1).$$

The Jacobian of $(\overline{\mu}, \overline{\nu})$ at (v,t) is given by the $2 \times (m-\lambda+1)$-matrix

$$\begin{pmatrix} 2v_1(e^{2t}-e^{-2t}) & 2v_2(e^{2t}-e^{-2t}) & \dots & 2v_{m-\lambda}(e^{2t}-e^{-2t}) & 2\delta^2 e^{-2t} + 2|v|^2(e^{2t}+e^{-2t}) \\ (1+2t\xi'(|v|^2)v_1^2)e^{t\cdot\xi(|v|^2)} & 2t\xi'(|v|^2)v_1v_2e^{t\cdot\xi(|v|^2)} & \dots & 2t\xi'(|v|^2)v_1v_{m-\lambda}e^{t\cdot\xi(|v|^2)} & e^{t\cdot\xi(|v|^2)}v_1\xi(|v|^2) \end{pmatrix}$$

For $i \in \{2, \dots, m-\lambda\}$ consider the 2×2-submatrix given by the first and the i-th column:

$$\begin{aligned} &\det \begin{pmatrix} 2v_1(e^{2t}-e^{-2t}) & 2v_i(e^{2t}-e^{-2t}) \\ (1+2t\xi'(|v|^2)v_1^2)e^{t\cdot\xi(|v|^2)} & 2t\xi'(|v|^2)v_1v_ie^{t\cdot\xi(|v|^2)} \end{pmatrix} \\ &= 2v_1(e^{2t}-e^{-2t})2t\xi'(|v|^2)v_1v_ie^{t\cdot\xi(|v|^2)} - 2v_i(e^{2t}-e^{-2t})(1+2t\xi'(|v|^2)v_1^2)e^{t\cdot\xi(|v|^2)} \\ &= 2v_i(e^{2t}-e^{-2t})e^{t\cdot\xi(|v|^2)}\left[2t\xi'(|v|^2)v_1^2 - (1+2t\xi'(|v|^2)v_1^2)\right] \\ &= -2v_i(e^{2t}-e^{-2t})e^{t\cdot\xi(|v|^2)}. \end{aligned}$$

This determinant vanishes if and only if $t = 0$ or $v_i = 0$. Thus, the rank of the Jacobian of $(\overline{\mu}, \overline{\nu})$ at (v,t) remains to be investigated only in the case that $t = 0$ or $v_2 = \dots = v_{m-\lambda} = 0$. In these cases we consider the 2×2-submatrix given by the first and the last column:

$$\begin{aligned} &\det \begin{pmatrix} 2v_1(e^{2t}-e^{-2t}) & 2\delta^2e^{-2t} + 2|v|^2(e^{2t}+e^{-2t}) \\ (1+2t\xi'(|v|^2)v_1^2)e^{t\cdot\xi(|v|^2)} & e^{t\cdot\xi(|v|^2)}v_1\xi(|v|^2) \end{pmatrix} \\ &= 2v_1(e^{2t}-e^{-2t})e^{t\cdot\xi(|v|^2)}v_1\xi(|v|^2) - (2\delta^2e^{-2t} + 2|v|^2(e^{2t}+e^{-2t}))(1+2t\xi'(|v|^2)v_1^2)e^{t\cdot\xi(|v|^2)}. \end{aligned}$$

If $t = 0$, then this determinant is further equal to $-(2\delta^2 + 4|v|^2)$, which is negative for all v. Now suppose that $v_2 = \dots = v_{m-\lambda} = 0$. In this case we have $|v|^2 = v_1^2$. Hence, the determinant vanishes if and only if the following term vanishes (extracting the global factor $2e^{t\cdot\xi(|v|^2)} \neq 0$):

$$v_1^2(e^{2t}-e^{-2t})\xi(v_1^2) - (\delta^2e^{-2t} + v_1^2(e^{2t}+e^{-2t}))(1+2t\xi'(v_1^2)v_1^2) \qquad (*).$$

If $v_1^2 < \varepsilon_0^2$, then $\xi(v_1^2) = 1$ and $\xi'(v_1^2) = 0$, so $(*)$ reduces to

$$\begin{aligned} v_1^2(e^{2t}-e^{-2t}) - (\delta^2e^{-2t} + v_1^2(e^{2t}+e^{-2t})) &= v_1^2\left[(e^{2t}-e^{-2t}) - (e^{2t}+e^{-2t})\right] - \delta^2e^{-2t} \\ &= -2v_1^2e^{-2t} - \delta^2e^{-2t} = -(2v_1^2+\delta^2)e^{-2t} < 0. \end{aligned}$$

Hence, we may assume that $\varepsilon_0^2 \leq v_1^2 < \varepsilon^2$ in the following. Choose $\delta > 0$ so small that

$$\log(1+\delta^2/\varepsilon_0^2) < \min\{\frac{1}{2\varepsilon^2 \max_{s\in\mathbb{R}}|\xi'(s)|}, \sinh^{-1}(1/4)\}.$$

(This is exactly the point in the proof where $\delta > 0$ has to be chosen sufficiently small.) Then it follows for all v_1 satisfying $\varepsilon_0^2 \leq v_1^2 < \varepsilon^2$ and for all t satisfying $0 \leq t \leq \log(\sqrt{|v|^2 + \delta^2}/|v|) = \log(\sqrt{1 + \delta^2/v_1^2}) = 1/2 \cdot \log(1 + \delta^2/v_1^2)$ that

- $1 + 2t\xi'(v_1^2)v_1^2 > 1/2$. In fact, it follows from $t \geq 0$ and
$$t \leq 1/2 \cdot \log(1 + \delta^2/v_1^2) \leq 1/2 \cdot \log(1 + \delta^2/\varepsilon_0^2) < 1/(4\varepsilon^2 \max_{s\in\mathbb{R}} |\xi'(s)|)$$
that $-4t\xi'(v_1^2)v_1^2 \leq 4t| - \xi'(v_1^2)|v_1^2 \leq 4t|\xi'(v_1^2)|\varepsilon^2 \leq 4\varepsilon^2 t \max_{s\in\mathbb{R}} |\xi'(s)| < 1$.
- $e^{2t} - e^{-2t} < 1/2$. In fact, we have $e^{2t} - e^{-2t} = 2\sinh(2t) < 1/2$ because
$$t \leq 1/2 \cdot \log(1 + \delta^2/v_1^2) \leq 1/2 \cdot \log(1 + \delta^2/\varepsilon_0^2) < 1/2 \cdot \sinh^{-1}(1/4).$$

Hence, $1 + 2t\xi'(v_1^2)v_1^2 > 1/2 > e^{2t} - e^{-2t} \geq 0$ (as $t \geq 0$). Therefore, combining the statements $v_1^2 \geq 0$, $e^{2t} - e^{-2t} \geq 0$, $\xi([0,\infty)) \subset [0,1]$, $\delta^2 e^{-2t} \geq 0$, $e^{2t} + e^{-2t} = 2\cosh(2t) \geq 2 \geq 1$ and $1 + 2t\xi'(v_1^2)v_1^2 \geq 0$ to show the first inequality, and using $v_1^2 > 0$ and $1 + 2t\xi'(v_1^2)v_1^2 > e^{2t} - e^{-2t}$ in the second inequality, we obtain the following estimate for the expression $(*)$:

$$\begin{aligned}
&v_1^2(e^{2t} - e^{-2t})\xi(v_1^2) - (\delta^2 e^{-2t} + v_1^2(e^{2t} + e^{-2t}))(1 + 2t\xi'(v_1^2)v_1^2) \\
&\leq v_1^2(e^{2t} - e^{-2t}) - v_1^2(1 + 2t\xi'(v_1^2)v_1^2) = v_1^2\left[(e^{2t} - e^{-2t}) - (1 + 2t\xi'(v_1^2)v_1^2)\right] \\
&< 0.
\end{aligned}$$

(ii). Let $\delta' \in (0,\delta]$. Suppose that $c \in R^\varepsilon_{-\delta'}$ is a critical point of $\nu|_{R^\varepsilon_{-\delta'}}$. Then it follows from $R^\varepsilon_{-\delta'} \subset \mu^{-1}(-\delta'^2)$ that c is also a critical point of $(\mu|_{H^\varepsilon_\delta}, \nu)$. (In fact, assuming that c is not a critical point of $(\mu|_{H^\varepsilon_\delta}, \nu)$, the kernel of the tangent map of $(\mu|_{H^\varepsilon_\delta}, \nu)$ at c, which is $\ker d_c\mu \cap \ker d_c\nu$, is of dimension $m-2$ by the dimension formula. In contradiction to that, $T_c R^\varepsilon_{-\delta'}$ is a $(m-1)$-dimensional subspace of $\ker d_c\mu \cap \ker d_c\nu$.) Hence, part (i) implies that $c \in 0 \times [-\delta,\delta] \times 0 \subset \mathbb{R}^\lambda \times \mathbb{R} \times \mathbb{R}^{m-\lambda-1}$. This results in the contradiction $-\delta'^2 = \mu(c) \geq 0$. Therefore, $\nu|_{R^-_{\varepsilon,\delta'}}$ has no critical points.

Let $x = (p,q) \in T^\varepsilon_-$. If $x \in H^{\varepsilon/3}_\delta$, then $\nu(x) = \nu_<(x) = q_1$ by definition of ν. If $x \in H^\varepsilon_\delta \setminus Z$, then, setting $(y,t) := (u,v,t) := (e^{-\theta(x)}p, e^{\theta(x)}q, -\theta(x))$, we obtain $\nu(x) = \nu_>(x) = \tilde{\nu}_>(u,v,t) = e^{t\cdot\xi(|v|^2)}v_1$. It follows from $x = (p,q) \in T^\varepsilon_-$ that $\theta(x) = 0$ by Lemma 7.2.1. Hence, $(y,t) = (x,0)$. Consequently, $\nu(x) = e^{t\cdot\xi(|v|^2)}v_1 = e^{0\cdot\xi(|q|^2)}q_1 = q_1$.

(iii). Let $\delta' \in (0,\delta]$. Let $x = (p,q) \in R^\varepsilon_{\delta'}$. First suppose that $x \notin (0 \times [-\delta,\delta] \times 0)$, i.e. $x \neq \{0\} \times \{\pm\delta'\} \times \{0\}$. Then it follows from part (i) that the linear map $d_x(\mu|_{H^\varepsilon_\delta}, \nu)\colon \mathbb{R}^m \to \mathbb{R}^2$ is surjective, so $\dim\ker d_x(\mu|_{H^\varepsilon_\delta}, \nu) = m-2$. As x is a nonsingular point of μ, we have $\dim\ker d_x\mu = m-1$. Hence it follows from $\ker d_x(\mu|_{H^\varepsilon_\delta}, \nu) = \ker d_x\mu \cap \ker d_x\nu \subset \mathbb{R}^m$ and $\dim\ker d_x\nu \in \{m-1, m\}$ that $\dim\ker d_x\nu = m-1$. Moreover, there exists a vector $w \in \ker d_x\mu = T_x R^\varepsilon_{\delta'}$ such that $w \notin \ker d_x\nu$. Therefore, x is a nonsingular point of $\nu|_{R^\varepsilon_{\delta'}}\colon R^\varepsilon_{\delta'} \to \mathbb{R}$. Now suppose that $x \in (0 \times [-\delta,\delta] \times 0)$, i.e. x is one of the points $x_\pm := \{0\} \times \{\pm\delta'\} \times \{0\} \in \mathbb{R}^\lambda \times \mathbb{R} \times \mathbb{R}^{m-1-\lambda}$. As $x_\pm$ are fold points of $(\mu|_{H^\varepsilon_\delta}, \nu)$ by property (i), we conclude that the linear map $d_{x_\pm}(\mu|_{H^\varepsilon_\delta}, \nu)\colon \mathbb{R}^m \to \mathbb{R}^2$ has rank 1, so $\dim\ker d_{x_\pm}(\mu|_{H^\varepsilon_\delta}, \nu) = m-1$. Hence it follows from $\ker d_{x_\pm}(\mu|_{H^\varepsilon_\delta}, \nu) = \ker d_{x_\pm}\mu \cap \ker d_{x_\pm}\nu \subset \ker d_{x_\pm}\mu = T_{x_\pm} R^\varepsilon_{\delta'}$ that $T_{x_\pm} R^\varepsilon_{\delta'} = \ker d_{x_\pm}(\mu|_{H^\varepsilon_\delta}, \nu) \subset \ker d_{x_\pm}\nu$ (because $T_{x_\pm} R^\varepsilon_{\delta'}$ and $\ker d_{x_\pm}(\mu|_{H^\varepsilon_\delta}, \nu)$ are both of dimension $m-1$). Therefore, $x_\pm$ are in fact critical points of $\nu|_{R^\varepsilon_{\delta'}}\colon R^\varepsilon_{\delta'} \to \mathbb{R}$. It remains to show that they are nondegenerate critical points of $\nu|_{R^{\varepsilon}_{\delta'}}$ (or of $\nu|_{R^{\varepsilon_0}_{\delta'}} = \nu_<|_{R^{\varepsilon_0}_{\delta'}}$ because $x_\pm \in H^{\varepsilon_0}_{\delta'} \cap \mu^{-1}(\delta'^2) = R^{\varepsilon_0}_{\delta'}$). For this purpose, we use the diffeomorphism

$$\phi^{\varepsilon_0}_{\delta'}\colon \operatorname{int}(\varepsilon_0 \cdot D^\lambda) \times S^{m-\lambda-1} \xrightarrow{\cong} R^{\varepsilon_0}_{\delta'}, \qquad \phi^{\varepsilon_0}_{\delta'}(u,v) = (u, \sqrt{|u|^2 + \delta'^2} \cdot v),$$

of Proposition 7.2.4, and the inverses of the stereographic projections

$$\sigma_\pm\colon \mathbb{R}^{m-\lambda-1} \xrightarrow{\cong} S^{m-\lambda-1} \setminus \{(\mp 1, 0, \ldots, 0)\},$$
$$w = (w_1, \ldots, w_{m-\lambda-1}) \mapsto \left(\mp\frac{|w|^2-1}{|w|^2+1}, \frac{2w_1}{|w|^2+1}, \ldots, \frac{2w_{m-\lambda-1}}{|w|^2+1}\right),$$

to calculate the Hessian of the composition

$$\nu_\pm := \nu_< \circ \phi^{\varepsilon_0}_{\delta'} \circ (\operatorname{id} \times \sigma_\pm)\colon \operatorname{int}(\varepsilon_0 \cdot D^\lambda) \times \mathbb{R}^{m-\lambda-1} \to \mathbb{R}$$

at the origin $0 \in \operatorname{int}(\varepsilon_0 \cdot D^\lambda) \times \mathbb{R}^{m-\lambda-1} \subset \mathbb{R}^{m-1}$. (Note that $(\phi^{\varepsilon_0}_{\delta'} \circ (\operatorname{id} \times \sigma_\pm))(0) = \phi^{\varepsilon_0}_{\delta'}(0, \pm 1, 0) = (0, \pm\delta', 0) = x_\pm$.) For all $w = (w_1, \ldots, w_{m-\lambda-1}) \in \mathbb{R}^{m-\lambda-1}$ and all $u \in \operatorname{int}(\varepsilon_0 \cdot D^\lambda)$ we have

$$\nu_\pm(u,w) = (\nu_< \circ \phi^{\varepsilon_0}_{\delta'})(u, \sigma_\pm(w)) = \nu_<(u, \sqrt{|u|^2+\delta'^2} \cdot \sigma_\pm(w)) = \mp\sqrt{|u|^2+\delta'^2} \cdot \left(1 - \frac{2}{|w|^2+1}\right).$$

For $i, j \in \{1, \ldots, \lambda\}$ one has the following partial derivatives:

$$\begin{aligned}
\partial_{u_i}\sqrt{|u|^2+\delta'^2} &= \frac{u_i}{\sqrt{|u|^2+\delta'^2}},\\
\partial^2_{u_i}\sqrt{|u|^2+\delta'^2} &= \partial_{u_i}\frac{u_i}{\sqrt{|u|^2+\delta'^2}} = \frac{1}{\sqrt{|u|^2+\delta'^2}} - \frac{u_i^2}{\sqrt{|u|^2+\delta'^2}^3},\\
\partial_{u_j}\partial_{u_i}\sqrt{|u|^2+\delta'^2} &= \partial_{u_j}\frac{u_i}{\sqrt{|u|^2+\delta'^2}} = -\frac{u_iu_j}{\sqrt{|u|^2+\delta'^2}^3} \qquad (i \neq j).
\end{aligned}$$

Moreover, for $i, j \in \{1, \ldots, m-\lambda-1\}$, one calculates

$$\begin{aligned}
\partial_{w_i}\left(1 - \frac{2}{|w|^2+1}\right) &= \frac{4w_i}{(|w|^2+1)^2},\\
\partial^2_{w_i}\left(1 - \frac{2}{|w|^2+1}\right) &= \partial_{w_i}\frac{4w_i}{(|w|^2+1)^2} = \frac{4}{(|w|^2+1)^2} - \frac{16w_i^2}{(|w|^2+1)^3},\\
\partial_{w_j}\partial_{w_i}\left(1 - \frac{2}{|w|^2+1}\right) &= \partial_{w_j}\frac{4w_i}{(|w|^2+1)^2} = -\frac{16w_iw_j}{(|w|^2+1)^3} \qquad (i \neq j).
\end{aligned}$$

All in all, the Hessian of $\nu_\pm$ at the origin $(u,w) = 0$ is given by the diagonal matrix

$$\operatorname{diag}\left(\pm 1/\delta', \quad \ldots, \quad \pm 1/\delta', \quad \mp 4\delta', \quad \ldots, \quad \mp 4\delta'\right).$$

Therefore, x_+ is a non-degenerate critical point of index $m-\lambda-1$ and x_- is a non-degenerate critical point of index λ.

Finally, $\nu(x_\pm) = \nu_<(x_\pm) = \pm\delta'$ because $x_\pm \in H^{\varepsilon_0}_\delta$.

(iv). Set $\varepsilon' := 2\varepsilon_0$. Suppose that $y = (u,v) \in \phi^\varepsilon_-(S^{\lambda-1} \times \{v \in \mathbb{R}^{m-\lambda};\ \varepsilon' < |v| < \varepsilon\}) \subset T^\varepsilon_-$ and $t \in \mathbb{R}$ such that $0 \leq t \leq \log(|u|/|v|)$. By Proposition 7.2.4 we have $\eta_y(t) = \eta(u,v,t) \in H^\varepsilon_\delta \setminus Z$.

Therefore, using $\xi(|v|^2) = 0$ (as $|v|^2 > \varepsilon'^2 = (2\varepsilon_0)^2$),

$$\nu(\eta_y(t)) = \nu_>(\eta(u, v, t)) = \tilde{\nu}_>(u, v, t) = e^{t \cdot \xi(|v|^2)} v_1 = v_1 \overset{(ii)}{=} \nu(\eta_y(0)).$$

□

7.3 Proof of Theorem 7.0.1

Let $\mathcal{C}$ denote the finite set of critical points of τ. By assumption, all critical points of τ have index λ, and $\tau(\mathcal{C}) = \{1/2\}$. Let ξ be a gradient-like vector field for τ (see [41, Definition 3.1, p. 20]). It is well-known that every point $w \in W$ lies on a uniquely determined maximally extended integral curve with respect to ξ, say

$$\gamma_w \colon I(w) \to W.$$

Here, $I(w) \subset \mathbb{R}$ denotes a suitable interval such that $0 \in I(w)$, and $\gamma_w \colon I(w) \to W$ is a smooth map such that $\gamma_w(0) = w$ and $\gamma_w'(t) = \xi(\gamma_w(t))$ for all $t \in I(w)$.

Remark 7.3.1. (i) Observe that, for fixed $t \in I(w)$, the shifted smooth curve

$$\gamma \colon I_t(w) := \{t' \in \mathbb{R};\ t + t' \in I(w)\} \longrightarrow W, \qquad t' \mapsto \gamma_w(t + t'),$$

is maximally extended with the properties $0 \in I_t(w)$, $\gamma(0) = \gamma_w(t)$ and $\gamma'(t') = \gamma_w'(t + t') = \xi(\gamma_w(t + t')) = \xi(\gamma(t'))$ for all $t' \in I_t(w)$. Hence, by uniqueness, we conclude that $I(\gamma_w(t)) = I_t(w)$ and $\gamma_{\gamma_w(t)} = \gamma$.

(ii) Note that, for all $w \in W$ and all $w' \in \gamma_w(I(w))$, we have

$$\gamma_{w'}(I(w')) = \gamma_w(I(w)).$$

(In fact, choose $t \in I(w)$ such that $w' = \gamma_w(t)$. Then, by part (i), every $t' \in I(w') = I(\gamma_w(t)) = \{t' \in \mathbb{R};\ t + t' \in I(w)\}$ satisfies $\gamma_{w'}(t') = \gamma_{\gamma_w(t)}(t') = \gamma_w(t + t')$. Therefore,

$$\begin{aligned}\gamma_{w'}(I(w')) &= \{\gamma_{w'}(t');\ t' \in I(w')\} = \{\gamma_w(t + t');\ t' \in \mathbb{R},\ t + t' \in I(w)\} \\ &= \{\gamma_w(t'');\ t'' \in I(w)\} = \gamma_w(I(w)).)\end{aligned}$$

For any subset $A \subset [0, 1]$ we define $W_A := \tau^{-1}(A)$. If $A = \{a\}$ consists of a single point, then we write $W_a := W_A$. Furthermore, let $W_< := W_{[0,1/2)}$ and $W_> := W_{(1/2,1]}$.

As $\mathcal{C} \subset W_{1/2}$, one can distinguish between the following three cases for a point $w \in W \setminus \mathcal{C}$ (compare the proof of [41, Theorem 4.1, p. 37 f.]):

- γ_w goes from W_0 to W_1. In this case, $I(w)$ is of the form $I(w) = [a, b]$, where

  $$\gamma_w(a) \in W_0, \qquad \gamma_w(b) \in W_1,$$

  and τ restricts to a diffeomorphism $\gamma_w(I(w)) \xrightarrow{\cong} [0, 1]$.
- γ_w goes from W_0 to some critical point of τ. In this case, $I(w) = [a, \infty)$, where

  $$\gamma_w(a) \in W_0, \qquad \lim_{t\to\infty} \gamma_w(t) \in \mathcal{C},$$

  and τ restricts to a diffeomorphism $\gamma_w(I(w)) \xrightarrow{\cong} [0, 1/2)$.
- γ_w goes from some critical point of τ to W_1. In this case, $I(w) = (-\infty, b]$, where

  $$\lim_{t\to-\infty} \gamma_w(t) \in \mathcal{C}, \qquad \gamma_w(b) \in W_1,$$

  and τ restricts to a diffeomorphism $\gamma_w(I(w)) \xrightarrow{\cong} (1/2, 1]$.

For every $c \in \mathcal{C}$ let $K(c)$ denote the union of all points in W that lie on integral curves with respect to ξ going to or from c (this includes the integral curve $\gamma_c = \mathrm{const}_c\colon \mathbb{R} \to \{c\}$). Note that $K_t(c) := K(c) \cap W_t \cong S^{\lambda-1}$ is the *left-hand sphere* (see [41, Definition 3.9, p. 28]) of c in W_t for all $t \in [0, 1/2)$. (As all critical points of τ are of index λ, all left-hand spheres are of dimension $\lambda - 1$.) Moreover, the sets $K(c) \subset W$, $c \in \mathcal{C}$, are compact and pairwise disjoint.

Let K denote the union of all $K(c)$, $c \in \mathcal{C}$, and let $K_t := K \cap W_t$ for $t \in [0, 1]$. Set $K_{\leq} := K \cap W_{[0,1/2]}$ and $K_{\geq} := K \cap W_{[1/2,1]}$.

For every $t \in [0, 1]$ let $\vartheta_t\colon W \setminus K \to \mathbb{R}$ be the map that assigns to $w \in W \setminus K$ the unique element $\vartheta_t(w) \in I(w) \subset \mathbb{R}$ such that $\tau(\gamma_w(\vartheta_t(w))) = t$, i.e. $\gamma_w(\vartheta_t(w)) = W_t \cap \gamma_w(I(w))$. It follows from the implicit function theorem that ϑ_t is smooth for fixed $t \in [0, 1]$ (compare the proof of [41, Assertion 4), pp. 53-54]). Therefore, we obtain for every $t \in [0, 1]$ a smooth map

$$\pi_t\colon W \setminus K \to W_t \setminus K_t, \qquad \pi_t(w) = \gamma_w(\vartheta_t(w)) \quad (= W_t \cap \gamma_w(I(w))).$$

For all $t, t' \in [0, 1]$ we have $\pi_{t'} \circ \pi_t = \pi_{t'}$. (In fact, fix $w \in W \setminus K$ and let $w' := \pi_t(w) = W_t \cap \gamma_w(I(w))$. Then, by Remark 7.3.1 (*ii*), $\pi_{t'}(w') = W_{t'} \cap \gamma_{w'}(I(w')) \overset{(ii)}{=} W_{t'} \cap \gamma_w(I(w)) = \pi_{t'}(w)$.) Thus, it follows from $\pi_t|_{W_t \setminus K_t} = \mathrm{id}_{W_t \setminus K_t}$ for all $t \in [0, 1]$ that for all $t, t' \in [0, 1]$ the restrictions

$$\pi_t|\colon W_{t'} \setminus K_{t'} \xrightarrow{\cong} W_t \setminus K_t,$$
$$\pi_{t'}|\colon W_t \setminus K_t \xrightarrow{\cong} W_{t'} \setminus K_{t'},$$

are mutually inverse diffeomorphisms. Hence, for every $t \in [0, 1]$ there is a diffeomorphism

$$\Pi_t\colon [0, 1] \times (W_t \setminus K_t) \xrightarrow{\cong} W \setminus K, \qquad \Pi_t(r, z) = \pi_r(z),$$

with inverse

$$\Pi_t^{-1}\colon W \setminus K \xrightarrow{\cong} [0, 1] \times (W_t \setminus K_t), \qquad \Pi_t^{-1}(w) = (\tau(w), \pi_t(w)).$$

(In fact, given $t \in [0, 1]$, the maps Π_t and Π_t^{-1} are well-defined and mutually inverse. Moreover, Π_t^{-1} is smooth because τ and π_t are smooth. To see that Π_t is smooth as well, it suffices by the inverse function theorem to show that the tangent map

$$d_w(\Pi_t^{-1}) = (d_w\tau, d_w\pi_t)\colon T_w(W \setminus K) \to T_{\tau(w)}[0, 1] \oplus T_{\pi_t(w)}(W_t \setminus K_t)$$

is for every point $w \in W \setminus K$ an isomorphism. Indeed, consider the direct sum decomposition

$$T_w(W \setminus K) = \mathbb{R}\gamma_w'(0) \oplus T_w(W_{\tau(w)} \setminus K_{\tau(w)}),$$

which results from the fact that $\tau \circ \gamma_w$ is an isomorphism $I(w) \xrightarrow{\cong} [0, 1]$. One finds that $d_w\tau$ has kernel $T_w(W_{\tau(w)} \setminus K_{\tau(w)})$, whereas $d_w\pi_t$ restricts to an isomorphism $T_w(W_{\tau(w)} \setminus K_{\tau(w)}) \xrightarrow{\cong} T_{\pi_t(w)}(W_t \setminus K_t)$. It follows directly that $d_w(\Pi_t^{-1})$ is an isomorphism.)

For every $t \in [0, 1/2)$ let $\vartheta_{<,t}\colon W_< \to \mathbb{R}$ be the smooth map that assigns to $w \in W_<$ the unique element $\vartheta_{<,t}(w) \in I(w) \subset \mathbb{R}$ such that $\tau(\gamma_w(\vartheta_{<,t}(w))) = t$, i.e. $\gamma_w(\vartheta_{<,t}(w)) = W_t \cap \gamma_w(I(w))$. Moreover, for every $t \in (1/2, 1]$ let $\vartheta_{>,t}\colon W_> \to \mathbb{R}$ be the smooth map that assigns to $w \in W_>$ the unique element $\vartheta_{>,t}(w) \in I(w) \subset \mathbb{R}$ such that $\tau(\gamma_w(\vartheta_{>,t}(w))) = t$, i.e. $\gamma_w(\vartheta_{>,t}(w)) =$

$W_t \cap \gamma_w(I(w))$. Note that by construction of ϑ_t we have $\vartheta_{<,t}|_{W_< \setminus K} = \vartheta_t|_{W_< \setminus K}$ for all $t \in [0,1/2)$ and $\vartheta_{>,t}|_{W_> \setminus K} = \vartheta_t|_{W_> \setminus K}$ for all $t \in (1/2,1]$.

We will briefly write $\pi := \pi_0$ and $\Pi := \Pi_0$.

Analogous to the construction of π_t and Π_t for $t \in [0,1]$, there exist smooth maps

$$\begin{aligned} &\pi_{<,t} \colon W_< \to W_t, && \pi_{<,t}(w) = \gamma_w(\vartheta_{<,t}(w)), && \text{for all } t \in [0,1/2), \\ &\pi_{>,t} \colon W_> \to W_t, && \pi_{>,t}(w) = \gamma_w(\vartheta_{>,t}(w)), && \text{for all } t \in (1/2,1], \end{aligned}$$

and diffeomorphisms

$$\begin{aligned} &\Pi_{<,t} \colon [0,1/2) \times W_t \xrightarrow{\cong} W_<, && \Pi_{<,t}(r,z) = \pi_{<,r}(z), && \text{for all } t \in [0,1/2), \\ &\Pi_{>,t} \colon (1/2,1] \times W_t \xrightarrow{\cong} W_>, && \Pi_{>,t}(r,z) = \pi_{>,r}(z), && \text{for all } t \in (1/2,1], \end{aligned}$$

with inverses

$$\begin{aligned} &\Pi_{<,t}^{-1} \colon W_< \xrightarrow{\cong} [0,1/2) \times W_t, && \Pi_{<,t}^{-1}(w) = (\tau(w), \pi_{<,t}(w)), && \text{for all } t \in [0,1/2), \\ &\Pi_{>,t}^{-1} \colon W_> \xrightarrow{\cong} (1/2,1] \times W_t, && \Pi_{>,t}^{-1}(w) = (\tau(w), \pi_{>,t}(w)), && \text{for all } t \in (1/2,1]. \end{aligned}$$

We will briefly write $\pi_< := \pi_{<,0}$, $\Pi_< := \Pi_{<,0}$ and $\pi_> := \pi_{>,1}$, $\Pi_> := \Pi_{>,1}$.

By construction, $\pi_{<,t}$ (respectively, $\pi_{>,t}$, $\Pi_{<,t}$, $\Pi_{>,t}$) coincides with π_t (respectively, π_t, Π_t, Π_t) whenever both are defined.

As $K_0 \subset W_0$ is a $(\lambda-1)$-dimensional smooth submanifold (namely the disjoint union of the left hand spheres $K_0(c)$, $c \in \mathcal{C}$), we may choose disjoint embeddings

$$\iota^0, \iota^1 \colon D^{m-1} \to W_0 \setminus K_0.$$

We will frequently use the notation $X := S^{m-2} = \partial D^{m-1}$.

For $j = 0,1$ we define the following cylinders in W:

$$V^j := \Pi([0,1] \times \iota^j(S^{m-2})) = \Pi([0,1] \times \iota^j(X)).$$

Moreover, define

$$V := W \setminus \Pi([0,1] \times (\iota^0(\operatorname{int} D^{m-1}) \sqcup \iota^1(\operatorname{int} D^{m-1}))).$$

For any subset $A \subset [0,1]$ let $V_A := V \cap W_A$ and, for $j = 0,1$, let $V_A^j := V^j \cap W_A$. If $A = \{a\}$ consists of a single point, then we write $V_a := V_A$ and $V_a^j := V_A^j$ for $j = 0,1$.

Note that (V_t, V_t^0, V_t^1) is a smooth manifold triad for every $t \in [0,1] \setminus \{1/2\}$. If $t < 1/2$, then $\Pi_<$ induces a diffeomorphism $(V_0, V_0^0, V_0^1) \xrightarrow{\cong} (V_t, V_t^0, V_t^1)$ via $v \mapsto \Pi_<(t,v)$, and if $t > 1/2$, then $\Pi_>$ induces a diffeomorphism $(V_1, V_1^0, V_1^1) \xrightarrow{\cong} (V_t, V_t^0, V_t^1)$ via $v \mapsto \Pi_>(t,v)$.

By definition of gradient-like vector fields (see [41, Definition 3.2, p. 20]), every critical point $c \in \mathcal{C}$ of the Morse function τ is the center of a chart

$$\psi_c \colon U_c \xrightarrow{\cong} U_c', \qquad \psi_c(c) = 0,$$

from an open neighbourhood $U_c \subset W$ of c to an open neighbourhood U'_c of $0 \in \mathbb{R}^m$ such that, in the notation introduced in the beginning of Section 7.2,

$$\tau \circ \psi_c^{-1} = \mu|_{U'_c} + 1/2,$$
$$d\psi_c \circ \xi \circ \psi_c^{-1} = v|_{U'_c}.$$

It follows from the uniqueness of integral curves that for all $w \in U_c$ and $x := \psi_c(w) \in U'_c$ the integral curve $\gamma_w\colon I(w) \to W$ with respect to ξ and the integral curve $\eta_x\colon \mathbb{R} \to \mathbb{R}^m$ with respect to v (see Section 7.2) correspond to each other via $\psi_c \circ \gamma_w = \eta_x$ on the component of 0 of $\gamma_w^{-1}(U_c) \subset I(w)$. Consequently, $\psi_c(K(c) \cap U_c) = Z \cap U'_c$. (Recall that $Z = (\mathbb{R}^\lambda) \cup (0 \times \mathbb{R}^{m-\lambda})$.)

Without loss of generality, we may assume that $U_c \cap U_{c'} = \emptyset$ for $c \neq c'$, $c, c' \in \mathcal{C}$, and $U_c \subset V \setminus \partial V$ for all $c \in \mathcal{C}$ and $j = 0, 1$.

By statements (i) and (ii) of Lemma 7.2.5 there exist $\varepsilon, \delta > 0$ such that

$$H_\delta^{2\varepsilon} \subset \bigcap_{c \in \mathcal{C}} U'_c,$$

and by statement (iii) of Lemma 7.2.5, this will still hold when we make ε or δ smaller.

Choose $\delta > 0$ so small that there exists a smooth map $\nu\colon H_\delta^\varepsilon \to \mathbb{R}$ with the properties listed in Proposition 7.2.6.

Setting $t_\pm := 1/2 \pm \delta^2$, note that $t_- + t_+ = 1$. Write $\pi_\pm := \pi_{t_\pm}$, $\Pi_\pm := \Pi_{t_\pm}$ and $(V_\pm, V_\pm^0, V_\pm^1) := (V_{t_\pm}, V_{t_\pm}^0, V_{t_\pm}^1)$, $V_{0-} := V_{[0,t_-]}$, $V_{-+} := V_{[t_-,t_+]}$, $V_{+1} := V_{[t_+,1]}$.

Note that for every $c \in \mathcal{C}$ we have $K(c) \cap V_{-+} \subset U_c$ and

$$\psi_c(K(c) \cap V_{-+}) = ((\delta \cdot D^k) \times 0) \cup (0 \times (\delta \cdot D^k)).$$

For every $c \in \mathcal{C}$ we fix $z_c \in (0,1)$ such that $z_c \neq z_{c'}$ for $c \neq c'$.

Using Lemma 7.1.3, we construct in the following an excellent Morse function

$$\zeta_-\colon (V_-, V_-^0, V_-^1) \to ([0,1], 0, 1)$$

with the following properties:

(ζ_-1) There exists a constant $C > 0$ such that for all $c \in \mathcal{C}$ we have

$$\zeta_-(\psi_c^{-1}(p,q)) = C \cdot q_1 + z_c, \qquad (p,q) \in T_-^\varepsilon.$$

(In particular, ζ_- has no critical points on $\psi_c^{-1}(T_-^\varepsilon)$.)

(ζ_-2) All indices of critical points of ζ_- are contained in $\{k-1, \ldots, m-k\}$.

(ζ_-3) For every $c \in \mathcal{C}$, z_c is not a critical value of ζ_-.

Proof. We wish to apply Lemma 7.1.3 to the smooth manifold triad $(Y, Y^0, Y^1) := (V_-, V_-^0, V_-^1)$ of dimension $n := m - 1 = \dim V_-$, to the chosen numbers $\{z_c\}_{c \in \mathcal{C}} \subset (0,1)$ and, setting $l := k - 1 \in \{3, \ldots, \lfloor \frac{m}{2} \rfloor - 1\} = \{3, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ and $d := \lambda - 1$, to the pairwise disjoint embeddings

$$\phi_c\colon S^{\lambda-1} \times \operatorname{int}(\varepsilon \cdot D^{m-\lambda}) \to V_-, \qquad c \in \mathcal{C},$$

defined for every $c \in \mathcal{C}$ by the composition

$$S^{\lambda-1} \times \mathrm{int}(\varepsilon \cdot D^{m-\lambda}) \xrightarrow{\phi_-^\varepsilon} T_-^\varepsilon \xrightarrow{\psi_c^{-1}|} U_c \cap V_- \hookrightarrow V_-,$$

where the diffeomorphism $\phi_-^\varepsilon \colon S^{k-1} \times \mathrm{int}(\varepsilon \cdot D^{m-\lambda}) \xrightarrow{\cong} T_-^\varepsilon$ is defined in Proposition 7.2.4. At this point it is important to note that we may assume $\lambda \in \{k, \dots, \lfloor \frac{m}{2} \rfloor\}$, so that $d = \lambda - 1 \in \{l, \dots, \lceil \frac{n}{2} \rceil - 1\}$ as required. (In fact, if the given $\lambda \in \{k, \dots, m-k\rfloor\}$ satisfies $\lambda > \lfloor \frac{m}{2} \rfloor$, then $m - \lambda < \lceil \frac{m}{2} \rceil$ and thus $m - \lambda \leq \lfloor \frac{m}{2} \rfloor$. But replacing τ by $1 - \tau$ in the statement of Theorem 7.0.1 does not affect the claims, but changes λ to $m - \lambda$.) Furthermore, note that V_-, V_-^0 and V_-^1 are simply connected because $(V_-, V_-^0, V_-^1) \cong (V_0, V_0^0, V_0^1)$, where $V_0^0 \cong V_0^1 \cong S^{m-2}$, and $V_0 = W_0 \setminus (\iota^0(\mathrm{int}\, D^{m-1}) \sqcup \iota^1(\mathrm{int}\, D^{m-1}))$, where W_0 is simply connected. As W_0 is even $(k-2)$-connected, it follows easily that

$$H_i(V_-, V_-^j) \cong H_i(V_0, V_0^j) = \tilde{H}_i(V_0, V_0^j) = \tilde{H}_i(V_0) = 0, \qquad i = 0, \dots, k-2, \quad j = 0, 1.$$

Hence, we obtain from Lemma 7.1.3 an excellent Morse function

$$f \colon (V_-, V_-^0, V_-^1) \to ([0,1], 0, 1)$$

with the following properties:

(*i*) f is (ϕ_c, z_c)-standard for all $c \in \mathcal{C}$, i.e. there exists $\varepsilon' \in (0, \varepsilon)$ such that

$$f(\phi_c(u, v)) = v_1 + z_c, \qquad (u, v) \in S^{\lambda-1} \times \mathrm{int}(\varepsilon' \cdot D^{m-\lambda}).$$

(*ii*) All indices of critical points of f are contained in $\{k-1, \dots, m-k\}$.

Setting $\zeta_- := f$, property (*ii*) will yield the desired property $(\zeta_- 2)$. However, to satisfy property $(\zeta_- 1)$ in addition, we have to precompose f with a suitable automorphism Ω of V_- (which restricts to the identity map outside a compact subset of $V_- \setminus \partial V_-$) that will be constructed next. Property $(\zeta_- 3)$ can finally be achieved afterwards by perturbing ζ_- slightly in a neighbourhood of its critical points as described in the proof of [41, Lemma 2.8, p. 17]. Of course, this little perturbation does not violate properties $(\zeta_- 1)$ and $(\zeta_- 2)$, and leaves ζ_- excellent.

Let us turn to the construction of the automorphism Ω of V_-. Set $C := \varepsilon'/\varepsilon \in (0,1)$, where $\varepsilon' \in (0, \varepsilon)$ is given by property (*i*).

Choose a diffeomorphism

$$\rho \colon \mathrm{int}(2\varepsilon \cdot D^{m-\lambda}) \xrightarrow{\cong} \mathrm{int}(2\varepsilon \cdot D^{m-\lambda})$$

such that $\rho(v) = C \cdot v$ for $|v| < \varepsilon$ and $\rho(v) = v$ for $|v| > 3\varepsilon/2$. (In fact, such a diffeomorphism ρ can be obtained by applying the *isotopy extension lemma* [22, Theorem 1.3, p. 180] to the isotopy of the compact submanifold $\varepsilon \cdot D^{m-\lambda} \subset \mathrm{int}(3\varepsilon/2 \cdot D^{m-\lambda})$ given at $s \in [0,1]$ by $v \mapsto (1 - s + s \cdot C) \cdot v$. The automorphism of $\mathrm{int}(3\varepsilon/2 \cdot D^{m-\lambda})$ thus obtained at $s = 1$ has compact support and can hence be extended to the desired automorphism ρ.)

The automorphism ρ gives rise to an automorphism

$$\omega_\rho := \phi_-^{2\varepsilon} \circ (\mathrm{id}_{S^{\lambda-1}} \times \rho) \circ (\phi_-^{2\varepsilon})^{-1} \colon T_-^{2\varepsilon} \xrightarrow{\cong} T_-^{2\varepsilon}$$

that restricts to the identity map outside the compact subset

$$\overline{T_-^{3\varepsilon/2}} = \phi_-^{2\varepsilon}(S^{\lambda-1} \times (3\varepsilon/2 \cdot D^{m-\lambda})) \subset T_-^{2\varepsilon}$$

and satisfies

$$\omega_\rho(p,q) = \phi_-^{\varepsilon}(\frac{p}{\sqrt{|q|^2+\delta^2}}, C \cdot q), \qquad (p,q) \in T_-^{\varepsilon} \subset T_-^{2\varepsilon} \qquad (\subset \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda}).$$

(Indeed, if $(p,q) \in T_-^{\varepsilon}$ and $(u,v) := (\phi_-^{2\varepsilon})^{-1}(p,q) = (\phi_-^{\varepsilon})^{-1}(p,q) \in S^{k-1} \times \mathrm{int}(\varepsilon \cdot D^{m-\lambda})$, then $(p,q) = \phi_-^{2\varepsilon}(u,v) = (\sqrt{|v|^2+\delta^2} \cdot u, v)$. As $|v| < \varepsilon$ implies $\rho(v) = C \cdot v$, we obtain

$$\omega_\rho(p,q) = \phi_-^{2\varepsilon}(u, \rho(v)) = \phi_-^{2\varepsilon}(\frac{p}{\sqrt{|q|^2+\delta^2}}, C \cdot q) = \phi_-^{\varepsilon}(\frac{p}{\sqrt{|q|^2+\delta^2}}, C \cdot q),$$

where the last equality holds since $|C \cdot q| = C \cdot |v| < |v| < \varepsilon$.)

Recall that $T_-^{2\varepsilon} \subset H_\delta^{2\varepsilon} \subset U'_c$ for every $c \in \mathcal{C}$, and the domains of the diffeomorphisms $\psi_c \colon U_c \xrightarrow{\cong} U'_c$ are pairwise disjoint and contained in $V \setminus \partial V$.

For every $c \in \mathcal{C}$ let ω_c denote the automorphism of $\psi_c^{-1}(T_-^{2\varepsilon})$ given by the composition of diffeomorphisms

$$\omega_c \colon \psi_c^{-1}(T_-^{2\varepsilon}) \xrightarrow{\psi_c|} T_-^{2\varepsilon} \xrightarrow{\omega_\rho} T_-^{2\varepsilon} \xrightarrow{(\psi_c)^{-1}|} \psi_c^{-1}(T_-^{2\varepsilon}).$$

Note that $\psi_c^{-1}(T_-^{2\varepsilon})$ is an open subset of $V_- \setminus \partial V_-$ for every $c \in \mathcal{C}$. (Indeed, $T_-^{2\varepsilon}$ is an open subset of $U'_c \cap \mu^{-1}(-\delta^2)$ by construction. Hence, $\psi_c^{-1}(T_-^{2\varepsilon})$ is an open subset of

$$\psi_c^{-1}(U'_c \cap \mu^{-1}(-\delta^2)) = \psi_c^{-1}((\tau \circ \psi_c^{-1})^{-1}(t_-)) = U_c \cap \tau^{-1}(t_-) = U_c \cap (V_- \setminus \partial V_-).)$$

As ω_c is an automorphism of $\psi_c^{-1}(T_-^{2\varepsilon})$ that restrict to the identity map outside the compact subset $\psi_c^{-1}(\overline{T_-^{3\varepsilon/2}}) \subset \psi_c^{-1}(T_-^{2\varepsilon})$, we obtain a well-defined automorphism

$$\Omega \colon (V_-, V_-^0, V_-^1) \xrightarrow{\cong} (V_-, V_-^0, V_-^1)$$

via extension of $\sqcup_{c \in \mathcal{C}} \omega_c$ by the identity map. (Recall that the subsets $\psi_c^{-1}(T_-^{2\varepsilon}) \subset V_- \setminus \partial V_-$ are open and pairwise disjoint for $c \in \mathcal{C}$.)

It remains to verify that the excellent Morse function

$$\zeta_- := f \circ \Omega \colon (V_-, V_-^0, V_-^1) \to ([0,1], 0, 1)$$

has the desired properties $(\zeta_- 1)$ and $(\zeta_- 2)$:

(ζ_-1) For all $c \in \mathcal{C}$ and all $(p,q) \in T^{\varepsilon}_-(\subset T^{2\varepsilon}_-)$ we have, by property (i) of f,

$$\begin{aligned}\zeta_-(\psi_c^{-1}(p,q)) &= f(\Omega(\psi_c^{-1}(p,q))) = f(\omega_c(\psi_c^{-1}(p,q))) = f(\psi_c^{-1}(\omega_\rho(p,q)))\\ &= f(\psi_c^{-1}(\phi^{\varepsilon}_-(\frac{p}{\sqrt{|q|^2+\delta^2}}, C\cdot q)))\\ &= f(\phi_c(\frac{p}{\sqrt{|q|^2+\delta^2}}, C\cdot q)) \overset{(i)}{=} C\cdot q_1 + z_c.\end{aligned}$$

(ζ_-2) This follows immediately from property (ii) of f.

□

Next, extend ζ_- to an excellent Morse function

$$\sigma_-\colon W_- \to \mathbb{R}$$

such that all indices of critical points of σ_- are contained in $\{k-1,\dots,m-k\}\cup\{0,m-1\}$. (In fact, recall that V_- can be obtained from W_- by deleting the interiors of two embedded disjoint $(m-1)$-balls $B^0, B^1 \subset W_-$. Therefore, one can use [41, Lemma 3.7, p. 26] to glue ζ_- and the Morse function

$$\zeta^j_-\colon (B^j,\partial B^j)\cong(D^{m-1},S^{m-2})\to(\mathbb{R},j), \qquad \zeta^j_-(x) = j + (-1)^j\cdot(|x|^2-1),$$

for $j=0,1$ along $V^j_- = \partial B^j$.)

In the last step we construct a smooth map

$$\sigma_{-+}\colon W_{-+}\to\mathbb{R}$$

with the following properties:

($\sigma_{-+}1$) If $t\in[t_-,t_+]$ and $t\neq 1/2$, then σ_{-+} restricts to a Morse function

$$\sigma_t := \sigma_{-+}|_{W_t}\colon W_t\to\mathbb{R}.$$

Moreover, there exists $t'_+\in(1/2,t_+]$ such that σ_t is excellent for all $t\in[t_-,t'_+]\backslash\{1/2\}$.

($\sigma_{-+}2$) $\tau|$ and σ_{-+} form the components of a fold map

$$(\tau|,\sigma_{-+})\colon W_{-+}\to[t_-,t_+]\times\mathbb{R}$$

whose fold points have all an absolute index contained in the set $\{\lfloor\frac{m}{2}\rfloor,\dots,m-k\}\cup\{m-1\}$.

(Note that, given σ_{-+}, the desired smooth map $\sigma\colon W\to\mathbb{R}$ can then be obtained by the following argument. Construct a diffeomorphism $\Phi\colon W'_{-+} := \tau^{-1}([t_-,t'_+]) \xrightarrow{\cong} W$ covered by a diffeomorphism $\phi\colon[t_-,t'_+]\xrightarrow{\cong}[0,1]$ (i.e. $\tau\circ\Phi = \phi\circ\tau|_{W_{-+}}$) such that $\phi(1/2)=1/2$. Then the smooth map $\sigma := \sigma_{-+}\circ\Phi^{-1}\colon W\to\mathbb{R}$ satisfies

$$(\mathrm{id}_{\mathbb{R}}\times\phi^{-1})\circ(\sigma,\tau) = (\sigma_{-+}\circ\Phi^{-1},\phi^{-1}\circ\tau) = (\sigma_{-+},\tau|_{W_{-+}})\circ\Phi^{-1}.$$

Hence, property ($\sigma_{-+}2$) implies the desired property (ii) for σ. Moreover, the desired property

(i) follows from $(\sigma_{-+}1)$.)

Proof. Fix $c \in \mathcal{C}$. Set $H(c) := \psi_c^{-1}(H_\delta^\varepsilon) \subset U_c$. Observe that $H(c)$ is an open subset of W_{-+} since H_δ^ε is an open subset of $\psi_c(U_c \cap V_{-+}) = U'_c \cap \mu^{-1}([-\delta^2, \delta^2])$ $(\subset \mathbb{R}^m)$ by Definition 7.2.3. Note that $H(c) \cap H(c') = \emptyset$ for $c \neq c'$, $c, c' \in \mathcal{C}$, as $U_c \cap U_{c'} = \emptyset$. Let ε' be taken from property (iv) of Proposition 7.2.6. Let $\overline{H_\delta^{\varepsilon'}}$ denote the closure of $H_\delta^{\varepsilon'}$ in $\mathbb{R}^m$ (equivalently, the closure of $H_\delta^{\varepsilon'}$ in $\mu^{-1}([-\delta^2, \delta^2])$). Note that $\overline{H_\delta^{\varepsilon'}}$ is bounded by Lemma 7.2.5 (i) and thus compact. Moreover, Definition 7.2.3 implies that $\overline{H_\delta^{\varepsilon'}} = \bigcap_{\tilde{\varepsilon} \in (\varepsilon', \varepsilon)} H_\delta^{\tilde{\varepsilon}}$. Hence, it follows from Lemma 7.2.5 (iii) that $\overline{H_\delta^{\varepsilon'}} = \bigcap_{\tilde{\varepsilon} \in (\varepsilon', \varepsilon)} H_\delta^{\tilde{\varepsilon}} \subset H_\delta^\varepsilon \subset U'_c$. Hence, $\hat{H}(c) := \psi_c^{-1}(\overline{H_\delta^{\varepsilon'}}) \subset V_{-+}$ is compact, and $\hat{H}(c) \subset H(c)$. Therefore, one obtains the open cover

$$W_{-+} = (W_{-+} \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)) \cup \bigsqcup_{c \in \mathcal{C}} H(c).$$

Define

$$\sigma_{-+} \colon W_{-+} \to \mathbb{R}, \qquad \sigma_{-+}(w) = \begin{cases} \sigma_-(\pi_-(w)), & \text{if } w \in W_{-+} \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c), \\ C \cdot \nu(\psi_c(w)) + z_c, & \text{if } w \in H(c) \text{ for some } c \in \mathcal{C}, \end{cases}$$

where $C > 0$ is the constant of property (ζ_-1) and ν is the map of Proposition 7.2.6.

First, one has to check that σ_{-+} is well-defined. (In fact, if $w \in H(c) \setminus \hat{H}(c) = \psi_c^{-1}(H_\delta^\varepsilon) \setminus \psi_c^{-1}(\overline{H_\delta^{\varepsilon'}}) = \psi_c^{-1}(H_\delta^\varepsilon \setminus \overline{H_\delta^{\varepsilon'}})$ for some $c \in \mathcal{C}$, then the point $x := \psi_c(w) \in H_\delta^\varepsilon \setminus \overline{H_\delta^{\varepsilon'}} \subset H_\delta^\varepsilon \setminus Z$ is by Proposition 7.2.4 of the form $x = \eta_y(t)$ for suitable $(p, q) = y \in T_-^\varepsilon \setminus \overline{T_-^\varepsilon}$ $(\subset T_-^\varepsilon \setminus Z)$ and $t \geq 0$, and we have $\eta_y([0, t]) \subset H_\delta^\varepsilon \setminus Z$. As $\eta_y(s) = (e^{-(s-t)}e^{-t}p, e^{s-t}e^t q) = \eta_x(s-t)$ for all $s \in \mathbb{R}$, this reads $\eta_x([-t, 0]) \subset H_\delta^\varepsilon \setminus Z$. By uniqueness of integral curves, we conclude that $(\psi_c^{-1} \circ \eta_x)|_{[-t,0]} = \gamma_w|_{[-t,0]}$. All in all, $\psi_c^{-1}(y) = \psi_c^{-1}(\eta_y(0)) = \psi_c^{-1}(\eta_x(-t)) = \gamma_w(-t)$ is the unique point contained in the intersection $W_- \cap \gamma_w(I(w))$. Therefore, $\pi_-(w) = \psi_c^{-1}(y)$. Hence, property ζ_-1 applied to $y \in T_-^\varepsilon$ yields

$$\sigma_-(\pi_-(w)) = \sigma_-(\psi_c^{-1}(y)) \overset{(\zeta_-1)}{=} C \cdot q_1 + z_c = C \cdot \nu(\psi_c(w)) + z_c.$$

The last equality holds since properties (iv) and (ii) of Proposition 7.2.6 imply that

$$\nu(\psi_c(w)) = \nu(x) = \nu(\eta_y(t)) \overset{(iv)}{=} \nu(\eta_y(0)) = \nu(y) \overset{(ii)}{=} q_1.)$$

As σ_{-+} is by definition smooth on the open subsets $W_{-+} \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)$ and $\bigsqcup_{c \in \mathcal{C}} H(c)$ of W_{-+}, we conclude that σ_{-+} is a smooth map.

It remains to check the desired properties for σ_{-+}:

$(\sigma_{-+}1)$. First suppose that $t \in [t_-, 1/2)$. We have to show that $\sigma_t := \sigma_{-+}|_{W_t}$ restricts on each of the open subsets $W_t \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)$ and $\bigsqcup_{c \in \mathcal{C}} H(c) \cap W_t$ of W_t to a smooth map that possesses only non-degenerate critical points. Recall that π_- restricts to a diffeomorphism $W_t \setminus K_t \xrightarrow{\cong} W_- \setminus K_{t_-}$. As $W_t \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)$ is an open subset of $W_t \setminus K_t$, we deduce that $\sigma_{-+}|_{W_t \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)} = \sigma_- \circ \pi_-|_{W_t \setminus \bigsqcup_{c \in \mathcal{C}} \hat{H}(c)}$ possesses only non-degenerate critical points. Moreover, note that these critical points are on pairwise different levels since σ_- is excellent. Next, by definition of σ_{-+},

$$\sigma_{-+}|_{H(c)\cap W_t} = C\cdot(\nu\circ\psi_c|_{H(c)\cap W_t}) + z_c, \qquad c\in\mathcal{C}.$$

Therefore, it follows from $\psi_c(H(c)\cap W_t) = H_\varepsilon^\delta \cap \mu^{-1}(t-1/2)$ and property (ii) of Proposition 7.2.6 that $\sigma_{-+}|_{H(c)\cap W_t}$ has no critical points. All in all, σ_t is for $t\in[t_-,1/2)$ an excellent Morse function with the desired properties.

Now suppose that $t\in(1/2,t_+]$. Again it suffices to show that $\sigma_t := \sigma_{-+}|_{W_t}$ restricts on each of the open subsets $W_t\setminus\bigsqcup_{c\in\mathcal{C}}\hat{H}(c)$ and $\bigsqcup_{c\in\mathcal{C}} H(c)\cap W_t$ of W_t to a smooth map that possesses only non-degenerate critical points. Analogous to the case $t\in[t_-,1/2)$ it follows that $\sigma_{-+}|_{W_t\setminus\bigsqcup_{c\in\mathcal{C}}\hat{H}(c)} = \sigma_-\circ\pi_-|_{W_t\setminus\bigsqcup_{c\in\mathcal{C}}\hat{H}(c)}$ possesses only non-degenerate critical points. Next, by definition of σ_{-+},

$$\sigma_{-+}|_{H(c)\cap W_t} = C\cdot(\nu\circ\psi_c|_{H(c)\cap W_t}) + z_c, \qquad c\in\mathcal{C}.$$

Therefore, it follows from $\psi_c(H(c)\cap W_t) = H_\varepsilon^\delta \cap \mu^{-1}(t-1/2)$ and property (iii) of Proposition 7.2.6 that $\sigma_{-+}|_{H(c)\cap W_t}$ possesses two critical points, and these are non-degenerate, namely one of index $m-\lambda-1$ and one of index λ. All in all, σ_t is a Morse function with the desired properties for $t\in(1/2,t_+]$.

It remains to show that there exists $t'_+\in(1/2,t_+]$ such that σ_t is excellent for all $t\in(1/2,t'_+]$. Recall that σ_- is excellent, and the levels of the critical points of σ_- are all different from the numbers z_c by (ζ_-3). Moreover, we have seen that the set of critical points of σ_t for $t\in(1/2,t_+]$ is the union of the set of critical points of σ_- and two more critical points for every $c\in\mathcal{C}$ that are arbitrarily near to z_c for $t\in(1/2,t_+]$ near $1/2$ by property (iii) of Proposition 7.2.6 (and are on different levels). Hence, by choosing $t'_+\in(1/2,t_+]$ sufficiently near to $1/2$, one can achieve that the critical levels of σ_t are pairwise different.

$(\sigma_{-+}2)$. Recall that Π_- restricts to a diffeomorphism

$$[t_-,t_+]\times(W_-\setminus K)\xrightarrow{\cong} W_{-+}\setminus K, \qquad (t,w)\mapsto\pi_t(w).$$

This can be further restricted to a diffeomorphism

$$\Pi_-|\colon [t_-,t_+]\times(W_-\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c))\xrightarrow{\cong} W_{-+}\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c), \qquad (t,w)\mapsto\pi_t(w).$$

(In fact, one has to show that $\Pi_-([t_-,t_+]\times(W_-\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c)))\subset W_{-+}\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c)$ and $\Pi_-^{-1}(W_{-+}\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c))\subset[t_-,t_+]\times(W_-\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c))$. Suppose that $(t,w)\in[t_-,t_+]\times(W_-\setminus K)$ satisfies $w' := \Pi_-(t,w)\in\hat{H}(c)\setminus K$ for some $c\in\mathcal{C}$. This implies that $x := \psi_c(w')\in\overline{H_\delta^{\varepsilon'}}\setminus Z = \bigcap_{\tilde\varepsilon\in(\varepsilon',\varepsilon)} H_\delta^{\tilde\varepsilon}\setminus Z$. By Proposition 7.2.4 there exists $(p,q) = y\in\bigcap_{\tilde\varepsilon\in(\varepsilon',\varepsilon)} T_-^{\tilde\varepsilon}\setminus Z$ such that $x\in\eta_y([0,\log(|p|/|q|)]) = \eta_y(\mathbb{R})\cap(\overline{H_\delta^{\varepsilon'}}\setminus Z)$. Let $t'\in[0,\log(|p|/|q|)]$ such that $\eta_y(t') = x$. As $\eta_y(s) = \eta_x(s-t')$ for all $s\in\mathbb{R}$, we have $\eta_x([-t',0])\subset\overline{H_\delta^{\varepsilon'}}\setminus Z$. By uniqueness of integral curves, we conclude that $(\psi_c^{-1}\circ\eta_x)|_{[-t',0]} = \gamma_{w'}|_{[-t',0]}$. All in all, $\psi_c^{-1}(y) = \psi_c^{-1}(\eta_y(0)) = \psi_c^{-1}(\eta_x(-t')) = \gamma_{w'}(-t')$ is the point $W_-\cap\gamma_{w'}(I(w')) = \pi_-(w') = \pi_-(\pi_t(w)) = w$. Therefore, $w = \psi_c^{-1}(y)\in W_-\cap(\hat{H}(c)\setminus K)$. This implies $\Pi_-([t_-,t_+]\times(W_-\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c)))\subset W_{-+}\setminus\bigcup_{c\in\mathcal{C}}\hat{H}(c)$. Conversely, suppose that $w\in W_{-+}\setminus K$ satisfies $(t',w') := \Pi_-^{-1}(w) = (\tau(w),\pi_-(w))\in[t_-,t_+]\times(\bigsqcup_{c\in\mathcal{C}}\hat{H}(c)\setminus K)$, i.e. $w' := \pi_-(w)\in\hat{H}(c)\setminus K$ for some $c\in\mathcal{C}$. This implies that $y := \psi_c(w')\in$

$\mu^{-1}(-\delta^2) \cap \overline{H_\delta^{\varepsilon'}} \setminus Z = \bigcap_{\tilde\varepsilon \in (\varepsilon', \varepsilon)} T_-^{\tilde\varepsilon} \setminus Z$. Hence, Proposition 7.2.4 implies that there exists $t \geq 0$ such that $\eta_y(t) \in \mu^{-1}(t' - 1/2) \cap (\overline{H_\delta^{\varepsilon'}} \setminus Z)$ and $\eta_y([0,t]) \subset \overline{H_\delta^{\varepsilon'}} \setminus Z$. By uniqueness of integral curves, we conclude that $(\psi_c^{-1} \circ \eta_y)|_{[0,t]} = \gamma_{w'}|_{[0,t]}$. Hence, $\gamma_{w'}(t) = \psi_c^{-1}(\eta_y(t)) \in W_{t'} \cap (\hat{H}(c) \setminus K)$ is the point $W_{t'} \cap \gamma_{w'}(I(w')) = \pi_{t'}(w') = \Pi_-(t', w') = w$. Therefore, $w = \gamma_{w'}(t) \in \hat{H}(c) \setminus K$. This implies $\Pi_-^{-1}(V_{-+} \setminus \bigcup_{c \in \mathcal{C}} \hat{H}(c)) \subset [t_-, t_+] \times (V_- \setminus \bigcup_{c \in \mathcal{C}} \hat{H}(c))$.)

The composition $(\tau|, \sigma_{-+}) \circ \Pi_-|$ is given at $(t, w) \in [t_-, t_+] \times (W_- \setminus \bigcup_{c \in \mathcal{C}} \hat{H}(c))$ by

$$(t, w) \mapsto (\tau|, \sigma_{-+}) \circ \Pi_-|(t, w) = (\tau(\pi_t(w)), \sigma_{-+}(\pi_t(w))) = (t, \sigma_-(\pi_{t_-}(\pi_t(w)))) = (t, \sigma_-(w))$$

and is hence the suspension of a smooth function with only non-degenerate critical points. As σ_- has only critical points of index contained in $\{k-1, \dots, m-k\} \cup \{0, m-1\}$, we conclude that the restriction of $(\tau|, \sigma_{-+})$ to the open subset $W_{-+} \setminus \bigcup_{c \in \mathcal{C}} \hat{H}(c)$ of W_{-+} is a fold map whose fold lines have all absolute index contained in $\{\lfloor \frac{m}{2} \rfloor, \dots, m-k\} \cup \{m-1\}$.

Next, we consider the restriction of $(\tau|, \sigma_{-+})$ to the open subset $H(c) \subset W_{-+}$ for some $c \in \mathcal{C}$. By definition, we have $\sigma_{-+} \circ \psi_c^{-1}|_{H_\delta^\varepsilon} = C \cdot \nu|_{H_\delta^\varepsilon} + z_c$. Moreover, $\tau \circ \psi_c^{-1} = \mu|_{U_c'} + 1/2$. Hence,

$$((\tau|, \sigma_{-+}) \circ \psi_c^{-1}|_{H_\delta^\varepsilon})(t, w) = (\mu|_{H_\delta^\varepsilon} + 1/2, C \cdot \nu|_{H_\delta^\varepsilon} + z_c).$$

This is up to the automorphism $(x, y) \mapsto (x + 1/2, C \cdot y + z_c)$ of $\mathbb{R}^2$ equal to $(\mu|_{H_\delta^\varepsilon}, \nu)$, which is by property (i) of Proposition 7.2.6 a fold map with a single fold line, whose absolute index is given by $\max\{\lambda, m-1-\lambda\} \in \{\lfloor \frac{m}{2} \rfloor, \dots, m-k\}$.

All in all, $(\tau|_{V_{-+}}, \sigma_{-+})$ is a fold map whose fold points have all absolute index contained in $\{\lfloor \frac{m}{2} \rfloor, \dots, m-k\} \cup \{m-1\}$.

□

This completes the proof of Theorem 7.0.1.

Chapter 8

Extending Boundary Conditions Generically over the Cylinder

Playing a major role in the proof of Theorem 10.1.3, the main result of the present chapter is the following

Theorem 8.0.1. *Fix integers* $m \geq 8$ *and* $k \in \{4, \dots, \lfloor \frac{m}{2} \rfloor\}$. *Let* M^{m-1} *be a closed connected smooth manifold of dimension* $m-1$. *Suppose that*

$$f, g \colon [0,1] \times M \to \mathbb{R}$$

are smooth maps such that $f_t := f(t,-)$ *and* $g_t := g(t,-)$ *are for every* $t \in [0,1]$ *excellent Morse functions* $M \to \mathbb{R}$ *with only critical points of index contained in* $\{k-1, \dots, m-k\} \cup \{0, m-1\}$. *Then there exists a generic smooth map*

$$F \colon [0,1] \times M \to \mathbb{R}^2$$

such that the absolute index of every fold point of F *is contained in the set* $\{\lfloor \frac{m}{2} \rfloor, \dots, m-k\} \cup \{m-1\}$, *and such that*

$$\begin{aligned} F(x,t) &= (f_t(x), t) \qquad (x,t) \in M \times [0, 1/4], \\ F(x,t) &= (g_t(x), t) \qquad (x,t) \in M \times [3/4, 1]. \end{aligned}$$

The preparation of the proof of Theorem 8.0.1 makes massively use of the content of Cerf's fundamental article "La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie" [9]. In particular, Cerf's valuable theorem [9, Théorème 2', V.2.1, p. 100] (see Theorem 8.2.6) is a major ingredient of our construction. Therefore, following [9], we recapitulate all the necessary notation in Section 8.2, and we give a careful exposition of the material of interest. Afterwards, the main purpose of Section 8.3 is to prove Corollary 8.3.11, which is the result from Cerf theory that flows into the proof of Theorem 8.0.1 (see Section 8.4).

8.1 Smoothing of Paths in a Diffeomorphism Group

The main result of the present section (see Corollary 8.1.4) states that the endpoints of any continuous path in the diffeomorphism group of a smooth manifold with boundary (equipped with the Whitney C^∞ topology) are isotopic. For the proof we exploit the general setting of [37], where spaces of smooth maps between smooth manifolds with corners are studied. A major insight is that this kind of spaces can be given the structure of a C^∞_c*-manifold* (see [37, Definition 9.1, p. 83]), which roughly means that they can be locally modeled on certain (in general infinite-dimensional) locally convex vector spaces. We make use of the explicit form of charts for the diffeomorphism group to construct the desired isotopy.

Let X and Y be smooth manifolds with corners. Let τ_{C^∞} denote the Whitney C^∞ topology (see [37, 4.4, page 33]) on $C^\infty(X,Y)$. Besides, $C^\infty(X,Y)$ can be equipped with the $F\mathfrak{D}$-topology $\tau_{F\mathfrak{D}}$ as defined in [37, Definition 4.10, page 40]. The advantage of the $F\mathfrak{D}$-topology is that one can equip $C^\infty(X,Y)$ (at least when $\partial Y = \emptyset$, see [37, Theorem 10.4, p. 91]) with the structure of a C^∞_c-manifold (compare [37, Remark 4.9, page 39 f.]).

Remark 8.1.1. The $F\mathfrak{D}$-topology has been designed by Michor to handle the case of a non-compact source manifold. We are eventually interested in the diffeomorphism group of a compact manifold. Then, the $F\mathfrak{D}$-topology coincides by definition with the (in general coarser) $\mathfrak{D}$-topology defined in [37, 4.7, p. 36]. Nevertheless, we do not assume compactness in the present section because the arguments are substantially the same.

Lemma 8.1.2. *The topological spaces* $(C^\infty(X,Y),\tau_{F\mathfrak{D}})$ *and* $(C^\infty(X,Y),\tau_{C^\infty})$ *have the same paths, i.e. a map* $[0,1]\to C^\infty(X,Y)$ *is continuous for the Whitney* C^∞ *topology if and only if it is continuous for the* $F\mathfrak{D}$*-topology.*

Proof. According to [37, Remark 4.11.1, page 40], it follows from [37, 4.4.3, p. 34] and [37, 4.7.7, p. 38] that a sequence $a_0, a_1, \dots$ converges to some a in $(C^\infty(X,Y),\tau_{C^\infty})$ if and only if $a_0, a_1, \dots$ converges to a in $(C^\infty(X,Y),\tau_{F\mathfrak{D}})$. The claim now results from applying the following fact to $S := C^\infty(X,Y)$, $\tau_1 := \tau_{C^\infty}$ and $\tau_2 := \tau_{F\mathfrak{D}}$ (or $\tau_2 := \tau_{C^\infty}$ and $\tau_1 := \tau_{F\mathfrak{D}}$):

Let S *be a set and let* τ_1 *and* τ_2 *be two topologies on* S *with the following property: If a sequence* $s_1, s_2, \dots$ *in* S *converges in* (S,τ_1) *to some* $s \in S$*, then* $s_1, s_2, \dots$ *also converges to* s *in* (S,τ_2)*. Then any path* $f\colon [0,1]\to S$ *in* (S,τ_1) *is also a path in* (S,τ_2)*.*

The above fact can be proven as follows. Let V be open in (S,τ_2). We have to show that $U := f^{-1}(V)$ is open in $[0,1]$.

Let $t \in U$. Suppose that for every integer $n>0$ there exists $t_n \in (t-1/n, t+1/n)\cap[0,1]$ such that $t_n \notin U$. Then $s_n := f(t_n) \notin V$.

Since $t_1, t_2, \dots$ converges to t in $[0,1]$ and $f\colon [0,1]\to (S,\tau_1)$ is continuous, and in particular sequentially continuous, we conclude that $s_1, s_2, \dots$ converges to $s := f(t)$ in (S,τ_1). Hence, $s_1, s_2, \dots$ converges to $s = f(t) \in V$ in (S,τ_2) by assumption. However, this contradicts the fact that $s_n \notin V$ for all n. Therefore, there exists an n such that $(t-1/n, t+1/n)\cap[0,1] \subset U$. This shows that U is open in $[0,1]$ as t was arbitrary. $\square$

Proposition 8.1.3. *Let X be a smooth manifold with boundary. Every $f_0 \in (\mathrm{Diff}(X), \tau_{F\mathfrak{D}})$ has an open neighbourhood U with the following property. For every $f_1 \in U$ there exists a smooth map*

$$\nu\colon [0,1] \times X \to X, \qquad \nu_t(x) := \nu(t,x),$$

such that $\nu_i = f_i$ for $i \in \{0,1\}$ and $\nu_t \in U$ for all $t \in [0,1]$.

Proof. Given a smooth manifold W with corners, the subset ${}^iTW \subset TW$ of all *inner* tangent vectors is defined as in [37, 2.6, p. 20]. (Note that, if W has no boundary, then ${}^iTW = TW$.) As pointed out in *loc.cit.*, TW is alway a manifold with corners, whereas ${}^iTW$ fails to be a manifold with corners in general. Nevertheless, the notion of smooth maps on ${}^iTW$ is still available (see [37, 10.1, p. 90]).

Recall that the concept of a *local addition* τ on W is introduced in [37, 10.1, p. 90] as a smooth map $\tau\colon {}^iTW \to W$ with the following properties:

(1) $(\pi_W, \tau)\colon {}^iTW \to W \times W$ is a diffeomorphism onto an open neighbourhood of the diagonal in $W \times W$. (Here, $\pi_W\colon TW \to W$ denotes the tangent bundle map.)
(2) $\tau(0_W(w)) = w$ for all $w \in W$. (Here, $0_W\colon W \to {}^iTW$ denotes the zero section.)

Now let Z denote a smooth manifold with corners, and let Y denote a smooth manifold without boundary. According to the proof of [37, Theorem 10.4, p. 91] a chart of the C^∞_c-structure of $C^\infty(Z,Y)$ centered at a given point f can be constructed as follows. Fix a local addition τ on Y. (This is always possible by [37, Lemma 10.1, p. 90].) An open neighbourhood of f in $(C^\infty(Z,Y), \tau_{F\mathfrak{D}})$ is defined by

$$\begin{aligned} U_f &:= \{g \in C^\infty(Z,Y);\ g \sim f \text{ and } g(z) \in \tau_{f(z)}({}^iT_{f(z)}Y) \text{ for all } z \in Z\} \\ &= \{g \in C^\infty(Z,Y);\ g \sim f \text{ and } (f,g)(Z) \subset (\pi_Y, \tau)({}^iTY)\}, \end{aligned}$$

where $f \sim g$ means that the set $\{z \in Z;\ f(z) \neq g(z)\}$ has compact closure in Z (see [37, Definition 4.10, p. 40]).

The *space of all vector fields along f with compact support* is defined as

$$\mathfrak{D}_f(Z,TY) := \{s \in C^\infty(Z,TY);\ \pi_Y \circ s = f,\ s \sim 0_Y \circ f\}.$$

Equipped with the $F\mathfrak{D}$-topology, $\mathfrak{D}_f(Z,TY)$ can be shown to be a certain locally convex vector space. In particular, its origin $0_Y \circ f$ possesses a neighbourhood basis consisting of convex open neighbourhoods.

The *canonical chart of $C^\infty(Z,Y)$ centered at f (and induced by τ)* is the homeomorphism

$$\varphi_f\colon U_f \xrightarrow{\cong} \mathfrak{D}_f(Z,TY), \qquad \varphi_f(g) = (\pi_Y, \tau)^{-1} \circ (f,g),$$

(i.e., $\varphi_f(g)(z) = \tau^{-1}_{f(z)} g(z)$ for all $z \in Z$), whose inverse is given by

$$\psi_f\colon \mathfrak{D}_f(Z,TY) \xrightarrow{\cong} U_f, \qquad \psi_f(s) = \tau \circ s.$$

Note that φ_f maps $f \in U_f$ to the origin $0_Y \circ f$ of $\mathfrak{D}_f(Z,TY)$.

In the present proof we are in particular interested in the case that $X = Y = Z$ is a smooth manifold with boundary. As explained in [37, 10.16, pp. 106-107], the case that Y has corners can be handled via the following modifications. The local addition $\tau\colon {}^iTY \to Y$ is chosen to be *boundary respecting* (see [37, Remark 10.3, p. 91]). In contrast to $\partial Y = \emptyset$, one cannot expect φ_f to be surjective because it can be shown that

$$\varphi_f(U_f) = \{s \in \mathfrak{D}_f(Z, TY);\ s(Z) \subset {}^iTY\}.$$

Let $C^\infty_{\text{nice}}(Z, Y)$ denote the subset of all $g \in C^\infty(Z, Y)$ such that $g^{-1}(\partial^j Y) = \partial^j Z$ for all $j \geq 0$ (see [37, page 107]). It can be shown that $\varphi_f(U_f \cap C^\infty_{\text{nice}}(Z, Y))$ is a certain closed linear subspace ${}^t\mathfrak{D}_f(Z, TY) \subset \mathfrak{D}_f(Z, TY)$. Hence, the restrictions of the maps φ_f to homeomorphisms

$$U_f \cap C^\infty_{\text{nice}}(Z, Y) \to {}^t\mathfrak{D}_f(Z, TY)$$

serve as charts of a C^∞_c-structure (without boundary) on $C^\infty_{\text{nice}}(Z, Y)$, see [37, Theorem 10.16, page 107]. In particular, if $X = Y = Z$, then the open subset $\text{Diff}(X) \subset C^\infty_{\text{nice}}(X, X)$ of all smooth automorphisms of X is a C^∞_c-manifold (without boundary).

To prove the claim, let $f_0 \in (\text{Diff}(X), \tau_{F\mathfrak{D}})$. As explained above, a chart in $C^\infty_{\text{nice}}(X, X)$ centered at f_0 is given by restriction of φ_{f_0} to the homeomorphism

$$U_{f_0} \cap C^\infty_{\text{nice}}(X, X) \to {}^t\mathfrak{D}_{f_0}(X, TX).$$

Since ${}^t\mathfrak{D}_{f_0}(X, TX)$ is a locally convex vector space, the open neighbourhood $\varphi_{f_0}(U_{f_0} \cap \text{Diff}(X))$ of the origin contains a convex open neighbourhood V of the origin $0 = 0_X \circ f_0$. Define

$$U := \varphi_{f_0}^{-1}(V) = \psi_{f_0}(V).$$

Given $f_1 \in U$, the desired smooth map $\nu\colon [0,1] \times X \to X$ is constructed as follows. Set $s_1 := \varphi_{f_0}(f_1) \in V$ and consider the well-defined map

$$\begin{aligned} &\nu\colon [0,1] \times X \to X, \\ &\nu_t(x) := \nu(t, x) := (\psi_{f_0}(t \cdot s_1))(x) = (\tau \circ (t \cdot s_1))(x) = \tau((t \cdot s_1)(x)) = \tau(t \cdot s_1(x)). \end{aligned}$$

Note that $t \cdot s_1 \in V$ for all $t \in [0,1]$ because V is convex and contains the points $\varphi_{f_0}(f_0) = 0$ and s_1. Therefore, $\nu_t \in \psi_{f_0}(V) = U$ for all $t \in [0,1]$, and $\nu_0 = \psi_{f_0}(0) = f_0$ and $\nu_1 = \psi_{f_0}(s_1) = f_1$. Finally, ν is smooth as the composition of the smooth maps $\tau\colon {}^iTX \to X$ and

$$[0,1] \times X \to {}^iTX, \qquad (t, x) \mapsto t \cdot s_1(x).$$

□

Corollary 8.1.4. *For any path* $\mu\colon [0,1] \to (\text{Diff}(X), \tau_{C^\infty})$ *there exists a smooth map*

$$\nu\colon [0,1] \times X \to X, \qquad \nu_t(x) := \nu(t, x),$$

such that $\nu_i = \mu(i)$ *for* $i \in \{0, 1\}$ *and* $\nu_t \in \text{Diff}(X)$ *for all* $t \in [0,1]$.

Proof. By Lemma 8.1.2, μ can be considered as a path in $(\text{Diff}(X), \tau_{F\mathfrak{D}})$. For every $t \in [0,1]$ let

U_t denote a neighbourhood of $\mu(t) \in (\mathrm{Diff}(X), \tau_{F\mathfrak{D}})$ with the property of Proposition 8.1.3. As $\bigcup_{t\in[0,1]} U_t$ is an open cover of the compact space $\mu([0,1])$, there exists a finite subset $T \subset [0,1]$ such that $\mu([0,1]) \subset \bigcup_{t\in T} U_t$. Define a finite graph Γ by taking T as the set of vertices, and by connecting two vertices $t, t' \in T$ by an edge if and only if $U_t \cap U_{t'} \neq \emptyset$. The connectedness of $\mu([0,1])$ implies that Γ is connected, too. Consequently, there exists a sequence $t_0, \ldots, t_r$ of elements in T such that $\mu(\varepsilon) \in U_{t_{\varepsilon \cdot r}}$ for $\varepsilon \in \{0,1\}$, and $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$ for $i \in \{0, \ldots, r-1\}$, say $g_i \in U_{t_i} \cap U_{t_{i+1}}$. For any two subsequent points p_0, p_1 in the sequence

$$\mu(0), \mu(t_0), g_0, \mu(t_1), g_1, \ldots, \mu(t_{r-1}), g_{r-1}, \mu(t_r), \mu(1)$$

there exists by Proposition 8.1.3 a smooth map

$$\lambda\colon [0,1] \times X \to X, \qquad \lambda_t(x) := \lambda(t,x),$$

such that $\lambda_i = p_i$ for $i \in \{0,1\}$ and $\lambda_t \in U$ for all $t \in [0,1]$. By means of an appropriate smooth map $[0,1] \to [0,1]$ one may in addition assume that $\lambda_t = \nu_\varepsilon$ for t near $\varepsilon \in \{0,1\}$. Finally, the smooth maps λ can be combined to desired smooth map ν. □

8.2 Cerf Theory

Throughout the present section, let (Y, Y^0, Y^1) denote a fixed smooth manifold triad (in the sense of [41, Definition 1.3, p. 2]) of dimension $n = \dim Y$.

The fundamtental object of interest is the space $\mathcal{F}$ of all smooth functions $(Y, Y^0, Y^1) \to ([0,1], 0, 1)$ without critical points on the boundary (see [9, I.3, p. 22]). ($\mathcal{F}$ and all spaces of smooth functions that occur in the present section are equipped with the Whitney C^∞ topology as defined in [37, 4.4, p. 33].) Note that $\mathcal{F}$ itself is contractible, being a convex subset of the real vector space of all real-valued functions on Y (see [9, p. 10]). Hence, one is rather interested in the study of certain natural subspaces of $\mathcal{F}$.

In [9, I.3, p. 22 ff.], Cerf defines a sequence $\mathcal{F}^0, \mathcal{F}^1, \dots, \mathcal{F}^j, \dots$ of subspaces of $\mathcal{F}$ that form the *natural stratification* of $\mathcal{F}$ (see [9, I.3, p. 23]). Particular attention is paid to the pair of strata $(\mathcal{F}^0, \mathcal{F}^1)$ which forms a *stratifiaction of codimension* 1 of $\mathcal{F}^0 \cup \mathcal{F}^1$ according to [9, I.3.1, p. 24]. (The concept of a stratification of codimension 1 is introduced in Definition 1 in [9, I.2.1, p. 17]).

A function in $\mathcal{F}$ is called *Morse function* if all its critical points are non-degenerate (see Definition 4 in [9, I.3.2, p. 25]).

Remark 8.2.1. By definition, a function in $\mathcal{F}$ maps $Y \setminus \partial Y$ not necessarily into $(0,1)$. Hence, the notion of Morse function adopted in [9, Definition I.3.2.4, p. 25] is slightly more general than that used in [41] as the image of a critical point is allowed to lie on the boundary of $[0,1]$. However, as such a critical point is necessarily definite, the two notions coincide for Morse functions without definite critical points.

Explicitly, $\mathcal{F}^0 \subset \mathcal{F}$ is the subspace of *excellent* Morse functions (see [9, I.3.1, p. 24]), i.e. Morse functions whose critical values lie on pairwise different levels.

Proposition 8.2.2. *Let $f_0, f_1 \in \mathcal{F}^0$ be excellent Morse functions that lie in the same path component of $\mathcal{F}^0$. Then there exist paths $\kappa\colon [0,1] \to \mathrm{Diff}(Y)$ with origin $\kappa_0 = \mathrm{id}_Y$ and $\lambda\colon [0,1] \to \mathrm{Diff}([0,1])$ with origin $\lambda_0 = \mathrm{id}_{[0,1]}$ such that $f_1 = \lambda_1 \circ f_0 \circ \kappa_1$.*

Proof. The group $\mathcal{G} := \mathrm{Diff}(Y, Y^0, Y^1) \times \mathrm{Diff}([0,1], 0, 1)$ operates from the left on $\mathcal{F}$ via $(g, g') \cdot f := g' \circ f \circ g^{-1}$ (see [9, I.3, p. 22]). Let $\mathcal{G}_e$ denote the (path) component of $\mathcal{G}$ containing the identity element $e := \mathrm{id}_Y \times \mathrm{id}_{[0,1]}$. In [9, I.3.2, p. 25] two elements of $\mathcal{F}$ are called *isotopic* if they lie in the same orbit of the action of $\mathcal{G}_e$ on $\mathcal{F}$.

By assumption, the Morse functions f_0 and f_1 lie in the same path component of $\mathcal{F}^0$, that is, they are contained in the same *cocell* of $\mathcal{F}$ in the sense of Definition 1 in [9, I.1.1, p. 15]. Hence, it follows from [9, I.3.2, p. 25] that they are isotopic, which means that $f_1 = (g, g') \cdot f_0 = g' \circ f_0 \circ g^{-1}$ for a suitable element $(g, g') \in \mathcal{G}_e$. Let $\Omega\colon [0,1] \to \mathcal{G}_e$ be a path between $\Omega(0) = e$ and $\Omega(1) = (g, g')$. Then the desired paths are defined by the components $\Omega = (\lambda, \kappa^{-1})$. □

The stratum $\mathcal{F}^1$ is the union $\mathcal{F}^1 = \mathcal{F}^1_\alpha \cup \mathcal{F}^1_\beta$ of two disjoint subspaces, where $\mathcal{F}^1_\alpha \subset \mathcal{F}$ denotes the subset of *functions of birth* and $\mathcal{F}^1_\beta \subset \mathcal{F}$ is the subset of *functions of crossing.* (For the precise definitions, see [9, p. 10] and [9, p. 24]).

Cerf studies certain paths in $\mathcal{F}^0 \cup \mathcal{F}^1$ that lie in $\mathcal{F}^0$ except for a discrete set of parameters at which they pass through $\mathcal{F}^1$. More precisely:

Definition 8.2.3. A *traversing path* in $\mathcal{F}^0 \cup \mathcal{F}^1$ (see Definition 1 in [9, I.2.1, p. 18]) is a continuous map $\gamma\colon [0,1] \to \mathcal{F}^0 \cup \mathcal{F}^1$ such that $\gamma^{-1}(\mathcal{F}^1)$ consists of a single parameter $t_0 \in (0,1)$, and such that for some (hence, any) choice of embeddings

$$\iota_-\colon ([0,1], 0, (0,1]) \to ([0,t_0], t_0, [0,t_0)),$$
$$\iota_+\colon ([0,1], 0, (0,1]) \to ([t_0,1], t_0, (t_0,1]),$$

the compositions $\gamma \circ \iota_-$ and $\gamma \circ \iota_+$ do not lie in the same path component of the space of all continuous maps $([0,1], 0, (0,1]) \to (\mathcal{F}^0 \cup \mathcal{F}^1, \gamma(t_0), \mathcal{F}^0)$ (equipped with the Whitney C^0 topology, which is by [37, 4.4, p. 33] nothing but the graph topology defined in [37, 3.2, p. 26]). A *good path* is a path that can be written as the concatenation of finitely many traversing paths. (Every path in $\mathcal{F}^0$ is also a good path, as it can formally be understood as the empty concatenation which is not excluded by the definition.)

Next, we introduce the space $\mathcal{T}$ of all traversing paths in $\mathcal{F}^0 \cup \mathcal{F}^1$, and the subspace $\mathcal{T}_f \subset \mathcal{T}$ of all traversing paths $\gamma \in \mathcal{T}$ with fixed origin $\gamma(0) = f \in \mathcal{F}^0$ (equipped with the Whitney C^0 topology).

A traversing path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$ either passes through $\mathcal{F}^1_\alpha$ (this case is studied in chapter III of [9]; γ is called a *path of birth/death* in [9, III.1.3, p. 66]) or through $\mathcal{F}^1_\beta$ (this case is studied in chapter II of [9]; γ is called a *path of* 1*-crossing* in the more general context of [9, II.3.1, p. 48]). In both cases, Cerf introduces specific smooth models of traversing paths called *elementary paths* as announced in [9, I.3.1, p. 24]. The elementary paths relative to $\mathcal{F}^1_\alpha$ are introduced as *elementary path of birth* in [9, III.1.2, p. 66] and *elementary path of death* in Definition 1 and 1' of [9, III.2.3, p. 71]. The elementary paths relative to $\mathcal{F}^1_\beta$ are introduced as *descending elementary path of* 1*-crossing* in [9, II.1.2, p. 42] and [9, II.3.1, p. 48].

As an application of Cerf's "lemme des chemins élémentaires" (see [9, I.2.2, p. 20]), any traversing path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$ can be related to an elementary path of the corresponding type by deforming γ through traversing paths rel origin. This fact underlies the following result:

Proposition 8.2.4. *If $f_0, f_1 \in \mathcal{F}^0$ can be connected by a traversing path in $\mathcal{F}^0 \cup \mathcal{F}^1$, then they can also be connected by a path that is the concatenation of an elementary path with origin f_0 and a path in $\mathcal{F}^0$ with endpoint f_1.*

Proof. Let γ be a traversing path in $\mathcal{F}^0 \cup \mathcal{F}^1$ such that $\gamma(i) = f_i$ for $i = 0,1$. We have to construct an elementary path γ_{el} with origin f_0 and a path γ_{reg} in $\mathcal{F}^0$ with endpoint f_1 such that $\gamma_{\mathrm{el}}(1) = \gamma_{\mathrm{reg}}(0)$. For this purpose we distinguish between the following two cases:

- γ passes through $\mathcal{F}^1_\alpha$.
  Following the argument in [9, III.1.3, p. 67], let $\mathcal{C}'$ denote the space of paths of birth (equipped with the Whitney C^0 topology, compare Lemma 2 in [9, I.2.1, p. 18]). (Note that $\mathcal{C}'$ is a union of path components of the space of traversing paths in $\mathcal{F}^0 \cup \mathcal{F}^1$.) Let $\mathcal{C}'_{f_0}$ denote the subspace of $\mathcal{C}'$ of paths of birth with origin f_0. Supposing that γ is a path of birth, we have $\gamma \in \mathcal{C}'_{f_0}$. By Proposition 1 in [9, III.1.3, p. 67] there exists a path $\Gamma\colon [0,1] \to \mathcal{C}'_{f_0}$ such that $\gamma_{\mathrm{el}} := \Gamma(0)$ is an elementary path of birth with origin f_0 and $\Gamma(1) = \gamma$. Let $\gamma_{\mathrm{reg}}\colon [0,1] \to \mathcal{F}^0$ be the path that is given at $t \in [0,1]$ by the endpoint of $\Gamma(t) \in \mathcal{C}'_{f_0}$. (Indeed, let us show that the map γ_{reg} is continuous. By [22, p. 34f] the Whitney C^0 topology (i.e. the graph topology, or *strong topology*) coincides

with the compact-open topology (or *weak topology*) on $\mathcal{C}'_{f_0}$ because the unit interval is compact. Since the unit interval is locally compact, [20, Proposition A.14 (a), p. 530] implies that the evaluation map $\mathrm{ev}_1 \colon \mathcal{C}'_{f_0} \to \mathcal{F}^0$, $\varphi \mapsto \varphi(1)$, is continuous. Hence, the composition $\gamma_{\mathrm{reg}} = \mathrm{ev}_1 \circ \Gamma$ is continuous, too.) In particular, the origin of γ_{reg} coincides with the endpoint of γ_{el}. Moreover, the endpoint of γ_{reg} is the endpoint of γ, namely f_1. In an analogous way one uses Proposition 2 in [9, III.2.3, p. 71] to construct γ_{el} and γ_{reg} in the case that γ is a path of death.

- γ passes through $\mathcal{F}^1_\beta$.
  Following the notation in [9, II.3.1, p. 48] and setting $p = 1$, let $\mathcal{C}_{1;f_0}$ denote the space of traversing paths of 1-crossing with origin f_0. In particular, $\gamma \in \mathcal{C}_{1;f_0}$. Moreover, let $\mathcal{E}_{1;f_0} \subset \mathcal{C}_{1;f_0}$ denote the subspace of descending elementary paths of 1-crossing with origin f_0. By [9, II.3.2, p. 50] we have

$$\pi_0(\mathcal{C}_{1;f_0}, \mathcal{E}_{1;f_0}) = 0.$$

  Consequently, there exists a path $\Gamma \colon [0,1] \to \mathcal{C}_{1;f_0}$ such that $\gamma_{\mathrm{el}} := \Gamma(0)$ is a descending elementary paths of 1-crossing with origin f_0 and $\Gamma(1) = \gamma$. Let $\gamma_{\mathrm{reg}} \colon [0,1] \to \mathcal{F}^0$ be the path that is given at $t \in [0,1]$ by the endpoint of $\Gamma(t)$. (Note that γ_{reg} is continuous by [20, Proposition A.14, p. 530] as the unit interval is locally compact.) In particular, the origin of γ_{reg} coincides with the endpoint of γ_{el}. Moreover, the endpoint of γ_{reg} is the endpoint of γ, namely f_1.

□

A Morse function $f \in \mathcal{F}$ is *ordered* (see [9, V.1.1, p. 95]) if and only if for every pair (c, c') of critical points of f such that the index of c is strictly smaller than the index of c', we have $f(c) < f(c')$. Note that there is no condition for critical values of the same index. The space of ordered excellent Morse functions is denoted by $\mathcal{O}^0 \subset \mathcal{F}^0$.

Proposition 8.2.5. *Every $f \in \mathcal{F}^0$ can be realized as the origin of a good path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$ whose endpoint $\gamma(1)$ lies in $\mathcal{O}^0$ such that the set of integers that occur as the index of a critical point of $\gamma(1)$ is contained in the set of integers that occur as the index of a critical point of f.*

Proof. In [9, V.1.3, p. 96] a traversing path with origin f_0 and endpoint f_1 is called *decreasing* if the number of inversions of f_1 is strictly smaller than the number of inversions of f_0. (Recall that a set $\{c_1, c_2\}$ of Morse singularities of f_0 of indices λ_1 and λ_2 is called an *inversion* of f_0 if $(f(c_1) - f(c_2))(\lambda_1 - \lambda_2) < 0$.) The desired good path is now obtained as the concatenation of the decreasing paths of 1-crossing obtained by an iterated application of statement $(*)$ in [9, V.1.3, p. 96], which says: If $f_0 \in \mathcal{F}^0$ has at least one inversion, then there exists a decreasing path of 1-crossings with origin f_0. □

Next, we specialize to the cylinder $Y = Y^0 \times [0,1]$, where Y^0 denotes a closed connected smooth manifold of dimension $n - 1 \geq 0$.

Given a pair of integers (i, q) such that $0 \leq i \leq n-1$ and $q \geq 0$, let $\mathcal{F}_{i,q} \subset \mathcal{O}$ denote the subspace of ordered Morse functions with precisely $2q$ critical points, where q critical points have index i and q critical points have index $i+1$ (see [9, V.2.1, p. 100]). Moreover, let $\mathcal{F}_{i,q;\alpha} \subset \mathcal{O}$ be the subspace of ordered functions whose set of critical points consists of a point of birth c of index i and $2q$ non-degenerate critical points, among which q are of index i and situated below the level of c, whereas q are of index $i+1$ and situated above the level of c. Define

$$\mathcal{F}_i := \bigcup_{q \geq 0} (\mathcal{F}_{i,q} \cup \mathcal{F}_{i,q;\alpha}) \qquad \subset \mathcal{O}.$$

It can be shown that $\mathcal{F}_i$ is an open subset of $\mathcal{F}$. Finally, set $\mathcal{F}_i^0 := \mathcal{F}_i \cap \mathcal{F}^0$ and $\mathcal{F}_i^1 := \mathcal{F}_i \cap \mathcal{F}^1$. Note that a good path in $\mathcal{F}_i^0 \cup \mathcal{F}_i^1$ as defined in [9, V.1.3, p. 96] is the same as a good path in $\mathcal{F}^0 \cup \mathcal{F}^1$ that lies entirely in $\mathcal{F}_i^0 \cup \mathcal{F}_i^1$.

With the above notation, Theorem $2'$ in [9, V.2.1, p. 100] states the following:

Theorem 8.2.6. *Let $Y = Y^0 \times [0,1]$. If $n \geq 6$, $\pi_1(Y^0) = 0$ and $2 \leq i \leq n-3$, then any pair of elements in $\mathcal{F}_i^0$ can be connected by a good path with values in $\mathcal{F}_i^0 \cup \mathcal{F}_i^1$.*

Recall from [9, p. 100f.] that $\mathcal{F}^0_{[a,b]}$ denotes for integers $0 \leq a < b \leq n$ the subspace of all $f \in \mathcal{O}^0$ such that $\mathcal{J}(f) \subset [a,b]$. Furthermore, for an integer $l \geq 0$, $\mathcal{F}^0_{[a,b];l}$ denotes the subspace of all $f \in \mathcal{F}^0_{[a,b]}$ with at most l critical points of index a.

Replacing the assumption that Y is a cylinder by the weaker assumption that (Y, Y^0) is highly connected, [9, Lemma 0, p. 101] admits the following direct generalization:

Lemma 8.2.7. *Suppose that $n \geq 6$. Suppose that the pair (Y, Y^0) is i-connected for some integer $0 \leq i \leq n-4$. Furthermore, assume that Y^0 is simply connected. Given integers $i+2 \leq j \leq n$ and $k \geq 1$, every good path of the form $([0,1], \{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j];k}, \mathcal{F}^0_{[i,j];k-1})$ is homotopic rel endpoints to a good path of the form $([0,1], \{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j];k-1}, \mathcal{F}^0_{[i,j];k-1})$.*

Proof. The proof is verbatim to the proof of [9, Lemma 0, p. 101]. In fact, the original assumption that Y is a cylinder is only used to justify statement $(**)$ in [9, p. 102]. If (Y, Y^0) is only known to be highly connected, then the statement still holds in a suitable range:

$(**)'$ Suppose that (Y, Y^0) is i-connected for some integer $0 \leq i \leq n-4$. Given integers $i+2 \leq j \leq n$ and $k \geq 1$, every point in $\mathcal{F}^0_{[i,j];k}$ is the origin of a *path of Smale* (see the definition in [9, V.2.3, p. 101]) with endpoint in $\mathcal{F}^0_{[i,j];k-1}$.

The existence of a path of Smale in statement $(**)'$ is granted by the proof of [57, Theorem 3, p. 601]. In fact, the path of Smale is just a one-parameter implementation of Smale's trick for trading critical points of a Morse function. A critical point of index i is traded for a critical point of index $i+2$ by first introducing a cancelling pair of critical points of subsequent indices $i+1$ and $i+2$ and then cancelling the critical point of index i with the critical point of index $i+1$. □

Based on the previous lemma, an adaption of the inductive structure of the proof of [9, Theorem 2', p. 100] yields the following

Theorem 8.2.8. *Suppose that $n \geq 7$. Assume that (Y, Y^0) and (Y, Y^1) are $(k-1)$-connected for some integer $0 \leq k \leq \lceil n/2 \rceil - 1$. (Note that there is no assumption for $k = 0$.) Furthermore, suppose that Y^0 and Y^1 are simply connected. Then, any pair of elements in $\mathcal{F}^0_{[k,n-k]}$ can be connected by a good path with values in $\mathcal{F}^0_{[k,n-k]} \cup \mathcal{F}^1_{[k,n-k]}$.*

Proof. According to Theorem [9, Theorem 1', p. 96], the given elements $f, f' \in \mathcal{F}^0_{[k,n-k]}$ can be joined by a good path γ of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[0,n]}, \mathcal{F}^0_{[k,n-k]})$. If $k=0$, then there is nothing else to show. Therefore, we may assume that $k \geq 1$ in the following.

Following the original proof of Theorem [9, Theorem 2', p. 100], there is a filtration

$$\mathcal{F}^0_{[i+1,j]} = \mathcal{F}^0_{[i,j];0} \subset \mathcal{F}^0_{[i,j];1} \subset \cdots \subset \mathcal{F}^0_{[i,j];l} \subset \cdots \subset \mathcal{F}^0_{[i,j]}, \qquad 0 \leq i < j \leq n.$$

Every good path of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j]}, \mathcal{F}^0_{[i+1,j]})$ has image in some $\overline{\mathcal{F}}^0_{[i,j];l}$. Therefore, noting that $k-1 \leq n-4$, iterated application of Lemma 8.2.7 to the $(k-1)$-connected pair (Y, Y^0) yields

(1) Given integers $0 \leq i \leq k-1$ and $i+2 \leq j \leq n$, every good path of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j]}, \mathcal{F}^0_{[i+1,j]})$ is homotopic rel endpoints to a good path of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i+1,j]}, \mathcal{F}^0_{[i+1,j]})$.

Similarly, exchanging the roles of Y^0 and Y^1, iterated application of Lemma 8.2.7 to the $(k-1)$-connected pair (Y, Y^1) yields (after changing the roles back)

(1') Given integers $n-k+1 \leq j \leq n$ and $0 \leq i \leq j-2$, every good path of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j]}, \mathcal{F}^0_{[i,j-1]})$ is homotopic rel endpoints to a good path of the form $([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,j-1]}, \mathcal{F}^0_{[i,j-1]})$.

Note that γ can be considered as a good path between f and f' of the form

$$\gamma_0 \colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[0,n]}, \mathcal{F}^0_{[1,n]}).$$

For $i = 0, \ldots, k-1$ (in increasing order) and $j = n$ we repeatedly apply (1) to the good path

$$\gamma_i \colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i,n]}, \mathcal{F}^0_{[i+1,n]})$$

to conclude that γ_i is homotopic rel endpoints to a good path of the form

$$([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i+1,n]}, \mathcal{F}^0_{[i+1,n]}).$$

If $i < k-1$, then this can be considered as a path of the form

$$\gamma_{i+1} \colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[i+1,n]}, \mathcal{F}^0_{[i+2,n]})$$

since the path still connects the given elements $f, f' \in \mathcal{F}^0_{[k,n-k]}$. If $i = k-1$, then we end up with a good path between f and f' of the form

$$\delta_0 \colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[k,n]}, \mathcal{F}^0_{[k,n-1]}).$$

Next, for $i = k$ and $j = n-k+1, \ldots, n$ (in decreasing order) we repeatedly apply $(1')$ to the good path

$$\delta_j \colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[k,j]}, \mathcal{F}^0_{[k,j-1]})$$

to conclude that δ_j is homotopic rel endpoints to a good path of the form

$$([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[k,j-1]}, \mathcal{F}^0_{[k,j-1]}).$$

If $j > n-k+1$, then this can be considered as a good path of the form

$$\delta_{j+1}\colon ([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[k,j-1]}, \mathcal{F}^0_{[k,j-2]})$$

since the path still connects the given elements $f, f' \in \mathcal{F}^0_{[k,n-k]}$. If $j = n-k+1$, then we end up with the desired good path between f and f' of the form

$$([0,1],\{0,1\}) \to (\overline{\mathcal{F}}^0_{[k,n-k]}, \mathcal{F}^0_{[k,n-k]}).$$

□

8.3 Adapted Homotopies

Throughout the present section, let (Y, Y^0, Y^1) denote a fixed smooth manifold triad (in the sense of [41, Definition 1.3, p. 2]) of dimension $n = \dim Y$.

The nature of our problem requires to consider smooth homotopies of the form

$$h\colon [0,1] \times (Y, Y^0, Y^1) \to ([0,1], 0, 1), \qquad (t, y) \mapsto h(t, y) =: h_t(y),$$

such that the following properties are satisfied:

(*i*) If $t \in [0,1]$ is near $i = 0, 1$, then h_t is an excellent Morse function

$$(Y, Y^0, Y^1) \to ([0,1], 0, 1).$$

(*ii*) For all $t \in [0,1]$, h_t has no critical points on $\partial Y = Y^0 \sqcup Y^1$.

(*iii*) The *track*

$$\hat{h}\colon [0,1] \times Y \to [0,1] \times [0,1], \qquad \hat{h}(t, y) = (t, h_t(y)),$$

of the homotopy h is a generic smooth map.

Definition 8.3.1. In the following, a homotopy h with the above properties (*i*) to (*iii*) will be called an *adapted homotopy* between the excellent Morse functions h_0 and h_1. Given an adapted homotopy h, let $\mathfrak{I}(h) \subset \mathbb{Z}$ denote the set of all integers that occur as the absolute index of a fold point of the track $\hat{h}$.

The definition of an adapted homotopy can of course be stated more generally by replacing the unit interval $[0,1]$ in the domain of h with any interval of the form $[a,b]$, $a < b$. Note that if $\xi\colon [a', b'] \xrightarrow{\cong} [a, b]$ is a diffeomorphism and $h\colon [a,b] \times Y \to [0,1]$ is an adapted homotopy, then $k := h \circ (\xi \times \mathrm{id}_Y)$ is also an adapted homotopy, and $\mathfrak{I}(k) = \mathfrak{I}(h)$. (Indeed, use that $\hat{k} = (\xi^{-1} \times \mathrm{id}_{[0,1]}) \circ \hat{h} \circ (\xi \times \mathrm{id}_Y)$.)

If $h\colon [a,b] \times Y \to [0,1]$ is an adapted homotopy and $h_{c'}$ is an excellent Morse function for all c' in a neighbourhood of some $c \in (a,b)$, then the restrictions $h_{\leq} := h|_{[a,c]\times Y}$ and $h_{\geq} := h|_{[c,b]\times Y}$ are adapted homotopies, and $\mathfrak{I}(h) = \mathfrak{I}(h_{\leq}) \cup \mathfrak{I}(h_{\geq})$. (Note that $\hat{h}_{\leq} = \hat{h}|_{[a,c]\times Y}$ and $\hat{h}_{\geq} = \hat{h}|_{[c,b]\times Y}$.) Conversely, if $h\colon [a,b] \times Y \to [0,1]$ is a smooth map and there exists $c \in (a,b)$ such that the restrictions $h_{\leq} := h|_{[a,c]\times Y}$ and $h_{\geq} := h|_{[c,b]\times Y}$ are adapted homotopies, then h is an adapted homotopy as well, and $\mathfrak{I}(h) = \mathfrak{I}(h_{\leq}) \cup \mathfrak{I}(h_{\geq})$.

Remark 8.3.2. Since adapted homotopies will be used as building blocks for the construction of certain generic smooth maps on topologically more complicated spaces than cylinders, it is necessary to have some control over the behaviour of the corresponding track near the boundaries of the smooth manifold $[0,1] \times Y$ with corners. In fact, property (*i*) refers to the behaviour of the track near $\{i\} \times Y$, $i = 0, 1$, whereas property (*ii*) is concerned with the behaviour of the track near $[0,1] \times Y^j$, $j = 0, 1$.

Given a Morse function $f\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$, let $\mathfrak{I}(f)$ denote the set of all integers of the form $\max\{i, n - i\}$, where i occurs as the index of a critical point of f.

Lemma 8.3.3. *Let* $h\colon [0,1] \times (Y, Y^0, Y^1) \to ([0,1], 0, 1)$ *be a smooth map such that* $h_t := h(t, -)$ *is a Morse function for every* $t \in [0,1]$ *and* h_t *is excellent for* t *near* $i = 0, 1$. *Then,* h *is an adapted homotopy whose track* $\hat{h}$ *is a fold map, and* $\mathfrak{I}(h) = \mathfrak{I}(h_t)$ *for every* $t \in [0,1]$.

Proof. It is clear that h satisfies properties (i) and (ii) of an adapted homotopy. Furthermore, Proposition 4.5.3 implies that h also satisfies property (iii). (In fact, $\hat{h}$ is a fold map by parts (i) and (ii) of Proposition 4.5.3.) Hence, h is an adapted homotopy.

It remains to show that $\mathfrak{J}(h) = \mathfrak{J}(h_t)$ for every $t \in [0,1]$. First, note that for every fold line $S \subset [0,1] \times Y$ of $\hat{h}$ the composition of $\hat{h}|_S$ with the projection $\pi\colon [0,1] \times [0,1] \to [0,1]$ to the first factor is a diffeomorphism $\pi \circ \hat{h}|_S \colon S \xrightarrow{\cong} [0,1]$. (In fact, it follows from Proposition 4.5.3$\,(i)$ that around any point $(t_0, y_0) \in S$, S is the zero locus of $(d_y h)(t,y) = 0$ in local coordinates y around $y_0 \in Y$. By the implicit function theorem, we can now solve y for t locally around (t_0, y_0) since $H(h_{t_0}(y_0))$ is non-degenerate by Proposition 4.5.3$\,(iii)$.) Given $t, t' \in [0,1]$, $t \neq t'$, a bijection between the critical points of the Morse functions h_t and $h_{t'}$ can be defined as follows. Every critical point y of h_t is mapped to the critical point y' of $h_{t'}$ that is uniquely determined by $(t', y') = S \cap \pi^{-1}(t')$, where S denotes the fold line of $\hat{h}$ that contains (t,y). (By parts (i) and (ii) of Proposition 4.5.3, (t,y) is in fact a fold point of $\hat{h}$, and y' is in fact a critical point of $h_{t'}$.) As the absolute index is constant along fold lines, the bijection thus obtained perserves the index of the critical points (at least its absolute value $\max\{i, n-i\}$). Consequently, $\mathfrak{J}(h) = \mathfrak{J}(h_t)$ for every $t \in [0,1]$. □

Corollary 8.3.4. *Any adapted homotopy* $h\colon [0,1] \times Y \to [0,1]$ *satisfies* $\mathfrak{J}(h_0) \cup \mathfrak{J}(h_1) \subset \mathfrak{J}(h)$.

Proof. Choose $a \in [0, 1/2)$ such that h_t is an excellent Morse function for all $t \in [0, 2a)$. Thus, $h_{\leq} := h|_{[0,a] \times Y}$ and $h_{\geq} := h|_{[a,1] \times Y}$ are adapted homotopies, and $\mathfrak{J}(h) = \mathfrak{J}(h_{\leq}) \cup \mathfrak{J}(h_{\geq})$. Hence, Lemma 8.3.3 implies that $\mathfrak{J}(h_0) = \mathfrak{J}(h_{\leq}) \subset \mathfrak{J}(h)$. Analogously, it follows that $\mathfrak{J}(h_1) \subset \mathfrak{J}(h)$. □

It is convenient to introduce a smooth version of concatenation for adapted homotopies. For every $\varepsilon \in (0,1)$ we fix once and for all a smooth map $\rho_\varepsilon \colon [0,1] \to [0,1]$ (see Figure 8.1) such that $\rho_\varepsilon(t) = t/2$ for $t \in [0, 1/2 - \varepsilon/2]$, $\rho_\varepsilon(t) = 1/2$ for $t \in [1/2 - \varepsilon/4, 1/2 + \varepsilon/4]$, $\rho_\varepsilon(t) = t/2 + 1/2$ for $t \in [1/2 + \varepsilon/2, 1]$ and $\rho'_\varepsilon(t) > 0$ for all $t \in [0, 1/2 - \varepsilon/4) \cup (1/2 + \varepsilon/4, 1]$. (Note that the condition on ρ'_ε can be achieved since $\rho_\varepsilon(1/2 - \varepsilon/2) < 1/2 < \rho_\varepsilon(1/2 + \varepsilon/2)$.)

Let $h, k\colon [0,1] \times Y \to [0,1]$ be adapted homotopies such that $h_1 = k_0$. Property (i) for adapted homotopies allows for choosing $\varepsilon \in (0, 1/2)$ such that h_t is an excellent Morse function for all $t \in [1-\varepsilon, 1]$ and k_t is an excellent Morse function for all $t \in [0, \varepsilon]$. In this situation the ε*-concatenation* of h and k is defined by

$$h *_\varepsilon k\colon [0,1] \times Y \to [0,1], \qquad (h *_\varepsilon k)(t,y) = \begin{cases} h(2\rho_\varepsilon(t), y), & t \in [0, 1/2), \\ k(2\rho_\varepsilon(t) - 1, y), & t \in [1/2, 1]. \end{cases}$$

Proposition 8.3.5. *The* ε*-concatenation* $l := h *_\varepsilon k$ *of two adapted homotopies* h *and* k *is again an adapted homotopy such that* $l_t = h_t$ *for* t *near* 0 *and* $l_t = k_t$ *for* t *near* 1. *Moreover,*

$$\mathfrak{J}(l) = \mathfrak{J}(h) \cup \mathfrak{J}(k).$$

Proof. By definition, $l_t = h_{2\rho_\varepsilon(t)}$ for $t \in [0, 1/2)$ and $l_t = k_{2\rho_\varepsilon(t) - 1}$ for $t \in [1/2, 1]$. Observing $l_t = h_1 = k_0$ for $t \in [1/2 - \varepsilon/4, 1/2 + \varepsilon/4]$, it follows that l is a smooth map of the form

$$l\colon [0,1] \times (Y, Y^0, Y^1) \to ([0,1], 0, 1).$$

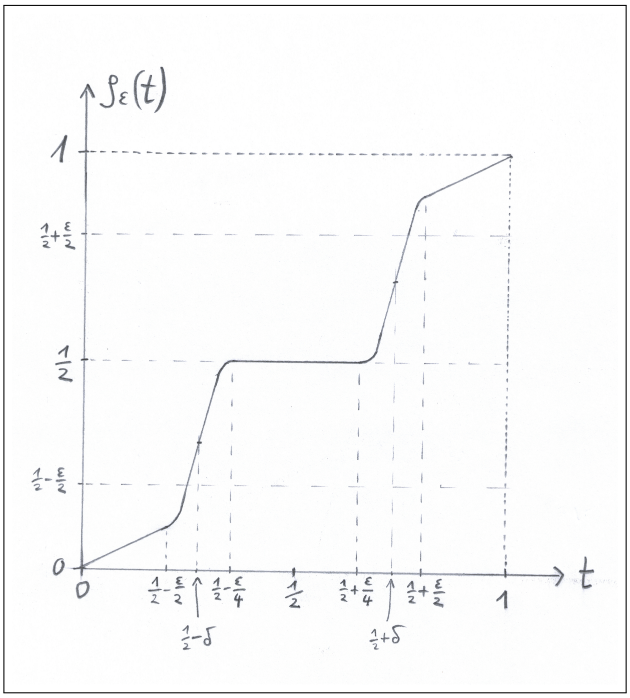

Figure 8.1: Graph of ρ_ε

Note that $l_t = h_{2\rho_\varepsilon(t)} = h_t$ for $t \in [0, 1/2 - \varepsilon/2]$ and $l_t = k_{2\rho_\varepsilon(t)-1} = k_t$ for $t \in [1/2 + \varepsilon/2, 1]$. As indicated in Figure 8.1 one may choose $\delta \in (\varepsilon/4, \varepsilon/2)$ such that

$$(1/2 - \delta, 1/2 + \delta) \subset \rho_\varepsilon^{-1}((1/2 - \varepsilon/2, 1/2 + \varepsilon/2)).$$

(This is possible since $\rho_\varepsilon([1/2 - \varepsilon/4, 1/2 + \varepsilon/4]) = \{1/2\}$.) Consequently, $\rho_\varepsilon([1/2 - \delta, 1/2]) \subset [1/2 - \varepsilon/2, 1/2]$ and $\rho_\varepsilon([1/2, 1/2 + \delta]) \subset [1/2, 1/2 + \varepsilon/2]$. Therefore, l_t is a Morse function for all $t \in [1/2 - \delta, 1/2 + \delta]$. (Indeed, by choice of $\varepsilon \in (0, 1/2)$, this follows for $t \in [1/2 - \delta, 1/2]$ from $l_t = h_{2\rho_\varepsilon(t)}$ and $2\rho_\varepsilon(t) \in [1 - \varepsilon, 1]$, and for $t \in [1/2, 1/2 + \delta]$ from $l_t = k_{2\rho_\varepsilon(t)-1}$ and $2\rho_\varepsilon(t) - 1 \in [0, \varepsilon]$.) Hence, Lemma 8.3.3 implies that $l|_{[1/2-\delta,1/2+\delta]\times Y}$ is an adapted homotopy, and $\mathfrak{J}(l|_{[1/2-\delta,1/2+\delta]\times Y}) = \mathfrak{J}(l_{1/2})$. Moreover,

$$l|_{[0,1/2-\delta]\times Y} = h \circ (2\rho_\varepsilon|_{[0,1/2-\delta]} \times \mathrm{id}_Y),$$
$$l|_{[1/2+\delta,1]\times Y} = k \circ (2\rho_\varepsilon|_{[1/2+\delta,1]} \times \mathrm{id}_Y),$$

are adapted homotopies such that $\mathfrak{J}(l|_{[0,1/2-\delta]\times Y}) = \mathfrak{J}(h)$ and $\mathfrak{J}(l|_{[1/2+\delta,1]\times Y}) = \mathfrak{J}(k)$. (In fact, note that $2\rho_\varepsilon|_{[0,1/2-\delta]}$ and $2\rho_\varepsilon|_{[1/2+\delta,1]}$ are diffeomorphisms onto their images in $[0, 1]$ because $\delta > \varepsilon/4$ and $\rho_\varepsilon'(t) > 0$ for all $t \in [0, 1/2 - \varepsilon/4) \cup (1/2 + \varepsilon/4, 1]$.) Consequently, l is an adapted homotopy, and

$$\mathfrak{J}(l) = \mathfrak{J}(h) \cup \mathfrak{J}(l_{1/2}) \cup \mathfrak{J}(k) = \mathfrak{J}(h) \cup \mathfrak{J}(k),$$

where the last equality follows from Corollary 8.3.4 since $l_{1/2} = k_0$.

□

Remark 8.3.6. Given an excellent Morse function $f\colon (Y, Y^0, Y^1) \to ([0, 1], 0, 1)$, the constant homotopy $\tilde{f}\colon [0, 1] \times Y \to [0, 1]$ given by $\tilde{f}_t := f$ for all $t \in [0, 1]$ is an adapted homotopy by Lemma 8.3.3. Hence, if h is any adapted homotopy with $h_1 = f$, then $k := h *_\varepsilon \tilde{f}$ is (for suitable $\varepsilon \in (0, 1)$) an adapted homotopy such that $k_t = h_t$ for t near 0, $k_t = f$ for t near 1, and $\mathcal{J}(k) = \mathcal{J}(h)$.

The object of the following considerations is to study the relationship of adapted homotopies to good paths in $\mathcal{F}^0 \cup \mathcal{F}^1$.

Given a good path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$, let

$$\mathcal{J}(\gamma) := \bigcup_{t\in\gamma^{-1}(\mathcal{F}^0)} \mathcal{J}(\gamma(t)) \subset \mathbb{Z}.$$

This definition implies that the concatenation $\gamma * \gamma'$ of two good paths γ, γ' in $\mathcal{F}^0 \cup \mathcal{F}^1$ with $\gamma(1) = \gamma'(0)$ satisfies $\mathcal{J}(\gamma * \gamma') = \mathcal{J}(\gamma) \cup \mathcal{J}(\gamma')$. Moreover, $\mathcal{J}(\overline{\gamma}) = \mathcal{J}(\gamma)$, where $\overline{\gamma}$ denotes the good path in $\mathcal{F}^0 \cup \mathcal{F}^1$ given by $\overline{\gamma}(t) = \gamma(1-t)$ for all $t \in [0,1]$.

Lemma 8.3.7. *If γ is a path in $\mathcal{F}^0$, then there exists an adapted homotopy $h\colon [0,1]\times Y \to [0,1]$ such that $h_t \in \mathcal{F}^0$ for all $t \in [0,1]$ and $h_i = \gamma(i)$ for $i = 0,1$. In particular, Lemma 8.3.3 implies that $\mathcal{J}(h) = \mathcal{J}(\gamma(i))$ for $i = 0,1$. Consequently, $\mathcal{J}(\gamma(t)) = \mathcal{J}(\gamma)$ for all $t \in [0,1]$.*

Proof. Fix $t \in [0,1]$. By Proposition 8.2.2, there exist paths $\kappa\colon [0,1] \to \mathrm{Diff}(Y)$ with origin $\kappa_0 = \mathrm{id}_Y$ and $\lambda\colon [0,1] \to \mathrm{Diff}([0,1])$ with origin $\lambda_0 = \mathrm{id}_{[0,1]}$ such that $\gamma(1) = \lambda_1 \circ \gamma(0) \circ \kappa_1$. Corollary 8.1.4 implies that there exist smooth maps $\tilde{\kappa}\colon [0,1] \times Y \to Y$ and $\tilde{\lambda}\colon [0,1] \times [0,1] \to [0,1]$ such that $\tilde{\kappa}_t := \tilde{\kappa}(t,-) \in \mathrm{Diff}(Y)$ and $\tilde{\lambda}_t := \tilde{\lambda}(t,-) \in \mathrm{Diff}([0,1])$ for all $t \in [0,1]$, and $\tilde{\kappa}(i,-) = \kappa_i$ and $\tilde{\lambda}(i,-) = \lambda_i$ for $i = 0,1$. Define the smooth map

$$h\colon [0,1] \times Y \to [0,1], \qquad h_t(y) := h(t,y) = (\tilde{\lambda}_t \circ \gamma(0) \circ \tilde{\kappa}_t)(y).$$

Since $h_0 = \gamma(0)$, $h_1 = \gamma(1)$, and h_t is an excellent Morse function for every $t \in [0,1]$, the remaining claims follow from Lemma 8.3.3. □

Lemma 8.3.8. *Every elementary path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$ induces an adapted homotopy*

$$h\colon [0,1] \times Y \to [0,1], \qquad h(t,y) = \gamma(t)(y).$$

Moreover, $\mathcal{J}(h) = \mathcal{J}(\gamma)$.

Proof. Let γ be an elementary path. We distinguish between the following two cases:

- γ is an elementary path of birth or death.
  It suffices to assume that γ is an elementary path of birth. (In fact, if γ is an elementary path of death, then the good path $\overline{\gamma}$ in $\mathcal{F}^0 \cup \mathcal{F}^1$ given by $\overline{\gamma}(t) = \gamma(1-t)$ for all $t \in [0,1]$ is by Definition 1 in [9, III.2.3, p. 71] an elementary path of birth. If h denotes the adapted homotopy induced by $\overline{\gamma}$, then $t \mapsto h_{1-t}$ is the desired adapted homotopy induced by γ.)
  Let i denote the index of γ.
  In the notation of [9, III.1.1, p. 65], the *model of birth* of index i is a suitable closed cylinder $B \times J \subset \mathbb{R}^{n-1} \times \mathbb{R}$ where $B = \mu \cdot D^{n-1}$ and $J = \mu \cdot D^1$ for some $\mu > 0$, and the *standard path of birth* of index i on $\mathbb{R}^n$ is a certain smooth map $b\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ such that the support of b is contained in the interior of $B \times J$, $\operatorname{supp} b \subset \varphi(\operatorname{int} B \times \operatorname{int} J)$.
  By [9, III.1.2, p. 66], there exists an embedding $\varphi\colon B \times J \to Y$ and an orientation preserving embedding $\varphi'\colon J \to [0,1]$ that misses the critical values of $\gamma(0)$ such that
  (a) $\gamma(t) = \gamma(0)$ on the complement $Y \setminus \varphi(B \times J)$.

(*b*) $\gamma(t) \circ \varphi = \varphi' \circ b_t$ for all $t \in [0,1]$.

As $\operatorname{supp} b \subset \varphi(\operatorname{int} B \times \operatorname{int} J)$, it follows from (a) and (b) that $h_t = h_0$ on the open subset $Y \setminus \varphi(\operatorname{supp} b) \subset Y$ for all $t \in [0,1]$. This implies that h is smooth on the open subset $[0,1] \times (Y \setminus \varphi(\operatorname{supp} b)) \subset [0,1] \times Y$. Note that $h_t(Y^i) = h_0(Y^i) = \{i\}$ for $i = 0,1$ and all $t \in [0,1]$ because $\partial Y \subset Y \setminus \varphi(\operatorname{supp} b)$. Furthermore, (b) implies that h is smooth on the open subset $[0,1] \times \varphi(\operatorname{int} B \times \operatorname{int} J) \subset [0,1] \times Y$ because b is smooth. Therefore, h is a smooth map. One has to check that h has the properties of an adapted homotopy:

(ii). It follows from $\partial Y \subset Y \setminus \varphi(\operatorname{supp} b)$ and $h_t = h_0 \in \mathcal{F}^0$ on $Y \setminus \varphi(\operatorname{supp} b)$ for every $t \in [0,1]$ that h_t has no critical points on $\partial Y = Y^0 \sqcup Y^1$.

(i). As $h_0 \in \mathcal{F}^0$ and $h_t|_{Y \setminus \varphi(\operatorname{supp} b)} = h_0|_{Y \setminus \varphi(\operatorname{supp} b)}$ for every $t \in [0,1]$, h_t has only non-degenerate critical points on the open subset $Y \setminus \varphi(\operatorname{supp} b) \subset Y$. Since $\operatorname{supp} b \subset \varphi(\operatorname{int} B \times \operatorname{int} J)$, it remains to consider h_t on the open subset $\varphi(\operatorname{int} B \times \operatorname{int} J) \subset Y$, where it is by construction of the form

$$h_t \circ \varphi \overset{(b)}{=} \varphi' \circ b_t = \varphi' \circ l_{\overline{\omega};t} \circ \psi^{-1}$$

for a certain automorphism ψ of $\mathbb{R}^n$. Here, l_t is the standard form for a *path of birth of index* i (see [9, III.1.1, p. 64]), and $l_{\overline{\omega};t}$ is a modified version of l_t that coincides with l_t in a neighbourhood of all its critical points and has compact support.

Note that l_t has no critical points for $t \in [0,1/2)$, $l_{1/2}$ has exactly one critical point at the origin that is degenerate, and l_t has two non-degenerate critical points for $t \in (1/2,1]$, which lie on different levels, and on levels different from those of h_0 (see [9, III.1.1, p. 65] and [9, III.1.2, p. 66]). Consequently, using (ii), h_t is an excellent Morse function for $t \neq 1/2$. In particular, property (i) for adapted homotopies holds.

(iii). Since $h_0 \in \mathcal{F}^0$ and $h_t|_{Y \setminus \varphi(\operatorname{supp} b)} = h_0|_{Y \setminus \varphi(\operatorname{supp} b)}$ for every $t \in [0,1]$, parts (i) and (ii) of Proposition 4.5.3 imply that $\hat{h}$ is a fold map on $[0,1] \times (Y \setminus \varphi(\operatorname{supp} b))$. Hence, it remains to consider $\hat{h}$ on on the open subset $\varphi(\operatorname{int} B \times \operatorname{int} J) \subset Y$, where $h_t \circ \varphi = \varphi' \circ l_{\overline{\omega};t} \circ \psi^{-1}$ as shown above. Since $l_{\overline{\omega};t}$ coincides with l_t in a neighbourhood of all its critical points, and l_t has only non-degenerate points for all $t \in [0,1]$ except for $l_{1/2}$ at the origin of $\mathbb{R}^n$, parts (i) and (ii) of Proposition 4.5.3 imply that $\hat{h}$ is a fold map on $\varphi(\operatorname{int} B \times \operatorname{int} J) \setminus \varphi(0)$. In a neighbourhood of the origin, the track of $l_{\overline{\omega};t}$, $(t,x) \mapsto (t, l_{\overline{\omega};t}(x)) = (t, l_t(x))$, is the standard form of the Whitney cusp. All in all, this shows that $\hat{h}$ is a generic smooth map.

Finally, $\mathcal{J}(h) = \mathcal{J}(\gamma)$. (In fact, as shown above, $h_t = \gamma(t)$ is an excellent Morse function for all $t \in [0,1] \setminus \{1/2\}$, and the non-degenerate critical points of $h_{1/2}$ are contained in $Y \setminus \varphi(\operatorname{supp} b)$, where $h_t|_{Y \setminus \varphi(\operatorname{supp} b)} = h_0|_{Y \setminus \varphi(\operatorname{supp} b)}$ for all $t \in [0,1]$. Hence, Lemma 8.3.3 implies that $\mathcal{J}(h) = \mathcal{J}(h_1) \cup \mathcal{J}(h_1)$. On the other hand, $\gamma^{-1}(\mathcal{F}^0) = [0,1] \setminus \{1/2\}$ and Lemma 8.3.3 imply that $\mathcal{J}(\gamma) = \mathcal{J}(h_1) \cup \mathcal{J}(h_1)$.)

- γ is a descending elementary path of 1-crossing.

  By Lemma 8.3.3, it suffices to show that h is smooth and that h_t is a Morse function for all $t \in [0,1]$ that is excellent for t near 0 and 1.

  By [9, II.3.1, p. 48] γ is in particular a *descending elementary path* for some critical point c of index i of the Morse function $\gamma_0 \in \mathcal{F}^0$ as defined in [9, II.1.2, p. 42]. Following the notation of [9, II.1.1, pp. 40-41], let M_i denote the *Morse model* of index i and let k_t denote the *standard descending path*, which is by construction smooth in t and has support in $\operatorname{int} M_i$.

By [9, II.1.2, p. 42] there exists an embedding $\varphi\colon M_i \to Y$ such that $\varphi(0) = c$ and an orientation preserving embedding $\varphi'\colon [-1,1] \to [0,1]$ such that

(a) $\gamma_t = \gamma_0$ on the complement $Y \setminus \varphi(M_i)$.

(b) $\gamma_t \circ \varphi = \varphi' \circ k_t$ for all $t \in [0,1]$.

As $\operatorname{supp} \overline{\omega} \subset M_i$ by construction, it follows from (a) and (b) that $\gamma_t = \gamma_0$ on the open subset $Y \setminus \varphi(\operatorname{supp} \overline{\omega}) \subset Y$. Hence, we conclude from $\gamma_0 \in \mathcal{F}^0$ that for every $t \in [0,1]$, γ_t has only non-degenerate critical points on $Y \setminus \varphi(\operatorname{supp} \overline{\omega})$ which lie all in the interior of Y and $\gamma_t(Y^i) = \{i\}$ for $i = 0, 1$.

As $\operatorname{supp} \overline{\omega} \subset M_i$, it remains to consider γ_t on $\varphi(M_i)$. It follows from Lemma 1 in [9, II.1.1, p. 41] that k_t has a single non-degenerate critical point in $\operatorname{int} M_i$ for all $t \in [0,1]$. All in all, it follows from (b) that γ_t is a Morse function for all $t \in [0,1]$. (In particular, γ_t has no critical points on the boundary of Y).

Finally, properties (a) and (b) imply that h is smooth because k_t has compact support in $\operatorname{int} M_i$.

□

The following theorem answers the question of when two given elements of $\mathcal{F}^0$ that can be connected by a good path can also be connected by an adapted homotopy.

Theorem 8.3.9. *If γ is a good path in $\mathcal{F}^0 \cup \mathcal{F}^1$, then there exists an adapted homotopy h between $h_0 = \gamma(0)$ and $h_1 = \gamma(1)$ such that $\mathcal{J}(h) = \mathcal{J}(\gamma)$.*

Proof. As every good path γ in $\mathcal{F}^0 \cup \mathcal{F}^1$ is the concatenation of a finite number of traversing paths, it suffices by Proposition 8.3.5 to prove the theorem in the following two special cases:

- γ is a path in $\mathcal{F}^0$.
  This case is considered in Lemma 8.3.7.
- γ is a traversing path in $\mathcal{F}^0 \cup \mathcal{F}^1$.
  Proposition 8.2.4 states that γ can be replaced by the concatenation $\gamma' * \gamma''$ of an elementary path γ' such that $\gamma'(0) = \gamma(0)$ and a path γ'' in $\mathcal{F}^0$ such that $\gamma''(0) = \gamma'(1)$ and $\gamma''(1) = \gamma(1)$. Moreover, Lemma 8.3.7 implies that $\mathcal{J}(\gamma) = \mathcal{J}(\gamma') \cup \mathcal{J}(\gamma'')$. By Lemma 8.3.8, γ' induces an adapted homotopy h' such that $h'_i = \gamma'(i)$ for $i = 0, 1$ and $\mathcal{J}(h') = \mathcal{J}(\gamma')$. As shown in the first special case of the present proof, there exists an adapted homotopy h'' between $h''_0 = \gamma''(0)$ and $h''_1 = \gamma''(1)$ such that $\mathcal{J}(h'') = \mathcal{J}(\gamma'')$. Hence, by Proposition 8.3.5, the desired adapted homotopy can be chosen to be $h := h' *_\varepsilon h''$ (for suitable $\varepsilon \in (0,1)$).

□

Next, we specialize to the case that Y is a cylinder.

Corollary 8.3.10. *Suppose that $(Y, Y^0, Y^1) = ([0,1] \times Y^0, 0 \times Y^0, 1 \times Y^1)$ is a cylinder of dimension $n \geq 6$, and let Y^0 be simply connected. Suppose that $f_0, f_1\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$ are excellent Morse functions without critical points of index different from $k-1$ and k for some integer $3 \leq k \leq n-2$. Then there exists an adapted homotopy h between $h_0 = f_0$ and $h_1 = f_1$ such that*

$$\mathcal{J}(h) \subset \{\max\{k-1, n+1-k\}, \max\{k, n-k\}\} =: J_k.$$

Proof. By Proposition 8.2.5, there exists for $i = 0, 1$ a good path γ_i in $\mathcal{F}^0 \cup \mathcal{F}^1$ with origin $\gamma_i(0) = f_i$ such that $\gamma_i(1) \in \mathcal{O}^0$ and $\mathcal{J}(\gamma_i) \subset J_k$. In particular, the endpoint $g_i := \gamma_i(1)$ has no critical points of index different from $k-1$ and k. As $Y = [0,1] \times Y^0$ is the cylinder, Lemma C.0.1 (a) implies that g_i has an equal number of critical points of index $k-1$ and k for $i = 0, 1$. Consequently, $g_i \in \mathcal{F}^0_{k-1}$. It follows from Theorem 8.2.6 for $i = k-1$ that there exists a good path $\gamma\colon [0,1] \to \mathcal{F}^0_{k-1} \cup \mathcal{F}^1_{k-1}$ between $\gamma(0) = g_0$ and $\gamma(1) = g_1$. In particular, $\mathcal{J}(\gamma) \subset J_k$. Therefore, the concatenation $\delta := \gamma_0 * \gamma * \overline{\gamma_1}$ is a good path from f_0 to f_1 such that $\mathcal{J}(\delta) = \mathcal{J}(\gamma_0) \cup \mathcal{J}(\gamma) \cup \mathcal{J}(\gamma_1) \subset J_k$. Finally, an application of Theorem 8.3.9 yields the desired adapted homotopy h. □

Similarly, there is also the following corollary of Theorem 8.2.8:

Corollary 8.3.11. *Suppose that (Y, Y^0, Y^1) is a smooth manifold triad of dimension $n = \dim Y \geq 7$ such that the pair (Y, Y^i) is $(l-1)$-connected for some integer $0 \leq l \leq \lceil n/2 \rceil - 1$, and Y^i is simply connected for $i = 0, 1$. Suppose that $f_0, f_1\colon (Y, Y^0, Y^1) \to ([0,1], 0, 1)$ are excellent Morse functions such that every critical point of f_i has index in $\{l, \ldots, n-l\}$ for $i = 0, 1$. Then there exists an adapted homotopy h between $h_0 = f_0$ and $h_1 = f_1$ such that*

$$\mathcal{J}(h) \subset \{\lceil n/2 \rceil, \ldots, n-l\} =: J_l.$$

Proof. By Proposition 8.2.5, there exists for $i = 0, 1$ a good path γ_i in $\mathcal{F}^0 \cup \mathcal{F}^1$ with origin $\gamma_i(0) = f_i$ such that $\gamma_i(1) \in \mathcal{O}^0$ and $\mathcal{J}(\gamma_i) \subset J_l$. In particular, the endpoint $g_i := \gamma_i(1)$ lies in $\mathcal{F}^0_{[l,n-l]}$ for $i = 0, 1$. It follows from Theorem 8.2.8 that there exists a good path $\gamma\colon [0,1] \to \mathcal{F}^0_{[l,n-l]} \cup \mathcal{F}^1_{[l,n-l]}$ between $\gamma(0) = g_0$ and $\gamma(1) = g_1$. In particular, $\mathcal{J}(\gamma) \subset J_l$. Therefore, the concatenation $\delta := \gamma_0 * \gamma * \overline{\gamma_1}$ is a good path from f_0 to f_1 such that $\mathcal{J}(\delta) = \mathcal{J}(\gamma_0) \cup \mathcal{J}(\gamma) \cup \mathcal{J}(\gamma_1) \subset J_l$. (Recall that $\overline{\gamma_1}$ denotes the good path in $\mathcal{F}^0 \cup \mathcal{F}^1$ given by $t \mapsto \gamma_1(1-t)$ for all $t \in [0,1]$.) Finally, an application of Theorem 8.3.9 yields the desired adapted homotopy h. □

8.4 Proof of Theorem 8.0.1

Applying Remark 8.3.6 to the adapted homotopies $f|_{[1/4,1/3]\times M}$ (at $t = 1/3$) and $g|_{[2/3,3/4]\times M}$ (at $t = 2/3$), it suffices to construct a generic smooth map

$$F\colon [1/3, 2/3] \times M \to \mathbb{R}^2,$$

such that the absolute index of every fold point of F is contained in $\{\lfloor \frac{m}{2} \rfloor, \dots, m-k\} \cup \{m-1\}$, and such that $F|_{[1/3,a]\times M} = \mathrm{id}_{[1/3,a]} \times f_{1/3}$ and $F|_{[b,2/3]\times M} = \mathrm{id}_{[b,2/3]} \times g_{2/3}$ for suitable $a \in (1/3, 1/2)$ and $b \in (1/2, 2/3)$.

In the following, by abuse of notation, we will write f_0 for $f_{1/3}$ and f_1 for $g_{2/3}$. Furthermore, without loss of generality, we will replace the interval $[1/3, 2/3]$ by $[0, 1]$. In particular, the desired F will be a generic smooth map $F\colon [0,1]\times M \to \mathbb{R}^2$ such that the absolute index of every fold point of F is contained in $\{\lfloor \frac{m}{2} \rfloor, \dots, m-k\} \cup \{m-1\}$, and such that $F|_{[0,a]\times M} = \mathrm{id}_{[0,a]} \times f_0$ and $F|_{[b,1]\times M} = \mathrm{id}_{[b,1]} \times g_1$ for suitable $a \in (0, 1/2)$ and $b \in (1/2, 1)$.

Lemma C.0.4 (a) implies that M is orientable, and by part (b) of the same Lemma, f_i has for $i = 0, 1$ exactly one critical point c_i^0 of index 0 and exactly one critical point c_i^1 of index $m - 1$.

Without loss of generality, we may for $i = 0, 1$ in addition assume that $f_i(c_i^j) = j$ for $j = 0, 1$. (In fact, by means of a suitable isotopy of $\mathbb{R}$, one can construct for $i = 0, 1$ an adapted homotopy h^i with $h^i_i = f_i$, and such that all $h^i_t := h^i(t, -)$, $t \in [0, 1]$, are excellent Morse functions with the same critical points of the same indices, and such that h^i_{1-i} satisfies $h^i_{1-i}(c_i^j) = j$ for $j = 0, 1$. The claim then follows from Proposition 8.3.5.)

For suitable $\varepsilon > 0$ there exist orientation preserving embeddings

$$\iota_i^j\colon 2\varepsilon \cdot D^{m-1} \to M, \qquad i, j \in \{0, 1\},$$

such that $\iota_i^j(0) = c_i^j$ and

$$(f_i \circ \iota_i^j)(x) = e^j(||x||^2) := j + (-1)^j ||x||^2, \qquad x \in 2\varepsilon \cdot D^{m-1}. \qquad (*)$$

In addition, possibly making $\varepsilon > 0$ smaller, there exists a diffeotopy $G\colon [0, 1] \times M \to M$ (i.e., G is a smooth map such that $G_0 = \mathrm{id}_M$, and $G_t := G(t, -)\colon M \to M$ is a diffeomorphism for every $t \in [0, 1]$) such that $G_1 \circ \iota_0^j = \iota_1^j$ for $j = 0, 1$. (In fact, to construct G, note first that there exists a diffeotopy of M that maps $\iota_0^j(0)$ to $\iota_1^j(0)$ for $j = 0, 1$. Hence, we may assume without loss of generality that $p^j := \iota_0^j(0) = \iota_1^j(0)$ for $j = 0, 1$. Note that $\iota_0^0 \sqcup \iota_0^1$ and $\iota_1^0 \sqcup \iota_1^1$ define partial tubular neighbourhoods of $\{p^0, p^1\}$ in M. As explained in [22, p. 109], $\iota_i^0 \sqcup \iota_i^1$ contains for $i = 0, 1$ a tubular neighbourhood ξ_i of $\{p^0, p^1\}$ such that $\xi_i = \iota_i^0 \sqcup \iota_i^1$ near $\{p^0, p^1\}$. Then, [22, Theorem 5.3, p. 112] implies that ξ_0 and ξ_1 are isotopic in the sense of Definition B.0.5. Note that, since $\iota_0^0 \sqcup \iota_0^1$ and $\iota_1^0 \sqcup \iota_1^1$ are both orientation preserving, we may achieve that the linear isomorphism $\xi_0 \to \xi_1$ described in property (ii) of Definition B.0.5 is the identity map. The claim now follows from the isotopy extension theorem [22, Theorem 1.3, p. 180].)

Set $g_i := f_i \circ G_i\colon M \to [0, 1]$ for $i = 0, 1$. By construction, g_1 has the same critical point of index 0 as $g_0 = f_0$, namely c_0^0, and the same critical point of index $m-1$ as $g_0 = f_0$, namely c_0^1. Moreover, $g_i \circ \iota_0^j = f_i \circ \iota_i^j$ for $i, j \in \{0,1\}$. As $f_i \circ \iota_i^j$ is independent of i by $(*)$, one can conclude that $g_0 \circ \iota_0^j = g_1 \circ \iota_0^j$ for $j = 0, 1$.

It suffices to construct the desired F with respect to g_0 and g_1 (instead of f_0 and f_1). In fact, if F has been constructed between g_0 and g_1, then one just has to precompose it with the diffeomorphism $[0,1] \times M \to [0,1] \times M$, $(t, y) \mapsto (t, G_t^{-1}(y))$.

Let $V := M \setminus (\iota_0^0(\varepsilon \cdot D^{m-1}) \cup \iota_0^1(\varepsilon \cdot D^{m-1}))$ and $V^j := \iota_0^j(\partial(\varepsilon \cdot D^{m-1})) \cong S^{m-2}$ for $j = 0, 1$. Finally, let $U^j := \iota_0^j(\varepsilon \cdot D^{m-1})$.

Note that g_i restricts to an excellent Morse function

$$h_i := g_i|_V : (V, V^0, V^1) \to ([\varepsilon^2, 1-\varepsilon^2], \varepsilon^2, 1-\varepsilon^2)$$

such that $\mathfrak{I}(h_i) = \mathfrak{I}(f_i) \setminus \{0, m-1\} \subset \{k-1, \ldots, m-k\}$.

Corollary 8.3.11 yields for $l := k-1$ an adapted homotopy

$$h : [0,1] \times (V, V^0, V^1) \to ([\varepsilon^2, 1-\varepsilon^2], \varepsilon^2, 1-\varepsilon^2)$$

between h_0 and h_1 such that $\mathfrak{I}(h) \subset \{k-1, \ldots, m-k\}$. (Note that the pair (V, V^j) is $(k-2)$-connected for $j = 0, 1$ by Lemma C.0.3 and the long exact sequence of homotopy groups for the pair (V, V^j) because $V^j \cong S^{m-2}$, and $\mathfrak{I}(h_0) =\subset \{k-1, \ldots, m-k\}$.) By Remark 8.3.6 we may in addition assume that $h_t = h_i$ for t near $i = 0, 1$.

There exists a diffeomorphism $\rho : \mathbb{R} \xrightarrow{\cong} (0,2)$ such that $\rho(r) = \sqrt{r+1}$ for $r \in (-1/2, 1/2)$. (In fact, let $\rho_0 : \mathbb{R} \xrightarrow{\cong} (-1, \infty)$ be a diffeomorphism such that $\rho_0(r) = r$ for $r > -1/2$. Furthermore, let $\rho_1 : (-1, \infty) \xrightarrow{\cong} (0,2)$ be a diffeomorphism such that $\rho_1(r) = \sqrt{r+1}$ for $r \in (-1, 1/2)$. Then the desired diffeomorphism is given by the composition $\rho := \rho_1 \circ \rho_0 : \mathbb{R} \xrightarrow{\cong} (0,2)$.)

For $j = 0, 1$, ρ gives rise to a tubular neighbourhood a^j of V^j in U^j via

$$a^j : \mathbb{R} \times V^j \to U^j, \qquad a^j(u, v) = \iota_0^j(\rho(u) \cdot (\iota_0^j)^{-1}(v)),$$

that restricts to a diffeomorphism

$$a^j| : ([0,\infty) \times V^j, 0 \times V^j) \xrightarrow{\cong} (V \cap U^j, V^j), \qquad a^j(u, v) = \iota_0^j(\rho(u) \cdot (\iota_0^j)^{-1}(v)).$$

Moreover, a^j gives rise to a tubular neighbourhood α^j of $[0,1] \times V^j$ in $[0,1] \times U^j$ via

$$\alpha^j : [0,1] \times \mathbb{R} \times V^j \to [0,1] \times U^j, \qquad \alpha^j(t, u, v) = (t, a^j(u, v)).$$

Fix $j \in \{0,1\}$. The composition

$$h^j := e^j \circ h| : [0,1] \times (V \cap U^j, V^j) \to ([\varepsilon^2, \infty), \varepsilon^2)$$

is a smooth map such that $h_t^j := h^j(t, -)$ has no critical points on V^j for every $t \in [0,1]$ and $h_t^j = h_i^j$ for t near $i = 0, 1$.

Moreover, using the diffeomorphism $a^j|$ mentioned above, we have for t near $i = 0, 1$ and

$(u, v) \in [0, 1/2) \times V^j$ that

$$(h_t^j \circ a^j)(u, v) = e^j(h_i(a^j(u, v)) = (e^j \circ h_i \circ \iota_0^j)(\rho(u) \cdot (\iota_0^j)^{-1}(v))$$
$$= (e^j \circ f_i \circ \iota_0^j)(\rho(u) \cdot (\iota_0^j)^{-1}(v)) = ||\rho(u) \cdot (\iota_0^j)^{-1}(v)||^2 = \varepsilon^2(u + 1).$$

Therefore, by Proposition B.0.1 (applied to $X := 0 \times V^j$, $Y := [0, \infty) \times V^j$, and $f_t := 1/\varepsilon^2(h_t^j \circ a^j|) - 1$), there exists (for suitable $\delta > 0$) a smooth map

$$b^j : [0, 1] \times [0, \delta) \times V^j \to [0, \infty) \times V^j, \qquad (t, u, v) \mapsto b^j(t, u, v) =: b_t^j(u, v),$$

such that b_t^j is a collar of $V^j \times 0$ in $V^j \times [0, \infty)$ for all $t \in [0, 1]$, $b_t^j(u, v) = (u, v)$ for t near $i = 0, 1$, and the composition $h_t^j \circ a^j \circ b_t^j : [0, \delta) \times V^j \to [\varepsilon^2, \infty)$ is of the form $(u, v) \mapsto \varepsilon^2(u+1)$ for all $t \in [0, 1]$.

Application of Proposition B.0.3 yields a smooth extension

$$\tilde{b}^j : [0, 1] \times (-\delta, \delta) \times V^j \to \mathbb{R} \times V^j$$

of b^j such that $\tilde{b}_t^j := \tilde{b}^j(t, -, -)$ is a tubular neighbourhood of $0 \times V^j$ in $\mathbb{R} \times V^j$ for all $t \in [0, 1]$, and $\tilde{b}_t^j(u, v) = (u, v)$ for t near $i = 0, 1$.

We apply Proposition B.0.7 (compare Remark B.0.8) to the cylinder $[0, 1] \times V^j$ and the (partial) tubular neighbourhood β^j of $[0, 1] \times V^j = [0, 1] \times 0 \times V^j$ in $[0, 1] \times \mathbb{R} \times V^j$ given by the track of $\tilde{b}^j$,

$$\beta^j : [0, 1] \times (-\delta, \delta) \times V^j \to [0, 1] \times \mathbb{R} \times V^j, \qquad \beta^j(t, u, v) = (t, \tilde{b}_t^j(u, v)),$$

to obtain for $i = 0, 1$ tubular neighbourhoods $(\tilde{k}_i^j, \xi^j)$ (where ξ^j denotes the trivial vector bundle $[0, 1] \times \mathbb{R} \times V^j \to [0, 1] \times V^j$) such that $\tilde{k}_0^j|_{[0,1]\times(-\delta'',\delta'')\times V^j} = \mathrm{id}_{[0,1]\times\mathbb{R}\times V^j}|_{[0,1]\times(-\delta'',\delta'')\times V^j}$ and $\tilde{k}_1^j|_{[0,1]\times(-\delta'',\delta'')\times V^j} = \beta^j|_{[0,1]\times(-\delta'',\delta'')\times V^j}$ for a suitable $\delta'' \in (0, \delta)$ (such that $[0, 1] \times (-\delta'', \delta'') \times V^j$ plays the role of the neighbourhood U in Proposition B.0.7), and an isotopy

$$\tilde{K}^j : [0, 1] \times [0, 1] \times \mathbb{R} \times V^j \to [0, 1] \times \mathbb{R} \times V^j$$

of tubular neighbourhoods from $(\tilde{k}_0^j, \xi^j)$ to $(\tilde{k}_1^j, \xi^j)$ such that $\tilde{K}_1^j = \tilde{k}_1^j$. It follows from $\beta^j(t, u, v) = (t, \tilde{b}_t^j(u, v)) = (t, u, v)$ for all $(t, u, v) \in [0, 1] \times (-\delta, \delta) \times V^j$ such that t is near $i = 0, 1$, that $\tilde{K}^j$ can be chosen to satisfy $\tilde{K}_s^j(t, u, v) = (t, u, v)$ for all $s \in [0, 1]$ and all $(t, u, v) \in [0, 1] \times (-\delta'', \delta'') \times V^j$ such that t is near $i = 0, 1$, say $t \in [0, t_0) \cup (1 - t_0, 1]$ for suitable $t_0 \in (0, 1/2)$.

Next we apply the *isotopy extension theorem* [22, Theorem 1.4, p. 180] to the open subset

$$U := [0, t_0) \times M \cup \alpha^0([0, 1] \times (-\delta'', \delta'') \times V^0) \cup \alpha^1([0, 1] \times (-\delta'', \delta'') \times V^1) \cup (1 - t_0, 1] \times M$$

of the manifold $[0, 1] \times M$ and the compact subset $A := [0, 1] \times V^0 \cup [0, 1] \times V^1 \cup 0 \times M \cup 1 \times M$ of U and the isotopy of U in $[0, 1] \times M$ given by

$$K : [0, 1] \times U \to [0, 1] \times M, \qquad K_s(t, x) = \begin{cases} (t, x), & \text{if } t \in [0, t_0) \cup (1 - t_0, 1], \\ \alpha^j(\tilde{K}_s^j(t, u, v)), & \text{if } x = a^j(u, v) \in a^j((-\delta'', \delta'') \times V^j). \end{cases}$$

Note that the image of $[0,1]\times U$ under the track $\hat{K}$ of K is an open subset of $[0,1]\times[0,1]\times M$. (In fact, the argument is analogous to that in the last part of the proof of Lemma B.0.4, where the role of K is played by G. For this purpose, we may assume without loss of generality that $K_s = K_i$ for s near $i \in \{0,1\}$. Then, all that remains to check is that $K_s\colon U \to [0,1]\times M$ is for every $s \in [0,1]$ an embedding such that $K_s(\partial U) \subset \partial([0,1]\times M)$. This holds because a^j, α^j and $\tilde{K}^j_s$ are for $j = 0,1$ embeddings, and $\tilde{K}^j_s$ restricts to the identity map $\partial U = \{0,1\}\times M = \partial([0,1]\times M)$.)

This yields a diffeotopy of $[0,1]\times M$ that agrees with K in a neighbourhood of $[0,1]\times A$ in $[0,1]\times U$, say on $[0,t_1)\times M \cup \alpha^0([0,1]\times(-\delta''',\delta''')\times V^0)\cup\alpha^1([0,1]\times(-\delta''',\delta''')\times V^1)\cup(1-t_1,1]\times M$ for suitable $\delta''' \in (0,\delta'')$ and $t_1 \in (0,t_0)$.

In particular, the endpoint of the diffeotopy is an automorphism

$$\Theta\colon ([0,1]\times M, 0\times M, 1\times M) \xrightarrow{\cong} ([0,1]\times M, 0\times M, 1\times M)$$

with the following properties:

(Θ1) Θ restricts to an automorphism of $([0,1]\times V, [0,1]\times V^0, [0,1]\times V^1)$.
(Θ2) Θ restricts to the identity on $t\times M$ for t near 0 and 1.
(Θ3) For $j = 0,1$ we have $(\Theta\circ\alpha^j)|_{[0,1]\times(-\delta''',\delta''')\times V^j} = (\alpha^j\circ\beta^j)|_{[0,1]\times(-\delta''',\delta''')\times V^j}$.

It follows from (Θ1) and (Θ2) that $H := \hat{h}\circ\Theta|_{[0,1]\times V}$ ($\hat{h}$ denotes the track of h) is a generic smooth map whose fold points are all of absolute index contained in $\{k-1,\dots,m-k\}$, and H is the suspension of $H_i = h_i$ in a cylinder neighbourhood of $i\times M$ in $[0,1]\times M$.

Moreover, by (Θ3), for all $(t,u,v) \in [0,1]\times[0,\delta''')\times V^j$, we have

$$\begin{aligned}(H\circ\alpha^j)(t,u,v) &= (h\circ\alpha^j\circ\beta^j)(t,u,v) = e^j(h^j(\alpha^j(t,b^j_t(u,v))))\\ &= e^j(h^j_t(a^j(b^j_t(u,v)))) = e^j(\varepsilon^2(u+1)).\end{aligned}$$

For $j = 0,1$ define the (definite) fold map

$$H^j\colon [0,1]\times U^j \to [0,1]\times\mathbb{R}, \qquad H^j(t,x) = (t, j+(-1)^j||(\iota^j_0)^{-1}(x)||^2) = (t, e^j(||(\iota^j_0)^{-1}(x)||^2)).$$

Then, for all $u \in [0,\delta''') \subset [0,1/2)$, $t \in [0,1]$ and $v \in V^j$, we use $\rho(u) = \sqrt{u+1}$ and $||(\iota^j_0)^{-1}(v)||^2 = \varepsilon^2$ to obtain

$$(H^j\circ\alpha^j)(t,u,v) = H^j(t,\iota^j_0(\rho(u)\cdot(\iota^j_0)^{-1}(v))) = (t, e^j(||\rho(u)\cdot(\iota^j_0)^{-1}(v)||^2)) = (t, e^j(\varepsilon^2(u+1))).$$

This shows that H^j and H coincide on a neighbourhood of $[0,1]\times V^j$ in $[0,1]\times V$.

Finally, the desired generic smooth map $F\colon [0,1]\times M \to [0,1]\times[0,1]$ is defined by $F|_{[0,1]\times V} = H$ and $F|_{[0,1]\times U^j} = H^j$.

This completes the proof of Theorem 8.0.1.

Chapter 9

Modifying Indefinite Fold Maps via Stein Factorization

Stein factorization is a powerful tool for the study of smooth mappings that has been introduced by Burlet and de Rham in [8]. It has been employed by Saeki (see e.g. [48] and [47]) for the study of special generic functions.

Let us recall the notion of Stein factorization of an arbitrary continuous map $f\colon X \to Y$ between topological spaces (see for instance [21, Definition 2.1]). Define an equivalence relation $\sim_f$ on X as follows. Two points $x_1, x_2 \in X$ are called equivalent, $x_1 \sim_f x_2$, if they are mapped by f to the same point $y := f(x_1) = f(x_2) \in Y$, and lie in the same connected component of $f^{-1}(y)$. The quotient map $\pi_f\colon X \to X/\sim_f$ gives rise to a unique set-theoretic factorization of f of the form

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle \pi_f}\downarrow & \nearrow{\scriptstyle \overline{f}} & \\ X/\sim_f & & \end{array} \quad .$$

(Note that $\overline{f}$ is well defined by setting $\overline{f}(\pi_f(x)) := f(x)$ since $\pi_f(x) = \pi_f(x')$ implies by definition of $\sim_f$ that x and x' lie in the same fiber of f.) If we equip $X_f := X/\sim_f$ with the quotient topology induced by the surjective map $\pi_f\colon X \to X_f$ (i.e. a subset $U \subset X_f$ is open in X_f if and only if $\pi_f^{-1}(U)$ is an open subset of X), then it follows that the maps π_f and $\overline{f}$ are continuous. Then, the above diagram is called the *Stein factorization* of f.

If the spaces X and Y and the map f carry more structure, then one can also expect to say more about their Stein factorization (see [29]).

If f happens to be a special generic map, then X_f even carries a smooth manifold structure (see [47]). A detailed account is worked out in [19].

Section 9.1 gives an individual proof of a version of this fact in the setting of indefinite fold lines (see Theorem 9.1.7). Afterwards, in Section 9.2, we formulate some techniques of modifying Stein factorizations. This culminates in a gluing theorem for Stein factorizations (see Theorem 9.2.4).

9.1 Stein Factorization for Indefinite Fold Maps

Let $f\colon X \to Y$ be a continuous map between topological spaces. Eventually, in Theorem 9.1.7, f will be a suitable fold map between certain smooth manifolds. In order to construct charts on X_f, we prepend the following convenient notion for subsets of X:

Definition 9.1.1. A subset $S \subset X$ is called *Stein adapted* for $f\colon X \to Y$ if $S \cap f^{-1}(y)$ forms a component of $f^{-1}(y)$ for every point $y \in f(S)$.

The next goal is to give conditions that suffice to show that X_f is a manifold. The method is to construct charts on X_f from suitable Stein adapted open subsets of X. For this purpose, U needs to be chosen in such a way that $f(U) \subset Y := \mathbb{R}^2$ is an embedded manifold with corners.

Lemma 9.1.2. *If $S \subset Y$ is Stein adapted for $f\colon X \to Y$, then*

(i) $S = \pi_f^{-1}(\pi_f(S))$.

(ii) $\overline{f}\colon X_f \to Y$ *restricts to a (continuous) bijective map* $\pi_f(S) \xrightarrow{\cong} f(S)$.

Proof. (i). It is clear that $S \subset \pi_f^{-1}(\pi_f(S))$. Conversely, given $x \in \pi_f^{-1}(\pi_f(S))$, there exists $s \in S$ such that $\pi_f(x) = \pi_f(s)$. By definition of the quotient map $\pi_f\colon X \to X_f$ this means that x and s lie in the same component C of $f^{-1}(y)$, where $y := f(x)$ $(= \overline{f}(\pi_f(x)) = \overline{f}(\pi_f(s)) = f(s))$. Since $S \subset Y$ is Stein adapted for f, it follows from $y = f(s) \in f(S)$ that $C = S \cap f^{-1}(y)$ because $s \in C \cap (S \cap f^{-1}(y))$. In particular, $x \in C \subset S$.

(ii). Observe that $\overline{f}\colon X_f \to Y$ restricts to a (continuous) surjective map $\pi_f(S) \to \overline{f}(\pi_f(S)) = f(S)$. To show injectivity, suppose that the elements $z, z' \in \pi_f(S)$ satisfy $\overline{f}(z) = \overline{f}(z')$. Writing $z = \pi_f(x)$ and $z' = \pi_f(x')$ for suitable $x, x' \in S$, this amounts to $y := f(x) = \overline{f}(z) = \overline{f}(z') = f(x') \in f(S)$. Consequently, since $S \subset Y$ is Stein adapted for f, the points $x, x' \in S \cap f^{-1}(y)$ lie in the same component of $f^{-1}(y)$. Hence, $z = \pi_f(x) = \pi_f(x') = z'$ by definition of the quotient topology on X_f. □

Corollary 9.1.3. *Suppose that X is compact and Y is a Hausdorff space. If there exists a finite cover $X = \bigcup_{i \in I} A_i$ by closed subsets $A_i \subset X$, $i \in I$, such that $A_i \subset X$ is Stein adapted for f for every $i \in I$, then $\overline{f}$ restricts to a closed map $\pi_f(A_i) \to f(A_i)$ for every $i \in I$, and π_f is a closed map.*

In particular, if X is a Hausdorff space, then X_f is a Hausdorff space by [36, Theorem 5.4, p. 252].

Proof. Fix $i \in I$. In order to show that $\overline{f}$ restricts to a closed map $\pi_f(A_i) \to f(A_i)$, let A be a closed subset of $\pi_f(A_i)$. Observe that $\pi_f(A_i)$ is a closed subset of X_f. (Indeed, as $A_i \subset X$ is a Stein adapted subset for f, Lemma 9.1.2 (i) implies that $A_i = \pi_f^{-1}(\pi_f(A_i))$. Hence, by definition of the quotient topology the claim follows.) Therefore, we conclude that $A' := \pi_f^{-1}(A)$ is a closed subset of X that is contained in $\pi_f^{-1}(\pi_f(A_i)) = A_i$ (note that π_f is surjective). As $f(A') = \overline{f}(\pi_f(\pi_f^{-1}(A))) = \overline{f}(A)$, it remains to show that $f(A')$ is a closed subset of $f(A_i)$. In fact, $f(A')$ is a closed subset of Y because f is a continuous map from the compact space X to the Hausdorff space Y. Moreover, $A' \subset A_i$ implies $f(A') \subset f(A_i)$, which shows that $f(A')$ is a closed subset of $f(A_i)$.

It remains to show that π_f is a closed map. If $B \subset X$ is a closed subset, then $B_i := B \cap A_i$ is a closed subset of X for every $i \in I$. As f is a continuous map from the compact space X to the Hausdorff space Y, $f(B_i)$ is a closed subset of Y for every $i \in I$. Therefore, $\overline{f}^{-1}(f(B_i))$ is a closed subset of X_f for all $i \in I$. Note that $\overline{f}^{-1}(f(B_i)) = \overline{f}^{-1}(\overline{f}(\pi_f(B_i))) = \pi_f(B_i)$ because $\pi_f(B_i) \subset \pi_f(A_i)$ and $\overline{f}$ restricts to a bijection $\pi_f(A_i) \xrightarrow{\cong} f(A_i)$ by Lemma 9.1.2 (ii) (recall that A_i is by assumption a Stein adapted subset of X for f for all $i \in I$). Finally, being the union of a finite number of closed subsets, $\pi_f(B) = \pi_f(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} \pi_f(B_i)$ is a closed subset of X_f. □

Corollary 9.1.4. *Suppose that there exists an open cover $X = \bigcup_{i\in I} U_i$ such that f restricts to an open map $U_i \to f(U_i)$ for every $i \in I$. If $U_i \subset X$ is Stein adapted for f for every $i \in I$, then $\overline{f}$ restricts to an open map $\pi_f(U_i) \to f(U_i)$ for every $i \in I$, and π_f is an open map.*

In particular, if X is second countable (e.g. if X is a manifold), then $X_f = \pi_f(X)$ is second countable as well.

Proof. Fix $i \in I$. In order to show that $\overline{f}$ restricts to an open map $\pi_f(U_i) \to f(U_i)$, let V be an open subset of $\pi_f(U_i)$. Observe that $\pi_f(U_i)$ is an open subset of X_f. (Indeed, as $U_i \subset X$ is a Stein adapted subset for f, Lemma 9.1.2 (i) implies that $U_i = \pi_f^{-1}(\pi_f(U_i))$. Hence, by definition of the quotient topology the claim follows.) Therefore, we conclude that $U := \pi_f^{-1}(V)$ is an open subset of X that is contained in $\pi_f^{-1}(\pi_f(U_i)) = U_i$ (note that π_f is surjective). As $f(U) = \overline{f}(\pi_f(\pi_f^{-1}(V))) = \overline{f}(V)$, it remains to show that $f(U)$ is an open subset of $f(U_i)$. This is in fact the case because f restricts by assumption to an open map $U_i \to f(U_i)$.

It remains to show that π_f is an open map. If $U \subset X$ is an open subset, then $U_i' := U \cap U_i$ is an open subset of U_i for every $i \in I$. As f restricts by assumption to an open map $U_i \to f(U_i)$, it follows that $f(U_i')$ is an open subset of $f(U_i)$. Recall that $\overline{f}$ restricts to a continuous map $\pi_f(U_i) \to \overline{f}(\pi_f(U_i)) = f(U_i)$. Therefore, $\overline{f}^{-1}(f(U_i'))$ is an open subset of $\pi_f(U_i)$ for all $i \in I$. Note that $\overline{f}^{-1}(f(U_i')) = \overline{f}^{-1}(\overline{f}(\pi_f(U_i'))) = \pi_f(U_i')$ because $\pi_f(U_i') \subset \pi_f(U_i)$ and $\overline{f}$ restricts to a bijection $\pi_f(U_i) \xrightarrow{\cong} f(U_i)$ by Lemma 9.1.2 (ii) (recall that U_i is by assumption a Stein adapted subset of X for f for all $i \in I$). As above, Lemma 9.1.2 (i) implies that $\pi_f(U_i)$ is an open subset of X_f. Thus, we have shown that $\pi_f(U_i')$ is an open subset of X_f for all $i \in I$. Finally, being the union of open subsets, $\pi_f(U) = \pi_f(\bigcup_{i\in I} U_i') = \bigcup_{i\in I} \pi_f(U_i')$ is an open subset of X_f. □

Lemma 9.1.5. *Let W be a compact smooth manifold with boundary ∂W of dimension $m := \dim W \geq 2$. Set $L := \mathbb{R} \times 0 \subset \mathbb{R}^2$. Suppose that $F\colon W \to \mathbb{R}^2$ is a fold map such that $L \cap F(\partial W) = \emptyset$ and $F|_{S(F)}$ is transverse to L. Then, $M := F^{-1}(L)$ is a closed submanifold of W such that $S(F) \pitchfork M$. Furthermore, for suitable $\varepsilon > 0$, there exists a tubular neighbourhood $M \times (-\varepsilon, \varepsilon)$ of $M = M \times 0$ in W on which F is of the form $F(x,t) = (f(x,t), t)$, $(x,t) \in M \times (-\varepsilon, \varepsilon)$.*

Proof. Let $\pi_L\colon \mathbb{R}^2 \to \mathbb{R}$ denote the projection to the second component. (In particular, $L = \pi_L^{-1}(0)$.) Then,

$$F_L := \pi_L \circ F\colon W \to \mathbb{R}$$

is a smooth map such that $F_L^{-1}(0) = M$. Moreover, as $F|_{S(F)}$ is transverse to L, it follows that 0 is a regular value of F_L. As W is compact and $F_L^{-1}(0) = M$, we conclude that there exists $\varepsilon_0 > 0$ such that $F_L^{-1}((-\varepsilon_0, \varepsilon_0)) \cap \partial W = \emptyset$, and every $t \in (-\varepsilon_0, \varepsilon_0)$ is a regular value of F_L. (In fact, as F_L is a submersion at every point in M, there exists a neighbourhood U of M in W such that F_L is a submersion on U. As $F_L(W \setminus U) \cup F_L(\partial W) \subset \mathbb{R} \setminus \{0\}$ is compact, there exists $\varepsilon_0 > 0$ such that $F_L^{-1}((-\varepsilon_0, \varepsilon_0)) \subset U \setminus \partial W$. Consequently, every $t \in (-\varepsilon_0, \varepsilon_0)$ is a regular value of F_L.) Fix $\varepsilon \in (0, \varepsilon_0)$. Then, [22, Exercise 5, p. 32] implies that $F_L^{-1}([-\varepsilon, \varepsilon])$ is a smooth submanifold of W of codimension 0 with boundary $F_L^{-1}(-\varepsilon) \sqcup F_L^{-1}(\varepsilon)$, and such that $F_L^{-1}([-\varepsilon, \varepsilon]) \cap \partial W = \emptyset$. As F_L restricts to a Morse function $(F_L^{-1}([-\varepsilon, \varepsilon]), F_L^{-1}(-\varepsilon), F_L^{-1}(\varepsilon)) \to ([-\varepsilon, \varepsilon], -\varepsilon, \varepsilon)$ without critical points, it follows that $F_L^{-1}([-\varepsilon, \varepsilon])$ is diffeomorphic to the cylinder $M \times [-\varepsilon, \varepsilon]$ in such a way that F_L becomes the projection to the second factor.

□

The following Lemma provides the local prototype of Stein factorization for indefinite fold maps.

Lemma 9.1.6. *Let M be a connected closed smooth manifold of dimension $m - 1 \geq 2$. Let F be a fold map of the form*

$$F = (f, \mathrm{pr}_{[0,1]})\colon M \times [0,1] \to \mathbb{R} \times [0,1], \qquad F(x,t) = (f(x,t), t),$$

without fold lines of absolute index $m-2$. Then the following statements hold:

(i) All fibers of F are connected.

(ii) $F(M \times [0,1]) \subset \mathbb{R}^2$ is the image of an embedding Φ of the form

$$\Phi = (\phi, \mathrm{pr}_{[0,1]})\colon [0,1] \times [0,1] \to \mathbb{R} \times [0,1].$$

(iii) F restricts to an open map $M \times [0,1] \to F(M \times [0,1])$.

Proof. For every $t \in [0,1]$ let $f_t\colon M \to \mathbb{R}$ be given by $f_t(x) = f(x,t)$. By Proposition 4.5.3, f_t is a fold map without critical points of index 1 and $m-2$ for all $t \in [0,1]$. (To apply the lemma formally correct, one has to extend F for some $\varepsilon > 0$ to a fold map $M \times (-\varepsilon, 1+\varepsilon) \to \mathbb{R}^2$ and then applies Lemma 9.1.5 to construct suitable tubular neighbourhoods of $M \times 0$ and $M \times 1$ in $M \times (-\varepsilon, 1+\varepsilon)$.) Hence, by Lemma C.0.4$(c)$, the fibers of f_t are connected for all $t \in [0,1]$. This implies property (i) because $F^{-1}(F(x,t)) = f_t^{-1}(f(x,t)) \times t$ for all $(x,t) \in M \times [0,1]$.

Concerning property (ii), observe that f_0 has precisely one minimum $\nu_0 \in M$ and one maximum $\mu_0 \in M$ by Lemma C.0.4(b). Each of these points lies on a definite fold line of F by Proposition 4.5.3. It follows from $\mathrm{pr}_{[0,1]} \circ F = \mathrm{pr}_{[0,1]}$ that these definite fold lines are given by the images of embeddings

$$\begin{aligned} \nu&\colon [0,1] \to M \times [0,1], \qquad \nu(0) = (\nu_0, 0), \\ \mu&\colon [0,1] \to M \times [0,1], \qquad \mu(0) = (\mu_0, 0), \end{aligned}$$

such that $\mathrm{pr}_{[0,1]} \circ \nu = \mathrm{id}_{[0,1]}$ and $\mathrm{pr}_{[0,1]} \circ \mu = \mathrm{id}_{[0,1]}$. Now Proposition 4.5.3 implies that f_t has for every $t \in [0,1]$ a minimum at $\nu(t) \in M \times t = M$ and a maximum at $\mu(t) \in M \times t = M$, and these are unique by Lemma C.0.4(b). Hence,

$$F(M \times [0,1]) = \{(s,t) \in \mathbb{R}^2;\ t \in [0,1],\ f(\nu(t)) \leq s \leq f(\mu(t))\}.$$

As $f(\nu(t)) < f(\mu(t))$ for all $t \in [0,1]$, the desired smooth map $\phi\colon [0,1]\times[0,1] \to \mathbb{R}$ can be chosen to be

$$\phi_b(a) := \phi(a,b) = a \cdot f(\nu(b)) + (1-a)\cdot f(\nu(b)).$$

Finally, property (iii) follows from property (ii) and Lemma C.0.4 (d). (In fact, it suffices to show that

$$\Phi^{-1}\circ F\colon M\times[0,1] \to [0,1]\times[0,1], \qquad (x,t)\mapsto ((\phi_t^{-1}\circ f_t)(x),t)$$

is an open map.)

□

The following theorem clarifies the structure of Stein factorization in the case of a fold map without fold lines of absolute index $m-2$ from a cobordism W^m into the plane.

Theorem 9.1.7. *Let (W, M_1, M_2) be a smooth manifold triad of dimension $m := \dim W \geq 3$. Suppose that $F\colon W \to \mathbb{R}^2$ is a fold map with the following poperties:*

- *(i) F has no fold lines of absolute index $m-2$.*
- *(ii) For suitable $\varepsilon > 0$ there exist collar neighbourhoods $M_1 \times [0,\varepsilon)$ and $M_2\times(1-\varepsilon,1]$ of $M_1\times 0 = M_1$ and $M_2 \times 1 = M_2$ in W such that*

$$F(x,t) = (f_t(x),t), \qquad (x,t)\in M_1\times[0,\varepsilon)\sqcup M_2\times(1-\varepsilon,1],$$

  where f_t is a Morse function $M_1\to\mathbb{R}$ for all $t\in[0,\varepsilon)$, and a Morse function $M_2\to\mathbb{R}$ for all $t\in(1-\varepsilon,1]$.

Consider the Stein factorization of F,

$$\begin{array}{ccc} W & \xrightarrow{\;F\;} & \mathbb{R}^2 \\ {\scriptstyle \pi_F}\downarrow & \nearrow_{\overline{F}} & \\ W_F & & \end{array}.$$

Then, W_F can be given the structure of a compact smooth manifold of dimension 2 with corners such that π_F is a fold map and $\overline{F}$ is a submersion. Furthermore, if $D(F)$ denotes the union of the definite fold lines of F, then the boundary of W_F decomposes as

$$\partial W_F = \pi_F(\partial W)\cup \pi_F(D(F)),$$

where $\pi_F(\partial W)\cap\pi_F(D(F)) = \pi_F(\partial W\cap D(F))$ is the set of corners of W_F, and π_F restricts to an embedding $D(F)\to\partial W_F$.

Proof. For every point $w\in W$, we will construct subsets $w\in V_w\subset A_w\subset U_w\subset W$ with the following properties:

(1) $U_w, V_w \subset W$ are open subsets and $A_w\subset W$ is a closed subset such that U_w, V_w and A_w are all Stein adapted for F.

(2) F restricts to an open map $U_w \to F(U_w)$.

(3) There exists a smooth manifold with corners Z_w of dimension 2 and an embedding $\alpha_w\colon Z_w \to \mathbb{R}^2$ such that $\alpha_w(Z_w) = F(U_w)$.

Then W_F can be given the structure of a 2-dimensional compact smooth manifold with corners as follows.

By property (1) we obtain an open cover $W = \bigcup_{w\in W} U_w$ by Stein adapted subsets $U_w \subset W$ such that F restricts to an open map $U_w \to F(U_w)$ for every $w \in W$ by (2). Hence, Lemma 9.1.2 (*ii*) and Corollary 9.1.4 imply that $\overline{F}$ restricts to a homeomorphism $\pi_F(U_w) \xrightarrow{\cong} F(U_w)$ for every $w \in W$. Invoking (3), we claim that a smooth atlas of W_F is given by

$$\{\varphi \circ \alpha_w^{-1} \circ \overline{F}|_{\overline{F}^{-1}(\alpha_w(A))};\, w \in W,\ \varphi\colon A \to B \text{ is a chart of } Z_w,$$
$$\text{where } A \subset Z_w \text{ and } B \subset [0,\infty) \times [0,\infty) \text{ are open subsets}\}.$$

(In fact, as π_F is an open map by Corollary 9.1.4, it is clear that the family $\{\overline{F}^{-1}(\alpha_w(A))\}_{w,A}$ forms an open cover of W_F. Let us check that coordinate changes are smooth. Let $\varphi\colon A \to B$ and $\varphi'\colon A' \to B'$ be charts of Z_w and $Z_{w'}$ for points $w, w' \in W$ such that the open subset $X := \overline{F}^{-1}(\alpha_w(A)) \cap \overline{F}^{-1}(\alpha_{w'}(A')) \subset W_F$ is nonempty. Note that $\overline{F}(X) = \alpha_w(A) \cap \alpha_{w'}(A')$ is an open subset of $F(U_w)$ and of $F(U_{w'})$, so $\alpha_w^{-1}(\overline{F}(X)) \subset Z_w$ and $\alpha_{w'}^{-1}(\overline{F}(X)) \subset Z_{w'}$ are open subsets. The coordinate change is of the form

$$(\varphi \circ \alpha_w^{-1} \circ \overline{F}|_X) \circ (\varphi' \circ \alpha_{w'}^{-1} \circ \overline{F}|_X)^{-1}\colon \varphi'(\alpha_{w'}^{-1}(\overline{F}(X))) \to \varphi(\alpha_w^{-1}(\overline{F}(X))),$$

where

$$\begin{aligned}(\varphi \circ \alpha_w^{-1} \circ \overline{F}|_X) \circ (\varphi' \circ \alpha_{w'}^{-1} \circ \overline{F}|_X)^{-1} &= \varphi \circ \alpha_w^{-1} \circ \overline{F}|_X \circ (\overline{F}|_X)^{-1} \circ \alpha_{w'} \circ \varphi'^{-1}|_{\varphi'(\alpha_{w'}^{-1}(\overline{F}(X)))} \\ &= \varphi \circ \alpha_w^{-1} \circ \alpha_{w'} \circ \varphi'^{-1}|_{\varphi'(\alpha_{w'}^{-1}(\overline{F}(X)))}.\end{aligned}$$

The latter expression is smooth since the restrictions

$$\alpha_w|\colon \alpha_w^{-1}(\overline{F}(X)) \to \mathbb{R}^2,$$
$$\alpha_{w'}|\colon \alpha_{w'}^{-1}(\overline{F}(X)) \to \mathbb{R}^2,$$

are embeddings with the same image $\overline{F}(X)$.) Note that Corollary 9.1.4 also implies that W_F is second countable since W is a manifold. Finally, one uses Corollary 9.1.3 to show that W_F is a Hausdorff space. (Indeed, note that W is a compact Hausdorff space and $\mathbb{R}^2$ is a Hausdorff space. From the open cover $W = \bigcup_{w\in W} V_w$ (see property (1)) we can extract a finite cover $W = \bigcup_{i\in I} V_{w_i}$ as W is compact. Therefore, still by property (1), we obtain a finite cover $W = \bigcup_{w\in W} A_w$ by closed subsets that are all Stein adapted for F. Hence, Corollary 9.1.3 implies that W_F is a Hausdorff space.)

Observe that by the above construction of a smooth atlas for W_F it is clear that $\overline{F}$ is locally an embedding (or equivalently, as $\dim W = 2$, an immersion). In particular, by commutativity of the Stein factorization diagram, π_F turns out to be smooth, and in particular a fold map since F is a fold map.

Given a point $w \in W$, let us construct subsets $w \in V_w \subset A_w \subset U_w \subset W$ with the desired properties (1) to (3). First consider the case $w \notin \partial W$. Fix an (affine) straight line $L \subset \mathbb{R}^2$

through the point $z := F(w)$ such that $F|_{S(F)}$ is transverse to L and $L \cap F(\partial W) = \emptyset$. (In fact, note that $(F|_{S(F)})^{-1}(z)$ is finite because $S(F)$ is compact. Let T denote the finite set of lines in $\mathbb{R}^2$ that are tangent to $F(S(F))$ at $F(c)$, $c \in (F|_{S(F)})^{-1}(z)$. Let V denote the open subset of $\mathbb{R}^2 \setminus \{z\}$ consisting of those points $v \in \mathbb{R}^2 \setminus \{z\}$ for which the line L_v through v and z is not parallel to $0 \times \mathbb{R} \subset \mathbb{R}^2$ and satisfies $L_v \cap F(\partial W) = \emptyset$. (This can only be done if $z \notin F(\partial W)$. Otherwise, if $z \in F(\partial W)$, then replace W by $W \setminus (M_1 \times [0, \varepsilon/2] \sqcup M_2 \times [1-\varepsilon/2, 1])$ in the present argument.) Consider the smooth map $\rho\colon V \to \mathbb{R}$ that maps every point $v \in V$ to the intersection of the line L_v and the line $0 \times \mathbb{R} \subset \mathbb{R}^2$. Then one can choose $L := L_v$, where $v \in V$ is chosen such that $\rho(v)$ is a regular value of $\rho \circ F|_{S(F)}$, and $L_v \notin T$.) By composition of F with an affine linear map $\mathbb{R}^2 \to \mathbb{R}^2$ we may assume that $L = \mathbb{R} \times 0 \subset \mathbb{R}^2$. Hence, by Lemma 9.1.5, $M := F^{-1}(L)$ is a closed submanifold of W such that $S(F) \pitchfork M$. Furthermore, for suitable $\delta > 0$, there exists a tubular neighbourhood $M \times (-2\delta, 2\delta)$ of $M = M \times 0$ in W on which F is of the form $F(x,t) = (f(x,t), t)$, $(x,t) \in M \times (-2\delta, 2\delta)$. Let M_w denote the component of M that contains w. Set $U_w := M_w \times (-\delta, \delta)$, $A_w := M_w \times [-\delta/2, \delta/2]$ and $V_w := M_w \times (-\delta/2, \delta/2)$. By Lemma 9.1.6, the fold map $F_w := F|_{M_w \times [-\delta,\delta]}$ without fold lines of absolute index $m-2$ has the following properties:

(a) All fibers of F_w are connected.

(b) $F_w(M_w \times [-\delta, \delta]) \subset \mathbb{R}^2$ is the image of an embedding Φ of the form

$$\Phi = (\phi, \mathrm{pr}_{[-\delta,\delta]})\colon [0,1] \times [-\delta, \delta] \to \mathbb{R} \times [-\delta, \delta].$$

(c) F_w restricts to an open map $M_w \times [-\delta, \delta] \to F_w(M_w \times [-\delta, \delta])$.

It remains to check the desired properties (1) to (3) for U_w, A_w and V_w. In fact, property (1) follows from property (a) because U_w, A_w and V_w are unions of fibers of F_w. Moreover, property (2) is an immediate consequence of property (c). Finally, property (3) follows from property (b).

If $w \in \partial W$, say $w \in M_1$, then one extends the fold map $F|_{M_1 \times [0,\varepsilon)}$ slightly to a fold map $F_w\colon M_1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ without fold lines of absolute index $m-2$. Setting $L := \mathbb{R} \times 0 \subset \mathbb{R}^2$, one can then proceed exactly as in the case $w \notin \partial W$. However, the correct choice of U_w, A_w and V_w will be the intersection of the choice above with $M_1 \times [0, \varepsilon)$. Moreover, the above embedding $M_1 \times (-\varepsilon, \varepsilon) \to F_w(M_1 \times (-\varepsilon, \varepsilon))$ must be restricted to an embedding of the form $M_1 \times [0, \varepsilon) \to F_w(M_1 \times [0, \varepsilon))$, which produces the desired charts for the corners of W_F.

□

The following result shows that the definite fold lines of a fold map without fold lines of absolute index $m-2$ behave in some sense rigid along m-dimensional cylinders. The proof demonstrates impressively that Stein factorization can help to reduce certain problems about fold maps to purely topological questions.

Corollary 9.1.8. *(cylindrical rigidity of definite fold lines)*
Let M be a connected closed smooth manifold of dimension $m-1 \geq 2$. Suppose that

$$F = (F_1, F_2) \colon W := M \times [0,1] \to \mathbb{R}^2$$

is a fold map with the following properties:

(*i*) *F has no fold lines of absolute index $m-2$.*
(*ii*) *There exists $\varepsilon \in (0, 1/2)$ such that, for all $t \in [0, \varepsilon] \sqcup [1-\varepsilon, 1]$, F_1 restricts to a Morse function $M = M \times t \to \mathbb{R}$, and $F_2(x,t) = t$ for all $x \in M$.*

Then the Stein factorization W_F of F (see Theorem 9.1.7) is diffeomorphic to $[0,1] \times [0,1]$. Consequently, F has exactly two definite fold lines, and these are intervals with one end in $M \times 0$ and one end in $M \times 1$.

Proof. The proof exploits the following well-known topological fact:

Suppose that X is a connected compact 2-dimensional topological manifold with boundary. If X has a boundary component that is contractible in X, then X is homeomorphic to the disc D^2.

Let $\pi_F \colon W \to W_F$ be the Stein factorization of F (see Theorem 9.1.7). W_F is a connected compact smooth manifold of dimension 2 with corners. Note that $\pi_F(M \times 0)$ and $\pi_F(M \times 1)$ are intervals that are contained in ∂W_F. Let C be the boundary component of W_F that contains the interval $\pi_F(M \times 0)$. We distinguish between the following two cases:

- $\pi_F(M \times 1)$ is not contained in C. In this case C is the union $C = \pi_F(M \times 0) \cup \pi_F(S)$ of two intervals along the endpoints, where S is a definite fold line of F. Since M is connected, there exists a continuous map $\alpha \colon S^1 \to W$ and two distinct points $a, b \in S^1$ such that one arc between a and b in S^1 is mapped homeomorphically to S and the other arc is mapped to $M \times 0$. Since M is simply connected by Lemma C.0.4 (e) (here, one uses that M admits a Morse function without critical points of index 1 or $m-2$, which follows from Definition 4.5.1 and Proposition 4.5.3), $W = M \times [0,1]$ is simply connected as well, so the loop α is contractible in W. Hence, the loop $\pi_F \circ \alpha$ is contractible in W_F. Then the topological fact above implies that W_F is homeomorphic to the disc D^2. However, ∂W_F must have at least two boundary components since $\pi_F(M \times 1)$ is by assumption not contained in C, contradiction.
- $\pi_F(M \times 1)$ is contained in C. In this case C is the union $C = \pi_F(M \times 0) \cup \pi_F(S) \cup \pi_F(M \times 1) \cup \pi_F(S')$ of four intervals along the endpoints, where S and S' are two definite fold lines of F. Since M is connected, we may assume that there exists a continuous map $\alpha \colon S^1 \to W$ and four pairwise distinct points $a, b, c, d \in S^1$ such that the arcs ab and cd are mapped homeomorphically to S and S', and the arcs bc and da are mapped to $M \times 0$ and $M \times 1$. The same argument as in the case above that $\pi_F(M \times 1)$ is not contained in C now yields that W_F is homeomorphic to the disc D^2. As ∂W_F contains four corners, it follows that W_F is diffeomorphic to $[0,1] \times [0,1]$.

□

Remark 9.1.9. Suppose that $m = 2$ in Corollary 9.1.8. Then property (i) is meaningless because 0 is not an admissible value for the absolute index of a fold line. Figure 9.1 gives a counterexample to the conclusion of Corollary 9.1.8.

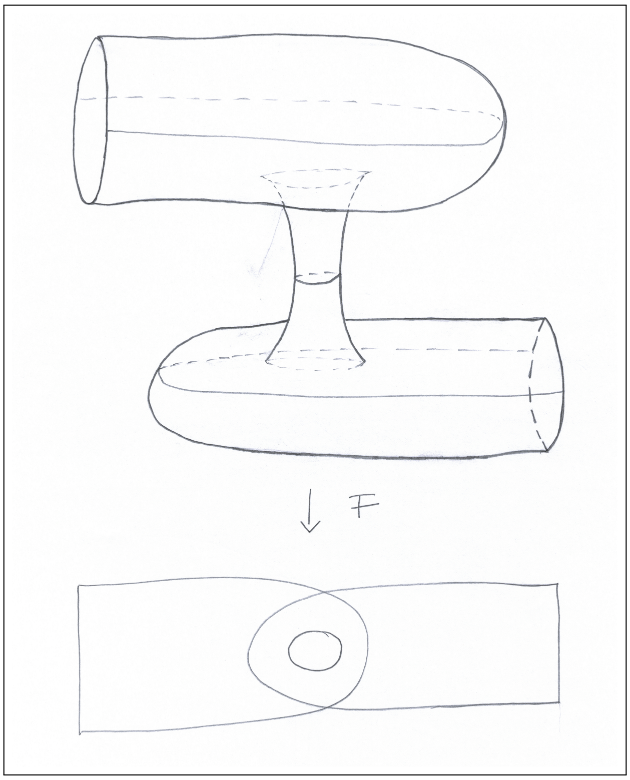

Figure 9.1: A fold map $F\colon S^1 \times [0,1] \to \mathbb{R}^2$

9.2 Local Modification of Stein Factorization

Lemma 9.2.1. *Given* $\varepsilon > \delta > 0$, *there exists a smooth map*

$$F = (F_1, F_2)\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^2, \qquad y = (t, x) = (t, x_1, \dots, x_n) \mapsto F(y),$$

with the following properties:

(i) F *is a special generic map such that* $S(F) = (\mathbb{R} \times 0) \setminus 0 \subset \mathbb{R} \times \mathbb{R}^n$.

(ii) The Stein factorization of F *(see Figure 9.2) is given by*

$$W_F = F(\mathbb{R}^{n+1} \setminus \{0\}) \subset \mathbb{R}^2.$$

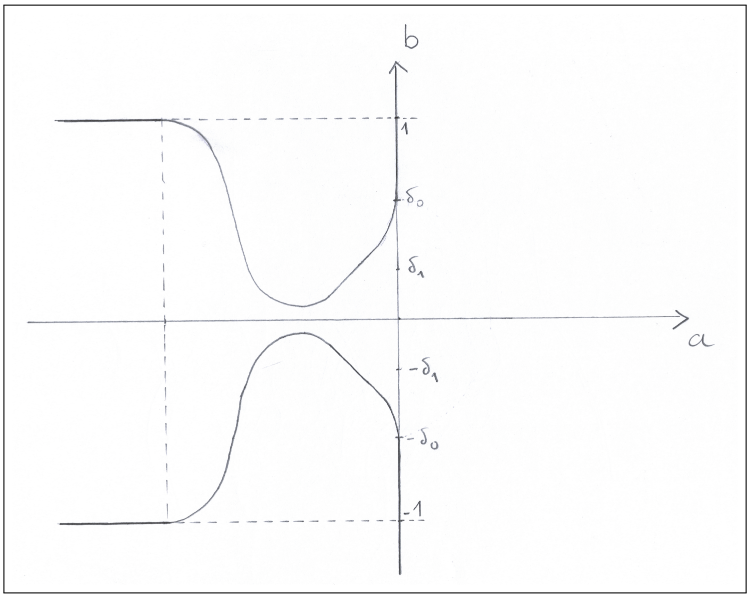

Figure 9.2: Stein factorization of the special generic map $F\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^2$

(iii) $F(y) = \begin{cases} (t, |x|^2), & \text{if } |y| > \varepsilon, \\ (t/|y|, |y| - c), & \text{if } |y| < \delta, \end{cases}$ *for a suitable constant* $c > 0$.

(iv) $\operatorname{sgn}(F_1(t,x)) = \operatorname{sgn}(t)$, *where* $\operatorname{sgn}(s) = \begin{cases} -1, & \text{if } s < 0, \\ 0, & \text{if } s = 0, \\ 1, & \text{if } s > 0, \end{cases}$ *for* $s \in \mathbb{R}$.

Proof. Choose $\varepsilon' \in (\delta, \varepsilon)$ and $\delta' \in (\delta, \varepsilon')$.

Choose smooth maps $\alpha, \beta, \gamma\colon \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\alpha(r) = \begin{cases} 0, & \text{if } r < \varepsilon', \\ 1, & \text{if } r > \varepsilon, \end{cases}$, $\alpha(r) > 0$ for $r > \varepsilon'$, and $\alpha'(r) \geq 0$ for all $r \in \mathbb{R}$. In particular, $\alpha(r) \geq 0$ for all $r \in \mathbb{R}$.
- $\beta(r) = \begin{cases} c - r, & \text{if } r < \varepsilon', \\ 0, & \text{if } r > \varepsilon, \end{cases}$ for some suitable constant $c > 0$, and $\beta'(r) \leq 0$ for all $r \in \mathbb{R}$. In particular, $\beta(r) \geq 0$ for all $r \in \mathbb{R}$.
- $\gamma(r) = \begin{cases} r, & \text{if } r < \delta, \\ 1, & \text{if } r > \delta', \end{cases}$, $\gamma(r) > 0$ for $r > 0$.

Define an open subset of $\mathbb{R}^2$ by

$$U := \{(p, q) \in \mathbb{R};\ p^2 + q > 0\}.$$

Let $\rho\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\rho(p, q) = \sqrt{p^2 + q}$. As $\rho(z) > 0$ for all $z \in U$ and $\gamma(r) > 0$ for all $r > 0$, the following smooth map is well-defined:

$$\Phi\colon U \to \mathbb{R}^2, \qquad z = (p, q) \mapsto \Phi(z) = (p/\gamma(\rho(z)), \alpha(\rho(z)) \cdot q - \beta(\rho(z))).$$

We will show that Φ restricts to an embedding on $D := (\mathbb{R} \times [0, \infty)) \setminus \{0\} \subset U$. For this purpose, define the open cover $U = U_{>\delta'} \cup U_{<\varepsilon'}$ by

$$\begin{aligned} U_{>\delta'} &:= \{z = (p, q) \in U;\ \rho(z) > \delta'\}, \\ U_{<\varepsilon'} &:= \{z = (p, q) \in U;\ \rho(z) < \varepsilon'\}. \end{aligned}$$

The Jacobian of Φ at a point $z = (p, q) \in D \cap U_{>\delta'}$ is of the form

$$J(\Phi, z) = \begin{pmatrix} 1 & * \\ 0 & \frac{\alpha'(\rho(z)) \cdot q}{2\rho(z)} + \alpha(\rho(z)) - \frac{\beta'(\rho(z))}{2\rho(z)} \end{pmatrix}$$

since $\gamma(\rho(z)) = 1$ for all $z \in U_{>\delta'}$. Hence, its determinant is given by

$$\det J(\Phi, z) = \frac{1}{2\rho(z)}(\alpha'(\rho(z)) \cdot q - \beta'(\rho(z))) + \alpha(\rho(z)).$$

Note that $q \geq 0$ (because $z \in D$), $\alpha' \geq 0$ and $-\beta' \geq 0$, $\rho(z) > 0$ and $\alpha(\rho(z)) \geq 0$. Hence, $\det J(\Phi, z) = 0$ implies that $\alpha(\rho(z)) = 0$ and $\beta'(\rho(z)) = 0$, simultaneously. However, $\alpha(\rho(z)) = 0$ implies that $\rho(z) \leq \varepsilon'$ by construction, which yields $\beta'(\rho(z)) = 1$ in contradiction to $\beta'(\rho(z)) = 0$. This shows that Φ restricts to a local diffeomorphism $U' \to \mathbb{R}^2$ on an open neighbourhood U' of $D \cap U_{>\delta'}$ in U.

It suffices to show that Φ is injective on $D \cap U_{>\delta'}$.

If $z = (p, q) \in U_{<\varepsilon'}$, then $\alpha(\rho(z)) = 0$ and $\beta(\rho(z)) = c - \rho(z)$ imply that Φ is of the form

$$\Phi(z) = (p/\rho(z), \rho(z) - c).$$

Define the smooth map

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad \Psi(a, b) = (a(b + c), (b + c)^2(1 - a^2)).$$

Note that if $z = (p, q) \in U_{<\varepsilon'}$, then

$$(\Psi \circ \Phi)(z) = \Psi(p/\rho(z), \rho(z) - c) = (p, \rho(z)^2 - p^2) = (p, q) = z.$$

In particular, $\Phi(U_{<\varepsilon'}) \subset V_{<\varepsilon'}$ for the open subset

$$V_{<\varepsilon'} := (\mathbb{R} \times (-c, \infty)) \cap \Psi^{-1}(U_{<\varepsilon'}) \subset \mathbb{R}^2.$$

By construction, every point $(a,b) \in V_{<\varepsilon'}$ satisfies $\Psi(a,b) \in U_{<\varepsilon'}$, and $b > -c$. Hence,

$$(\Phi \circ \Psi)(a,b) = \Phi(a(b+c),(b+c)^2(1-a^2)) = (a(b+c)/|b+c|, |b+c|-c) = (a,b).$$

Consequently, Φ restricts to a diffeomorphism $U_{<\varepsilon'} \xrightarrow{\cong} V_{<\varepsilon'}$ whose inverse is given by the restriction of Ψ.

All in all, we have shown that Φ restricts to a submersion $D \to \mathbb{R}^2$. Next let us show that Φ is injective on D. (This will imply that Φ restricts to an embedding $D \to \mathbb{R}^2$.) Let $z = (p,q)$ and $z' = (p',q')$ be two points in D such that $\Phi(z) = \Phi(z')$. Setting $r := \rho(z)$ and $r' := \rho(z')$, this implies $p/\gamma(r) = p'/\gamma(r')$ and $\alpha(r) \cdot q - \beta(r) = \alpha(r') \cdot q - \beta(r')$. In order to show $z = z'$ we distinguish between the following three cases:

- $z, z' \in U_{<\varepsilon'}$. As Φ restricts to a diffeomorphism $U_{<\varepsilon'} \xrightarrow{\cong} V_{<\varepsilon'}$, it follows in this case from $\Phi(z) = \Phi(z')$ that $z = z'$.
- Exactly one of the points z and z' is contained in $U_{<\varepsilon'}$, say $z \in U_{<\varepsilon'}$ and $z' \notin U_{<\varepsilon'}$. Hence, $r \in (0,\varepsilon')$ and $r' \geq \varepsilon'$. It follows from $\beta(s) = c - s$ for $s < \varepsilon'$ and $\beta'(s) \leq 0$ for all $s \in \mathbb{R}$ that $\beta(r) > \beta(r')$. Using $\alpha(r) = 0$ (since $r < \varepsilon'$), $\alpha(r') \geq 0$ and $q' \geq 0$ (recall that $z' \in D$), we obtain $-\beta(r) = \alpha(r') \cdot q - \beta(r') \geq -\beta(r')$ in contradiction to $\beta(r) > \beta(r')$.
- $z, z' \notin U_{<\varepsilon'}$. In particular, $r, r' \geq \varepsilon' > \delta'$. Thus, it follows from $\gamma(r) = \gamma(r') = 1$ that $p = p'$. Therefore, it suffices to show that $r = r'$. Without loss of generality, we assume that $r < r'$, i.e. $q < q'$. It follows from $r' > r \geq \varepsilon'$ that $\alpha(r') > 0$. First suppose that $r = \varepsilon'$. In this case, we have $\alpha(r) = 0$. Consequently, using $q' > q \geq 0$ (because $q \in D$), we obtain $-\beta(r) = \alpha(r') \cdot q - \beta(r') > -\beta(r')$ in contradiction to $\beta'(s) \leq 0$ for all $s \in \mathbb{R}$. Therefore, we may assume that $r > \varepsilon'$ in the following. Then, $\alpha(r) > 0$ and $q < q'$ yield $\alpha(r) \cdot q < \alpha(r) \cdot q'$. Finally, the facts $q' \geq 0$ (since $z' \in D$), $\alpha(r) \leq \alpha(r')$ and $-\beta(r) \leq -\beta(r')$ (recall that $\alpha'(s) \geq 0$ and $\beta'(s) \leq 0$ for all $s \in \mathbb{R}$) imply

$$\alpha(r) \cdot q - \beta(r) < \alpha(r) \cdot q' - \beta(r) \leq \alpha(r') \cdot q - \beta(r')$$

  in contradiction to $\alpha(r) \cdot q - \beta(r) = \alpha(r') \cdot q - \beta(r')$. Consequently, $z = z'$ in this case.

Finally, define F to be the composition of Φ with the map $\Delta\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^2$, $\Delta(t,x) = (t,|x|^2)$:

$$F\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^2, \qquad y = (t,x) = (t,x_1,\ldots,x_n) \mapsto F(y) = \Phi(t,|x|^2).$$

Note that this is a well-defined smooth map since the image of Δ is just $D = (\mathbb{R} \times [0,\infty)) \setminus \{0\}$. It remains to check the desired properties (i) to (ii):

(i). This follows from $F = \Psi \circ \Delta$ since the special generic map $\Delta\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^2$, $\Delta(t,x) = (t,|x|^2)$, satisfies $S(\Delta) = (\mathbb{R} \times 0) \setminus 0 \subset \mathbb{R} \times \mathbb{R}^n$, and Φ restricts to an embedding on $D = \Delta(\mathbb{R}^{n+1} \setminus \{0\})$.

(ii). The Stein factorization of Δ is given by $D = (\mathbb{R} \times [0,\infty)) \setminus \{0\}$, the upper half-plane minus the origin. Its boundary $\mathbb{R} \setminus \{0\}$ is transformed by the embedding $\Phi|_D$ to the boundary shown in Figure 9.2.

(iii). If the point $y = (t,x) = (t,x_1,\ldots,x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ satisfies $\rho(t,|x|^2) = \sqrt{t^2+|x|^2} =$

$|y| > \varepsilon$, then $\alpha(|y|) = 1$, $\beta(|y|) = 0$ and $\gamma(|y|) = 1$ imply that

$$F(y) = \Phi(t, |x|^2) = (t/\gamma(|y|), \alpha(|y|) \cdot |x|^2 - \beta(|y|)) = (t, |x|^2).$$

If $\rho(t, |x|^2) = |y| < \delta$, then $\alpha(|y|) = 0$, $\beta(|y|) = c - |y|$ and $\gamma(|y|) = |y|$ imply that

$$F(y) = \Phi(t, |x|^2) = (t/\gamma(|y|), \alpha(|y|) \cdot |x|^2 - \beta(|y|)) = (t/|y|, |y| - c).$$

(iv). It suffices to note that $F_1(y) = t/\gamma(|y|)$ for all $y = (t, x) = (t, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$.

□

Lemma 9.2.1 has the following consequences which we formulate in two Propositions:

Proposition 9.2.2. *Let (W, M_1, M_2) be a smooth manifold triad of dimension $m := \dim W \geq 3$. Suppose that $F\colon W \to \mathbb{R}^2$ is a fold map with the poperties (i) and (ii) of Theorem 9.1.7. Let $w, w' \in W \setminus \partial W$, $w \neq w'$, be two definite fold points of F. Let U and U' be open chart neighbourhoods of w and w' diffeomorphic to the open ball $\{y \in \mathbb{R}^m;\ |y| < \varepsilon\} \subset \mathbb{R}^m$ for some $\varepsilon > 0$, and let V, V' be open chart neighbourhoods of $F(w)$ and $F(w')$ in $\mathbb{R}^2$ that are orientation preservingly diffeomorphic to open neighbourhoods of the origin in $\mathbb{R}^2$, such that $F(U) \subset V$ and $F(U') \subset V'$, and in each of these coordinates F takes the form $(t, x) \mapsto (t, |x|^2)$. Set $W_0 := W \setminus (U \cup U')$, and let $W' := W/\sim$ denote the smooth manifold obtained by 0-surgery, identifying $r \cdot y_0 \sim (\delta - r) \cdot y_0$, $r \in (0, \delta)$, $|y_0| = 1$, in the balls U and U' for some $\delta > 0$. Then there exists a fold map $F'\colon W' \to \mathbb{R}^2$ such that $F'|_{W_0} = F|_{W_0}$, $F'|_{W' \setminus W_0}$ is a fold map with only definite fold lines, and $W'_{F'}$ is obtained from W_F by attaching a 1-handle between the points $F(w)$ and $F(w')$ in W_F.*

Proof. Consider the Stein factorization $\pi_F\colon W \to W_F$ of F. Apply Lemma 9.2.1 for $\varepsilon > \delta > 0$ to replace π_F in the given local coordinates on $U \setminus \{w\}$ and $U' \setminus \{w'\}$ with the map constructed there. By construction, the modified map π_F takes the form

$$(t, x) = y = r \cdot y_0 \mapsto (t/r, r - c) = ((y_0)_1, r - c), \qquad r \in (0, \delta), |y_0| = 1,$$

in the given local coordinates around w and w'. Hence, when identifying $r \cdot y_0 \sim (\delta - r) \cdot y_0$, one can also identify $((y_0)_1, r - c) \sim ((y_0)_1, (\delta - c) - r)$ in $\tilde{W} := W_F \setminus \{\pi_f(w), \pi_F(w')\}$ to obtain a well-defined smooth map $W' \to \tilde{W}/\sim$. Note that $\tilde{W}/\sim$ is the result of attaching a 1-handle between the points $F(w)$ and $F(w')$ in W_F. Furthermore, $\tilde{W}/\sim$ can be identified to be the Stein factorization $W'_{F'}$ of a suitable extension of $F|_{W_0}$ to a fold map $F'\colon W' \to \mathbb{R}^2$ such that $F'|_{W' \setminus W_0}$ is a fold map with only definite fold lines. □

Proposition 9.2.3. *Fix integers $n \geq 6$ and $l \in \{2, \dots, \lceil \frac{n}{2} \rceil - 1\}$. For $i = 1, 2$ let $f_i \colon M_i^n \to \mathbb{R}$ be an excellent Morse function without critical points of index in $\{1, \dots, l-1\} \cup \{n-l+1, \dots, n-1\}$ on a connected closed smooth n-dimensional manifold M_i^n. (In particular, Lemma C.0.4 (a) implies that M_1 and M_2 are orientable. Moreover, by Lemma C.0.4 (b), f_i has for $i = 1, 2$ exactly one critical point of index 0 and exactly one critical point of index n.) For $i = 1, 2$ and $\lambda \in \{0, \dots, n\}$ let $\nu_i^{(\lambda)}$ denote the number of critical points of f_i of index λ.*

There exists a compact smooth manifold W^{n+1} of dimension $n+1$ with boundary diffeomorphic to $\partial W \cong (-M_1) \sharp M_1 \sharp M_2 \sharp (-M_2)$, and a fold map $F \colon W \to \mathbb{R}^2$ with the following properties:

(i) For suitable $\varepsilon > 0$ there exists a collar neighbourhood $\partial W \times [0, \varepsilon)$ of $\partial W \times 0 = \partial W$ in W on which F is of the form

$$F(x, t) = (f_t(x), t), \qquad (x, t) \in \partial W \times [0, \varepsilon),$$

where $f_t \colon \partial W \to \mathbb{R}$ is an excellent Morse function for all $t \in [0, \varepsilon)$.

(ii) The number $\nu_0^{(\lambda)}$ of critical points of f_0 of index $\lambda \in \{0, \dots, n\}$ is given by

$$\nu_0^{(\lambda)} = \nu_1^{(\lambda)} + \nu_1^{(n-\lambda)} + \nu_2^{(\lambda)} + \nu_2^{(n-\lambda)}.$$

(iii) All fold lines of F have absolute index in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and the Stein factorization W_F of F (see Theorem 9.1.7) is diffeomorphic to the half-disc

$$\{(x, y) \in \mathbb{R}^2,\ x^2 + y^2 \leq 1,\ x \geq 0\}.$$

Proof. As all Stein factorizations considered in this proof coincide with the images of the underlying fold maps in the plane, we will not make a formal distinction.

For $i = 1, 2$ consider the fold map

$$F_i := f_i \times \mathrm{id}_{[0,1]} \colon W_i := M_i \times [0, 1] \to \mathbb{R}^2.$$

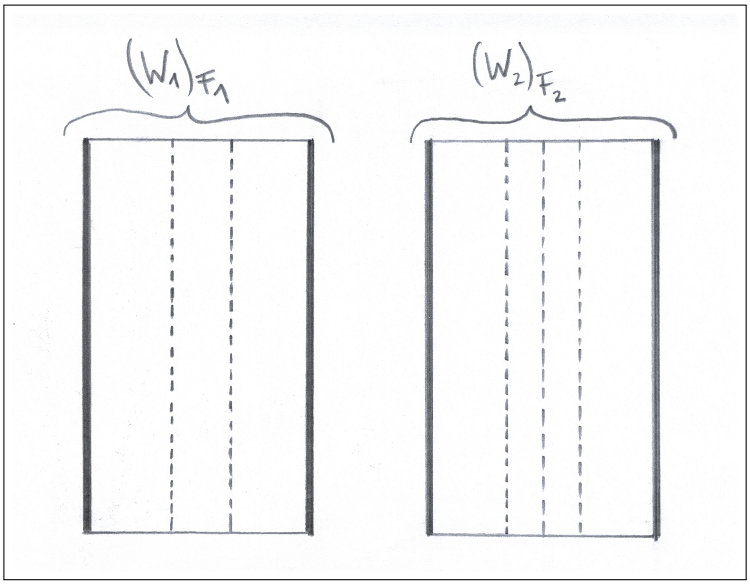

Figure 9.3: Stein factorizations $(W_i)_{F_i}$, $i = 1, 2$

Figure 9.3 shows their Stein factorizations $(W_i)_{F_i} \cong f_i(M_i) \times [0, 1] \subset \mathbb{R}^2$ (see Theorem 9.1.7), where the bold parts of the boundary represent the image of the definite fold lines of F_i in W_{F_i}, and the dashed lines indicate the image of the indefinite fold lines of F_i in W_{F_i}. (The same conventions will hold in all figures featuring Stein factorization.)

Let $V := W_1 \natural W_2$ denote the the boundary connected sum of W_1 and W_2 with respect to the boundary components $M_i \times 0$ of W_i for $i = 1, 2$ (see [27, Addendum, pp. 507-508]). Note that V has three boundary components, and these are diffeomorphic to $-M_1$, $-M_2$ and $M_1 \sharp M_2$. Using Lemma 9.2.1 (iv), arguments analogous to those in the proof of Proposition 9.2.2 show that one can construct a fold map $G\colon V \to \mathbb{R}^2$ whose Stein factorization $V_G \subset \mathbb{R}^2$ is given by Figure 9.4. In particular, all fold lines of G have absolute index in $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$.

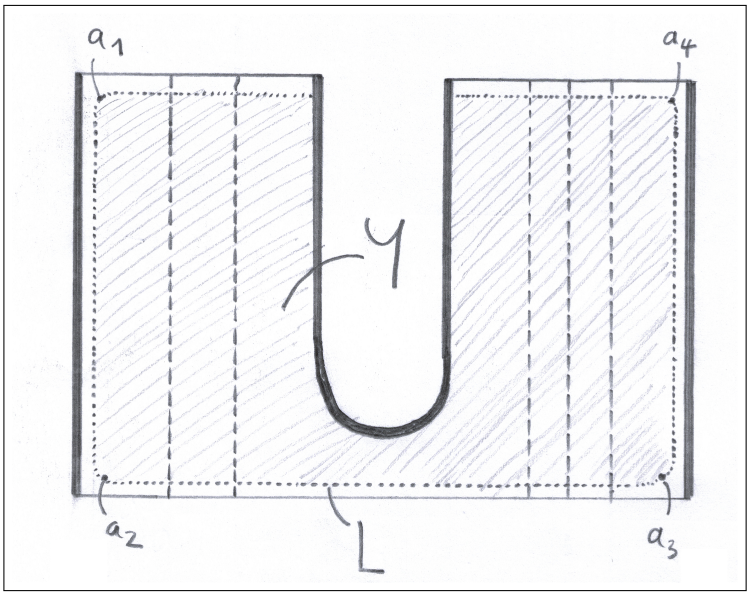

Figure 9.4: Cutting the Stein factorization V_G along L

We cut V_G along the dotted smooth curve $L \cong [0,1]$ indicated in Figure 9.4. As L is transverse to $G(S(G))$, the preimage $G^{-1}(L)$ is a closed smooth submanifold of V that is diffeomorphic to $(-M_1)\sharp M_1 \sharp M_2 \sharp (-M_2)$. (In fact, fix a diffeomorphism $L \cong [0,1]$ and introduce four points $a_1, \ldots, a_4 \in [0,1]$ as indicated in Figure 9.4. Then there are diffeomorphisms

$$G^{-1}([0,a_2)) \cong (-M_1) \setminus \mathrm{pt}, \qquad G^{-1}((a_3,1]) \cong (-M_2) \setminus \mathrm{pt},$$
$$G^{-1}((a_1,a_4)) \cong (M_1 \sharp M_2) \setminus \{\mathrm{pt}, \mathrm{pt}'\},$$
$$G^{-1}((a_1,a_2)) \cong G^{-1}((a_3,a_4)) \cong (0,1) \times S^{n-1}.$$

Finally, note that the direct sum operation is associative up to orientation preserving diffeomorphism.) The result of cutting V_G along L is evidently a smooth 2-dimensional manifold with corners with two components. Let Y denote the component that corresponds to the shaded region in Figure 9.4. Note that $W := G^{-1}(Y)$ is a compact smooth manifold of dimension $n+1$ with boundary $\partial W \cong (-M_1)\sharp M_1 \sharp M_2 \sharp (-M_2)$. Furthermore, G restricts to a fold map $F := G|_W \colon W \to \mathbb{R}^2$ whose Stein factorization W_F is diffeomorphic to Y. It is obvious from Figure 9.5 that Y satisfies property (iii). As L is transverse to $G(S(G))$, there exists by Lemma 9.1.5 a collar $\partial W \times [0,\varepsilon)$ of $\partial W \times 0 = \partial W \subset W$ such that $x \mapsto F_1(x,t) := f_t(x)$ is an excellent Morse function for all $t \in [0,\varepsilon)$ and $F_2(x,t) = t$ for all $(x,t) \in \partial W \times [0,\varepsilon)$, which proves (i). By choice of L, it is clear that the numbers of critical points of f_0 are as required in (ii).

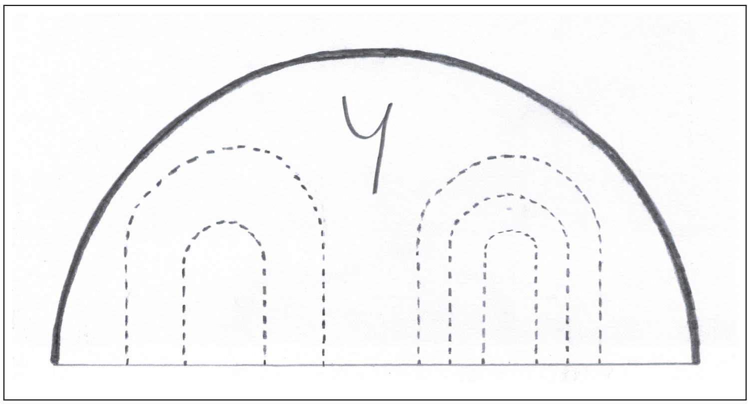

Figure 9.5: $W_F \cong Y$ is diffeomorphic to the half-disc

□

The following result implements the gluing of fold maps with suitable boundary conditions, which is needed in the proof of the inclusion $G_n^l \subset \overline{C}_n^{l-1}$ (see Section 10.3.2):

Theorem 9.2.4. *Let* (W_i, M_i, N_i), $i = 1, 2$, *be smooth manifold triads of the same dimension* $m := \dim W_i \geq 8$. *Suppose that* N_1 *and* M_2 *are connected and diffeomorphic to each other.*

Let $F_i \colon W_i \to \mathbb{R}^2$, $i = 1, 2$, *be fold maps with only fold lines of absolute index contained in* $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1-l\} \cup \{m-1\}$ *for some* $l \in \{2, \ldots, \lfloor \frac{m}{2} \rfloor - 1\}$ *such that, for suitable* $\varepsilon > 0$ *and* $i = 1, 2$, *there exist collar neighbourhoods* $M_i \times [0, \varepsilon)$ *and* $N_i \times (1-\varepsilon, 1]$ *of* $M_i \times 0 = M_i$ *and* $N_i \times 1 = N_i$ *in* W_i *on which* F_i *is of the form*

$$
\begin{aligned}
F_i(x,t) &= ((f_i)_t(x), t), \qquad && (x,t) \in M_i \times [0, \varepsilon), \\
F_i(x,t) &= ((g_i)_t(x), t), \qquad && (x,t) \in N_i \times (1-\varepsilon, 1],
\end{aligned}
$$

where $(f_i)_t$ *is an excellent Morse function* $M_i \to \mathbb{R}$ *for all* $t \in [0, \varepsilon)$ *and* $(g_i)_t$ *is an excellent Morse function* $N_i \to \mathbb{R}$ *for all* $t \in (1-\varepsilon, 1]$.

Suppose that $(g_1)_1$ *and* $(f_2)_0$ *have the same number of critical points of index* λ *for every* $\lambda \in \{0, \ldots, m-1\}$.

Let $h \colon N_1 \xrightarrow{\cong} M_2$ *be a diffeomorphism and let* $W := W_1 \cup_h W_2$ *be the gluing along the boundary. Suppose that* W *is equipped with the unique smoothness structure (see [41, Lemma 3.7, p. 26]) determined by the requirement that* $M := h(N_1) = M_2 \subset W$ *has a closed tubular neighbourhood* $M \times [-\varepsilon/2, \varepsilon/2] \subset W$ *such that the canonical embeddings* $\varphi_i \colon W_i \to W$ *satisfy*

$$
\begin{aligned}
\varphi_1(n_1, t) &= (h(n_1), t-1), \qquad && (n_1, t) \in N_1 \times [1-\varepsilon/2, 1] \ (\subset W_1), \\
\varphi_2(m_2, t) &= (m_2, t), \qquad && (m_2, t) \in M_2 \times [0, \varepsilon/2] \ (\subset W_2).
\end{aligned}
$$

Then the fold maps F_i *can be "glued" to a fold map* $F \colon W \to \mathbb{R}^2$ *with the following properties:*

(i) $F \circ \varphi_1|_{W_1 \setminus (N_1 \times [1-\varepsilon/4, 1])} = F_1|_{W_1 \setminus (N_1 \times [1-\varepsilon/4, 1])}$ *and* $F \circ \varphi_2|_{W_2 \setminus (M_2 \times [0, \varepsilon/4])} = F_2|_{W_2 \setminus (M_2 \times [0, \varepsilon/4])}$.

(ii) The Stein factorizations behave under gluing as $W_F \cong (W_1)_{F_1} \cup_{F_1(N_1) \cong F_2(M_2)} (W_2)_{F_2}$.

(iii) F *has only fold lines of absolute index contained in* $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1-l\} \cup \{m-1\}$.

Proof. Property (i) tells us how to define F on the open subset $W \setminus (M \times [-\varepsilon/4, \varepsilon/4])$ of W.

Therefore, it suffices to construct a fold map

$$F_M\colon M \times [-\varepsilon/2, \varepsilon/2] \to \mathbb{R}^2$$

with the following properties:

$(i)'$ $F_M \circ \varphi_1|_{N_1 \times [1-\varepsilon/2, 1-\varepsilon/8)} = F_1|_{N_1 \times [1-\varepsilon/2, 1-\varepsilon/8)}$ and $F_M \circ \varphi_2|_{M_2 \times (\varepsilon/8, \varepsilon/2]} = F_2|_{M_2 \times (\varepsilon/8, \varepsilon/2]}$.
$(ii)'$ The Stein factorization of F_M is diffeomorphic to $[0,1] \times [-\varepsilon/2, \varepsilon/2]$.
$(iii)'$ F_M has only fold lines of absolute index contained in $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1-l\} \cup \{m-1\}$.

The desired fold map F will then be given by

$$F\colon W \to \mathbb{R}^2, \qquad F(w) = \begin{cases} F_M(w), & \text{if } w \in M \times [-\varepsilon/2, \varepsilon/2] \ (\subset W), \\ F_1(\varphi_1^{-1}(w)), & \text{if } w \in \varphi_1(W_1 \setminus (N_1 \times [1-\varepsilon/8, 1])), \\ F_2(\varphi_2^{-1}(w)), & \text{if } w \in \varphi_2(W_2 \setminus (M_2 \times [0, \varepsilon/8])). \end{cases}$$

(In fact, note that F is a well-defined smooth map by property $(i)'$. Moreover, F is a fold map that satisfies property (iii) because F_M is a fold map that satisfies property $(iii)'$ and F_i, $i = 1, 2$, are by assumption fold maps with only fold lines of absolute index contained in $\{\lceil \frac{m-1}{2} \rceil, \ldots, m-1-l\} \cup \{m-1\}$. Furthermore, observe that F satisfies property (i) by construction. Finally, property (ii) follows from property $(ii)'$ and the definition of Stein factorization.)

By Theorem 8.0.1 there exists a generic smooth map

$$G_M\colon M \times [-\varepsilon/2, \varepsilon/2] \to \mathbb{R} \times [-\varepsilon/2, \varepsilon/2]$$

such that $G_M \circ \varphi_1|_{N_1 \times [1-\varepsilon/2, 1-\varepsilon/8)} = F_1|_{N_1 \times [1-\varepsilon/2, 1-\varepsilon/8)}$ and $G_M \circ \varphi_2|_{M_2 \times (\varepsilon/8, \varepsilon/2]} = F_2|_{M_2 \times (\varepsilon/8, \varepsilon/2]}$, and such that the absolute indices of indefinite fold points of G_M are all contained in the set $\{\lfloor \frac{m}{2} \rfloor, \ldots, m-1-l\} \cup \{m-1\}$. The generic smooth map G_M has an even number of cusps by Corollary 6.2.4. As the Morse functions g_1 and f_2 have the same number of critical points of index λ for every $\lambda \in \{0, \ldots, m-1\}$ by assumption, we may eliminate all cusps by Proposition 6.1.3 to obtain the desired fold map F_M. Indeed, properties $(i)'$ and $(iii)'$ are satisfied by construction. Hence, property $(ii)'$ follows from cylindrical rigidity of definite fold lines (see Corollary 9.1.8).

□

Chapter 10

Detecting Exotic Spheres via Indefinite Folds

10.1 Introduction and Statement of Results

In recent years, various results in the literature have been pointing to a deep relationship between surgery theory and the theory of singularities of smooth mappings. This perspective is spectacularly underlined by a theorem due to Saeki [47] (see Theorem 10.1.1 below) which provides a link between the study of exotic differentiable structures on spheres and the theory of fold maps with only definite fold singularities, also known as special generic maps. Going further, the purpose of the present chapter is to investigate in this context the role that is played by indefinite fold singularities for the detection of exotic spheres.

Recall that a *special generic function* on a closed smooth manifold P of dimension ≥ 1 is a Morse function $f\colon P \to \mathbb{R}$ with only minima and maxima. Note that the existence of a special generic function on P is a strong condition which already implies that P is *homeomorphic* to a finite disjoint union of standard spheres (see [47, Lemma 2.2]). Furthermore, a smooth map $F\colon Q \to \mathbb{R}^2$ defined on a smooth manifold Q (possibly with boundary) of dimension ≥ 2 is called a *fold map* if for every singular point q of F in Q there exist suitable coordinate systems around q and $F(q)$ in which F takes the form

$$(t, x_1, \dots, x_n) \mapsto (t, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2),$$

where $n+1$ denotes the dimension of Q and $i \in \{0, \dots, n\}$ is a suitable integer. (If q lies on the boundary of Q, then we require F to be in some boundary chart around q the restriction to the upper half space of a fold map defined on an open subset of $\mathbb{R}^{n+1}$.) For $\partial Q = \emptyset$ it is a well-known fact that the singular locus of the fold map F is a 1-dimensional smooth submanifold of Q that is closed as a subset. The *absolute index* of F at the singular point q,

$$\max(i, n-i) \in \{\lceil \frac{n}{2} \rceil, \dots, n\},$$

is intrinsically defined and is constant along components of the singular locus (so-called “fold lines”). Fold lines of absolute index n are referred to as *definite* fold lines, otherwise they are called *indefinite* fold lines. Finally, F is a *special generic map* if all of its fold lines are definite.

Throughout the present chapter, let $n \geq 6$ denote a fixed integer. The protagonists of Saeki's theorem are two abelian groups dependent on n that we shall introduce next.

Let Θ_n denote the well-known *group of homotopy n-spheres* as defined for instance in [27]. Its elements are represented by oriented homotopy n-spheres subject to the equivalence relation of h-cobordism. The group law is given by taking the connected sum of two representatives, the identity element is the class of the standard sphere with its standard orientation, and inverses are given by reversing the orientation of a representative. This makes Θ_n into an abelian group.

The *(oriented) cobordism group of special generic functions* $\tilde{\Gamma}(n,1)$ is defined as the set

$$\begin{aligned}\tilde{\Gamma}(n,1) := \{(M^n, f);\ & M^n \text{ closed oriented smooth manifold of dimension } n,\\ & f\colon M \to \mathbb{R} \text{ special generic function}\}/\sim,\end{aligned}$$

where two admissible pairs (M_1^n, f_1) and (M_2^n, f_2) are equivalent, $(M_1^n, f_1) \sim (M_2^n, f_2)$, if there exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = M_1 \sqcup -M_2$, and a special generic map $F\colon W \to \mathbb{R}^2$ such that

$$\begin{aligned}F|_{M_1 \times [0,\varepsilon)} &= f_1 \times \mathrm{id}_{[0,\varepsilon)}\colon M_1 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),\\ F|_{M_2 \times (1-\varepsilon,1]} &= f_2 \times \mathrm{id}_{(1-\varepsilon,1]}\colon M_2 \times (1-\varepsilon,1] \to \mathbb{R} \times (1-\varepsilon,1],\end{aligned}$$

where $M_1 \times [0,\varepsilon)$ and $M_2 \times (1-\varepsilon, 1]$ are collar neighbourhoods of $M_1 \times 0 = M_1$ and $M_2 \times 1 = M_2$ in W for suitable $\varepsilon > 0$. (Note that $\sim$ is indeed an equivalence relation: symmetry is immediate from the definition, transitivity is ensured by the form of F near the boundaries, and reflexivity $(M^n, f) \sim (M^n, f)$ is obtained by considering the suspension $F = f \times \mathrm{id}_{[0,1]}$ on the cylinder $W = M \times [0,1]$.) The composition law is given by disjoint union, $[(M_1^n, f_1)] + [(M_2^n, f_2)] := [(M_1^n \sqcup M_2^n, f_1 \sqcup f_2)]$ (observe that the manifolds in admissible pairs are not required to be connected), the identity element is the class $[(\emptyset, f_\emptyset)]$ of the unique function $f_\emptyset$ from the empty set to $\mathbb{R}$, and inverses are given by $-[(M^n, f)] := [(-M^n, -f)]$. This gives $\tilde{\Gamma}(n,1)$ the structure of an abelian group.

Saeki's theorem now provides an isomorphism between these two groups:

Theorem 10.1.1. *For $n \geq 6$, the group homomorphism*

$$\Phi\colon \Theta_n \xrightarrow{\cong} \tilde{\Gamma}(n,1)$$

given by $\Phi([\Sigma^n]) = [(\Sigma^n, f)]$ for any choice of special generic map $f\colon \Sigma \to \mathbb{R}$ is an isomorphism.

Let us remark that the tilde in the notation for $\tilde{\Gamma}(n,1)$ reflects the fact that the manifolds in admissible pairs are equipped with an orientation. Neglecting orientations, there exists an unoriented version of Saeki's theorem involving the analogously defined cobordism group $\Gamma(n,1)$ of special generic functions, but our focus lies on the oriented version.

Philosophically, Theorem 10.1.1 asserts that special generic maps are closely related to the study of exotic differentiable structures on manifolds (see [47, Remark 3.6]). One might wonder whether indefinite fold lines do also measure any interesting features related to exotic smooth structures. And if so, what kind of information about exotic spheres do they detect? The present chapter is devoted to give first answers to this type of questions which recently came to the fore in the context of the so-called *aggregate invariant* that has been defined by Banagl within his framework of positive TFTs [4]. In the light of Theorem 10.1.1, the aggregate invariant is designed to distinguish the standard sphere from exotic spheres.

The major idea is now to relax the equivalence relation $\sim$ that occurs in the definition of the group $\tilde{\Gamma}(n,1)$ by means of an additional parameter

$$l \in \{1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$$

which controls the permitted values for the absolute index of fold lines, and to use the resulting equivalence relation $\sim_l$ to define the set

$$\begin{aligned} \tilde{\Gamma}_l(n,1) := \{(M^n, f);\ & M^n \text{ closed oriented smooth manifold of dimension } n, \\ & f\colon M \to \mathbb{R} \text{ special generic function}\}/\sim_l . \end{aligned}$$

More precisely, two admissible pairs (M_1^n, f_1) and (M_2^n, f_2) are equivalent, $(M_1^n, f_1) \sim_l (M_2^n, f_2)$, if there exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = M_1 \sqcup -M_2$, and a fold map $F\colon W \to \mathbb{R}^2$ whose fold lines are now allowed to have their absolute index contained in $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$ (this is the only novelty compared to the definition of $\sim$), and such that

$$\begin{aligned} F|_{M_1 \times [0,\varepsilon)} &= f_1 \times \mathrm{id}_{[0,\varepsilon)}\colon M_1 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon), \\ F|_{M_2 \times (1-\varepsilon,1]} &= f_2 \times \mathrm{id}_{(1-\varepsilon,1]}\colon M_2 \times (1-\varepsilon,1] \to \mathbb{R} \times (1-\varepsilon,1], \end{aligned}$$

where $M_1 \times [0,\varepsilon)$ and $M_2 \times (1-\varepsilon,1]$ are suitable collar neighbourhoods of $M_1 \times 0 = M_1$ and $M_2 \times 1 = M_2$ in W for some $\varepsilon \in (0,1/2)$. (Note that $\sim_l$ is indeed an equivalence relation because symmetry, transitivity and reflexivity hold for the same reasons as for $\sim$. As far as reflexivity is concerned, note that the suspension of a special generic map is still an $\sim_l$-admissible fold map on the cylinder!) The definition of the group structure on $\tilde{\Gamma}_l(n,1)$ is literally the same as that on $\tilde{\Gamma}(n,1)$: the composition law is given by disjoint union, $[(M_1^n, f_1)] + [(M_2^n, f_2)] := [(M_1^n \sqcup M_2^n, f_1 \sqcup f_2)]$, the identity element is the class $[(\emptyset, f_\emptyset)]$ of the unique function $f_\emptyset$ from the empty set to $\mathbb{R}$, and inverses are given by $-[(M^n, f)] := [(-M^n, -f)]$.

Remark 10.1.2. For the lowest value $l = 1$ results by Ikegami [24] imply that $\tilde{\Gamma}_1(n,1) = 0$ (see Proposition 10.3.1). However, the calculation of the groups $\tilde{\Gamma}_l(n,1)$ for general l has the difficulty that it requires the construction of fold maps into the plane whose fold lines have absolute indices in a prescribed set. Well-known methods from the literature such as Eliashberg's machinery [14] are not applicable here since they produce fold maps with no control over the set of occuring absolute indices.

Observe that decreasing the parameter l makes $\sim_l$ coarser, i.e. if $l' < l$, then $(M_1^n, f_1) \sim_l (M_2^n, f_2)$ implies $(M_1^n, f_1) \sim_{l'} (M_2^n, f_2)$. Hence, the highest value $l = \lceil \frac{n}{2} \rceil - 1$ corresponds to the finest of the equivalence relations $\sim_l$. For the choice $l = \lceil \frac{n}{2} \rceil - 1$ one obtains

$$\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\} = \begin{cases} \{k, k+1\} \cup \{n\}, & n = 2k \text{ even}, \\ \{k+1\} \cup \{n\}, & n = 2k+1 \text{ odd}. \end{cases}$$

In particular, if $n = 2k$ is even, then the set $\{k\} \cup \{n\}$ is not included (it would have corresponded to the choice $l = \lceil \frac{n}{2} \rceil = k$).

As $(M_1^n, f_1) \sim (M_2^n, f_2)$ implies $(M_1^n, f_1) \sim_l (M_2^n, f_2)$ for all $l \in \{1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, there is for every such l a natural epimorphism of abelian groups given by

$$\pi_l \colon \tilde{\Gamma}(n,1) \twoheadrightarrow \tilde{\Gamma}_l(n,1), \qquad \pi_l([(M^n, f)]) = [(M^n, f)],$$

whose kernel

$$\ker \pi_l = \{[(M^n, f)] \in \tilde{\Gamma}(n,1);\ (M^n, f) \sim_l (\emptyset, f_\emptyset)\}$$

corresponds under the isomorphism Φ of Theorem 10.1.1 to the following subgroup of Θ_n:

$$G_n^l := \{[\Sigma^n] \in \Theta_n; (\Sigma^n, f) \sim_l (\emptyset, f_\emptyset) \text{ for some (any) choice of special generic function } f \colon \Sigma \to \mathbb{R}\}.$$

(Note that the condition for an element $[\Sigma^n] \in \Theta_n$ to lie in G_n^l does not depend on the chosen special generic function $f \colon \Sigma \to \mathbb{R}$. Indeed, if $f, g \colon \Sigma \to \mathbb{R}$ are special generic functions, then $(\Sigma^n, f) \sim (\Sigma^n, g)$ by [46, Lemma 3.1, p. 4]. Hence, $(\Sigma^n, f) \sim_l (\Sigma^n, g)$. Consequently, $(\Sigma^n, f) \sim_l (\emptyset, f_\emptyset)$ if and only if $(\Sigma^n, g) \sim_l (\emptyset, f_\emptyset)$.) Explicitly, an element $[\Sigma^n] \in \Theta_n$ lies in G_n^l if and only if there exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ whose fold lines are allowed to have an abosolute index contained in $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$, and such that there exists a special generic function $f \colon \Sigma \to \mathbb{R}$ with

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$

where $\Sigma \times [0, \varepsilon)$ is a collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for suitable $\varepsilon > 0$.
As $G_n^l \subset G_n^{l-1}$ for all $l \in \{2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, we have a filtration of Θ_n by subgroups

$$G_n^{\lceil \frac{n}{2} \rceil - 1} \subset \cdots \subset G_n^2 \subset G_n^1 \subset \Theta_n.$$

In order to study this filtration whose definition is strongly related to fold maps into the plane, we introduce two more filtrations of Θ_n by subgroups. For every $l \in \{0, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ we define

$$C_n^l := \{[\Sigma^n] \in \Theta_n; \exists \text{ compact oriented smooth manifold } W^{n+1} \text{ of dimension } n+1 \text{ such that } \partial W = \Sigma \text{ and } W \text{ is } l\text{-connected}\},$$

$$\overline{C}_n^l := \{[\Sigma^n] \in \Theta_n; \exists \text{ compact oriented smooth manifold } W^{n+1} \text{ of dimension } n+1 \text{ such that } \partial W = \Sigma,\ W \text{ is } l\text{-connected and } \chi(W) \equiv 1 \pmod 2\}.$$

Observe that these are indeed subgroups of Θ_n. To show this for C_n^l, observe that $[S^n] \in C_n^l$ since the standard sphere bounds the standard $(n+1)$-ball which is contractible. Moreover, if $[\Sigma^n] \in C_n^l$, then $-[\Sigma^n] = [-\Sigma^n] \in C_n^l$ because if Σ^n bounds W, then $-\Sigma^n$ bounds $-W$. Finally, to see that C_n^l is closed under composition, let Σ_i^n be for $i = 1, 2$ a homotopy n-sphere that bounds an l-connected compact oriented smooth manifold W_i^{n+1} of dimension $n+1$. By [27, Addendum, pp. 507-508] the connected sum $\Sigma := \Sigma_1 \sharp \Sigma_2$ is bounded by the boundary connected sum $W := W_1 \natural W_2$. By construction, W is a compact oriented smooth manifold with $\partial W = \Sigma$. Moreover, W is l-connected being homotopy equivalent to $W_1 \vee W_2$. (Indeed, first use the Seifert van Kampen theorem to show that W is simply connected. Afterwards, use [20, Corollary 2.25, p. 126] to show that the homology groups of W vanish up to degree l. Finally, the Hurewicz theorem implies that W is l-connected.) The same reasoning can be applied to show that $\overline{C}_n^l$ is a subgroup of Θ_n. In fact, one only has to check in addition that W has odd Euler characteristic if this is true for W_1 and W_2. As $W \simeq W_1 \vee W_2$, this follows from the formula $\chi(W) = \chi(W_1) + \chi(W_2) - \chi(W_1 \cap W_2)$ (see [20, Exercise 21, p. 157]).

Therefore, we have filtrations of Θ_n by subgroups

$$C_n^{\lceil \frac{n}{2} \rceil - 1} \subset \cdots \subset C_n^1 \subset C_n^0 \subset \Theta_n,$$
$$\overline{C}_n^{\lceil \frac{n}{2} \rceil - 1} \subset \cdots \subset \overline{C}_n^1 \subset \overline{C}_n^0 \subset \Theta_n.$$

Now we are prepared to state the main theorem of the present chapter:

Theorem 10.1.3. *For $n \geq 6$ every $l \in \{1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ we have $\overline{C}_n^l \subset G_n^l \subset \overline{C}_n^{l-1}$ in Θ_n.*

Theorem 10.1.3 has the following non-trivial application to a concrete family of homotopy spheres that can be obtained from a plumbing construction (see [30, p. 162]), namely the *Kervaire spheres*. Recall that Kervaire spheres occur in dimensions of the form $n \equiv 1 \pmod 4$, and there is a unique Kervaire sphere of dimension n for every such dimension. For $n \equiv 1 \pmod 4$ it is well-known (see [35, Corollary 6.43, p. 136]) that

$$bP_{n+1} \cong \begin{cases} \mathbb{Z}/2, & \text{if } n \neq 2^j - 3 \text{ for all integers } j, \\ 0, & \text{if } n \in \{5, 13, 29, 61\}. \end{cases}$$

If $bP_{n+1} \cong \mathbb{Z}/2$, then the unique exotic sphere in bP_{n+1} is the n-dimensional Kervaire sphere. If $bP_{n+1} \cong 0$, then the Kervaire sphere of dimension n is diffeomorphic to the standard sphere.

From the perspective of fold maps, Theorem 10.1.3 implies the following result (see Section 10.4).

Corollary 10.1.4. *Suppose that the integer n satisfies $n \equiv 13 \pmod{16}$ and $n \geq 237$. If $bP_{n+1} \cong \mathbb{Z}/2$ (this holds whenever $n \neq 2^j - 3$ for all integers j), then for any exotic n-sphere Σ^n (i.e. $[\Sigma^n] \neq [S^n]$ in Θ_n) the following statements are equivalent:*

(i) Σ^n is the Kervaire sphere of dimension n, i.e. the unique exotic sphere in bP_{n+1}.

(ii) There exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ with $S(F)$ having a single closed component, namely of absolute index $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and such that there exists a special generic function $f \colon \Sigma \to \mathbb{R}$ with

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$

where $\Sigma \times [0, \varepsilon)$ is a collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for suitable $\varepsilon > 0$.

Moreover, $bP_{n+1} \subsetneq \Theta_n$ holds (at least) for $n \in \{237, 285, 333, 381, 445, 461, 477\}$, which shows that indefinite fold lines of middle absolute index $\lceil \frac{n}{2} \rceil$ can in fact distinguish the Kervaire sphere from other exotic n-spheres in these dimensions.

Finally, the results of this chapter have remarkable consequences on the study of Banagl's *aggregate invariant*. Recall from [4, Section 10, p. 81f.] that the definition of the aggregate invariant $\mathfrak{A}(\Sigma^n)$ of a homotopy sphere Σ^n, $n \geq 6$, involves a priori the choice of special generic maps $f_S \colon S^n \to \mathbb{R}$ and $f_\Sigma \colon \Sigma^n \to \mathbb{R}$. The following result is shown in Section 10.5:

Proposition 10.1.5. (*a*) *The value* $\mathfrak{A}(\Sigma^n) \in Q$ *of the aggregate invariant on a homotopy sphere* Σ^n *of dimension* $n \geq 6$ *is independent of the choice of* f_S *and* f_Σ. *Moreover,* $\mathfrak{A}$ *induces a well-defined map* $\mathfrak{A}\colon \Theta_n \to Q$ *by setting* $\mathfrak{A}([\Sigma^n]) := \mathfrak{A}(\Sigma^n)$ *for every class* $[\Sigma^n] \in \Theta_n$.

(*b*) *There exists a map* $\mathfrak{a}\colon \Theta_n \to \mathbb{N}_0$ *with the following properties for every* $[\Sigma^n] \in \Theta_n$:

- (*i*) $\mathfrak{A}([\Sigma^n]) = (\sum_{i \geq \mathfrak{a}([\Sigma^n])} q^i, 0, \sum_{i \geq \mathfrak{a}([\Sigma^n])} q^i) \in \mathbb{B}[[q]] \oplus \mathbb{B}[[q]] \oplus \mathbb{B}[[q]] \overset{\phi^{-1}}{\cong} Q(H_{2,2}) \subset Q$, *where the isomorphism* ϕ *is defined in the proof of [4, Corollary 10.4, p. 85].*
- (*ii*) *If* $\Sigma^n \ncong S^n$, *then* $\mathfrak{a}([\Sigma^n]) > 0$. *Moreover,* $\mathfrak{a}([S^n]) = 0$.
- (*iii*) *If* $[\Sigma^n] \in G^l_n$, *then* $\mathfrak{a}([\Sigma^n]) \leq (n-l) - \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n}{2} \rfloor + 1 - l$.

Finally, Theorem 10.1.3 and Proposition 10.1.5 imply that, unlike Milnor's λ-invariant, the aggregate invariant cannot detect individual exotic 7-spheres. (Use $bP_8 = \Theta_7$ in Corollary 10.5.1.)

10.2 About Groups of Homotopy Spheres

The following Proposition 10.2.1 is an immediate consequence of Wall's work [56] on smooth highly connected almost closed manifolds of even dimension. In his study of the diffeomorphism classification of such manifolds, Wall defines a homomorphism

$$\mathfrak{o}\colon \mathcal{G}_k \to \Theta_{2k-1},$$

where $\mathcal{G}_k$ denotes the Grothendieck group of k-spaces (see [56, pp. 169-171]). Furthermore, he gives a complete calculation of $\mathcal{G}_k$ that distinguishes between seven cases with respect to k (see [56, pp. 171-177]). The subgroup $\mathfrak{o}\mathcal{G}_k \subset \Theta_{2k-1}$ given by the image of the homomorphism $\mathfrak{o}$ is of particular interest to us. This group seems to be not fully understood in the literature (see [53]). However, an important observation is that $bP_{2k} \subset \mathfrak{o}\mathcal{G}_k$. The group $\mathfrak{o}\mathcal{G}_k$ can be characterized (at least for $k \neq 8$) via Morse theory as follows:

Proposition 10.2.1. *Let Σ^{2k-1} be a homotopy sphere for some integer $k > 3$. If $k \neq 8$, then the following statements are equivlalent:*

- *(i) The element $[\Sigma^{2k-1}] \in \Theta_{2k-1}$ is contained in the subgroup $\mathfrak{o}\mathcal{G}_k \subset \Theta_{2k-1}$.*
- *(ii) There exists a smooth manifold triad $(W^{2k}, S^{2k-1}, \Sigma^{2k-1})$ that admits a Morse function*

$$(W^{2k}, S^{2k-1}, \Sigma^{2k-1}) \to ([0,1], 0, 1)$$

with an even number of critical points that are all of index k.

For $k = 8$ we only have the weaker statement that $[\Sigma^{15}] \in bP_{16}$ implies (ii), whereas the implication $(ii) \Rightarrow (i)$ is still valid.

Proof. $(i) \Rightarrow (ii)$. Suppose that $[\Sigma^{2k-1}] \in \mathfrak{o}\mathcal{G}_k$ for $k \neq 8$ and $[\Sigma^{15}] \in bP_{16}$ for $k = 8$. In both cases, it suffices to construct a $(k-1)$-connected smooth manifold V^{2k} with boundary Σ^{2k-1} such that the rank of the finitely generated free abelian group $H_k(V^{2k})$ is *even.* (In fact, having constructed such a manifold V^{2k}, the proof of (ii) can be completed as follows. Deleting a small $2k$-ball in the interior of V^{2k} yields a smooth manifold triad $(W^{2k}, S^{2k-1}, \Sigma^{2k-1})$. As W^{2k} is still $(k-1)$-connected, Lemma C.0.3 then implies that there exists a Morse function $(W^{2k}, S^{2k-1}, \Sigma^{2k-1}) \to ([0,1], 0, 1)$ with only critical points of index k. By [41, page 36], the number of critical points of this Morse function can be interpreted as the rank of $H_k(W^{2k}, S^{2k-1}) = H_k(V^{2k})$, and is thus even.) For the construction of V^{2k} we distinguish between the following two cases (note that $bP_8 = \mathfrak{o}\mathcal{G}_4 = \Theta_7$ for $k = 4$):

- $k \notin \{4, 8\}$. In this case we conclude from the construction of the homomorphism $\mathfrak{o}$ (see [56, pp. 169-171]) that there exists a $(k-1)$-connected smooth manifold V^{2k} with boundary Σ^{2k-1}. It then follows from Wall's calculation of $\mathcal{G}_k$ in [56, Theorem 2, page 176] that the rank r of the k-space $H_k(V^{2k})$ corresponding to the almost closed smooth manifold V^{2k} is even. (Note that we avoid case (2) in Wall's theorem since $k \notin \{4, 8\}$.)
- $k \in \{4, 8\}$. (In fact, this argument works whenever $[\Sigma^{2k-1}] \in bP_{2k}$ and k is even.) In this case the homotopy sphere Σ^{2k-1} can be realized as the boundary of a compact parallelizable connected smooth manifold V^{2k}. By [27, Theorem 5.5, page 514] we can additionally assume that V^{2k} is $(k-1)$-connected. (Note that the concepts of parallelizability and s-parallelizability coincide for a smooth connected manifold with nonempty boundary by

[27, Lemma 3.4, page 509].) Using that k is even, [33, Theorem 3.3, page 73] implies that the *signature* τ of V^{2k} is divisible by 8. (Recall that τ is by definition the signature of the integral symmetric bilinear form

$$H^k(V^{2k}, \Sigma^{2k-1}) \times H^k(V^{2k}, \Sigma^{2k-1}) \to \mathbb{Z},$$
$$(x, y) \mapsto \langle x \cup y, [V^{2k}] \rangle,$$

where $[V^{2k}] \in H_{2k}(V^{2k}, \Sigma^{2k-1})$ denotes the fundamental class determined by the orientation of V^{2k}. The signature of V^{2k} is given by $\tau = a - b$, where a (b) is the number of positive (negative) diagonal entries when the above integral symmetric bilinear form is diagonalized over $\mathbb{R}$.) As τ is in particular even, it now follows from case (2) of Wall's calculation of $\mathcal{G}_k$ in [56, Theorem 2, page 176] that the rank r of the k-space $H_k(V^{2k})$ corresponding to the almost closed smooth manifold V^{2k} is even.

(ii) $\Rightarrow$ *(i)*. By [41, page 36], W^{2k} is homotopy equivalent to $S^{2k-1} \bigcup_{i=1}^r D^k$. Therefore, W^{2k} is $(k-1)$-connected. (In fact, it follows from Seifert-van Kampen's theorem that W^{2k} is simply connected, compare [41, Remark 1), page 70]. Afterwards, one can use the Hurewicz theorem.) By gluing W^{2k} and a $2k$-disc D^{2k} along the common boundary S^{2k-1}, we hence obatin a $(k-1)$-connected smooth manifold V^{2k} with boundary Σ^{2k-1}. Consequently, $[\Sigma^{2k-1}] \in \mathfrak{o}\mathcal{G}_k$.

□

The next proposition collects properties of the groups $\overline{C}_n^l$ and C_n^l that are implied by results from the literature. However, the entire calculation of these groups seems not accessible.

Proposition 10.2.2. *For* $l \in \{1, \dots, \lceil \frac{n}{2} \rceil - 1\}$ *the following statements hold:*

(i) $\overline{C}_n^l \subset C_n^l$, *and equality holds (at least) in the following cases:*
- $l = 1$
- $n \equiv 0, 1, 2 \pmod 4$
- $n \geq 7$, $n \neq 15$, $n \equiv 3 \pmod 4$ *and* $l = (n-1)/2$

(ii) $C_n^1 = \Theta_n$.

(iii) $\overline{C}_n^l \subset \overline{C}_n^{l-1}$ *and* $C_n^l \subset C_n^{l-1}$ *for* $l \geq 2$, *and equality holds (at least) if*

$$l \equiv 3, 5, 6, 7 \pmod 8.$$

(iv) $bP_{n+1} \subset C_n^{\lceil \frac{n}{2} \rceil - 1}$, *and equality holds (at least) in the following cases:*
- $n + 1 \in \{2k, 2k+1\}$ *for some integer* $k > 10$ *with* $k \equiv 2 \pmod 8$
- $n + 1 = 2k$ *for some integer* $k \geq 113$ *with* $k \equiv 1 \pmod 2$
- $n + 1 = 2k + 1$ *for some integer* $k \geq 113$ *with* $k \not\equiv 1 \pmod 8$

Proof. *(i)*. The inclusion $\overline{C}_n^l \subset C_n^l$ holds by definition. We prove equality in the following three cases:

- $l = 1$. It suffices to show that $\overline{C}_n^1 = \Theta_n$ because $\overline{C}_n^1 \subset C_n^1 \subset \Theta_n$ then implies $\overline{C}_n^1 = C_n^1 = \Theta_n$. Note that this also shows *(ii)*. In order to establish $\overline{C}_n^1 = \Theta_n$, let $[\Sigma^n] \in \Theta_n$. Choose a special generic function $f \colon \Sigma \to \mathbb{R}$. It follows from Proposition 10.3.1 that $[(\Sigma^n, f)] = 0 \in \tilde{\Gamma}_1(n, 1)$. Explicitly, there exists an oriented compact smooth manifold

W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F\colon W \to \mathbb{R}^2$ such that

$$F|_{\Sigma\times[0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)}\colon M_1 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),$$

where $\Sigma\times[0,\varepsilon)$ is a suitable collar neighbourhood of $\Sigma\times 0 = \Sigma$ in W for some $\varepsilon \in (0,1/2)$. Thus, Proposition 6.2.3 implies that $\chi(W) \equiv 1 \pmod 2$ since F has no cusps and $S(F)$ has only one component diffeomorphic to $[0,1]$. It remains to make W 1-connected by a finite sequence of surgeries (see [40, Theorem 3, p. 49]). Note that this does not affect $\partial W = \Sigma$ and $\chi(W) \equiv 1 \pmod 2$. Hence, $[\Sigma^n] \in \overline{C}^1_n$.

- $n \equiv 0,1,2 \pmod 4$. If $[\Sigma^n] \in C^l_n$, then there exists an l-connected compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$. It suffices to show that $\chi(W) \equiv 1 \pmod 2$. As Σ^n is homeomorphic to S^n, $\hat{W} := W \cup_{\partial W \cong \partial D^{n+1}} D^{n+1}$ is a closed oriented topological manifold of dimension $n+1$, and $\chi(\hat{W}) = \chi(W) + \chi(D^{n+1}) - \chi(S^n) \equiv \chi(W) + 1 \pmod 2$. A closed oriented topological manifold M^m can only have odd Euler characteristic if its dimension m is divisible by 4. (In fact, if the dimension of M is odd, then the Euler characteristic of M is zero by [20, Corollary 3.37, p. 249]. Now suppose that $m = 4k+2$ for a suitable integer k. In this case it is well-known that the rank of $H^{k+1}(M;\mathbb{Z})$ is even (for instance, see the argument on [20, p. 252]). Hence, $\chi(M)$ is even as a consequence of Poincare duality.) As $n+1 \not\equiv 0 \pmod 4$ by assumption, it follows that $\chi(\hat{W})$ is even, which means that $\chi(W)$ is odd. In consequence, $[\Sigma^n] \in \overline{C}^l_n$.
- $n \geq 7$, $n \neq 15$, $n \equiv 3 \pmod 4$ and $l = (n-1)/2$. Note that $C^l_n = \mathfrak{o}\mathcal{G}_{l+1} \subset \Theta_n$. Hence, if $[\Sigma^n] \in C^l_n$, then the proof of the part $(ii) \Rightarrow (i)$ of Proposition 10.2.1 implies for $k := (n+1)/2$ that Σ bounds an $l = (k-1)$-connected smooth manifold V^{n+1} which is of the form $V = W \cup_{S^n} D^{n+1}$, where $(W^{2k}, S^{2k-1}, \Sigma^{2k-1})$ is a smooth manifold triad that admits a Morse function $(W^{2k}, S^{2k-1}, \Sigma^{2k-1}) \to ([0,1],0,1)$ with an even number of critical points. Lemma 6.2.1 implies that $0 \equiv \chi(W) - \chi(S^n) \equiv \chi(W) \pmod 2$. Thus, $\chi(V) = \chi(W \cup_{S^n} D^{n+1}) = \chi(W) + \chi(D^{n+1}) - \chi(S^n) \equiv 1 \pmod 2$. As a result, $[\Sigma^n] \in \overline{C}^l_n$.

(ii). The proof of statement (ii) is given in the proof of the case $l=1$ in statement (i).

(iii). The inclusions $\overline{C}^l_n \subset \overline{C}^{l-1}_n$ and $C^l_n \subset C^{l-1}_n$ hold by definition for $l \geq 2$. We have to show that $\overline{C}^l_n = \overline{C}^{l-1}_n$ whenever $l \equiv 3,5,6,7 \pmod 8$. (The equality $C^l_n = C^{l-1}_n$ can be shown analogously by ignoring the condition on the Euler characteristic.) If $[\Sigma^n] \in \overline{C}^{l-1}_n$, then Σ bounds an $(l-1)$-connected compact smooth manifold W^{n+1} of dimension $n+1$ such that $\chi(W) \equiv 1 \pmod 2$. Fix a CW-structure on W. In particular, W is $(l-1)$-parallelizable (i.e., parallelizable over the $(l-1)$-skeleton as defined in [40, p. 49]). The obstruction to make W l-parallelizable vanishes if $\pi_{l-1}(SO(n)) = 0$. (In fact, let $\alpha\colon (D^l, S^{l-1}) \to (W, W^{l-1})$ be the characteristic map of any l-cell of the CW-structure of W. As D^l is contractible, we may choose an isomorphism $\alpha^*(TW) \cong D^l \times \mathbb{R}^n$. Moreover, as $TW|_{W^{l-1}}$ is trivial, we may choose an isomorphism $(\partial\alpha)^*(TW) \cong S^{l-1} \times \mathbb{R}^n$, where $\partial\alpha\colon S^{l-1} \to W^{l-1}$ denotes the restriction of α.) By Bott periodicity (see the proof of [27, Theorem 3.1, p. 508]), this is indeed the case since $l \equiv 3,5,6,7 \pmod 8$ and $2 \leq l \leq n-2$. Finally, if W is l-parallelizable, W can be made l-connected by a finite sequence of surgeries by [40, Theorem 3, p. 49]. Note that this does not affect $\partial W = \Sigma$ and $\chi(W) \equiv 1 \pmod 2$. Hence, $[\Sigma^n] \in \overline{C}^1_n$.

(iv). The inclusion $bP_{n+1} \subset C^{\lceil \frac{n}{2} \rceil - 1}_n$ holds since any parallelizable compact smooth manifold W^m of dimension $m = n+1$ can be made $(\lceil \frac{n}{2} \rceil - 1) = (\lfloor \frac{m}{2} \rfloor - 1)$-connected by a finite sequence of surgeries without changing ∂W. Then a result by Stolz (see [53, Theorem B, p. XIX]) implies for $m := n+1$ that equality holds in each of the following cases:

- $n+1 \in \{2k, 2k+1\}$ for some integer $k > 10$ with $k \equiv 2 \pmod 8$.
  This follows from part (i) of [53, Theorem B, p. XIX] applied to $d \in \{0,1\}$.
- $n+1 = 2k$ for some integer $k \geq 113$ with $k \equiv 1 \pmod 2$.
  This follows from part (ii) of [53, Theorem B, p. XIX] for $d = 0$. Note that $m \not\equiv 0 (\mathrm{mod}\, 4)$ because we have chosen k to be odd.
- $n+1 = 2k+1$ for some integer $k \geq 113$ with $k \not\equiv 1 \pmod 8$.
  This follows from part (ii) of [53, Theorem B, p. XIX] for $d = 1$. Note that one has to require $k \not\equiv 1 \pmod 8$ then, and $m = n+1 = 2k+1 \not\equiv 0 (\mathrm{mod}\, 4)$ is satisfied.

□

Remark 10.2.3. Note that Crowley (see the proof of [10, Lemma 2.16, p. 22 f.]) observes that equality in statement (iv) of Proposition 10.2.2 holds whenever $n+1 = 2k$, $k > 1$, $k \equiv 0, 4, 6, 7 \pmod 8$.

10.3 Proof of Theorem 10.1.3

Recall that $n \geq 6$ is a fixed integer. Theorem 10.1.3 claims that, for all $l \in \{1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$,

$$\overline{C}_n^l \subset G_n^l \subset \overline{C}_n^{l-1}.$$

This claim comprises two statements about how certain subgroups of Θ_n are included in each other. Therefore, the proof of Theorem 10.1.3 also decomposes into two halves, which turn out to differ massively in character. The proof of the inclusion $\overline{C}_n^l \subset G_n^l$ (see Section 10.3.1) requires the controlled construction of a fold map on a given highly connected cobordism, whereas the proof of the inclusion $G_n^l \subset \overline{C}_n^{l-1}$ (see Section 10.3.2) starts out with a certain fold map on a quite arbitrary cobordism and exploits the given properties of the fold map in order to improve the connectedness of the cobordism via specific modifications.

The validity of Theorem 10.1.3 for $l = 1$ follows from the following proposition. (In fact, Proposition 10.3.1 implies that $G_n^1 = \Phi^{-1}(\ker \pi_1) = \Phi^{-1}(\tilde{\Gamma}(n,1)) = \Theta_n$ by definition of G_n^1. Furthermore, properties (ii) and (i) of Proposition 10.2.2 imply that $\Theta_n \overset{(ii)}{=} C_n^1 \overset{(i)}{=} \overline{C}_n^1 \subset \overline{C}_n^0 = \Theta_n$.)

Proposition 10.3.1. $\tilde{\Gamma}_1(n,1) = 0$.

Proof. Let $[(M^n, f)] \in \tilde{\Gamma}_1(n,1)$, i.e., $f \colon M \to \mathbb{R}$ is a special generic function on the closed oriented smooth manifold M of dimension n. It follows from [24, Theorem 2.8, p. 215] (if $n \not\equiv 1 \pmod 4$) and [24, Theorem 2.9, p. 215] (if $n \equiv 1 \pmod 4$) that $f \colon \Sigma \to \mathbb{R}$ represents the identity element in the cobordism group $\mathcal{M}_n$ of Morse functions on oriented manifolds of dimension n. (In fact, note that $\tilde{\Psi}([f]) = [\Sigma^n] = 0 \in \Omega_n$ in the n-dimensional oriented cobordism group Ω_n since any homotopy sphere bounds an oriented compact smooth manifold by [4, Lemma 10.1, p. 81]. Moreover, $\tilde{\Phi}([f]) = 0 \in \mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$ because f has exactly two critical points, one of index 0 and one of index n. Furthermore, if $n = 4k+1$ for some integer k, then $\Lambda([f]) = \sigma([f]) - \sigma(M, \mathbb{Q}) = 0 \in \mathbb{Z}/2$ because $\sigma([f]) = \sum_{\lambda=0}^{2k} C_\lambda(f) = 1 \pmod 2$ (see [24, Definition 2.5, p. 214]) and $\sigma(M, \mathbb{Q}) = \sum_{i=0}^{2k} \dim H_i(\Sigma; \mathbb{Q}) = 1 \pmod 2$ (see [24, Definition 2.6, p. 214]).) Hence, by definition of $\mathcal{M}_n$ (see [24, Definition 2.1, p. 212]), there exists an oriented compact smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ such that

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon M_1 \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),$$

where $\Sigma \times [0,\varepsilon)$ is a suitable collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for some $\varepsilon \in (0, 1/2)$. This shows that $[(M^n, f)] = 0 \in \tilde{\Gamma}_1(n,1)$. □

The statement of Theorem 10.1.3 obviously holds for $n = 6$ because in this case, $\Theta_6 = 0$. Therefore, we may assume that $n \geq 7$ and $l \in \{2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ in the following. It then suffices to prove Theorem 10.1.3 for $l \in \{3, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ ($\neq \emptyset$ for $n \geq 7$). (Indeed, the case $l = 2$ will follow from $\overline{C}_n^3 \subset G_n^3$ (which is the case $l = 3$) and $\Theta_n = \overline{C}_n^1$ (see the case $l = 1$) in combination with property (iii) of Proposition 10.2.2: $\overline{C}_n^2 \overset{(iii)}{=} \overline{C}_n^3 \subset G_n^3 \subset G_n^2 \subset \Theta_n = \overline{C}_n^1$.)

From now on we may assume $n \geq 7$ and $l \in \{3, \ldots, \lceil \frac{n}{2} \rceil - 1\}$ in the proof of Theorem 10.1.3.

10.3.1 Proof of the Inclusion $\overline{C}^l_n \subset G^l_n$

It is claimed that $\overline{C}^l_n \subset G^l_n$ for all integers $n \geq 7$ and $l \in \{3, \ldots, \lceil \frac{n}{2} \rceil - 1\}$.

Explicitly, given an element $[\Sigma^n] \in \overline{C}^l_n$, one has to show that $(\Sigma^n, f_\Sigma) \sim_l (\emptyset, f_\emptyset)$ for some (and hence, any) special generic function $f_\Sigma \colon \Sigma \to \mathbb{R}$. Fix special generic functions

$$f_\Sigma \colon \Sigma \to \mathbb{R}, \qquad f_S \colon S^n \to \mathbb{R}.$$

Then it suffices to show that $(\Sigma^n, f_\Sigma) \sim_l (S^n, f_S)$. (In fact, Theorem 10.1.1 implies that $[(S^n, f_S)] = \Phi([S^n]) = [(\emptyset, f_\emptyset)]$ in $\tilde{\Gamma}(n, 1)$. Thus, $(S^n, f_S) \sim (\emptyset, f_\emptyset)$ and therefore $(S^n, f_S) \sim_l (\emptyset, f_\emptyset)$.) By definition of the equivalence relation $\sim_l$, one has to construct a pair (W, F) consisting of

- a compact oriented smooth manifold W of dimension $n+1$ with boundary $\partial W = \Sigma \sqcup -S^n$, and
- a fold map $F \colon W \to \mathbb{R}^2$ all of whose fold lines have absolute index contained in $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$, and such that

$$F|_{\Sigma \times [0,\varepsilon)} = f_\Sigma \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$
$$F|_{S^n \times (1-\varepsilon,1]} = f_S \times \mathrm{id}_{(1-\varepsilon,1]} \colon S^n \times (1-\varepsilon, 1] \to \mathbb{R} \times (1-\varepsilon, 1],$$

  on suitable collar neighbourhoods $\Sigma \times [0, \varepsilon)$ and $S^n \times (1-\varepsilon, 1]$ of $\Sigma \times 0 = \Sigma$ and $S^n \times 1 = S^n$ in W, $\varepsilon \in (0, 1/2)$.

As far as the construction of W is concerned, the assumption $[\Sigma^n] \in \overline{C}^l_n$ ensures by definition of $\overline{C}^l_n$ the existence of an l-connected compact oriented smooth manifold W' of dimension $n+1$ with boundary $\partial W' = \Sigma$ such that $\chi(W')$ is odd. Deleting a small open $(n+1)$-ball from $W' \setminus \partial W'$, we obtain an l-connected compact oriented smooth manifold W of dimension $n+1$ with boundary $\partial W = \Sigma^n \sqcup -S^n$ and even Euler characteristic $\chi(W)$. Take this manifold W to be the desired cobordism.

The strategy for the construction of F is to construct first a generic smooth map $F' \colon W \to \mathbb{R}^2$ all of whose fold points have absolute index contained in $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$, and such that

$$F'|_{\Sigma \times [0,\varepsilon)} = f_\Sigma \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$
$$F'|_{S^n \times (1-\varepsilon,1]} = f_S \times \mathrm{id}_{(1-\varepsilon,1]} \colon S^n \times (1-\varepsilon, 1] \to \mathbb{R} \times (1-\varepsilon, 1],$$

on suitable collar neighbourhoods $\Sigma \times [0, \varepsilon)$ and $S^n \times (1-\varepsilon, 1]$ of $\Sigma \times 0 = \Sigma$ and $S^n \times 1 = S^n$ in W, $\varepsilon \in (0, 1/2)$. Once F' is constructed, Proposition 6.1.3 can be applied as follows to the smooth manifold triad (W, Σ^n, S^n) and the generic smooth map $F' \colon W \to \mathbb{R}^2$ to produce the desired fold map F by elimination of all cusps of F'. First of all, W is by construction connnected and has dimension $n + 1 \geq 8 \geq 3$. Note that the number c of cusps of F' is even by Proposition 6.2.3. (Indeed, this follows from $\chi(W) \equiv c + k \pmod 2$ since $\chi(W)$ is even by construction and $k = 2$ because each of the special generic functions f_Σ and f_S has exactly two critical points, one minimum and one maximum, by Lemma C.0.4(b).) Choose $\mathcal{R}$ to be the union of components of $S(F')$ that contain at least one definite fold point. (Note that every $R \in \mathcal{R}$ is in fact a fold line of F'. In fact, if R contains at least one cusp of F', then $A_{F'}(R) \subset \{\lceil \frac{n}{2} \rceil, \ldots, n\}$ is a *nice* subset in the sense of Definition 6.1.1. However, $n \in A_{F'}(R)$

would imply $n-1 \in A_{F'}(R)$ in contradiction to $A_{F'}(R) \subset \{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$ and $l \geq 3 > 1$.) Then one necessarily has to choose $\mathcal{P} = \emptyset$. (In fact, as all critical points of f_S and f_Σ are minima and maxima, they are all definite fold points of F' and do thus lie on components of $S(F')$ that are contained in $\mathcal{R}$.) Finally, set $r = 1$ and choose the nice subset $A_1 := \{\lceil \frac{n}{2} \rceil, \dots, n-l\}$. (Note that if a component of $S(F')$ is not contained in $\mathcal{R}$, then all its fold points have absolute index in A_1 by construction of $\mathcal{R}$.) Finally, Proposition 6.1.3 yields a fold map $F \colon W \to \mathbb{R}^2$ all of whose fold points have absolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$ by properties (ii) and (iii). Moreover, property (i) implies that F has the desired boundary conditions.

It remains to construct the smooth generic map F'. For this purpose, W will first be cut into a sequence of cobordisms such that on each piece one can carefully construct with the help of Theorem 7.0.1 a fold map into the plane all of whose fold points have absolute indices that lie in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$. In a second step these individual fold maps will be joined near the common boundary components of subsequent cobordisms to produce the generic smooth map F' on W with the correct behaviour near the boundary of W and with the correct index constraints.

By Lemma C.0.3 (set $m := n+1$ and $k := l+1$) and the alternate version of the *final rearrangement theorem* (see [41, Theorem 4.8, p. 44]) there exists a self-indexing Morse function

$$\tau \colon (W, \Sigma^n, S^n) \to ([-1/2, n+1+1/2], -1/2, n+1+1/2)$$

such that all critical points of τ are contained in $\tau^{-1}([l+1, n-l])$, i.e. all indices of critical points of τ lie in $\{l+1, \dots, n-l\}$. (Note that this set is always nonempty since $l \leq \lceil \frac{n}{2} \rceil - 1 = \lfloor \frac{n+1}{2} \rfloor - 1 \leq \frac{n+1}{2} - 1 = \frac{n-1}{2}$ implies that $l+1 \leq n-l$.)

Note that $\tau^{-1}(t)$ is an $(l-1)$-connected closed manifold for all $t \in [-1/2, n+1+1/2] \setminus \mathbb{Z}$. (Indeed, this is certainly true for the homotopy spheres $\Sigma = \tau^{-1}(-1/2)$ and $S^n = \tau^{-1}(n+1+1/2)$. By an argument analogous to [41, Remark 1), p. 70], the Seifert van Kampen theorem implies that $\tau^{-1}(t)$ is simply connected for all $t \in [-1/2, n+1+1/2] \setminus \mathbb{Z}$. This uses that $\lambda, n+1-\lambda \geq 3$ for all $\lambda \in \{l+1, \dots, n-l\}$ and that W is simply connected. The claim now follows from the Hurewicz theorem since by [44, Proposition 4.19(iii), p. 56] the effect of a p-surgery on a closed smooth manifold of dimension $d \geq p+2$ does not affect its homology groups in dimensions strictly below $\min(p, d-p-1)$. In our case, $d = n$ and $p \in \{l, \dots, n-l-1\}$.)

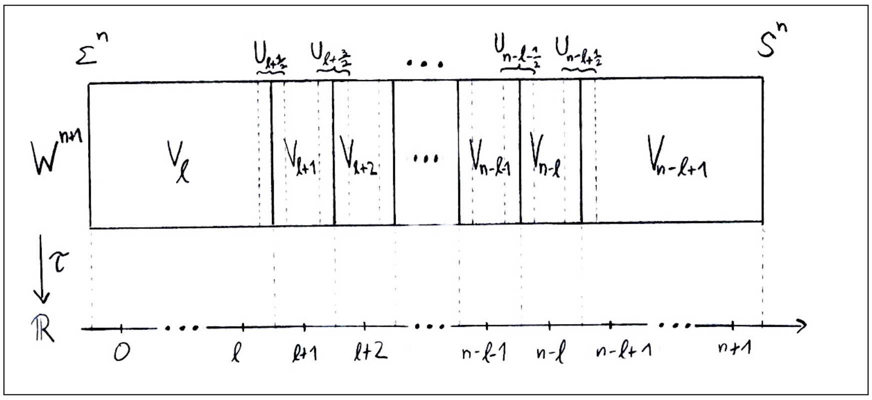

Figure 10.1: Decomposition of W

For any subset $A \subset \mathbb{R}$ define $W_A := \tau^{-1}(A)$. If $A = \{a\}$ consists of one element $a \in \mathbb{R}$, then write $W_a := W_A$. For every $\lambda \in \{l+1, \dots, n-l\}$ define the smooth manifold triad (see

Figure 10.1)

$$(V_\lambda, W_{\lambda-1/2}, W_{\lambda+1/2}) := (W_{[\lambda-1/2,\lambda+1/2]}, W_{\lambda-1/2}, W_{\lambda+1/2}).$$

Moreover, define the smooth manifold triads (see Figure 10.1)

$$(V_l, W_{-1/2}, W_{l+1/2}) := (W_{[-1/2,l+1/2]}, W_{-1/2}, W_{l+1/2}),$$
$$(V_{n-l+1}, W_{n-l+1/2}, W_{n+1+1/2}) := (W_{[n-l+1/2,n+1+1/2]}, W_{n-l+1/2}, W_{n+1+1/2}).$$

By construction, τ restricts for every $\lambda \in \{l+1, \dots, n-l\}$ to a Morse function

$$\tau_\lambda \colon (V_\lambda, W_{\lambda-1/2}, W_{\lambda+1/2}) \to ([\lambda - 1/2, \lambda + 1/2], \lambda - 1/2, \lambda + 1/2)$$

with only critical points of index λ, all contained in $\tau^{-1}(\lambda)$. Moreover, τ restricts to Morse functions

$$\tau_l \colon (V_l, W_{-1/2}, W_{l+1/2}) \to ([-1/2, l+1/2], -1/2, l+1/2),$$
$$\tau_{n-l+1} \colon (V_{n-l+1}, W_{n-l+1/2}, W_{n+1+1/2}) \to ([n-l+1/2, n+1+1/2], n-l+1/2, n+1+1/2).$$

without critical points. Therefore, we may fix diffeomorphisms

$$\Xi_l \colon (V_l, W_{-1/2}, W_{l+1/2}) \xrightarrow{\cong} \Sigma^n \times ([-1/2, l+1/2], -1/2, l+1/2),$$
$$\Xi_{n-l+1} \colon (V_{n-l+1}, W_{n-l+1/2}, W_{n+1+1/2}) \xrightarrow{\cong} S^n \times ([n-l+1/2, n+1+1/2], n-l+1/2, n+1+1/2),$$

such that $\mathrm{pr}_2 \circ \Xi_l = \tau_l$ and $\mathrm{pr}_2 \circ \Xi_{n-l+1} = \tau_{n-l+1}$.

All in all, there is the following decomposition of W, where gluing is performed along common boundary components of subsequent cobordisms:

$$W = V_l \cup_{W_{l+1/2}} V_{l+1} \cup_{W_{l+1+1/2}} V_{l+2} \cup \dots \cup V_{n-l-1} \cup_{W_{n-l-1/2}} V_{n-l} \cup_{W_{n-l+1/2}} V_{n-l+1}.$$

For $m := n+1 \geq 8$, $k := l+1 \in \{4, \dots, \lfloor \frac{m}{2} \rfloor\}$ and every $\lambda \in \{l+1, \dots, n-l\} = \{k, m-k\}$, there exists by Theorem 7.0.1 a smooth map

$$\sigma_\lambda \colon V_\lambda \to \mathbb{R}$$

that restricts for every $t \in [\lambda - 1/2, \lambda + 1/2] \setminus \{\lambda\}$ to an excellent Morse function $\tau^{-1}(t) \to \mathbb{R}$, and such that σ_λ and τ_λ form the components of a fold map

$$F_\lambda := (\sigma_\lambda, \tau_\lambda) \colon V_\lambda \to \mathbb{R}^2$$

all of whose fold lines have absolute index in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$.

Furthermore, define the fold maps

$$F_l := (f_\Sigma \times \mathrm{id}_{[-1/2,l+1/2]}) \circ \Xi_l \colon V_l \to \mathbb{R} \times [-1/2, l+1/2],$$
$$F_{n-l+1} := (f_S \times \mathrm{id}_{[n-l+1/2,n+1+1/2]}) \circ \Xi_{n-l+1} \colon V_{n-l+1} \to \mathbb{R} \times [n-l+1/2, n+1+1/2].$$

For every integer $\mu \in \{l, \dots, n-l\}$ define the smooth manifold triad (see Figure 10.1)

$$(U_{\mu+1/2}, W_{\mu+1/4}, W_{\mu+3/4}) := (W_{[\mu+1/4,\mu+3/4]}, W_{\mu+1/4}, W_{\mu+3/4}).$$

As τ restricts for every integer $\mu \in \{l, \dots, n-l\}$ to a Morse function

$$(U_{\mu+1/2}, W_{\mu+1/4}, W_{\mu+3/4}) \to ([\mu+1/4, \mu+3/4], \mu+1/4, \mu+3/4)$$

without critical points, we may fix a diffeomorphism

$$\Xi_{\mu+1/2} \colon (U_{\mu+1/2}, W_{\mu+1/4}, W_{\mu+3/4}) \xrightarrow{\cong} W_{\mu+1/2} \times ([\mu+1/4, \mu+3/4], \mu+1/4, \mu+3/4)$$

such that $\mathrm{pr}_2 \circ \Xi_{\mu+1/2} = \tau|_{U_{\mu+1/2}}$.

Fix $\mu \in \{l, \dots, n-l\}$. Consider the restrictions

$$\begin{aligned} F_\mu|_{V_\mu \cap U_{\mu+1/2}} &\colon (V_\mu \cap U_{\mu+1/2}, W_{\mu+1/4}, W_{\mu+1/2}) \to \mathbb{R}^2, \\ F_{\mu+1}|_{V_{\mu+1} \cap U_{\mu+1/2}} &\colon (V_{\mu+1} \cap U_{\mu+1/2}, W_{\mu+1/2}, W_{\mu+3/4}) \to \mathbb{R}^2. \end{aligned}$$

In general, these maps not fit together along the common boundary $W_{\mu+1/2}$. Under the diffeomorphism $\Xi_{\mu+1/2}$ these maps correspond to maps

$$\begin{aligned} G^-_{\mu+1/2} &\colon W_{\mu+1/2} \times ([\mu+1/4, \mu+1/2], \mu+1/4, \mu+1/2) \to \mathbb{R}^2, \\ G^+_{\mu+1/2} &\colon W_{\mu+1/2} \times ([\mu+1/2, \mu+3/4], \mu+1/2, \mu+3/4) \to \mathbb{R}^2, \end{aligned}$$

such that

$$\begin{aligned} G^-_{\mu+1/2} \circ \Xi_{\mu+1/2}|_{W_{\mu+1/2} \times [\mu+1/4,\mu+1/2]} &= F_\mu|_{V_\mu \cap U_{\mu+1/2}}, \\ G^+_{\mu+1/2} \circ \Xi_{\mu+1/2}|_{W_{\mu+1/2} \times [\mu+1/2,\mu+3/4]} &= F_{\mu+1}|_{V_{\mu+1} \cap U_{\mu+1/2}}. \end{aligned}$$

It follows from the construction of F_μ that

$$\begin{aligned} \mathrm{pr}_2 \circ G^-_{\mu+1/2} &= \mathrm{pr}_2 \circ F_\mu|_{V_\mu \cap U_{\mu+1/2}} \circ \Xi^{-1}_{\mu+1/2}|_{W_{\mu+1/2} \times [\mu+1/4,\mu+1/2]} \\ &= \tau \circ \Xi^{-1}_{\mu+1/2}|_{W_{\mu+1/2} \times [\mu+1/4,\mu+1/2]} \\ &= \mathrm{pr}_2 \,. \end{aligned}$$

Analogously, it follows from the construction of $F_{\mu+1}$ that $\mathrm{pr}_2 \circ G^+_{\mu+1/2} = \mathrm{pr}_2$.

Moreover, $\mathrm{pr}_1 \circ G^-_{\mu+1/2}$ restricts for every $t \in [\mu+1/4, \mu+1/2]$ to an excellent Morse function

$$W_{\mu+1/2} = W_{\mu+1/2} \times t \to \mathbb{R}$$

that has only critical points of index contained in the set $\{l, \dots, n-l\} \cup \{0, n\}$.

Similarly, $\mathrm{pr}_1 \circ G^+_{\mu+1/2}$ restricts for every $t \in [\mu+1/2, \mu+3/4]$ to an excellent Morse function

$$W_{\mu+1/2} = W_{\mu+1/2} \times t \to \mathbb{R}$$

that has only critical points of absolute index contained in the set $\{l, \dots, n-l\} \cup \{0, n\}$.

Hence, Theorem 8.0.1 applied to $m := n+1 \geq 8$ and $k := l+1 \in \{4, \ldots, \lfloor \frac{m}{2} \rfloor\}$ implies that there exists a generic smooth map

$$G_{\mu+1/2}\colon W_{\mu+1/2} \times ([\mu+1/4, \mu+3/4], \mu+1/4, \mu+3/4) \to \mathbb{R}^2$$

that agrees with $G^-_{\mu+1/2}$ near $W_{\mu+1/2} \times (\mu+1/4)$ and with $G^+_{\mu+1/2}$ near $W_{\mu+1/2} \times (\mu+3/4)$, and such that every fold point of $G_{\mu+1/2}$ has absolute index contained in the set $\{\lceil \frac{n}{2} \rceil, \ldots, n-l\} \cup \{n\}$.

All in all, the generic smooth map F' with the desired properties can be defined as

$$F'\colon W \to \mathbb{R}^2, \qquad F'(w) = \begin{cases} (G_{\mu+1/2} \circ \Xi^{-1}_{\mu+1/2})(w), & \text{if } w \in U_{\mu+1/2}, \mu \in \{l, \ldots, n-l\}, \\ F_\lambda(w), & \text{else, where } w \in V_\lambda, \lambda \in \{l, \ldots, n-l+1\}. \end{cases}$$

This completes the proof of the inclusion $\overline{C}^l_n \subset G^l_n$ in Theorem 10.1.3.

10.3.2 Proof of the Inclusion $G_n^l \subset \overline{C}_n^{l-1}$

Recall that the claim $G_n^l \subset \overline{C}_n^{l-1}$ has to be shown for all integers $n \geq 7$ and $l \in \{3, \dots, \lceil \frac{n}{2} \rceil - 1\}$. Let $[\Sigma^n] \in G_n^l$. As Σ^n is a homotopy sphere, one may fix a special generic function $f \colon \Sigma \to \mathbb{R}$. By definition of G_n^l there exists a pair (W, F) consisting of a compact connected oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ whose fold lines have an abosolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and such that

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0,\varepsilon) \to \mathbb{R} \times [0,\varepsilon),$$

where $\Sigma \times [0,\varepsilon)$ is a collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for suitable $\varepsilon > 0$.

Consider the Stein factorization of F (see Definition 9.1.1), which can be expressed in the diagram

$$\begin{array}{ccc} W & \xrightarrow{\;F\;} & \mathbb{R}^2 \\ {\scriptstyle \pi_F}\downarrow & \nearrow_{\overline{F}} & \\ W_F & & \end{array}$$

.

By Theorem 9.1.7, W_F can be given the structure of a compact smooth manifold of dimension 2 with corners such that π_F is a fold map and $\overline{F}$ is a submersion. Furthermore, if $D(F)$ denotes the union of the definite fold lines of F, then the boundary of W_F decomposes as

$$\partial W_F = \pi_F(\Sigma) \cup \pi_F(D(F)),$$

where $\pi_F(\Sigma) \cap \pi_F(D(F)) = \pi_F(\Sigma \cap D(F))$ is the set of corners of W_F, and π_F restricts to an embedding $D(F) \to \partial W_F$. Since $\overline{F}$ is locally an embedding, the properties of F imply that $\pi_F \colon W \to W_F$ is a stable fold map whose fold lines have all absolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and $S(\pi_F) \cap \Sigma = S(F) \cap \Sigma = D(F) \cap \Sigma = D(\pi_F) \cap \Sigma$.

Let $X_0 := \{(x,y) \in \mathbb{R};\ x^2 + y^2 \leq 1,\ y \geq 0\}$ denote the unit half disc in the upper half plane. Given an integer $g > 0$, let X_g denote a fixed smooth manifold of dimension 2 with corners that is obtained from the half disc X_0 by smoothly attaching g handle pairs (h_j, h_j'), $j \in \{1, \dots, g\}$, to the part $y > 0$ as shown in Figure 10.2. By construction, X_g is naturally equipped with an immersion $\xi_g \colon X_g \to \mathbb{R}^2$.

By Proposition 9.2.2 we may assume that ∂W_F is connected, where $D(\rho)$ denotes the union of definite fold lines of ρ. Moreover, as $\overline{F}$ is locally an embedding into $\mathbb{R}^2$, W_F is orientable. Hence, the classification of compact oriented smooth surfaces implies that there exists for a suitable integer $g \geq 0$ a diffeomorphism

$$\Xi \colon W_F \xrightarrow{\cong} X_g =: X$$

such that the immersion $\xi := \xi_g \colon X \to \mathbb{R}^2$ restricts to a diffeomorphism $\Xi(\pi_F(\Sigma)) \xrightarrow{\cong} [-1,1] \times 0$. The properties of π_F imply that the composition

$$\rho := \Xi \circ \pi_F \colon W \to X$$

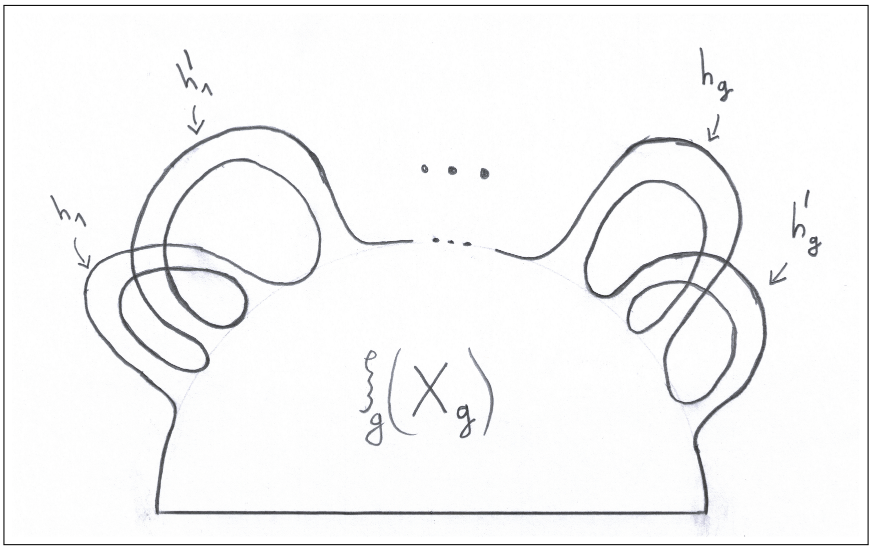

Figure 10.2: $\xi_g(X_g)$

is a fold map whose fold lines have all absolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and $S(\rho) \cap \Sigma = D(\rho) \cap \Sigma$. Moreover, the boundary of X decomposes as

$$\partial X = \rho(\Sigma) \cup \rho(D(\rho)),$$

where $\rho(\Sigma) = \xi^{-1}([-1,1] \times 0)$, $\rho(\Sigma) \cap \rho(D(\rho)) = \rho(\Sigma \cap D(\rho)) = \{\xi^{-1}(-1,0), \xi^{-1}(1,0)\}$ is the set of corners of X, and ρ restricts to an embedding $D(\rho) \to \partial X$. The properties of ρ imply that the composition

$$G := \xi \circ \rho \colon W \to \mathbb{R}^2$$

is a fold map whose fold lines have all absolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and $S(G) \cap \Sigma = D(G) \cap \Sigma$. Concerning the behaviour of G near Σ, there exists a collar neighbourhood $\Sigma \times [0,\varepsilon)$ of $\Sigma = \Sigma \times 0$ in W on which G is of the form

$$G(x,t) = (h(x,t),t), \qquad (x,t) \in \Sigma \times [0,\varepsilon),$$

where $x \mapsto h_t(x) := h(x,t)$ is a special generic function on Σ for all $t \in [0,\varepsilon)$. (In fact, set $L := \mathbb{R} \times 0$. As $G(\Sigma) \subset L$, $G|_{S(G)}$ is transverse to L, and $S(G) \cap \Sigma = D(G) \cap \Sigma$, the claim follows from Lemma 9.1.5. In order to apply the lemma formally, one first has to extend G from a collar neighbourhood $\Sigma \times [0,\infty)$ of $\Sigma = \Sigma \times 0$ in W to a fold map on $\Sigma \times (-\varepsilon,\infty)$ for some $\varepsilon > 0$, compare Remark 3.1.3.)

Note that the following diagram can be considered as the Stein factorization of G:

$$\begin{array}{ccc} W & \xrightarrow{G} & \mathbb{R}^2 \\ {\scriptstyle\rho}\downarrow & \nearrow_{\xi} & \\ X & & \end{array}$$

.

As in Saeki's proof of [47, Lemma 3.3], our strategy is to reduce this general situation inductively in g to the special case that $g = 0$ by a careful modification of the pair (W,G). More precisely, every induction step modifies (W,G) to a pair (W',F') with the following properties:

- $F'\colon W' \to \mathbb{R}^2$ is a fold map whose fold lines have an abosolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$,
- $\partial W' = \Sigma$, and $F'|_{\Sigma \times [0,\varepsilon)} = G|_{\Sigma \times [0,\varepsilon)}$ on a suitable collar neighbourhood $\Sigma \times [0,\varepsilon)$ of $\Sigma \times 0 = \Sigma$ in W' for suitable $\varepsilon > 0$,
- $W'_{F'} \cong X_{g-1}$.

The role of the new pair (W, G) with g reduced to $g-1$ is adopted by the pair (W', G') obtained from (W', F') by identifying $W'_{F'} \cong X_{g-1}$ in the same way as the old pair (W, G) was obtained from (W, F) above.

Note that $g = 0$ will imply the claim $[\Sigma^n] \in \overline{C}_n^{l-1}$ because W is then an $(l-1)$-connected compact oriented smooth manifold of dimension $n+1$ such that $\partial W = \Sigma$ and $\chi(W) \equiv 1 \pmod 2$. In fact, W is a compact oriented smooth manifold of dimension $n+1$ such that $\partial W = \Sigma$. Moreover, Proposition 6.2.3 implies for $(W, W_0, W_1) = (W, \Sigma, \emptyset)$ and the fold map $G\colon W \to \mathbb{R}^2$ that $\chi(W) \equiv 1 \pmod 2$. (Indeed, in order to obtain the correct behaviour of G near the boundary Σ of W, one applies Remark 8.3.6 to the collar neighbourhood $\Sigma \times [0,\varepsilon)$ of $\Sigma = \Sigma \times 0$ in W on which $G(x,t) = (h(x,t), t)$ for all $(x,t) \in \Sigma \times [0,\varepsilon)$. Note that this modification of G does not change the number $c = 0$ of cusps of the fold map G and the number k of components of $S(G)$ that are diffeomorphic to $[0,1]$. Hence, $\chi(W) \equiv c + k \pmod 2$. Note that k is half of the number of critical points of the special generic function $h_0 = G|\colon \Sigma \to \mathbb{R} \times 0 = \mathbb{R}$. As special generic functions on a connected closed manifold have exactly two critical points by Lemma C.0.4, it follows that $k = 1$.) It remains to show that W is $(l-1)$-connected. For this purpose, one applies Lemma 6.2.2 to $(W, W_0, W_1) = (W, \Sigma, \emptyset)$ and the fold map $G\colon W \to \mathbb{R}^2$ (see Figure 10.3). (In order to obtain the correct behaviour of G near the boundary Σ of W, one applies Remark 8.3.6 to the collar neighbourhood $\Sigma \times [0,\varepsilon)$ of $\Sigma = \Sigma \times 0$ in W on which $G(x,t) = (h(x,t), t)$ for all $(x,t) \in \Sigma \times [0,\varepsilon)$. Analogous to the proof of Lemma 8.3.3 one can show for the modified map G that the image of $S(G) \cap (\Sigma \times [0,\varepsilon))$ under G in the plane is nowhere tangent to the x-axis. This fact will be used in the argument below.)

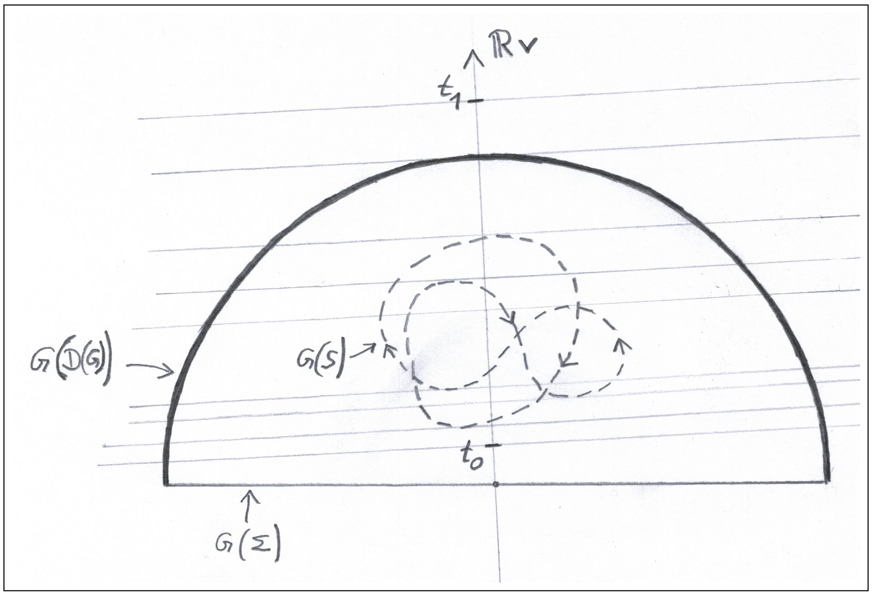

Figure 10.3: Critical levels of τ', where S denotes an indefinite fold line of G

By part $(b)(i)$ of Lemma 6.2.2 there exists a linear projection $\pi_v\colon \mathbb{R}^2 \to \mathbb{R}$ (for suitable $v \in S^1$)

such that the composition $\tau := \pi_v \circ G\colon W \to \mathbb{R}$ restricts to a Morse function $\tau'\colon (W', W_0', W_1') \to ([t_0, t_1], t_0, t_1)$. (By choosing $v \in S^1$ sufficiently near to the north pole $(0,1)$ one can achieve that the image of $S(G) \cap (\Sigma \times [0, \varepsilon))$ under G in the plane is nowhere tangent to $\mathbb{R} \cdot v^\perp$ since it is nowhere tangent to the x-axis.)

Now one exploits parts $(b)(i)$ and $(b)(ii)$ of Lemma 6.2.2 to show that all critical points of τ' have index in $\{l, \dots, n+1-l\} \cup \{n+1\}$. In fact, by part $(b)(i)$, every critical point of τ' of index $j \in \{1, \dots, l-1\}$ (respectively, $j \in \{n+2-l, \dots, n\}$) is a fold point of G of absolute index $\max\{j-1, n+1-j\} = n+1-j \geq n+2-l$ or $\max\{j, n-j\} = n-j \geq n+1-l$ (respectively, $\max\{j-1, n+1-j\} = j-1 \geq n+1-l$ or $\max\{j, n-j\} = j \geq n+2-l$). (Use $l \leq \lceil \frac{n}{2} \rceil - 1$ for the calculation of the maxima.) Since all fold lines of G have absolute index contained in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, one concludes that all critical points of τ' have index in $\{0,1\} \cup \{l, \dots, n+1-l\} \cup \{n, n+1\}$ such that those critical points of τ' of index in $\{0,1\} \cup \{n, n+1\}$ are definite fold points of G. But by part $(b)(ii)$ the critical points of $\tau' = \pi_v \circ G|_{W'}$ are also the critical points of the Morse function

$$\tau'' := \tau'|_{S(G) \cap W'} = \pi_v \circ G|_{S(G) \cap W'}\colon S(G) \cap (W', W_0', W_1') \to ([t_0, t_1], t_0, t_1).$$

Note that G restricts to an immersion $G|_{S(G)}$, and the image $G(D(G))$ is a half-circle because $X = X_0$. (As G was modified on $\Sigma \times [0, \varepsilon)$, the shape of X will deviate from a half-circle near the x-axis. However, as the image of $S(G) \cap (\Sigma \times [0, \varepsilon))$ under G in the plane is by choice of v nowhere tangent to $\mathbb{R} \cdot v^\perp$, it follows that τ'' has no critical points on $S(G) \cap (\Sigma \times [0, \varepsilon))$.) Hence, Figure 10.3 shows that τ'' has a unique critical point on $D(G) \cap W'$, and this is obviously the global maximum of τ'. All in all, if $c \in W'$ is a critical point of τ' whose index is contained in $\{0,1\} \cup \{n, n+1\}$, then $c \in D(G)$ implies that c is the unique critical point of τ'' on $D(G) \cap W'$ and thus the unique critical point of τ' of index $n+1$. Hence, by Lemma 6.2.2$(a)(i)$, there exists a Morse function $(W, \Sigma) \to ([0, \infty), 0)$ whose critical points have index in $\{l, \dots, n+1-l\} \cup \{n+1\}$. Finally, an argument analogous to the proof of the implication $(ii) \Rightarrow (i)$ in Lemma C.0.3 shows that W is $(l-1)$-connected.)

In the case $g > 0$, let us explain the procedure that modifies (W, G) to a pair (W', F') with the desired properties. Consider the handle pair $(h, h') := (h_1, h_1')$ of $X = X_g$.

Theorem 3.4.14 implies that we may additionally assume that ρ is a *stable* fold map, i.e. ρ has only double points with normal crossings. Then, we may introduce smooth curves $L \cong [0,1]$ in h and $L' \cong [0,1]$ in h' corresponding to straight lines $\xi(L)$ and $\xi(L')$ in $\mathbb{R}^2$ (see Figure 10.4) such that L and L' are transverse to $\rho(S(\rho))$ and miss all double points of ρ. Note that $M := \rho^{-1}(L)$ and $M' := \rho^{-1}(L')$ are connected closed smooth manifolds of dimension n, and ρ restricts to excellent Morse functions $f\colon M \to L \cong [0,1] \subset \mathbb{R}$ and $f'\colon M' \to L' \cong [0,1] \subset \mathbb{R}$ with only critical points of index contained in $\{0, l, \dots, n-l, n\}$. It follows from Lemma C.0.4(e) that M and M' are simply connected (and in particular orientable). Moreover, by Lemma C.0.4(b), f and f' have exactly one critical point of index 0 and n each. Let ν_λ and ν_λ' denote the number of critical points of f and f' of index $\lambda \in \{l, \dots, n-l\}$.

Next, consider the smooth curve $K \cong [0,1]$ in X, where $\xi(K)$ is indicated as a dashed curve in Figure 10.4. Then the submanifold $P := \rho^{-1}(K)$ of W is an oriented closed smooth manifold of dimension n that is diffeomorphic to $M \sharp M' \sharp (-M) \sharp (-M')$. (In fact, fix a diffeomorphism $K \cong [0,1]$ and introduce six points $a_1, \dots, a_6 \in [0,1]$ as indicated in Figure 10.4. Then there

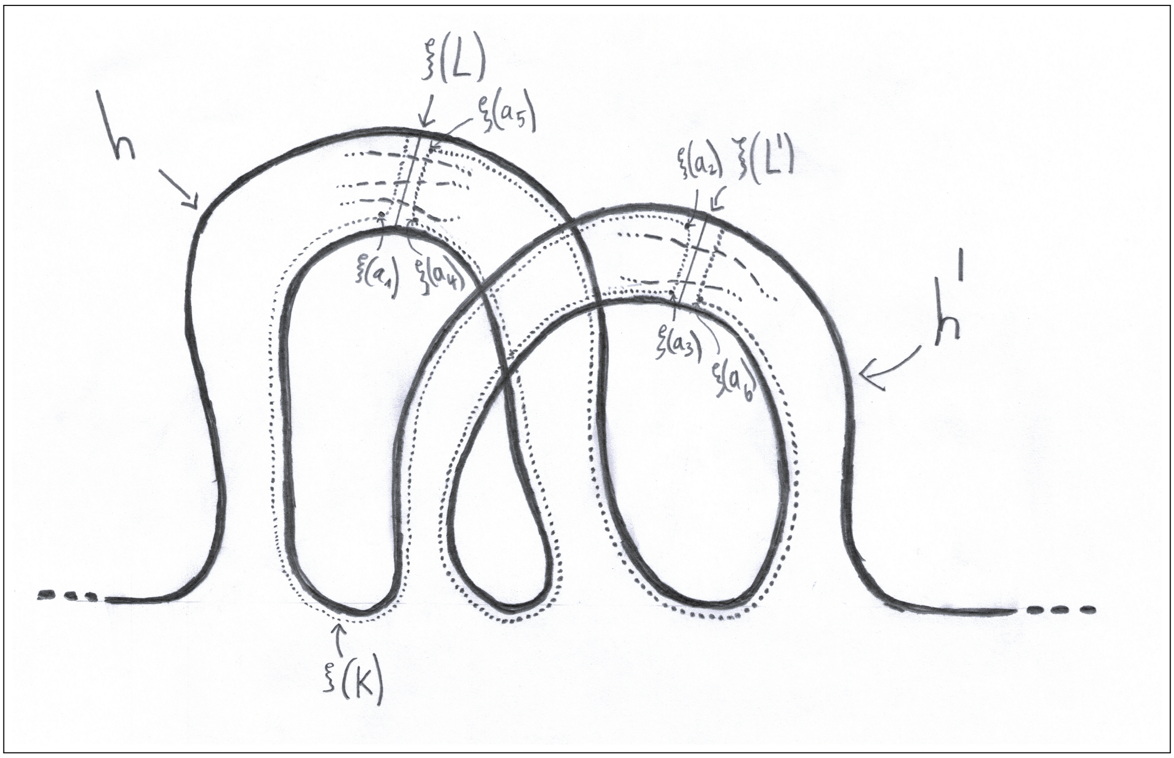

Figure 10.4: Cutting a handle pair of X

are diffeomorphisms

$$\rho^{-1}([0,a_2)) \cong M\setminus \mathrm{pt}, \qquad \rho^{-1}((a_3,a_6)) \cong M\setminus\{\mathrm{pt},\mathrm{pt}'\},$$
$$\rho^{-1}((a_1,a_4)) \cong M'\setminus\{\mathrm{pt},\mathrm{pt}'\}, \qquad \rho^{-1}((a_5,1]) \cong M'\setminus \mathrm{pt},$$
$$\rho^{-1}((a_1,a_2)) \cong \rho^{-1}((a_3,a_4)) \cong \rho^{-1}((a_5,a_5)) \cong (0,1)\times S^{n-1}.$$

Finally, note that the direct sum operation is associative up to orientation preserving diffeomorphism.) Furthermore, ρ restricts to an excellent Morse function $P\to K\cong[0,1]$ whose set of critical points consists of $\nu_\lambda+\nu'_\lambda+\nu_{n-\lambda}+\nu'_{n-\lambda}$ critical points of index $\lambda\in\{l,\dots,n-l\}$ and exactly one critical point of index 0 and n.

The result of cutting X along K is a smooth 2-dimensional manifold with corners. Let Y denote the arising component that contains $\rho(\Sigma)$. Note that Y is diffeomorphic via a diffeomorphism $\Psi\colon Y\xrightarrow{\cong} Z$ to the smooth 2-manifold with corners Z shown in Figure 10.5 that is immersed in $\mathbb{R}^2$ via an immersion $\zeta\colon Z\to\mathbb{R}^2$.

Note that $V:=\rho^{-1}(Y)$ is a connected oriented smooth manifold of dimension $n+1$ with boundary $\Sigma\sqcup P$, and the composition $\Psi\circ\rho$ restricts to a fold map $H\colon V\to\mathbb{R}^2$ such that $H^{-1}(\mathbb{R}\times 1)=P$.

Using Lemma 9.1.5, there exists a collar neighbourhood $P\times(1-\varepsilon,1]$ of P in V and a suitable $\varepsilon>0$ on which H is of the form

$$H(x,t)=(r(x,t),t), \qquad (x,t)\in P\times[0,\varepsilon),$$

where $x\mapsto r_t(x):=r(x,t)$ is an excellent Morse function on P for all $t\in[0,\varepsilon)$.

Application of Proposition 9.2.3 to the excellent Morse functions $f\colon M\to\mathbb{R}$ and $f'\colon M'\to$

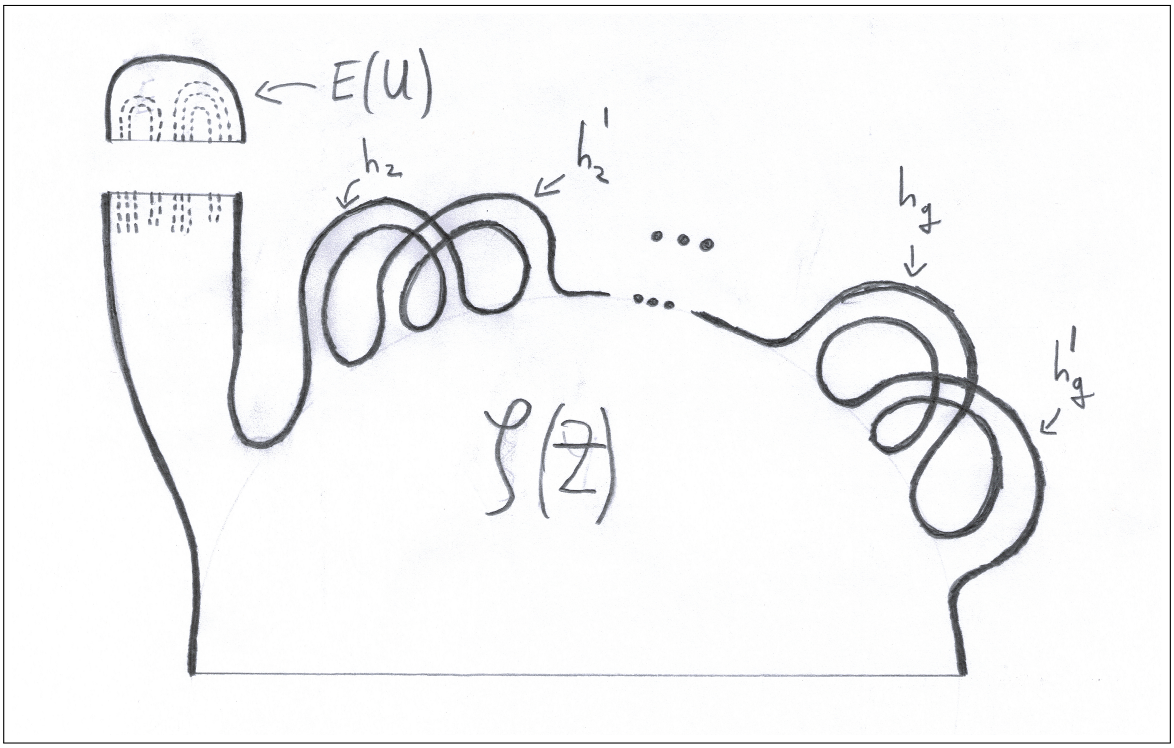

Figure 10.5: Gluing of fold maps

$\mathbb{R}$ yields a pair (U, E) (see Figure 10.5) consisting of a compact smooth manifold U^{n+1} of dimension $n+1$ with boundary $\partial U \cong (-M)\sharp M\sharp M'\sharp(-M')$ and a fold map $E\colon U \to \mathbb{R}^2$ with the following properties:

(i) For suitable $\varepsilon > 0$ there exists a collar neighbourhood $\partial U \times [0, \varepsilon)$ of $\partial U \times 0 = \partial U$ in U on which E is of the form

$$E(x, t) = (e_t(x), t), \qquad (x, t) \in \partial U \times [0, \varepsilon),$$

where e_t is an excellent Morse function $\partial U \to \mathbb{R}$ for all $t \in [0, \varepsilon)$.

(ii) The number of critical points of e_0 of index $\lambda \in \{0, \dots, n\}$ is given by

$$\nu_\lambda + \nu_{n-\lambda} + \nu'_\lambda + \nu'_{n-\lambda}.$$

(iii) All fold lines of E have absolute index in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and the Stein factorization U_E of E (see Theorem 9.1.7) is diffeomorphic to the unit half-disc X_0.

By commutativity of the connected sum operation, there exists a diffeomorphism $\Phi\colon \partial U \xrightarrow{\cong} P$. Therefore, by Theorem 9.2.4 we may glue the fold maps $E\colon U \to \mathbb{R}^2$ and $H\colon V \to \mathbb{R}^2$ along $\partial U \cong P$ to obtain the desired fold map $F'\colon W' \to \mathbb{R}^2$.

This completes the proof of the inclusion $G_n^l \subset \overline{C}_n^{l-1}$ in Theorem 10.1.3.

10.4 Detecting Kervaire Spheres via Indefinite Fold Singularities

We give here the proof of Corollary 10.1.4.

Proof. By Proposition 10.2.2 (iv), the equality $bP_{n+1} = C_n^{\lceil \frac{n}{2} \rceil - 1}$ holds because n is by assumption of the form $n+1 = 2k$ for a suitable odd integer $k \geq 113$. (In fact, writing $n = 16a + 13$ for a suitable integer a, one obtains $k = \frac{n+1}{2} = 8a + 7 \equiv 1 \pmod 2$. Moreover, $n \geq 237$ implies that $k = \frac{n+1}{2} \geq 113$.) Setting $l := \lceil \frac{n}{2} \rceil - 1 = k - 1 = 8a + 6 \equiv 6 \pmod 8$, Proposition 10.2.2 (iii) implies that $\overline{C}_n^l = \overline{C}_n^{l-1}$. Furthermore, Proposition 10.2.2 (i) implies that $\overline{C}_n^l = C_n^l$ and $\overline{C}_n^{l-1} = C_n^{l-1}$ because $n \equiv 1 \pmod 4$. All in all, $bP_{n+1} = C_n^{\lceil \frac{n}{2} \rceil - 1} = \overline{C}_n^{\lceil \frac{n}{2} \rceil - 1} = \overline{C}_n^{\lceil \frac{n}{2} \rceil - 2} = C_n^{\lceil \frac{n}{2} \rceil - 2}$.
Invoking Theorem 10.1.3 for n and $l = \lceil \frac{n}{2} \rceil - 1$ as above, one obtains $\overline{C}_n^l \subset G_n^l \subset \overline{C}_n^{l-1}$ in Θ_n. Hence, one finds

$$G_n^{\lceil \frac{n}{2} \rceil - 1} = bP_{n+1}.$$

Since $n \equiv 1 \pmod 4$ and n is not of the form $2^j - 3$ for an integer $j \geq 1$ by assumption, it follows from [35, Corollary 6.43, p. 136] that $bP_{n+1} \cong \mathbb{Z}/2$. The unique non-trivial element of bP_{n+1} is represented by the *Kervaire spere*, say Σ^n. Therefore, as $[\Sigma^n] \neq [S^n]$ in Θ_n, statement (i) holds if and only if $[\Sigma^n] \in bP_{n+1}$, or equivalently $[\Sigma^n] \in G_n^{\lceil \frac{n}{2} \rceil - 1}$. This holds if and only if there exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ whose fold lines are allowed to have an abosolute index contained in $\{\lceil \frac{n}{2} \rceil, \ldots, n - l\} \cup \{n\} = \{k\} \cup \{n\}$ (recall that $n = 2k - 1$ and $l = k - 1$), and such that there exists a special generic function $f \colon \Sigma \to \mathbb{R}$ with

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$

where $\Sigma \times [0, \varepsilon)$ is a collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for suitable $\varepsilon > 0$.

By a modification of W and F, one may assume that $S(F)$ has precisely one closed component, and this has absolute index k. (Indeed, closed components of $S(F)$ of absolute index n can be absorbed in the components of $S(F)$ diffeomorphic to $[0, 1]$ (which have necessarily absolute index n) by Proposition 9.2.2. Finally, the closed components of $S(F)$ of absolute index k can be connected to a single one by Proposition 6.1.3.)

It remains to check that $bP_{n+1} \subsetneq \Theta_n$ for $n \in \{237, 285, 333, 381, 445, 461, 477\}$. Indeed, it follows from the classification of homotopy spheres (see [35, Theorem 6.1, p. 123f]) that

$$\operatorname{coker} J_n \cong \Theta_n / bP_{n+1},$$

where $J_n \colon \pi_n(SO) \to \pi_n^s$ denotes the J-homomorphism. Therefore, it suffices to show that $\operatorname{coker} J_n$ is non-trivial. But Bott periodicity (see the proof of [27, Theorem 3.1, p. 508]) implies that $\pi_n(SO) = 0$ because $n \equiv 5 \pmod 8$. Hence, $\operatorname{coker} J_n \cong \pi_n^s$. Finally, [45, Table A3.5, p. 370ff] proves that $\pi_n^s \neq 0$ for the desired values of n. $\square$

10.5 The Aggregate Invariant of a Homotopy Sphere

First, we give here a proof of Proposition 10.1.5.

Proof. Theorem 3.4.9 allows the calculation of the state sum of a given cobordism by means of fold *maps* rather than fold *fields*. Since the aggregate invariant is defined in terms of state sums, we do not have to care about whether a fold map is a fold field in the following.

(a). Fix a homotopy sphere Σ^n of dimension $n \geq 6$. Given two pairs of special generic functions

$$f_S^{(1)}, f_S^{(2)} \colon S^n \to \mathbb{R}, \qquad f_\Sigma^{(1)}, f_\Sigma^{(2)} \colon \Sigma^n \to \mathbb{R},$$

the proof of [47, Lemma 3.1] implies that there exist smooth maps

$$F_S \colon S^n \times [0,1] \to \mathbb{R} \times [0,1], \qquad F_\Sigma \colon S^n \times [0,1] \to \mathbb{R} \times [0,1],$$

with only definite fold lines such that, for suitable $\varepsilon \in (0, 1/2)$,

$$\begin{aligned} F_S|_{S^n \times [0,\varepsilon]} &= f_S^{(2)} \times \mathrm{id}_{[0,\varepsilon]}, \qquad & F_S|_{S^n \times [1-\varepsilon,1]} &= f_S^{(1)} \times \mathrm{id}_{[1-\varepsilon,1]}, \\ F_\Sigma|_{\Sigma^n \times [0,\varepsilon]} &= f_\Sigma^{(1)} \times \mathrm{id}_{[0,\varepsilon]}, \qquad & F_\Sigma|_{\Sigma^n \times [1-\varepsilon,1]} &= f_\Sigma^{(2)} \times \mathrm{id}_{[1-\varepsilon,1]}. \end{aligned}$$

Let $W \in \mathrm{Cob}(S^n, \Sigma)$ be any oriented cobordism from S^n to Σ. Then it follows from cylindrical rigidity (see Corollary 9.1.8) of F_S and F_Σ that

$$Z_W(\overline{f}_S^{(1)}, \overline{f}_\Sigma^{(1)}) = Z_W(\overline{f}_S^{(2)}, \overline{f}_\Sigma^{(2)}).$$

Consequently,

$$\mathfrak{A}(\Sigma^n) = \sum_{\overline{f}_\Sigma \in C_2(\Sigma)} \sum_{W \in \mathrm{Cob}(S^n, \Sigma)} Z_W(\overline{f}_S, \overline{f}_\Sigma) = \sum_{W \in \mathrm{Cob}(S^n, \Sigma)} Z_W(\overline{f}_S, \overline{f}_\Sigma) \in Q$$

does not depend on the choice of f_S and f_Σ. Finally, if $[\Sigma_1^n] = [\Sigma_2^n]$ in Θ_n, then we claim that $\mathfrak{A}(\Sigma_1^n) = \mathfrak{A}(\Sigma_2^n)$. In fact, let V be an h-cobordism between Σ_1 and Σ_2. Given two special generic functions

$$f_1 \colon \Sigma_1 \to \mathbb{R}, \qquad f_2 \colon \Sigma_2 \to \mathbb{R},$$

the proof of [47, Lemma 3.1] implies that there exists a smooth map

$$F \colon V \cong \Sigma_1 \times [0,1] \to \mathbb{R} \times [0,1]$$

with only definite fold lines such that, for suitable $\varepsilon \in (0, 1/2)$,

$$F|_{\Sigma_1 \times [0,\varepsilon]} = f_1 \times \mathrm{id}_{[0,\varepsilon]}, \qquad F|_{\Sigma_2 \times [1-\varepsilon,1]} = f_2 \times \mathrm{id}_{[1-\varepsilon,1]},$$

on suitable collar neighbourhoods of $\Sigma_1 \subset V$ and $\Sigma_2 \subset V$. Fix a special generic function

$f_S\colon S^n \to \mathbb{R}$. Then it follows from cylindrical rigidity (see Corollary 9.1.8) of F that

$$\begin{aligned}\mathfrak{A}(\Sigma_1^n) &= \sum_{W_1 \in \mathrm{Cob}(S^n,\Sigma_1)} Z_{W_1}(\overline{f}_S, \overline{f}_1) = \sum_{W_1 \in \mathrm{Cob}(S^n,\Sigma_1)} \sum_{F_1 \in \mathcal{F}(W_1; f_S, f_1)} Y(\mathbb{S}(F_1)) \\ &= \sum_{W_2 \in \mathrm{Cob}(S^n,\Sigma_2)} \sum_{F_2 \in \mathcal{F}(W_2; f_S, f_2)} Y(\mathbb{S}(F_2)) = \sum_{W_2 \in \mathrm{Cob}(S^n,\Sigma_2)} Z_{W_2}(\overline{f}_S, \overline{f}_2) = \mathfrak{A}(\Sigma_2^n).\end{aligned}$$

(b). Concerning property (i), note that the first and third component of $\mathfrak{A}([\Sigma^n]) \in \mathbb{B}[[q]] \oplus \mathbb{B}[[q]] \oplus \mathbb{B}[[q]]$ are equal and of the desired form by Proposition 9.2.2. (In fact, the attachment of handles to analogous to Proposition 9.2.2 allows us to perform surgery on the definite fold lines of a fold map without changing the indefinite fold lines. Therefore, given a fold map F that is relevant to the calculation of $\mathfrak{A}(\Sigma^n)$, one can exchange the open Brauer morphisms $\supset\subset$ and $=$ in $\mathbb{S}(F)$ without changing the number of loops in $\mathbb{S}(F)$, or introduce precisely one more loop without changing the open part of the Brauer morphism $\mathbb{S}(F)$.) Furthermore, the middle component vanishes because the non-reduced index is always constant along fold lines, which is not the case for a fold line that connects the minimum of f_S with the maximum of f_Σ (or vice versa). Property (ii) is a reformulation of the results of the proofs of [4, Theorem 10.2, p. 81] and [4, Proposition 10.3, p. 84]. Finally, in order to show property (iii), suppose that $[\Sigma^n] \in G_n^l$. By definition of G_n^l there exists a pair (W, F), where W is a (connected) oriented compact smooth manifold of dimension $n+1$ such that $\partial W = \Sigma$, and $F\colon W \to \mathbb{R}^2$ is a fold map whose fold lines have absolute index in $\{\lceil \frac{n}{2} \rceil, \dots, n-l\} \cup \{n\}$, and such that F is the suspension of a special generic function $\Sigma \to \mathbb{R}$ in a suitable collar neighbourhood of $\Sigma \subset W$. By Proposition 9.2.2 we may in addition assume that F has no definite loops. Then, application of Proposition 6.1.3 relative to the set $\mathcal{R}$ consisting of the unique definite fold line of F and to the nice subset $A_1 := \{\lceil \frac{n}{2} \rceil, \dots, n-l\}$ of $\{\lceil \frac{n}{2} \rceil, \dots, n\}$, we may modify F in a compact subset of $W \setminus \partial W$ in such a way that the resulting fold map has exactly $(n-l) - \lceil \frac{n}{2} \rceil + 1 = \lfloor \frac{n}{2} \rfloor + 1 - l$ loops. Hence, property (i) implies that $\mathfrak{a}([\Sigma^n]) \leq \lfloor \frac{n}{2} \rfloor + 1 - l$. □

Combining Proposition 10.1.5 and Theorem 10.1.3, one obtains the following

Corollary 10.5.1. *Suppose that* $n \geq 7$ *and* $n \neq 15$. *Then,* $\mathfrak{a}([\Sigma^n]) = 1$ *for all* $[S^n] \neq [\Sigma^n] \in bP_{n+1} \subset \Theta_n$.

Proof. As $bP_{n+1} = \{[S^n]\}$ for n even, we may assume that n is odd. In this case properties (iv) and (i) of Proposition 10.2.2 imply for $l = \lceil \frac{n}{2} \rceil - 1 = (n-1)/2$ that

$$bP_{n+1} \overset{(iv)}{\subset} C_n^{(n-1)/2} \overset{(i)}{=} \overline{C}_n^{(n-1)/2}.$$

Therefore, $bP_{n+1} \subset \overline{C}_n^{(n-1)/2} \subset G_n^{(n-1)/2}$ by Theorem 10.1.3. Finally, if $[S^n] \neq [\Sigma^n] \in bP_{n+1} \subset \Theta_n$, then properties (ii) and (iii) of Proposition 10.1.5 (b) imply that

$$1 \overset{(ii)}{\leq} \mathfrak{a}([\Sigma^n]) \overset{(iii)}{\leq} \lfloor \frac{n}{2} \rfloor + 1 - l = (n-1)/2 + 1 - (n-1)/2 = 1.$$

□

10.6 Outlook

Finally, let us indicate some remarks and open questions that arise from our results:

- Are there more cases than that presented in Proposition 10.2.2 in which the groups $\overline{C}_n^l$ and C_n^l can be calculated? Is there an example where $\overline{C}_n^l \subsetneq C_n^l$ in Proposition 10.2.2 (i)?
- What is the acutal size of the group G_n^l in Theorem 10.1.3? At present, no example is known to the author where $\overline{C}_n^l \subsetneq G_n^l$ or $G_n^l \subsetneq \overline{C}_n^{l-1}$ for some pair (n, l). It would be desirable to have $\overline{C}_n^l = G_n^l$.
- What happens for the value $l = \lceil \frac{n}{2} \rceil$ in Theorem 10.1.3? (Here, one has to extend the definition of G_n^l and $\overline{C}_n^l$ to $l = \lceil \frac{n}{2} \rceil$ in the obvious way.) If n is odd, then the statement of Theorem 10.1.3 holds trivially because $G_n^l = 0$ by Theorem 10.1.1 and also $\overline{C}_n^l = 0$. If $n = 2k$ is even, then the inclusion $\overline{C}_n^l \subset G_n^l$ is trivial since $\overline{C}_n^l = 0$ as almost closed k-connected $(2k+1)$-manifolds are contractible, so the boundary is the standard sphere. However, concerning the other inclusion, the author can currently only show $G_n^l \subset \overline{C}_n^{l-2}$ for the value $l = \lceil \frac{n}{2} \rceil$, which is basically due to the fact that one needs critical points whose indices are in two subsequent dimensions to apply the Smale trick.
- Consider a homotopy sphere Σ^n with the following property. There exists a pair (W^{n+1}, F) consisting of a compact oriented smooth manifold W^{n+1} of dimension $n+1$ such that $\partial W = \Sigma$, and a fold map $F \colon W \to \mathbb{R}^2$ with a single closed fold line, and such that there exists a special generic function $f \colon \Sigma \to \mathbb{R}$ with

$$F|_{\Sigma \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)} \colon \Sigma \times [0, \varepsilon) \to \mathbb{R} \times [0, \varepsilon),$$

  where $\Sigma \times [0, \varepsilon)$ is a collar neighbourhood of $\Sigma \times 0 = \Sigma$ in W for suitable $\varepsilon > 0$. Are there similar conclusions to that of Corollary 10.1.4? (Note that the difference to Corollary 10.1.4 is that the absolute index of the unique closed fold line of F here is not required to have absolute index $\lceil \frac{n}{2} \rceil$.)
- If one refines Banagl's positive TFT based on fold maps in such a way that fold loops of different absolute indices are counted separately (which would require a certain enrichment of the Brauer category that assigns numbers to loops and arcs), then Corollary 10.1.4 implies that the redefined aggregate invariant can detect the Kervaire sphere in certain dimensions.

Appendix

Appendix A

Transversality

The underlying reference for the present chapter is [17].

In the following, X and Y denote smooth manifolds (without boundary). The purpose of the following sections is to prove Proposition A.3.2. This can be considered as a relative version of the Thom transversality theorem and allows to extend transversality conditions in a jet space of smooth maps $X \to Y$ over compact subsets.

A.1 Transversality

In the present section we recall the fundamental principle of *transversality*. We start with the definitions given in [17, Definition II.4.1, page 50] and [17, Definition II.4.8, page 54]):

Definition A.1.1. Let $f \colon X \to Y$ be smooth and let W be a submanifold of Y.

(a) Given a point $x \in X$, we say that f *intersects* W *transversally at* x (and write $f \pitchfork W$ at x) if either (i) $f(x) \notin W$ or (ii) $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$.
(b) If A is a subset of X, then we say that f *intersects* W *transversally on* A (and write $f \pitchfork W$ on A) if $f \pitchfork W$ at x for all $x \in A$.
(c) If B is a subset of W, then we say that f *intersects* W *transversally on* B (and write $f \pitchfork W$ on B) if $f \pitchfork W$ on $f^{-1}(B)$.
(d) Finally, we say that f *intersects* W *transversally* (and write $f \pitchfork W$) if $f \pitchfork W$ on X.

The following Lemma is an immediate consequence of Definition A.1.1(a):

Lemma A.1.2. *Let* $f \colon X \to Y$ *be smooth and let* W *be a submanifold of* Y*.*

(a) Assume that $x \in X$ *is a point and* $W' \subset W$ *is an open subset. (Note that* W' *is also a submanifold of* Y*.) Then the following holds:*
(1) If $f \pitchfork W$ *at* x*, then* $f \pitchfork W'$ *at* x*.*
(2) If $f \pitchfork W'$ *at* x *and* $f(x) \in W'$*, then* $f \pitchfork W$ *at* x*.*
(b) Assume that $x \in X'$*, where* X' *is an open subset of* X*. Let* $f' \colon X' \to Y$ *denote the restriction of* f *to* X'*. Then* $f \pitchfork W$ *at* x *if and only if* $f' \pitchfork W$ *at* x*.*

Proof. (a). Note that if $f(x) \in W'$, then $T_{f(x)}W' = T_{f(x)}W$ in $T_{f(x)}Y$ as W' is open in W.

(1). We may assume that $f(x) \in W'$. Since $f \pitchfork W$ at x by assumption and $f(x) \in W' \subset W$, Definition A.1.1(a) implies that $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX) = T_{f(x)}W' + (df)_x(T_xX)$. Hence, $f \pitchfork W'$ at x by Definition A.1.1(a).

(2). Since $f \pitchfork W'$ at x and $f(x) \in W'$ by assumption, Definition A.1.1(a) implies that $T_{f(x)}Y = T_{f(x)}W' + (df)_x(T_xX) = T_{f(x)}W + (df)_x(T_xX)$. Hence, $f \pitchfork W$ at x by Definition A.1.1(a).

(b). Set $y := f(x) = f'(x)$. If $y \notin W$, then we have $f \pitchfork W$ at x and $f' \pitchfork W$ at x by Definition A.1.1(a). If $y \in W$, then Definition A.1.1(a) implies that $f \pitchfork W$ at x if and only if $T_yY = T_yW + (df)_x(T_xX)$. Equivalently, $T_yY = T_yW + (df')_x(T_xX')$. (Note that $(df)_x(T_xX) = (df')_x(T_xX')$ as X' is an open subset of X.) The previous equality holds if and only if $f' \pitchfork W'$ at x.

□

Lemma A.1.3. *Let $f\colon X \to Y$ be smooth and let W be a submanifold of Y. Assume that $x \in X$ is a point such that $f \pitchfork W$ at x and $f(x) \in W$. Then there exists an open neighbourhood $U \subset X$ of x such that $f \pitchfork W$ on U.*

Proof. Let $\pi\colon \mathbb{R}^m = \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^{m-k}$ denote the projection to the first factor. Since $W^k \subset Y^m$ is a submanifold, there exists a chart $\phi\colon V \to V' \subset \mathbb{R}^m$ around $f(x) \in Y$ such that $W \cap V = \phi^{-1}(V' \cap N)$, where $N := \{z \in \mathbb{R}^m;\ z_1 = \cdots = z_{m-k} = 0\} = \pi^{-1}(0)$. Being the composition of submersions, the map $\psi\colon V \to \mathbb{R}^{m-k}$ defined by $\psi(y) = \pi(\phi(y))$ for all $y \in V$ is a submersion. Note that $W \cap V = \psi^{-1}(0)$. (In fact, $z \in V$ satisfies $\psi(z) = \pi(\phi(z)) = 0$ if and only if $\phi(z) \in V' \cap \pi^{-1}(0) = V' \cap N$. Equivalently, $z \in \phi^{-1}(V' \cap N) = W \cap V$.) By [17, Lemma II.4.3, page 52], the composition $g := \psi \circ f$, which is defined as a map $g\colon f^{-1}(V) \to \mathbb{R}^{m-k}$, is a submersion at x. Hence, there exists an open subset $U \subset X$ such that $x \in U \subset f^{-1}(V)$ and such that g is a submersion at all points of U. It remains to show that $f \pitchfork W$ on U. We fix a point $x' \in U$ and have to show that $f \pitchfork W$ at x'. By Definition A.1.1(a), we can assume that $f(x') \in W$. By construction, $f(x') \in f(U) \subset f(f^{-1}(V)) \subset V$, so V is an open neighbourhood of $f(x')$ in Y. Recall that the submersion $\psi\colon V \to \mathbb{R}^{m-k}$ satisfies $W \cap V = \psi^{-1}(0)$. Applying [17, Lemma II.4.3, page 52] again, we obtain $f \pitchfork W$ at x' since $f(x') \in W$ and $g = \psi \circ f$ is a submersion at x'. □

A.2 Transversality in Jet Manifolds

Lemma A.2.1. *Let A be a closed subset of Y and let W be a submanifold of Y such that $A \subset W$. Then*

(a) $\{f \in C^\infty(X,Y);\ f(X) \cap A = \emptyset\}$,

(b) $\{f \in C^\infty(X,Y);\ f \pitchfork W \text{ on } A\}$

is an open subset of $C^\infty(X,Y)$ in the Whitney C^1 topology, and thus, C^∞ topology.

Proof. (a). Let $\beta\colon J^1(X,Y) \to Y$ be the target map. Since A is a closed subset of Y, $U := \beta^{-1}(Y \setminus A)$ is an open subset of $J^1(X,Y)$. By definition of the Whitney C^1 topology, see [17, Definition II.3.1(iii), page 42], we conclude that

$$M(U) := \{f \in C^\infty(X,Y);\ (j^1 f)(X) \subset U\}$$

is an open subset of $C^\infty(X,Y)$ in the Whitney C^1 topology. Set

$$T := \{f \in C^\infty(X,Y);\ f(X) \cap A = \emptyset\}.$$

By construction, $M(U) = T$. (In fact, given $f \in C^\infty(X,Y)$, we have $f \in T$ if and only if $f(x) \in Y \setminus A$ for all $x \in X$. Since $\beta((j^1 f)(x)) = f(x)$ for all $x \in X$, we have $f \in T$ if and only if $(j^1 f)(x) \in \beta^{-1}(Y \setminus A) = U$ for all $x \in X$. Hence, $f \in T$ if and only if $f \in M(U)$.) This completes the proof of (a).

(b). The argument is an adaption of the proof of [17, Proposition II.4.5, page 52]. For completeness, we present it in full detail.

In the following, let $\sigma \in J^1(X,Y)$ be a 1-jet with source $x \in X$ and target $y \in Y$ and let $f\colon X \to Y$ be a representative of σ. (In particular, $y = f(x)$, and the linear map $(df)_x\colon T_xX \to T_yY$ does not depend on the representative f.) Define a subset $U \subset J^1(X,Y)$ as follows. We require that $\sigma \in U$ if and only if either $y \notin A$ or $y \in A$ and $T_yY = T_yW + (df)_x(T_xX)$. We set

$$M(U) := \{f \in C^\infty(X,Y);\ (j^1 f)(X) \subset U\},$$
$$T := \{f \in C^\infty(X,Y);\ f \pitchfork W \text{ on } A\}.$$

By construction, $T = M(U)$. (In fact, given $f \in C^\infty(X,Y)$, we have $f \in T$ if and only if $f \pitchfork W$ at x for all $x \in f^{-1}(A)$ (see Definition A.1.1(c)). Since $A \subset W$, this is satisfied if and only if $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$ for all $x \in X$ with $f(x) \in A$. Equivalently, $(j^1 f)(x) \in U$ for all $x \in X$. This holds if and only if $f \in M(U)$.)

By definition of the Whitney C^1 topology, see [17, Definition II.3.1(iii), page 42], it suffices to show that U is an open subset of $J^1(X,Y)$. (Then $T = M(U)$ will be an open subset of $C^\infty(X,Y)$ in the Whitney C^1 topology, which completes the proof of (b).)

Consider the complement

$$V := J^1(X,Y) \setminus U = \{\sigma \in J^1(X,Y);\ y \in A \text{ and } T_{f(x)}Y \neq T_{f(x)}W + (df)_x(T_xX)\},$$

where σ denotes a 1-jet with source $x \in X$ and target $y \in Y$, and $f\colon X \to Y$ is a representative of σ. It suffices to construct for every $\sigma \in J^1(X,Y)$ an open neighbourhood $T_\sigma \subset J^1(X,Y)$ of σ such that the intersection $V \cap T_\sigma$ is a closed subset of T_σ. (In fact, this implies that U is an open subset of $J^1(X,Y)$ as follows. Note that for every $\sigma \in J^1(X,Y)$ the complement of $V \cap T_\sigma$ in T_σ is given by $U \cap T_\sigma$. By construction, $U \cap T_\sigma$ is an open subset of T_σ, and hence an open subset of $J^1(X,Y)$. Therefore, the union

$$\bigcup_{\sigma \in J^1(X,Y)} (U \cap T_\sigma) = U \cap \Big(\bigcup_{\sigma \in J^1(X,Y)} T_\sigma \Big) = U \cap J^1(X,Y) = U$$

is also an open subset of $J^1(X,Y)$.)

Let $\sigma \in J^1(X,Y)$ be a 1-jet with source $x \in X$ and target $y \in Y$ and let $f\colon X \to Y$ be a representative of σ. Let us construct the desired open neighbourhood T_σ of σ in $J^1(X,Y)$. If $y \notin A$, then σ is an element of $T_\sigma := \beta^{-1}(Y \setminus A)$. Note that T_σ is an open subset of $J^1(X,Y)$ because A is a closed subset of Y by assumption. Obviously, $V \cap T_\sigma = \emptyset$, which is a closed subset of T_σ. This completes the construction of T_σ in the case $y \notin A$. Next, we assume that $y \in A$. Set $n := \dim X$, $m := \dim Y$ and $k := \dim W$. Since W is a submanifold of Y and $y \in W$, there exists a chart $\varphi\colon Y' \xrightarrow{\cong} Y'' \subset \mathbb{R}^m$ around y such that $W \cap Y' = \varphi^{-1}(Y'' \cap N)$, where $N := \{z \in \mathbb{R}^m;\ z_1 = \cdots = z_{m-k} = 0\}$. Let $\psi\colon X' \xrightarrow{\cong} X'' \subset \mathbb{R}^n$ be a chart around x such that $f(X') \subset Y'$. By definition of the manifold structure on $J^1(X,Y)$ in the proof of [17, Theorem II.2.7(1), page 40], a chart around σ is given by

$$\xi\colon J^1(X',Y') \xrightarrow{\cong} J^1(X'',Y'') = X'' \times Y'' \times \operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^m),$$
$$\xi(\sigma') = (\psi(x'), \varphi(y'), d\varphi_{y'} \circ df'_{x'} \circ (d\psi_{x'})^{-1}),$$

where $\sigma' \in J^1(X',Y')$ is a 1-jet with source $x' \in X'$ and target $y' \in Y'$, and $f'\colon X' \to Y'$ is a representative of σ'. In particular, $T_\sigma := J^1(X',Y')$ is an open neighbourhood of $\sigma \in J^1(X,Y)$. It remains to show that $V \cap T_\sigma$ is a closed subset of T_σ.

Let $\sigma' \in T_\sigma \subset J^1(X,Y)$ be a 1-jet with source $x' \in X'$ and target $y' \in Y'$ and let $f'\colon X \to Y$ be a representative of σ'. We have $\sigma' \in V$ if and only if $y' \in A$ and $T_{y'}Y \neq T_{y'}W + (df')_{x'}(T_{x'}X)$, i.e. $f'(x') \in A$ and f' does *not* intersect W transversally at x'. Let $\pi\colon \mathbb{R}^m = \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^{m-k}$ be the projection to the first factor. The submersion $\phi\colon Y' \to \mathbb{R}^{m-k}$ defined by $\phi(z) = \pi(\varphi(z))$ for all $z \in Y'$ satisfies $\phi^{-1}(0) = W \cap Y'$. (In fact, $z \in Y'$ satisfies $\phi(z) = \pi(\varphi(z)) = 0$ if and only if $\varphi(z) \in Y'' \cap \pi^{-1}(0) = Y'' \cap N$. Equivalently, $z \in \varphi^{-1}(Y'' \cap N) = W \cap Y'$.) By [17, Lemma II.4.3, page 52], we obtain

$$V \cap T_\sigma = \{\sigma' \in T_\sigma;\ y' \in A \text{ and } \phi \circ f'\colon f'^{-1}(Y') \to \mathbb{R}^{m-k} \text{ is } \textit{not} \text{ a submersion at } x'\}.$$

Note that $\phi \circ f'$ is *not* a submersion at x' if and only if $d(\phi \circ f')_{x'} = \pi \circ d\varphi_{y'} \circ df'_{x'}$ has rank $< m-k$. Equivalently, $\eta(\xi(\sigma')) \in F$, where

$$\eta\colon J^1(X'',Y'') \to \operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^{m-k}), \qquad \eta(x,y,B) = \pi \circ B,$$

and $F := \{B \in \operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^{m-k});\ \operatorname{rank} B < m-k\}$. All in all, we have shown

$$\xi(V \cap T_\sigma) = X'' \times \varphi(A \cap Y') \times \eta^{-1}(F).$$

$\varphi(A \cap Y')$ is a closed subset of Y'' as A is a closed subset of Y. Moreover, note that F is a closed subset of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k})$. (Indeed, by [17, Proposition II.5.3, page 60] the complement $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k}) \setminus F = L^0(\mathbb{R}^n, \mathbb{R}^{m-k})$ is a submanifold of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m-k})$ of codimension $(n - q + r)(m - k - q + r) = 0$, where $r = 0$ and $q = \min(n, m - k)$.) All in all, we conclude that $V \cap T_\sigma$ is a closed subset of T_σ. □

Remark A.2.2. Part (b) of the previous Lemma is used in the proof of the Thom transversality theorem [17, Theorem II.4.9, page 54]. It is a slight generalization of [17, Proposition II.4.5, page 52]. In fact, [17, Proposition II.4.5, page 52] is the special case that W is closed as a subset of Y and $A := W$.

Corollary A.2.3. *Let A be a closed subset of $J^k(X, Y)$ and let W be a submanifold of $J^k(X, Y)$ such that $A \subset W$. Then*

(a) $\{f \in C^\infty(X, Y);\ j^k(f)(X) \cap A = \emptyset\}$,

(b) $\{f \in C^\infty(X, Y);\ j^k(f) \pitchfork W \text{ on } A\}$

is an open subset of $C^\infty(X, Y)$ in the Whitney C^∞ topology.

Proof. It follows from Lemma A.2.1 that

$$S := \{g \in C^\infty(X, J^k(X, Y));\ g(X) \cap A = \emptyset\},$$
$$T := \{g \in C^\infty(X, J^k(X, Y));\ g \pitchfork W \text{ on } A\}$$

are open subsets of $C^\infty(X, J^k(X, Y))$ in the Whitney C^∞ topology. Since

$$j^k \colon C^\infty(X, Y) \to C^\infty(X, J^k(X, Y)), \qquad g \mapsto j^k(g),$$

is continuous in the Whitney C^∞ topology by [17, Proposition II.3.4, page 46],

$$(j^k)^{-1}(S) = \{f \in C^\infty(X, Y);\ j^k(f)(X) \cap A = \emptyset\},$$
$$(j^k)^{-1}(T) = \{f \in C^\infty(X, Y);\ j^k(f) \pitchfork W \text{ on } A\}$$

are open subsets of $C^\infty(X, Y)$ in the C^∞ topology. □

A.3 An Application of the Thom Transversality Theorem

The following Lemma collects some elementary statements which will be used in the proof of Proposition A.3.2.

Lemma A.3.1. *(a) Let $U \subset W \subset X$, where W is a submanifold of X and U is an open subset of W. Then $\overline{W}^X \setminus U$ is a closed subset of X.*

(b) Let $C \subset U \subset X$, where C is compact and U is open in X. Then there exists an open subset $V \subset X$ such that $C \subset V$ and $\overline{V}^X$ is a compact subset of U.

(c) Let K and A be disjoint subsets of a metric space (Z, d), where K is compact and A is a closed subset of Z. Then there exists $\delta > 0$ such that $d(z, a) \geq \delta$ for all $z \in K$ and all $a \in A$.

Proof. (a). It suffices to construct an open subset $V \subset X$ such that $\overline{W}^X \cap V = U$. (In fact, $\overline{W}^X \setminus U = \overline{W}^X \setminus (\overline{W}^X \cap V) = \overline{W}^X \setminus V$ will then be a closed subset of X.)

Every point $x \in W$ has an open neighbourhood $V_x \subset X$ such that $V_x \cap W$ is a closed subset of V_x. (In fact, since W^k is a submanifold of X^n, there exists a chart $\phi\colon V \to V' \subset \mathbb{R}^n$ of X around any given point $x \in W$ such that $V \cap W = \phi^{-1}(V' \cap N)$, where $N := \{z \in \mathbb{R}^n;\ z_{k+1} = \cdots = z_n = 0\}$. Noting that N is a closed subset of $\mathbb{R}^n$, we may take $V_x := V$.) Fix a point $x \in W$. By construction, $U_x := V_x \setminus W$ is an open subset of V_x (and thus, of X) such that $W \cap U_x = \emptyset$. Thus, $\overline{W}^X \cap U_x = \emptyset$. Since $V_x = U_x \cup (W \cap V_x)$, this implies $\overline{W}^X \cap V_x = \overline{W}^X \cap W \cap V_x = W \cap V_x$. Hence, $V' := \bigcup_{x \in W} V_x$ is an open subset of X such that

$$\overline{W}^X \cap V' = \overline{W}^X \cap \Big(\bigcup_{x \in W} V_x\Big) = \bigcup_{x \in W} \overline{W}^X \cap V_x = \bigcup_{x \in W} W \cap V_x = W \cap \Big(\bigcup_{x \in W} V_x\Big) = W \cap V'.$$

Since U is an open subset of W, there exists an open subset $U' \subset X$ such that $U = W \cap U'$. Finally, using $W \subset V'$, the open subset $V := V' \cap U'$ of X satisfies

$$\overline{W}^X \cap V = \overline{W}^X \cap V' \cap U' = W \cap V' \cap U' = W \cap U' = U.$$

(b). V can be constructed in the following way. For every $x \in C$ we choose a compact neighbourhood C_x of x in U and an open subset $U_x \subset U$ such that $x \in U_x \subset C_x$. Consequently, U_x is an open neighbourhood of x in X such that $\overline{U_x}^X \subset C_x$. (Note that C_x is a closed subset of X, since C_x compact and X is a Hausdorff space.) As C is compact, we can extract a finite subcovering $\bigcup_{x \in \Sigma} U_x$ from the open covering $C \subset \bigcup_{x \in C} U_x$. We can conclude that $V := \bigcup_{x \in \Sigma} U_x$ is an open subset of X which contains C. Furthermore, $\overline{V}^X \subset \bigcup_{x \in \Sigma} \overline{U_x}^X \subset \bigcup_{x \in \Sigma} C_x \subset U$. (Note that the first inclusion uses the finiteness of Σ. In fact, $\bigcup_{x \in \Sigma} \overline{U_x}^X$ is a closed subset of X.) Finally, $\overline{V}^X$ is compact, being a closed subset of the compact space $\bigcup_{x \in \Sigma} C_x$.

(c). If there is no such δ, then there exist sequences $z_1, z_2, \ldots$ in K and $a_1, a_2, \ldots$ in A such that $d(z_i, a_i) < \frac{1}{i}$ for all integers $i > 0$. Since K is compact and metrizable, we may assume that $z_1, z_2, \ldots$ has a limit point $z \in K$ by passing to a subsequence. The triangle inequality $d(a_i, z) \leq d(a_i, z_i) + d(z_i, z)$ shows that z is also the limit point of the sequence $a_1, a_2, \ldots$. Since A is a closed subset of Z, we conclude that $z \in A$. This yields the contradiction $z \in K \cap A = \emptyset$. □

The following Lemma is essentially an application of [17, Corollary II.4.11, page 56], which in turn follows from the proof of the Thom transversality theorem [17, Theorem II.4.9, page 54]. Roughly speaking, assume that $f\colon X \to Y$ satisfies a certain transversality condition in a jet space on the complement of a compact subspace in X. In this situation, the following result shows that f can be approximated rel boundary conditions by a map $g\colon X \to Y$ which satisfies this transversality condition on all of X.

Proposition A.3.2. *Let $f\colon X \to Y$ be smooth, $C \subset X$ a compact subspace and $W \subset J^k(X,Y)$ a submanifold such that the k-jet $j^k(f)\colon X \to J^k(X,Y)$ has the following properties:*

(1) $j^k(f) \pitchfork W$ on $X \setminus C$.
(2) $\overline{W}^{J^k(X,Y)} \cap j^k(f)(X \setminus C) = W \cap j^k(f)(X \setminus C)$.

Given an open subset $U \subset X$ such that $C \subset U$ and $\overline{U}^X$ is compact and an open neighbourhood N of $f \in C^\infty(X,Y)$ (in the C^∞ topology), there exists a smooth map $g \in N$ such that $g|_{X\setminus U} = f|_{X\setminus U}$ and $j^k(g) \pitchfork W$.

Proof. Using Lemma A.3.1(b), we choose open subsets $V, V' \subset X$ such that $C \subset V$, $\overline{V}^X \subset V'$ and $\overline{V'}^X \subset U$. By assumption, $C' := \overline{U}^X$ is compact. Note that $C'' := C' \setminus V'$ is a compact subspace of $X \setminus C$, as V' is an open subset of the compact space C' and $C \subset V'$. Hence, $K := j^k(f)(C'')$ is a compact subspace of $j^k(f)(X \setminus C)$. It follows from (2) that $R := j^k(f)^{-1}(W) \cap (X \setminus C)$ is a closed subset of $X \setminus C$. (In fact, let $x_1, x_2, \dots$ be a sequence in R with limit point $x \in X \setminus C$. It suffices to show that $x \in R$ as the manifold $X \setminus C$ is a first-countable topological space. Application of the continuous map $j^k(f)$ yields the sequence $j^k(f)(x_1), j^k(f)(x_2), \dots$ with limit point $j^k(f)(x)$ in $J^k(X,Y)$. As $j^k(f)(x_1), j^k(f)(x_2), \dots$ is a sequence in W by definition of R, we conclude that $j^k(f)(x) \in \overline{W}^{J^k(X,Y)}$. Hence, it follows from $x \in X \setminus C$ and (2) that $j^k(f)(x) \in \overline{W}^{J^k(X,Y)} \cap j^k(f)(X \setminus C) \subset W$. Therefore, $x \in j^k(f)^{-1}(W) \cap (X \setminus C) = R$.) In particular, the intersection $C'' \cap R = C'' \cap j^k(f)^{-1}(W)$ of the compact subspace $C'' \subset X \setminus C$ and the closed subset $R \subset X \setminus C$ is compact. Thus, $L := j^k(f)(C'' \cap j^k(f)^{-1}(W)) = j^k(f)(C'') \cap W = K \cap W$ is compact.

Recall that the source map $\alpha\colon J^k(X,Y) \to X$ satisfies $\alpha \circ j^k(l) = \mathrm{id}_X$ for all $l \in C^\infty(X,Y)$. We have $\alpha(L) \subset \alpha(K) = (\alpha \circ j^k(f))(C'') = C'' = C' \setminus V' \subset X \setminus \overline{V}^X =: \check{V}$. Consequently, $L \subset W \cap \alpha^{-1}(\check{V}) =: \check{W}$. Using Lemma A.3.1(b), we choose an open subset $Z \subset W$ such that $L \subset Z$ and $\overline{Z}^W$ is a compact subset of $\check{W}$. (Indeed, we have $L \subset \check{W} \subset W$, where L is compact and $\check{W} = W \cap \alpha^{-1}(\check{V})$ is an open subset of the manifold W.) Note that $\overline{Z}^W$ is a closed subset of $J^k(X,Y)$, because $\overline{Z}^W$ is compact and $J^k(X,Y)$ is a Hausdorff space. Hence, Corollary A.2.3(b) implies that

$$T := \{h \in C^\infty(X,Y);\ j^k(h) \pitchfork \check{W} \text{ on } \overline{Z}^W\}$$

is an open subset of $C^\infty(X,Y)$ in the C^∞ topology. Note that $f \in T$. (In fact, by Definition A.1.1, we have to show that $j^k(f) \pitchfork \check{W}$ at x for all $x \in j^k(f)^{-1}(\overline{Z}^W)$. Given such an x, we observe $j^k(f)(x) \in \overline{Z}^W \subset \check{W}$. As $x \in j^k(f)^{-1}(\check{W}) \subset j^k(f)^{-1}(\alpha^{-1}(\check{V})) = (\alpha \circ j^k(f))^{-1}(\check{V}) = \check{V} = X \setminus \overline{V}^X \subset X \setminus C$, we have $j^k(f) \pitchfork W$ at x by assumption (1). Since $\check{W} = W \cap \alpha^{-1}(\check{V})$ is an open subset of W and $j^k(f)(x) \in \check{W}$, Lemma A.1.2(a) yields $j^k(f) \pitchfork \check{W}$ at x.)

We choose a metric d on the smooth manifold $J^k(X,Y)$ such that d is compatible with the topology on $J^k(X,Y)$. (This is possible by [17, Lemma I.5.9, page 24].) Choose $\delta > 0$ such that $d(a,b) \geq \delta$ for all $a \in K$ and all $b \in W \setminus Z$. (Indeed, we can apply Lemma A.3.1(c) to the subsets K and $A := \overline{W}^{J^k(X,Y)} \setminus Z$ of the metric space $(J^k(X,Y), d)$. Note that K is compact and A is a closed subset of $J^k(X,Y)$ by Lemma A.3.1(a). Furthermore, K and A are disjoint, since $K \subset j^k(f)(X \setminus C)$ implies $K \cap \overline{W}^{J^k(X,Y)} = K \cap W = L \subset Z$ by (2).) By [17, page 43],

$$B_\delta(f) := \{h \in C^\infty(X,Y);\ d(j^k(f)(x), j^k(h)(x)) < \delta \quad \forall\, x \in X\}$$

is an open neighbourhood of f in $C^\infty(X,Y)$ in the C^k, and thus C^∞, topology.

Set $W' := W \cap \alpha^{-1}(V')$, which is an open subset of W. By [17, Corollary II.4.11, page 56], there exists a smooth map $g\colon X \to Y$ such that $g \in N \cap T \cap B_\delta(f)$, $j^k(g) \pitchfork W'$ and $g|_{X\setminus U} = f|_{X \setminus U}$. (In fact, $W' = W \cap \alpha^{-1}(V')$ is a submanifold of $J^k(X,Y)$ such that $\overline{\alpha(W')}^X \subset \overline{V'}^X \subset U$. Moreover, by construction, $N \cap T \cap B_\delta(f)$ is an open neighbourhood of $f \in C^\infty(X,Y)$.) In particular, we have $g \in N$ and $g|_{X\setminus U} = f|_{X\setminus U}$. It remains to show that $j^k(g) \pitchfork W$. We fix a point $x \in X$ and have to show that $j^k(g) \pitchfork W$ at x. By Definition A.1.1(a), we may assume that $j^k(g)(x) \in W$. We distinguish the following three cases:

- $x \in V'$. It follows from $(\alpha \circ j^k(g))(x) = x \in V'$ that $j^k(g)(x) \in W \cap \alpha^{-1}(V') = W'$. As $j^k(g) \pitchfork W'$ at x by construction of g and W' is an open subset of W, Lemma A.1.2(a) implies that $j^k(g) \pitchfork W$ at x.
- $x \in X \setminus C'$. Since $x \in X \setminus C' \subset X \setminus C$, we have $j^k(f) \pitchfork W$ at x by assumption (1). As $g|_{X\setminus C'} = f|_{X\setminus C'}$ and $X \setminus C'$ is an open subset of X, it follows from Lemma A.1.2(b) that $j^k(g) \pitchfork W$ at x. (Note that $j^k(f)|_{X\setminus C'} = j^k(g)|_{X\setminus C'}$.)
- $x \in C' \setminus V' = C''$. It follows from $g \in B_\delta(f)$ that

  $$d(j^k(f)(x), j^k(g)(x)) < \delta.$$

  As $j^k(f)(x) \in j^k(f)(C'') = K$ and $j^k(g)(x) \in W$, we obtain $j^k(g)(x) \in Z$ by the choice of δ. In particular, $x \in j^k(g)^{-1}(\overline{Z}^W)$. Hence, $g \in T$ implies that $j^k(g) \pitchfork \check{W}$ at x. It follows from $x \in C' \setminus V' \subset X \setminus \overline{V}^X = \check{V} = (\alpha \circ j^k(g))^{-1}(\check{V}) = j^k(g)^{-1}(\alpha^{-1}(\check{V}))$ that $j^k(g)(x) \in W \cap \alpha^{-1}(\check{V}) = \check{W}$. Since $\check{W}$ is an open subset of W, Lemma A.1.2(a) yields $j^k(g) \pitchfork W$ at x.

□

A.4 Transversality in Vector Bundles

Recall from Section 2.2.1 the notion of a Whitney stratified subspace of a smooth manifold. Proposition A.3.2 can be generalized to Whitney stratified subspaces of the jet space:

Corollary A.4.1. *Let $f\colon X \to Y$ be smooth, $C \subset X$ a compact subspace, and $W \subset J^k(X,Y)$ a Whitney stratified subspace such that the k-jet $j^k(f)\colon X \to J^k(X,Y)$ satisfies $j^k(f) \pitchfork W$ on $X \setminus C$. Then, given an open neighbourhood $U \subset X$ of C, there exists a smooth map $g\colon X \to Y$ such that $g|_{X\setminus U} = f|_{X\setminus U}$ and $j^k(g) \pitchfork W$.*

Proof. Let S_i denote the stratum of W of dimension $0 \le i \le n := \dim W$. (Hence, S_i is the union of all i-dimensional pieces of W.) Note that the closed subset $W_i := \bigcup_{j=0}^{i} S_j \subset J^k(X,Y)$ is a Whitney stratified space of dimension i. (The pieces of W_i are unions of components of the S_j.)

Using Proposition A.3.2, we construct inductively for $i = 0,\dots,n$ a compact subset $C_i \subset U$ and a smooth map $f_i\colon X \to Y$ with the following properties:

$(1)_i$ $f_i|_{X\setminus U} = f|_{X\setminus U}$,
$(2)_i$ $j^k(f_i) \pitchfork W$ on $X \setminus C_i$,
$(3)_i$ $j^k(f_i)(C_i) \cap W_i = \emptyset$.

Note that the smooth map $g := f_n\colon X \to Y$ will have the desired properties since $g|_{X\setminus U} = f|_{X\setminus U}$ by property $(1)_n$, and $j^k(g) \pitchfork W$ follows from properties $(2)_n$ and $(3)_n$ as $W_n = W$.

To construct f_0, we apply Corollary A.4.1 to the smooth map $f\colon X \to Y$, the compact subspace $C \subset X$, the submanifold $W_0 \subset J^k(X,Y)$, and an open subset $U_0 \subset X$ such that $C \subset U_0$ and such that $\overline{U}_0^X \subset U$ is compact. (Note that the required properties (1) and (2) of Corollary A.4.1 are satisfied since $j^k(f) \pitchfork W_0$ holds on $X \setminus C$ by assumption, and W_0 is a closed subset of $J^k(X,Y)$.) Hence, Corollary A.4.1 implies that there exists a smooth map $f_0\colon X \to Y$ such that $f_0|_{X\setminus U_0} = f|_{X\setminus U_0}$ and $j^k(f_0) \pitchfork W_0$. As $j^k(f_0) \pitchfork W$ on $j^k(f_0)^{-1}(W_0)$, there exists by Proposition 2.2.2 an open neighbourhood $V_0 \subset X$ of $j^k(f_0)^{-1}(W_0)$ such that $j^k(f_0) \pitchfork W$ on V_0. Set $C_0 := \overline{U}_0^X \setminus V_0$, which is a compact subset of U. It remains to check the desired properties $(1)_0$, $(2)_0$ and $(3)_0$. Property $(1)_0$ follows from $f_0|_{X\setminus U_0} = f|_{X\setminus U_0}$ since $\overline{U}_0^X \subset U$. In order to check property $(2)_0$, note that $X \setminus C_0 = V_0 \cup X \setminus \overline{U}_0^X$. Note that $j^k(f_0) \pitchfork W$ on V_0 by choice of V_0. Furthermore, $f_0|_{X\setminus \overline{U}_0^X} = f|_{X\setminus \overline{U}_0^X}$ implies that $j^k(f_0)|_{X\setminus \overline{U}_0^X} = j^k(f)|_{X\setminus \overline{U}_0^X}$, so it follows from $X \setminus \overline{U}_0^X \subset X \setminus C$ and $j^k(f) \pitchfork W$ on $X \setminus C$ that $j^k(f_0) \pitchfork W$ on $X \setminus \overline{U}_0^X$. Finally, property $(3)_0$ holds since $j^k(f_0)^{-1}(W_0) \cap C_0 \subset V_0 \cap C_0 = \emptyset$ by construction of C_0.

For $i \ge 1$, the construction of f_i from f_{i-1} is as follows. Let $X_i \subset U$ be an open neighbourhood of C_{i-1}. By property $(3)_{i-1}$ one may additionally assume that $X_i \cap j^k(f_{i-1})^{-1}(W_{i-1}) = \emptyset$ because $W_{i-1} \subset J^k(X,Y)$ is a closed subset. Hence, Lemma A.2.1 (a) implies that $N_i := \{h \in C^\infty(X_i,Y);\ j^k(h)^{-1}(W_{i-1}) = \emptyset\}$ is an open neighbourhood in $C^\infty(X_i,Y)$ (in the Whitney C^∞ topology) of the restriction $h_i := f_{i-1}|\colon X_i \to Y$. Apply Proposition A.3.2 to the smooth map $h_i\colon X_i \to Y$, the compact subspace $C_{i-1} \subset X_i$, the submanifold $S_i \subset J^k(X,Y)$, and an open subset $U_i \subset X_i$ such that $C_{i-1} \subset U_i$ and such that $\overline{U}_i^X \subset X_i$ is compact. (Note that the required properties (1) and (2) are satisfied. Indeed, $j^k(h_i) \pitchfork S_i$ holds on $X_i \setminus C_{i-1}$ by property $(2)_{i-1}$. Moreover, $j^k(h_i)^{-1}(W_{i-1}) = X_i \cap j^k(f_{i-1})^{-1}(W_{i-1}) = \emptyset$ implies that $j^k(h_i)(X_i \setminus C_{i-1}) \cap \overline{S_i}^{J^k(X_i,Y)} = j^k(h_i)(X_i \setminus C_{i-1}) \cap S_i$ because $\overline{S_i}^{J^k(X_i,Y)} \setminus S_i \subset W_{i-1}$.) Thus, there exists a smooth map $g_i\colon X_i \to Y$ with $g_i \in N_i$ such that $g_i|_{X_i\setminus U_i} = h_i|_{X_i\setminus U_i}$ and $j^k(g_i) \pitchfork S_i$. As $j^k(g_i) \pitchfork W$ on $j^k(g_i)^{-1}(S_i)$, there exists by Proposition 2.2.2 an open neighbourhood $V_i \subset X_i$ of $j^k(g_i)^{-1}(S_i)$ such that $j^k(g_i) \pitchfork W$ on V_i. Let us define the pair (C_i, f_i). Set $C_i := \overline{U}_i^X \setminus V_i$, which is a compact subset of X_i, and hence of U. Note that $X = X_i \cup (X \setminus \overline{U}_i^X)$ is an open cover of X. Define the smooth map

$$f_i\colon X \to Y, \qquad f_i(x) = \begin{cases} g_i(x), & x \in X_i, \\ f_{i-1}(x), & x \in X \setminus \overline{U}_i^X. \end{cases}$$

(Indeed, f_i is a well-defined smooth map since the smooth maps g_i and f_{i-1} coincide with h_i on $X_i \cap (X \setminus \overline{U}_i^X) = X_i \setminus \overline{U}_i^X \subset X_i \setminus U_i$.) It remains to check the desired properties $(1)_i$, $(2)_i$ and $(3)_i$. Property $(1)_i$ follows from $f_i|_{X\setminus\overline{U}_i^X} = f_{i-1}|_{X\setminus\overline{U}_i^X}$ and property $(1)_{i-1}$ since $\overline{U}_0^X \subset U$. In order to check property $(2)_i$, note that $X \setminus C_i = V_i \cup (X \setminus \overline{U}_i^X)$. We have $j^k(f_i) \pitchfork W$ on V_i because $f_i|_{V_i} = g_i|_{V_i}$, and $j^k(g_i) \pitchfork W$ on V_i. Moreover, $j^k(f_i) \pitchfork W$ on $X \setminus \overline{U}_i^X$ because $f_i|_{X\setminus\overline{U}_i^X} = f_{i-1}|_{X\setminus\overline{U}_i^X}$ and $j^k(f_{i-1}) \pitchfork W$ on $X \setminus C_{i-1}$ ($\supset X \setminus \overline{U}_i^X$) by property $(2)_{i-1}$. Finally, property $(3)_i$ holds because $g_i \in N_i$ and $j^k(g_i)^{-1}(S_i) \subset V_i$ imply that

$$j^k(f_i)^{-1}(W_i) \cap C_i = j^k(g_i)^{-1}(W_i) \cap C_i = (j^k(g_i)^{-1}(W_{i-1}) \cup j^k(g_i)^{-1}(S_i)) \cap C_i \subset V_i \cap C_i = \emptyset.$$

□

Lemma A.4.2. *Let $f\colon X \to Y$ be a submersion of smooth manifolds. Given a Whitney stratified subspace $M \subset X$ and a smooth submanifold $N \subset Y$, the following statements hold:*

(*a*) *The smooth submanifold $f^{-1}(N) \subset X$ satisfies $f^{-1}(N) \pitchfork M$ in X if and only if the restriction $f|\colon M \to Y$ is transverse to $N \subset Y$.*

(*b*) *Suppose that the conditions of part (a) hold. Let $g\colon Y \to X$ be a smooth section of f such that the restriction $g|\colon N \to f^{-1}(N)$ is transverse to $f^{-1}(N) \cap M \subset f^{-1}(N)$. Then, for every smooth submanifold $N \subset N' \subset Y$, there exists an open neighbourhood $f^{-1}(N) \subset V \subset f^{-1}(N')$ such that $V \pitchfork M$ in X, and such that the restriction $g|\colon g^{-1}(V) \to V$ is transverse to $V \cap M$ on N.*

Proof. (*a*). It suffices to assume that M is a smooth manifold. One has to show the equivalence of the following two statements for every point $p \in M$ with $q := f(p) \in N$:

(*i*) $T_p f^{-1}(N) + T_pM = T_pX$.

(*ii*) $df_p(T_pM) + T_qN = T_qY$.

(*i*) ⇒ (*ii*). Let $v \in T_qY$. Since f is a submersion, there exists $w \in T_pX$ such that $v = df_p(w)$. By (*i*) there exist $w_N \in T_p f^{-1}(N)$ and $w_M \in T_pM$ such that $w = w_N + w_M$. Hence,

$$v = df_p(w) = df_p(w_N) + df_p(w_M) \in df_p(T_p f^{-1}(N)) + df_p(T_pM) \subset df_p(T_pM) + T_qN.$$

(*ii*) ⇒ (*i*). Conversely, let $w \in T_pX$. By (*ii*) there exist $w_M \in T_pM$ and $v_N \in T_qN$ such that $df_p(w) = df_p(w_M) + v_N$. Hence,

$$w - w_M \in (df_p)^{-1}(v_N) \subset (df_p)^{-1}(T_qN) = T_p f^{-1}(N).$$

(b). It suffices to assume that M is a smooth manifold. By means of Proposition 2.2.2, it follows from the assumption $f^{-1}(N) \pitchfork M$ in X from part (a) that there exists an open neighbourhood $f^{-1}(N) \subset V \subset f^{-1}(N')$ such that $V \pitchfork M$ in X. Note that $g^{-1}(V)$ is an open neighbourhood of N in N' since g is a section of f (thus restricting to a map $g|\colon N' \to f^{-1}(N')$).

It remains to show that, for every point $q \in N$ with $p := g(q) \in M$,

$$d_q g(T_q g^{-1}(V)) + T_p(V \cap M) = T_p V.$$

For this purpose, let $w \in T_p V$. Since $T_p f^{-1}(N) + T_p M = T_p X$ by the conditions of part (a), there exist $w_N \in T_p f^{-1}(N) \subset T_p V$ and $w_M \in T_p M$ such that $w = w_N + w_M$. Note that $w_M = w - w_N \in T_p V$ implies that $w_M \in T_p V \cap T_p M = T_p(V \cap M)$. Next, it follows from $d_q g(T_q N) + T_p(f^{-1}(N) \cap M) = T_p f^{-1}(N)$ (which holds by assumption on g) that there exist $v_N \in T_q N$ and $w_\cap \in T_p(f^{-1}(N) \cap M) \subset T_p(V \cap M)$ such that $w_N = d_q g(v_N) + w_\cap$. Hence,

$$w = w_N + w_M = d_q g(v_N) + (w_\cap + w_M).$$

This is the desired composition since $d_q g(v_N) \in d_q g(T_q N) \subset d_q g(T_q N') = d_q g(T_q g^{-1}(V))$ and $w_\cap + w_M \in T_p(V \cap M)$. □

The following Proposition deals with the extension of sections of a smooth vector bundle that are transverse to a given Whitney stratified subspace of the total space. Note that the case of a product bundle is covered by the case $k = 0$ and $Y = \mathbb{R}^r$ in Corollary A.4.1 since $J^0(X, \mathbb{R}^r) = X \times \mathbb{R}^r$.

Proposition A.4.3. *Let $\pi\colon E \to X$ be a smooth vector bundle. Suppose that $W \subset E$ is a Whitney stratified subspace. Let $C \subset X$ be a compact subspace, and let $f\colon X \setminus C \to E$ be a smooth section of π such that $f \pitchfork W$. Then, given an open subset $U \subset X$ such that $C \subset U$, there exists a smooth section $F\colon X \to E$ of π such that $F \pitchfork W$ and $F|_{X \setminus U} = f|_{X \setminus U}$.*

Proof. During the proof, we adopt the notation of [62] for smooth triangulations of smooth manifolds as defined there.

There exists a compact codimension 0 submanifold with boundary $V \subset U$ such that $C \subset V \setminus \partial V$, and a smooth triangulation (K, η) of V extending a smooth triangulation (L, λ) of ∂V such that $\pi|_S$ is transverse to $\eta|_{\operatorname{int}\sigma}$ for every stratum S of W and every simplex $\sigma \in K$. (Indeed, V with its desired smooth triangulation can be constructed as follows. By [43, Theorem 10.6, p.103f] there exists a compact codimension 0 submanifold with boundary $V \subset U$ such that $C \subset V \setminus \partial V$, and a smooth triangulation (K, η) of V extending a smooth triangulation (L, λ) of ∂V. The task is to modify V in such a way that the required transversality conditions hold. Note that [62, Proposition 2, p. 2] generalizes to a finite number of smooth maps $h_j\colon Y_j \to X_0$, $j = 1, \ldots, d$, which is essentially possible because the transversality condition is achieved by choosing a regular value of a certain smooth map in the course of the proof of [62, Lemma 6, p. 4]. Using our generalizations, we then follow the proof of the main theorem indicated in [62, Section 2, p. 2] noting that the diffeomorphisms ψ_σ that are chosen have support in stars of the barycentric subdivision of K. Hence, if the originial smooth triangulation (K, η) of V is fine enough, then the finally obtained diffeomorphism $\psi\colon U \to U$ will have the property that $C \subset \psi(V) \setminus \psi(\partial V)$. Finally, replacing V by $\psi(V)$ and (K, η) by $(K, \psi \circ \eta)$, the required transversality properties will be satisfied.) Consequently, Lemma A.4.2 (a) implies that $\pi^{-1}(\operatorname{int}\sigma) \pitchfork W$ for every simplex $\sigma \in K$. (By abuse of notation, we write $\operatorname{int}\sigma$ instead of $\eta(\operatorname{int}\sigma)$ etc.) Thus, if $P_i := |K_i| \setminus (|K_{i-1}| \cup |L|) \subset \operatorname{int} V$ is the i-dimensional smooth submanifold given by the union of the interiors of the i-simplices of $K \setminus L$, and $\pi_i \colon E_i \to P_i$ denotes the restriction of $\pi \colon E \to X$ to P_i, then $E_i \pitchfork W$, and the transverse intersection $W_i := E_i \cap W$ is a Whitney stratified subspace of E_i.

For $i = -1, \ldots, m$ (where $m = \dim V$) we construct inductively a pair (U_i, F_i) consisting of

- an open neighbourhood U_i of $|K_i| \cup |L| \cup (U \setminus V) = |K_i| \cup (U \setminus \operatorname{int} V)$ in U, and
- a smooth section $F_i \colon U_i \to E$ of π

with the following properties:

$(1)_i$ $F_i \pitchfork W$.

$(2)_i$ For every $j \in \{0, \ldots, i\}$, the restriction $F_i| \colon P_j \to E_j$ is transverse to $W_j \subset E_j$.

$(3)_i$ F_i coincides with f in a neighbourhood of $U \setminus \operatorname{int} V$ in U.

Finally, since $U_m = U$, the desired smooth section $F \colon X \to E$ of π can be chosen to be the gluing of $F_m \colon U_m = U \to E$ and $f|_{X \setminus V}$ along the open subset $U_m \cap (X \setminus V) = U \setminus V \subset X$.

Initially, set $(U_{-1}, F_{-1}) := (U \setminus C, f|_{U \setminus C})$. Supposing that the pair (U_i, F_i) has been constructed for some $i \in \{-1, \ldots, m-1\}$, the pair (U_{i+1}, F_{i+1}) can be constructed as follows.

Let $\sigma \in K \setminus L$ be an $(i+1)$-simplex. It suffices to extend F_i in the desired way over a tubular neighbourhood of $\operatorname{int}\sigma \subset \operatorname{int} V$ rel an open neighbourhood of $\partial\sigma$ in U_i. Repetition of this process for every $(i+1)$-simplex of $K \setminus L$ then yields the desired map F_{i+1}.

Since (K, η) is a smooth triangulation of V, the embedding $i_\sigma \colon \Delta^{i+1} \to U$ of the standard $(i+1)$-simplex $\Delta^{i+1} \subset \mathbb{R}^{i+1}$ that corresponds to σ is the restriction of an embedding of an open neighbourhood of $\Delta^{i+1} \subset \mathbb{R}^{i+1}$ into U whose image is an $(i+1)$-dimensional smooth submanifold $\Sigma \subset U$. For every $j \in \{0, \ldots, i\}$, property $(2)_i$ allows us to apply Lemma A.4.2 (b) to the following constellation:

$$\begin{aligned} f \colon X \to Y \quad &\leftrightarrow \quad \pi| \colon \pi^{-1}(U_i) \to U_i, \\ M \subset X \quad &\leftrightarrow \quad W \cap \pi^{-1}(U_i) \subset \pi^{-1}(U_i), \\ N \subset N' \subset Y \quad &\leftrightarrow \quad U_i \cap P_j \subset U_i \cap \Sigma \subset U_i, \\ g \colon Y \to X \quad &\leftrightarrow \quad F_i| \colon U_i \to \pi^{-1}(U_i). \end{aligned}$$

(Indeed, note that $\pi^{-1}(U_i) \to U_i$ is a submersion, and $W \cap \pi^{-1}(U_i)$ is a Whitney stratified subspace of $\pi^{-1}(U_i)$. Furthermore, condition (a), i.e. transversality of $\pi| \colon W \cap \pi^{-1}(U_i) \to U_i$ to $U_i \cap P_j \subset U_i$, holds by choice of the triangulation (K, η). Finally, since $f^{-1}(N)$ corresponds to $\pi^{-1}(U_i \cap P_j) = \pi^{-1}(U_i) \cap E_j$, the requirement on the smooth section $F_i| \colon U_i \to \pi^{-1}(U_i)$ of $\pi| \colon \pi^{-1}(U_i) \to U_i$ is satisfied by property $(2)_i$.) Hence, there exists an open neighbourhood

$$\pi^{-1}(U_i) \cap E_j \subset V_j \subset \pi^{-1}(U_i \cap \Sigma)$$

such that $V_j \pitchfork (W \cap \pi^{-1}(U_i))$ in $\pi^{-1}(U_i)$, and such that the restriction $F_i| \colon F_i^{-1}(V_j) \to V_j$ is transverse to $V_j \cap W \cap \pi^{-1}(U_i)$ on $U_i \cap P_j$. Therefore, the open subset

$$V_\cup := \bigcup_{j=0}^{i} V_j \qquad \subset \pi^{-1}(U_i \cap \Sigma)$$

is transverse to $W \cap \pi^{-1}(U_i)$ in $\pi^{-1}(U_i)$, and the restriction $F_i|\colon F_i^{-1}(V_\cup) \to V_\cup$ is transverse to $V_\cup \cap W \cap \pi^{-1}(U_i)$ on $U_i \cap \bigcup_{j=0}^{i} P_j$. In particular, $F_i^{-1}(V_\cup)$ is an open neighbourhood of $\partial\sigma$ ($\subset U_i \cap \bigcup_{j=0}^{i} P_j$) in $U_i \cap \Sigma$. Thus, by Proposition 2.2.2, there exists an open neighbourhood $U_\cup$ of $\partial\sigma$ in $F_i^{-1}(V_\cup)$ such that the restriction $F_i|\colon F_i^{-1}(V_\cup) \to V_\cup$ is on $U_\cup$ transverse to

$$V_\cup \cap W \cap \pi^{-1}(U_i) = V_\cup \cap \pi^{-1}(U_i \cap \Sigma) \cap W \cap \pi^{-1}(U_i) = V_\cup \cap \pi^{-1}(U_i \cap \Sigma) \cap W.$$

Therefore, $O_\cup := \operatorname{int}\sigma \cap U_\cup$ is an open subset of $\operatorname{int}\sigma$ such that the complement $\operatorname{int}\sigma \setminus O_\cup$ is compact, and the restriction $F_i|\colon O_\cup \to V_\cup$ is transverse to

$$V_\cup \cap \pi^{-1}(U_i \cap \Sigma) \cap W = V_\cup \cap \pi^{-1}(U_i \cap \Sigma) \cap W \cap \pi^{-1}(O_\cup) = V_\cup \cap W \cap \pi^{-1}(O_\cup).$$

(Here one uses that F_i is a section of π.) Now, $\pi^{-1}(\operatorname{int}\sigma) \pitchfork W$ by construction of the triangulation (K, η) of V, so the restriction $F_i|\colon O_\cup \to \pi^{-1}(\operatorname{int}\sigma)$ is transverse to the Whitney stratified subspace $W \cap \pi^{-1}(\operatorname{int}\sigma)$ of $\pi^{-1}(\operatorname{int}\sigma)$. (Note that $F_i(O_\cup) \subset V_\cup$.)

Hence, by Corollary A.4.1 (applied for $k = 0$; note that π restricts to a product bundle on the contractible subspace $\operatorname{int}\sigma \subset U$), there exists a smooth section $g_\sigma\colon \operatorname{int}\sigma \to \pi^{-1}(\operatorname{int}\sigma)$ of $\pi|\colon \pi^{-1}(\operatorname{int}\sigma) \to \operatorname{int}\sigma$ that is transverse to $W \cap \pi^{-1}(\operatorname{int}\sigma)$, and coincides with F_i outside a compact subset of $\operatorname{int}\sigma$.

Finally, use a smooth partition of unity to extend g_σ to a section $f_\sigma\colon T_\sigma \to E$ of π over some tubular neighbourhood $T_\sigma \cong \operatorname{int}\sigma \times \mathbb{R}^{m-i-1}$ of $\operatorname{int}\sigma \subset \operatorname{int} V$ in such a way that f_σ coincides with F_i in an open neighbourhood of $\partial\sigma$ in U_i. (Note that the restriction of E to T_σ is isomorphic to the trivial bundle $T_\sigma \times \mathbb{R}^r$, where r denotes the rank of E.) Now, Lemma A.4.2 (b) implies that $f_\sigma \pitchfork W$ on $\operatorname{int}\sigma$ because $f_\sigma| = g_\sigma\colon \operatorname{int}\sigma \to \pi^{-1}(\operatorname{int}\sigma)$ is transverse to $W \cap \pi^{-1}(\operatorname{int}\sigma)$. (Note that one can work with $N' = Y$ and $V = X$ in the notation of the Lemma.) Hence, Proposition 2.2.2 implies that $f_\sigma \pitchfork W$ on an open neighbourhood of $\operatorname{int}\sigma$ in T_σ. All in all, the resulting section F_{i+1} will have the desired properties by construction of the extensions f_σ of F_i over tubular neighbourhoods of the $(i+1)$-simplices σ of $K \setminus L$. $\square$

Appendix B

Collar and Tubular Neighbourhoods

Throughout the present subsection, let X denote a closed smooth manifold.

Proposition B.0.1. *Suppose that X (compact) is the boundary of a smooth manifold Y. Let*

$$f\colon [0,1]\times(Y,X)\to([0,\infty),0),\qquad (t,y)\mapsto f(t,y)=:f_t(y),$$

be a smooth map such that f_t has no critical points on X for all $t\in[0,1]$ and $f_t=f_i$ for t near $i=0,1$. Then there exists (for suitable $\varepsilon>0$) a smooth map

$$\kappa\colon [0,1]\times[0,\varepsilon)\times X\to Y,\qquad (t,u,x)\mapsto\kappa(t,u,x)=:\kappa_t(u,x),$$

such that κ_t is a collar of X in Y for all $t\in[0,1]$, $\kappa_t=\kappa_i$ for t near $i=0,1$, and the composition $f_t\circ\kappa_t\colon[0,\varepsilon)\times X\to[0,\infty)$ is of the form $(u,x)\mapsto u$ for all $t\in[0,1]$.

Moreover, if $(Y,X)=([0,\infty)\times X, 0\times X)$ and $f_t=\mathrm{pr}_{[0,\infty)}$ for t near $i=0,1$, then one can achieve that $\kappa_t=\mathrm{id}_{[0,\infty)\times X}|_{[0,\varepsilon)\times X}$ for t near $i=0,1$.

Remark B.0.2. The assumption that $f_t=f_i$ for t near $i=0,1$ could be eliminated (together with the analogous requirement for κ_t), but we keep it to avoid manifolds with corners in the proof.

Proof. It suffices to construct the desired κ_t for $t\in(0,1)$.

Let $c\colon[0,\infty)\times X\to Y$ be a collar of X in Y (see [22, Theorem 6.1, p. 113] and [22, Theorem 2.1, p. 152]). By the assumptions on f, the composition

$$g:=f\circ(\mathrm{id}_{(0,1)}\times c)\colon(0,1)\times[0,\infty)\times X\to[0,\infty),\qquad (t,u,x)\mapsto g(t,u,x)=:g_t(u,x),$$

is a smooth map such that $g((0,1)\times 0\times X)=f((0,1)\times c(0\times X))=f((0,1)\times X)=\{0\}$ and such that $g_t=f_t\circ c$ has no critical points on $0\times X$ for all $t\in(0,1)$. Consequently, the smooth vector fields $\xi:=0\times\partial_u\times 0$ on $(0,1)\times[0,\infty)\times X$ and $\upsilon:=\partial_u\times 0$ on $[0,\infty)\times X$ satisfy

$$\xi(g)(t,u,x)=\frac{d}{du'}g(t,u',x)|_{u'=u}=\frac{d}{du'}g_t(u',x)|_{u'=u}=\upsilon(g_t)(u,x)$$

for all $(t,u,x)\in(0,1)\times[0,\infty)\times X$, and

$$\xi(g)(t,0,x)=\upsilon(g_t)(0,x)>0,\qquad (t,x)\in(0,1)\times X.$$

(Indeed, concerning the second claim, note that

$$\xi(g)(t,0,x) = \frac{d}{du} g_t(u,x)|_{u=0} = \lim_{u \searrow 0} \frac{g_t(u,x)}{u} \geq 0$$

for all $(t,x) \in (0,1) \times X$. The tangent space of $[0,\infty) \times X$ at $(0,x)$ is the direct sum of the tangent space of $0 \times X$ at $(0,x)$ and $\mathbb{R}\upsilon(0,x)$. As $g_t(0 \times X) = \{0\}$ and g_t has no critical points on $0 \times X$, it follows that $\xi(g)(t,0,x) = \upsilon(g_t)(0,x) = dg_t(\upsilon(0,x)) \neq 0$.)

As X is compact and g_t is independent of t for t near $i = 0,1$, it follows from $\xi(g)|_{(0,1)\times 0 \times X} > 0$ that there exists $\delta > 0$ such that $\xi(g)|_Z > 0$, where $Z := (0,1) \times [0,\delta) \times X$.

Fix $t \in (0,1)$. The restriction $h_t := g_t|_{[0,\delta)\times X} \colon [0,\delta) \times X \to [0,\infty)$ is a submersion because

$$\upsilon(h_t)(u,x) = \upsilon(g_t)(u,x) = \xi(g)(t,u,x) > 0, \qquad (u,x) \in [0,\delta) \times X.$$

In consequence, $h_t^{-1}(0) = 0 \times X$. (In fact, if $(u,x) \in [0,\delta) \times X$ satisfies $h_t(u,x) = 0$ and $u > 0$, then the smooth map $\gamma\colon [0,u] \to \mathbb{R}$ given by $\gamma(u') = h_t(u',x)$ takes the value 0 at both boundary points. Hence, by Rolle's theorem, there exists $u_0 \in (0,u)$ such that $0 = \gamma'(u_0) = \upsilon(h_t)(u_0,x) > 0$, a contradiction.) Similarly, one obtains $\min g((0,1) \times \delta \times X) > 0$. (In fact, if the point $(t,x) \in (0,1) \times X$ satisfies $g(t,\delta,x) = 0$, then the the smooth map $\gamma\colon [0,\delta] \to \mathbb{R}$ given by $\gamma(u) = g_t(u,x)$ takes the value 0 at both boundary points. Hence, by Rolle's theorem, there exists $u_0 \in (0,\delta)$ such that $0 = \gamma'(u_0) = \upsilon(g_t)(u_0,x) > 0$, a contradiction.) If we choose $\varepsilon \in (0, \min g((0,1) \times \delta \times X))$, then the preimage $V_t := h_t^{-1}([0,\varepsilon])$ satisfies

$$V_t = (g_t|_{[0,\delta]\times X})^{-1}([0,\varepsilon]).$$

In conclusion, V_t is compact (being a closed subset of the compact set $[0,\delta] \times X$). Moreover, as V_t is the preimage of $[0,\varepsilon]$ under the submersion h_t and $h_t(0 \times X) = 0$, Exercise 5 in [22, p. 32] implies that V_t is a submanifold of $[0,\delta) \times X$. All in all, V_t is a cobordism from $V_t^0 := h_t^{-1}(0) = 0 \times X$ to $V_t^1 := h_t^{-1}(\varepsilon)$.

By construction, the submersion h_t restricts to a Morse function

$$l_t \colon (V_t, V_t^0, V_t^1) \to ([0,\varepsilon], 0, \varepsilon)$$

without critical points. Moreover, it follows from $\upsilon(h_t)(u,x) > 0$ for all $(u,x) \in [0,\delta) \times X$ that

$$\upsilon_t := \frac{1}{\upsilon|_{V_t}(l_t)} \cdot \upsilon|_{V_t}$$

is a gradient-like vector field for l_t on V_t that satisfies $\upsilon_t(l_t) = 1$.

Hence, the proof of [41, Theorem 3.4, pp. 21-23] yields a diffeomorphism of the form

$$k_t \colon [0,\varepsilon] \times V_t^0 \xrightarrow{\cong} V_t, \qquad k_t(u,(0,x)) = \psi^t_{(0,x)}(u),$$

where

$$\psi^t_{(0,x)} \colon [0,\varepsilon] \to V_t$$

denotes for every $x \in X$ the integral curve with respect to the vector field υ_t (i.e. $\partial_u \psi^t_{(0,x)}(u) = \upsilon_t(\psi^t_{(0,x)}(u))$ for all $u \in [0,\varepsilon]$) which is uniquely determined by requiring that it passes through

$(0,x)$ and satisfies $l_t(\psi^t_{(0,x)}(u)) = u$ for all $u \in [0,\varepsilon]$. Equivalently, $\psi^t_{(0,x)}$ is uniquely determined by $\psi^t_{(0,x)}(0) = (0,x)$. (In fact, concerning the ($\Rightarrow$) direction, if $u \in [0,\varepsilon]$ is such that $\psi^t_{(0,x)}(u) = (0,x)$, then application of l_t yields $u = 0$. Conversely, if $\psi^t_{(0,x)}(0) = (0,x)$, then $\psi^t_{(0,x)}$ passes through $(0,x)$ and the composition $a := l_t \circ \psi^t_{(0,x)}$ satisfies $a(0) = 0$ and $a'(u) = dl_t(d\psi^t_{(0,x)}(\partial_u|_u)) = dl_t(\upsilon_t(\psi^t_{(0,x)}(u))) = \upsilon_t(l_t)(\psi^t_{(0,x)}(u)) = 1$.)

Let $h := g|_Z$. (Note that $h_t = h(t,-,-)$.) As $\xi|_Z(h) = \xi(g)|_Z > 0$, we can define the smooth vector field $\zeta := \frac{1}{\xi|_Z(h)} \cdot \xi|_Z$ on Z.

Let $\rho_t \colon V_t \to Z$, $\rho_t(u,x) = (t,u,x)$. (This is well-defined since V_t is contained in $[0,\delta) \times X$ for all $t \in (0,1)$.) Note that $\zeta \circ \rho_t = 0 \times \upsilon_t$. (In fact, if $(u,x) \in V_t$ and $z := \rho_t(u,x) \in Z$, then $\xi|_Z(h)(z) = \xi(g)(z) = \upsilon(g_t)(u,x) = dg_t(\upsilon(u,x)) = dl_t(\upsilon(u,x))$ and $\xi(z) = 0 \times \upsilon(u,t)$ imply

$$\zeta(\rho_t(u,x)) = \zeta(z) = \frac{1}{\xi|_Z(h)(z)} \cdot \xi(z) = 0 \times (\frac{1}{dl_t(\upsilon(u,x))} \cdot \upsilon(u,t)) = 0 \times (\upsilon_t)(u,x).)$$

By [22, page 151] there exists an open neighbourhood Ω of $0 \times Z$ in $[0,\infty) \times Z$ on which the *flow* of ζ is defined as a smooth map

$$\eta \colon \Omega \to Z.$$

(Following the remarks in [22, page 151] about the flow of a vector field on a manifold with boundary, one first extends ζ over the double $\tilde{Z}$ of Z. The flow of this extension $\tilde{\zeta}$ is then a smooth map of the form $\tilde{\eta} \colon \tilde{\Omega} \to \tilde{Z}$, where $\tilde{\Omega}$ is an open neighbourhood of $0 \times \tilde{Z}$ in $\mathbb{R} \times \tilde{Z}$ such that for every $z \in \tilde{Z}$ the intersection $\tilde{\Omega} \cap (\mathbb{R} \times z)$ is of the form $\tilde{J}(z) \times z$, where $\tilde{J}(z)$ is the maximally extended open interval around 0 on which the integral curve of $\tilde{\zeta}$ at z is defined. As ζ is by construction nowhere tangent to $\partial Z = (0,1) \times 0 \times X$ and points into Z, $\tilde{\eta}$ restricts to the desired smooth map $\eta \colon \Omega \to Z$, where $\Omega := \tilde{\Omega} \cap ([0,\infty) \times Z)$.) In particular, for every $z \in Z$ the intersection $\Omega \cap (\mathbb{R} \times z)$ is of the form $J(z) \times z$, where $J(z)$ is an interval of the form $J(z) = [0, a(z))$ such that the smooth curve $\eta^z \colon J(z) \to Z$, $\eta^z(s) = \eta(s,z)$, is an integral curve of ζ at z (i.e., $\eta^z(0) = z$ and $\partial_s \eta^z(s) = \zeta(\eta^z(s))$ for all $s \in J(z)$). Moreover, $J(z)$ is the maximally extended interval of the form $[0, a(z))$ on which the integral curve of ζ at z is defined. Given $(t,x) \in (0,1) \times X$, the curve $\gamma \colon [0,\varepsilon] \to Z$, $\gamma(u) = (t, \psi^t_{(0,x)}(u))$, satisfies $\gamma(0) = (t, \psi^t_{(0,x)}(0)) = (t,0,x)$ and

$$\partial_u(t, \psi^t_{(0,x)}(u)) = 0 \times \upsilon_t(\psi^t_{(0,x)}(u)) = \zeta(\rho_t(\psi^t_{(0,x)}(u))) = \zeta(t, \psi^t_{(0,x)}(u)).$$

Hence, $[0,\varepsilon] \subset J(t,0,x)$ and $(t, \psi^t_{(0,x)}(u)) = \eta^{(t,0,x)}(u)$ for all $u \in [0,\varepsilon]$. Therefore, $[0,\varepsilon] \times (0,1) \times 0 \times X \subset \Omega$. Let $\pi \colon Z \to [0,\delta) \times X$, $\pi(t,u,x) = (u,x)$. Define the smooth map

$$k \colon (0,1) \times [0,\varepsilon) \times X \to [0,\delta) \times X, \qquad k(t,u,x) := \pi(\eta^{(t,0,x)}(u)) = \psi^t_{(0,x)}(u) = k_t(u,(0,x)).$$

Finally, a smooth map is defined by the composition

$$\kappa := c \circ k \colon (0,1) \times [0,\varepsilon) \times X \to Y, \qquad (t,u,x) \mapsto \kappa(t,u,x) =: \kappa_t(u,x).$$

It remains to check the desired properties:

- For every $t \in (0,1)$, $\kappa_t = c \circ k_t$ is a collar of X in Y since it is the composition of embeddings, and $\kappa_t(0,x) = c(k(t,0,x)) = c(\psi^t_{(0,x)}(0)) = c(0,x) = x$ for all $x \in X$.

- For t near $i = 0, 1$, κ_t is independent of t. (In fact, for t near $i = 0, 1$, l_t and hence υ_t is independent of t. Thus, the integral curve $\psi^t_{(0,x)}$ of υ_t is also independent of t for all $x \in X$, which implies the claim because $\kappa_t = c \circ \psi^t_{(0,x)}(u)$.)
- Finally, for every $t \in (0, 1)$ and all $(u, x) \in [0, \varepsilon) \times X$ we have

$$(f_t \circ \kappa_t)(u, x) = f_t(c(\psi^t_{(0,x)}(u))) = g_t(\psi^t_{(0,x)}(u)) = l_t(\psi^t_{(0,x)}(u)) = u,$$

noting that $g_t|_{V_t} = h_t|_{V_t} = l_t|_{V_t}$.

Finally suppose that $(Y, X) = ([0, \infty) \times X, 0 \times X)$ and $f_t = \mathrm{pr}_{[0,\infty)}$ for t near $i = 0, 1$. Under these assumptions, one may choose the collar c to be the identity map of Y, which implies that $g = f$. In particular, $\upsilon(g_t) = 1$ and $V_t = [0, \varepsilon] \times X$, $l_t = \mathrm{pr}_{[0,\varepsilon]}$, $\upsilon_t = \upsilon|_{V_t}$, for t near $i = 0, 1$. Hence, $\psi^t_{(0,x)}$ can be chosen to be given by $u \mapsto (u, x)$ for t near $i = 0, 1$. Thus, $\kappa_t = k_t = \mathrm{id}_{[0,\infty)\times X}|_{[0,\varepsilon)\times X}$ for t near $i = 0, 1$.

□

It is well-known that any collar of $0 \times X$ in $[0, \infty) \times X$ can be extended to a tubular neighbourhood of $0 \times X$ in $\mathbb{R} \times X$. The following proposition shows that an isotopy of such collars can always be extended to an isotopy of tubular neighbourhoods between given extensions of both ends of the isotopy.

Proposition B.0.3. *Suppose that*

$$f \colon [0, 1] \times [0, \infty) \times X \to [0, \infty) \times X, \qquad (t, u, x) \mapsto f(t, u, x) =: f_t(u, x),$$

is a smooth map such that f_t *is a collar of* $X = 0 \times X$ *in* $[0, \infty) \times X$ *for all* $t \in [0, 1]$, *and* $f_t = f_i$ *for* t *near* $i = 0, 1$. *If* $\overline{f}_i \colon \mathbb{R}\times X \to \mathbb{R}\times X$ *are extensions of* f_i *to tubular neighbourhoods of* $X = 0 \times X$ *in* $\mathbb{R} \times X$ *for* $i = 0, 1$, *then* f *extends to a smooth map*

$$\tilde{f} \colon [0, 1] \times \mathbb{R} \times X \to \mathbb{R} \times X, \qquad (t, u, x) \mapsto \tilde{f}(t, u, x) =: \tilde{f}_t(u, x),$$

such that $\tilde{f}_t$ *is a tubular neighbourhood of* $X = 0 \times X$ *in* $\mathbb{R} \times X$ *for all* $t \in [0, 1]$, *and* $\tilde{f}_t = \overline{f}_i$ *for* t *near* $i = 0, 1$.

Proof. In order to avoid the appearance of corners, we will prove a statement equivalent to the proposition, replacing the closed unit interval $[0, 1]$ in the domain of f by the open unit interval $(0, 1)$.

We will work with the *track* of the isotopy f (see [22, p. 178]) given by

$$\hat{f} \colon (0, 1) \times [0, \infty) \times X \to (0, 1) \times [0, \infty) \times X, \qquad \hat{f}(t, u, x) = (t, f(t, u, x)).$$

Note that $\hat{f}|_{(0,1)\times[0,1)\times X}$ is an embedding. (In fact, it is an injective immersion because f_t is an embedding for every $t \in [0, 1]$. It is also a homeomorphism onto its image since X is compact, f_t is independent of t for t near $i = 0, 1$, and the image is a Hausdorff space.) Thus, after a permutation of factors $\hat{f}|_{(0,1)\times[0,1)\times X}$ can be interpreted as a collar

$$F \colon [0, 1) \times V \to [0, \infty) \times V$$

of $(0,1)\times X =: V = 0\times V$ in $[0,\infty)\times V$ because $\hat{f}(t,0,x) = (t, f(t,0,x)) = (t,0,x)$.

It suffices to extend F to a partial tubular neighbourhood

$$\tilde{F}\colon (-1,1)\times V \to \mathbb{R}\times V, \qquad (u,t,x)\mapsto \tilde{F}(u,t,x) =: \tilde{F}_t(u,x),$$

of $V = 0\times V$ in $\mathbb{R}\times V$ such that $\mathrm{pr}_{(0,1)}\circ\tilde{F} = \mathrm{pr}_{(0,1)}$ (i.e. $\tilde{F}$ is *level-preserving* in the sense of [22, p. 178]) and $\mathrm{pr}_{\mathbb{R}\times X}\circ\tilde{F}_t = \overline{f}_i$ for t near $i = 0,1$. The desired map $\tilde{f}$ will then be given by $\tilde{f}_t := \mathrm{pr}_{\mathbb{R}\times X}\circ\tilde{F}_t$ for all $t\in(0,1)$.

In the construction of $\tilde{F}$ we will use a smooth partition of unity to glue vector fields that are induced by local extensions of F. The flow of the resulting vector field will then be used to construct the desired level-preserving extension $\tilde{F}$.

Let $\varepsilon\in(0,1)$ such that $f_t = f_0$ for $t\in(0,\varepsilon)$ and $f_t = f_1$ for $t\in(1-\varepsilon,1)$. Fix $\delta\in(0,\varepsilon)$.

Fix a point $v = (t,x)\in(\delta,1-\delta)\times X\subset V$. As X is a smooth manifold, we may choose $r>0$ and open neighbourhoods $J\subset(\delta,1-\delta)$ of t and $U\subset X$ of x such that $F|_{[0,r)\times J\times U}$ extends to a smooth map

$$\tilde{F}^{(v)}\colon Y^{(v)} := (-r,r)\times J\times U\to\mathbb{R}\times V$$

such that $\mathrm{pr}_{(0,1)}\circ\tilde{F}^{(v)} = \mathrm{pr}_J$. As $\tilde{F}^{(v)}$ has maximal rank at $(0,v)$, it follows from the inverse function theorem that for a sufficiently small choice of $Y^{(v)}$, $\tilde{F}^{(v)}$ becomes an embedding. In particular, the image $Z^{(v)} := \tilde{F}^{(v)}(Y^{(v)})$ is an open neighbourhood of $F(0,v) = (0,v)$ in $\mathbb{R}\times(\delta,1-\delta)\times X$. In addition, setting $J_0 := (0,\varepsilon)$ and $J_1 := (1-\varepsilon,1)$, and extension of $F|_{[0,\infty)\times J_i\times X}$ is given for $i = 0,1$ by the embedding

$$\tilde{F}^{(i)} := (\mathrm{pr}_{\mathbb{R}}\circ\overline{f}_i, \mathrm{pr}_{J_i}, \mathrm{pr}_X\circ\overline{f}_i)\colon Y^{(i)} := \mathbb{R}\times J_i\times X\to\mathbb{R}\times V.$$

Note that $\mathrm{pr}_{(0,1)}\circ\tilde{F}^{(i)} = \mathrm{pr}_{J_i}$. Moreover, the image $Z^{(i)} := \tilde{F}^{(i)}(Y^{(i)})$ is an open neighbourhood of $0\times J_i\times X$ in $\mathbb{R}\times J_i\times X$.

We thus obtain a family $\{\tilde{F}^{(i)}\}_{i=0,1}\cup\{\tilde{F}^{(v)}\}_{v\in(\delta,1-\delta)\times X}$ of embeddings that extend F locally around all points of $0\times V\subset\mathbb{R}\times V$. In general, these embeddings will not fit together to an embedding of an open neighbourhood of $0\times V$ into $\mathbb{R}\times V$, so we pass on to vector fields which can be glued via a partition of unity.

For every $v\in(\delta,1-\delta)\times X$ let $\xi^{(v)} := d\tilde{F}^{(v)}(\partial_u\times 0\times 0)$ denote the smooth vector field on the open subset $Z^{(v)} := \tilde{F}^{(v)}((-r,r)\times J\times U)\subset\mathbb{R}\times(\delta,1-\delta)\times X$ which is induced via $\tilde{F}^{(v)}$ by the vector field $\partial_u\times 0\times 0$ on $(-r,r)\times J\times U$. Hence, $d\,\mathrm{pr}_{(0,1)}(\xi^{(v)}) = d(\mathrm{pr}_{(0,1)}\circ\tilde{F}^{(v)})(\partial_u\times 0\times 0) = d\,\mathrm{pr}_J(\partial_u\times 0\times 0) = 0$. In addition, for $i = 0,1$ let $\xi^{(i)} := d\tilde{F}^{(i)}(\partial_u\times 0\times 0)$ denote the smooth vector field on the open subset $Z^{(i)} := \tilde{F}^{(i)}(\mathbb{R}\times J_i\times X)\subset\mathbb{R}\times J_i\times X$ which is induced via $\tilde{F}^{(i)}$ by the vector field $\partial_u\times 0\times 0$ on $\mathbb{R}\times J_i\times X$. By construction, $d\,\mathrm{pr}_{(0,1)}(\xi^{(i)}) = 0$.

Let $\{\lambda^{(i)}\}_{i=0,1}\cup\{\lambda^{(v)}\}_{v\in(\delta,1-\delta)\times X}$ be a smooth partition of unity subordinate to the open cover $\{Z^{(i)}\}_{i=0,1}\cup\{Z^{(v)}\}_{v\in(\delta,1-\delta)\times X}$ of $Z := Z^{(0)}\cup Z^{(1)}\cup_{v\in(\delta,1-\delta)\times X} Z^{(v)}$. Note that Z is an open neighbourhood of $0\times V$ in $\mathbb{R}\times V$. Define a smooth vector field ξ on Z via

$$\xi = \lambda^{(0)}\xi^{(0)} + \lambda^{(1)}\xi^{(1)} + \sum_{v\in(\delta,1-\delta)\times X}\lambda^{(v)}\xi^{(v)}.$$

If $\xi_{\geq} := dF(\partial_u\times 0)$ denotes the smooth vector field on the open subset $Z_{\geq} := F([0,\infty)\times V)\subset$

$[0,\infty)\times V$ which is induced via F by the vector field $\partial_u \times 0$ on $[0,\infty)\times V$, then $\xi = \xi_{\geq}$ on $Z \cap Z_{\geq} = F(Y \cap ([0,\infty)\times V))$. (In fact, note that $\xi^{(v)} = \xi_{\geq}$ on $Z^{(v)} \cap Z_{\geq}$ for all $v \in (\delta, 1-\delta)\times X$ and $\xi^{(i)} = \xi_{\geq}$ on $Z^{(i)} \cap Z_{\geq}$ for $i = 0, 1$.)

It follows from $\operatorname{supp}\lambda^{(v)} \subset Z^{(v)} \subset \mathbb{R}\times(\delta, 1-\delta)\times X$ for all $v \in (\delta, 1-\delta)\times X$ that $\xi = \xi^{(0)}$ on $Z \cap (\mathbb{R}\times(0,\delta)\times X)$ and $\xi = \xi^{(1)}$ on $Z \cap (\mathbb{R}\times(1-\delta, 1)\times X)$. Moreover, note that $d\,\mathrm{pr}_{(0,1)}(\xi) = 0$.

There exists an open neighbourhood Ω of $0 \times Z$ in $\mathbb{R}\times Z$ on which the *flow* of ξ (see [22, page 151]) is defined as a smooth map

$$\eta\colon \Omega \to Z.$$

In particular, if $z \in Z$ and $\varepsilon > 0$ is so small that $(-\varepsilon, \varepsilon)\times z \subset \Omega$, then the smooth curve $\eta^z\colon (-\varepsilon,\varepsilon) \to Z$, $\eta^z(u) = \eta(u, z)$, satisfies $\eta^z(0) = z$ and $d\eta^z(\partial_u|_u) = \xi(\eta^z(u))$ for all u.

Note that $\Omega_0 := \Omega \cap (\mathbb{R}\times 0 \times V)$ is an open neighbourhood of $0\times 0\times V$ in $\mathbb{R}\times 0\times V$. (In fact, Ω is open in $\mathbb{R}\times Z$ which is open in $\mathbb{R}\times\mathbb{R}\times V$. Moreover, $0\times 0\times V \subset 0\times Z \subset \Omega$.) As X is compact, there exists $\rho > 0$ such that $(-\rho,\rho)\times 0 \times [\delta/2, 1-\delta/2]\times X \subset \Omega_0$ and $(-\rho,\rho)\times[\delta/2, 1-\delta/2]\times X \subset Y$ (recall that Y is an open neighbourhood of $0\times V$ in $\mathbb{R}\times V$).

We define the smooth map

$$\tilde{F}\colon (-\rho,\rho)\times[\delta/2, 1-\delta/2]\times X \to \mathbb{R}\times V, \qquad \tilde{F}(u,v) = \eta(u,0,v) = \eta^{(0,v)}(u).$$

It remains to check that $\tilde{F}$ possesses the desired properties:

- $\tilde{F}$ agrees with F on $[0,\rho)\times[\delta/2, 1-\delta/2]\times X$.
  In fact, given $v \in [\delta/2, 1-\delta/2]\times X$, consider the smooth curve

  $$\delta^v\colon [0,\rho) \to \mathbb{R}\times V, \quad \delta^v(u) = F(u,v).$$

  Note that $\delta^v(0) = F(0,v) = (0,v)$ and $d\delta^v(\partial_u|_u) = dF((\partial_u \times 0)|_{(u,v)}) = \xi_{\geq}(\delta^v(u)) = \xi(\delta^v(u))$ for all $u \in [0,\rho)$ (using that $(u,v) \in Y \cap ([0,\infty)\times V)$). Consequently, $\delta^v = \eta^{(0,v)}|_{[0,\rho)}$ by local uniqueness of integral curves. All in all, $\tilde{F}(u,v) = \eta^{(0,v)}(u) = \delta^v(u) = F(u,v)$ for all $u \in [0,\rho)$.
- $\tilde{F}$ is level-preserving, i.e. $\mathrm{pr}_{(0,1)} \circ \tilde{F} = \mathrm{pr}_{[\delta/2, 1-\delta/2]}$.
  Fix $v := (t,x) \in [\delta/2, 1-\delta/2]\times X$. Then the smooth curve

  $$\delta^v\colon (-\rho,\rho) \to (0,1), \qquad \delta^{(v)}(u) = \mathrm{pr}_{(0,1)}(\eta^{(0,v)}(u)),$$

  satisfies $\delta^v(0) = \mathrm{pr}_{(0,1)}(\eta^{(0,v)}(0)) = \mathrm{pr}_{(0,1)}(0,v) = t$ and $d\delta^v(\partial_u|_u) = d\,\mathrm{pr}_{(0,1)}(d\eta^{(0,v)}(\partial_u|_u)) = d\,\mathrm{pr}_{(0,1)}(\xi(\eta^{(0,v)}(u))) = 0$ for all $u \in (-\rho,\rho)$. Hence, it is a constant curve, and therefore $\mathrm{pr}_{(0,1)}(\tilde{F}(u,v)) = \mathrm{pr}_{(0,1)}(\eta^{(0,v)}(u)) = \delta^v(u) = \delta^v(0) = t$ for all $u \in (-\rho,\rho)$.
- $\tilde{F}$ agrees with $\tilde{F}^{(0)}$ on $(-\rho,\rho)\times(\delta/2,\delta)\times X$ and with $\tilde{F}^{(1)}$ on $(-\rho,\rho)\times(1-\delta, 1-\delta/2)\times X$.
  Fix $v \in (\delta/2,\delta)\times X$ and consider the smooth curve

  $$\delta^v\colon (-\rho,\rho) \to \mathbb{R}\times V, \qquad \delta^v(u) = \tilde{F}^{(0)}(u,v).$$

  Note that $\delta^v(0) = \tilde{F}^{(0)}(0,v) = F(0,v) = (0,v)$ and $d\delta^v(\partial_u|_u) = d\tilde{F}^{(0)}((\partial_u \times 0)|_{(u,v)}) = \xi^{(i)}(\tilde{F}^{(0)}(u,v)) = \xi(\tilde{F}^{(0)}(u,v)) = \xi(\delta^v(u))$ for all $u \in (-\rho,\rho)$ (using that $\tilde{F}^{(0)}(u,v) \in Z \cap (\mathbb{R}\times(0,\delta)\times X)$). Consequently, $\delta^v = \eta^{(0,v)}|_{(-\rho,\rho)}$ by local uniqueness of integral curves. All in all, $\tilde{F}(u,v) = \eta^{(0,v)}(u) = \tilde{F}^{(0)}(u,v) = F(u,v)$ for all $u \in (-\rho,\rho)$.

Analogously, one shows that $\tilde{F}$ agrees with $\tilde{F}^{(1)}$ on $(-\rho,\rho) \times (1-\delta, 1-\delta/2) \times X$.

To produce the desired map $\tilde{F} \colon \mathbb{R} \times V \to \mathbb{R} \times V$, one extends the above map $\tilde{F}$ via $\tilde{F}^{(0)}$ on $\mathbb{R} \times (0,\delta) \times X$ and $\tilde{F}^{(1)}$ on $\mathbb{R} \times (1-\delta,1) \times X$ to a smooth level-preserving map defined on $(\mathbb{R} \times (0,\delta) \times X) \cup ((-\rho,\rho) \times [\delta, 1-\delta] \times X) \cup (\mathbb{R} \times (1-\delta,1) \times X)$. Finally, one precomposes with a suitable embedding of $\mathbb{R} \times V$ into this set that restricts to the identity map on $[0,\infty) \times V$.

□

Let M be a smooth submanifold with boundary of a smooth manifold V with boundary. Following [22, p. 109], a *tubular neighbourhood* of M in V is a pair (f,ξ) consisting of a smooth vector bundle $\xi = (p, E, M)$ and an embedding $f \colon E \to V$ such that $f|_M = \mathrm{id}_M$ (where M is identified with the zero section of ξ) and $f(E)$ is an open subset of V.

The following Lemma supplies a useful characterization of the openness of $f(E)$ in V:

Lemma B.0.4. *Let P and Q be smooth manifolds with boundary such that* $\dim P = \dim Q$. *Suppose that $\varphi \colon P \to Q$ is an embedding. Then, $\varphi(P)$ is an open subset of Q if and only if $\varphi(\partial P) \subset \partial Q$.*

Proof. Note that $A := \varphi(P)$ is a smooth submanifold of Q with boundary $\partial A = \varphi(\partial P)$ in the sense of [22, p. 30]. (To show this, adapt the proof of [22, Theorem 3.1, p. 21].)

The condition $\partial A \subset \partial Q$ holds if and only if $\partial A = A \cap \partial Q$. (In fact, suppose that $x \in A \cap \partial Q$. Then [22, Exercise 2, p. 31] implies that $x \in \partial Q$ cannot lie in $\varphi(P \setminus \partial P)$ as φ is an embedding. Thus, $x \in A \setminus \varphi(P \setminus \partial P) = \partial A$.) The latter is equivalent to the statement that A is a *neat* submanifold of Q since φ is an embedding and $\dim P = \dim Q$. (In fact, note that $T_x A$ is not contained in $T_x(\partial Q)$ for all $x \in \partial A$ and use [22, p. 31]). Finally, a submanifold of Q is neat if and only if it admits a tubular neighbourhood by [22, Theorem 6.3, p. 114] and [22, p. 109]. As the codimension of A in Q is zero, having a tubular neighbourhood in Q just means for A to be an open subset of Q. □

Recall the notion of *isotopy of tubular neighbourhoods* defined in [22, p. 111f]:

Definition B.0.5. Let $(f_i, \xi_i = (p_i, E_i, M))$ be a tubular neighbourhood of M in V for $i = 0, 1$. An *isotopy of tubular neighbourhoods* from (f_0, ξ_0) to (f_1, ξ_1) is a rel M isotopy (recall from [22, p. 111] that the track of an isotopy is required to be an embedding)

$$F \colon [0,1] \times E_0 \to V$$

from E_0 to V such that the following properties hold:

- (i) $F_0 = f_0$.
- (ii) $F_1(E_0) = f_1(E_1)$, and $f_1^{-1} F_1 \colon E_0 \to E_1$ is an isomorphism $\xi_0 \to \xi_1$ of vector bundles.
- (iii) $\hat{F}([0,1] \times E_0)$ is an open subset of $[0,1] \times V$.

We will content ourselves with the following sufficient criterion for the existence of an isotopy of tubular neighbourhoods:

Lemma B.0.6. *Assume that M is compact. Let $(f_i, \xi = (p, E, M))$ be tubular neighbourhoods of M in V for $i = 0, 1$. Given a smooth map $F \colon [0,1] \times E \to V$ such that $(F_t := F(t,-), \xi)$*

is a tubular neighbourhood of M *in* V *for all* $t \in [0,1]$ *and such that* $F_i = f_i$ *for* $i = 0,1$*, there exist tubular neighbourhoods* $(\tilde{f}_i, \xi)$ *of* M *in* V *for* $i = 0,1$ *with the following properties:*

(*i*) *There exists a neighbourhood* U *of* M *in* E *such that* $\tilde{f}_i|_U = f_i|_U$ *for* $i = 0,1$*.*

(*ii*) *There exists an isotopy of tubular neighbourhoods*

$$G\colon [0,1] \times E \to V$$

from $(\tilde{f}_0, \xi)$ *to* $(\tilde{f}_1, \xi)$ *such that* $G_1 = \tilde{f}_1$ *and* $G|_{[0,1]\times U} = F|_{[0,1]\times U}$*.*

Proof. Choose a smooth map $\rho\colon [0,1] \to [0,1]$ that maps a neighbourhood of i in $[0,1]$ to i for $i = 0,1$. Moreover, using that M is compact, choose a tubular neighbourhood (g, ξ) of M in E such that $g\colon E \to E$ restricts to the identity map in a neighbourhood U of M in E and such that the closure of $g(E)$ in E is compact. We claim that the smooth map

$$G := F \circ (\rho \times g)\colon [0,1] \times E \to V$$

is an isotopy of tubular neighbourhoods from $(\tilde{f}_0, \xi) := (f_0 \circ g, \xi)$ to $(\tilde{f}_1, \xi) := (f_1 \circ g, \xi)$ in the sense of Definition B.0.5. By choice of g, $(\tilde{f}_i, \xi)$ are tubular neighbourhoods of M in V for $i = 0,1$ that satisfy claim (i) for U as above. Hence, it remains to show that G is a rel M isotopy that satisfies properties (i) to (iii) of an isotopy of tubular neighbourhoods.

In order to show that G is an isotopy, we have to show that the track $\hat{G}\colon [0,1]\times E \to [0,1]\times V$ is an embedding (see [22, p. 111]). As $G_t := G(t,-) = F_{\rho(t)} \circ g$ is an embedding for all $t \in [0,1]$, it follows that the track $\hat{G}$ is an injective immersion. To see that $\hat{G}$ restricts to a homeomorphism $[0,1]\times E \to \hat{G}([0,1]\times E)$, we set $H := F\circ(\rho\times \mathrm{id}_E)\colon [0,1]\times E \to V$ and write $\hat{G} = \hat{H}\circ(\mathrm{id}_{[0,1]} \times g)$. Now note that $\mathrm{id}_{[0,1]} \times g$ restricts to a homeomorphism $[0,1] \times E \to [0,1] \times g(E)$ as g is an embedding. Moreover, the track $\hat{H}\colon [0,1] \times E \to [0,1] \times V$ is injective because $H_t := H(t,-) = F_{\rho(t)}$ is an embedding for all $t \in [0,1]$, and thus restricts to a homeomorphism $K \to \hat{H}(K)$ for any compact subset $K \subset [0,1] \times E$ (noting that $[0,1] \times V$ is a Hausdorff space). Hence, taking K to be the closure of $[0,1]\times g(E)$ in $[0,1]\times E$, it follows that $\hat{G}$ restricts to a homeomorphism $[0,1] \times E \to \hat{G}([0,1] \times E)$. All in all, G is a rel M isotopy. (It is clear that $G_t = F_{\rho(t)} \circ g$ leaves M pointwise fixed for all $t \in [0,1]$.)

Note that G satisfies properties (i) and (ii) of an isotopy of tubular neighbourhoods because $G_i = F_{\rho(i)} \circ g = f_i \circ g$ for $i = 0,1$. It remains to check property (iii), which states that $\hat{G}([0,1] \times E)$ is an open subset of $[0,1] \times V$. By choice of ρ we have $G_t = G_i$ for t near $i = 0,1$. Hence, G can be extended to a smooth map $D\colon \mathbb{R} \times E \to V$ by setting $D_t := G_0$ for $t < 0$ and $D_t := G_1$ for $t > 1$. Since $\hat{D}(\mathbb{R} \times E) \cap ([0,1] \times V) = \hat{G}([0,1] \times E)$, it suffices to show that $\hat{D}(\mathbb{R} \times E)$ is an open subset of $\mathbb{R} \times V$. For this purpose, note that $\hat{D}$ is an embedding. (In fact, $\hat{D}$ is an injective immersion since D_t is an embedding for all $t \in [0,1]$. Furthermore, $\hat{D}|\colon \mathbb{R} \times E \to \hat{D}(\mathbb{R} \times E)$ is a homeomorphism. To show this, choose $\varepsilon > 0$ so small that $G_t = G_0$ for $t \in [0,\varepsilon)$ and $G_t = G_1$ for $t \in (1-\varepsilon, 1]$. Now it suffices to note that the following restrictions of $\hat{D}|$ to open subsets of domain and codomain are homeomorphisms:

$$\mathrm{id}_{(-\infty,\varepsilon)} \times G_0|\colon (-\infty,\varepsilon) \times E \xrightarrow{\cong} (-\infty,\varepsilon) \times G_0(E) = \hat{D}(\mathbb{R} \times E) \cap ((-\infty,\varepsilon) \times V),$$
$$\hat{G}|\colon (0,1) \times E \xrightarrow{\cong} \hat{G}((0,1) \times E) = \hat{D}(\mathbb{R} \times E) \cap ((0,1) \times V),$$
$$\mathrm{id}_{(1-\varepsilon,\infty)} \times G_1|\colon (1-\varepsilon,\infty) \times E \xrightarrow{\cong} (1-\varepsilon,\infty) \times G_1(E) = \hat{D}(\mathbb{R} \times E) \cap ((1-\varepsilon,\infty) \times V).)$$

Hence, by Lemma B.0.4, it suffices to show that $\hat{D}(\mathbb{R}\times\partial E)\subset\mathbb{R}\times\partial V$. Equivalently, $G_t(\partial E)\subset\partial V$ for all $t\in[0,1]$. The latter holds by Lemma B.0.4 as $G_t\colon E\to V$ is for all $t\in[0,1]$ an embedding such that $G_t(E)$ is open in V.

□

Proposition B.0.7. *Consider the cylinder $Z:=X\times[0,1]$. Suppose that*

$$f=(f_Z=(f_X,f_{[0,1]}),f_{\mathbb{R}})\colon(Z\times\mathbb{R},Z\times(0,\infty))\to(Z\times\mathbb{R},Z\times(0,\infty))$$

is a tubular neighbourhood of $Z=Z\times 0$ in $Z\times\mathbb{R}$ which is for t near $i=0,1$ of the form

$$f(x,t,u)=(f_X(x,i,u),t,f_{\mathbb{R}}(x,i,u)),\qquad (x,u)\in X\times\mathbb{R}.$$

If ξ denotes the trivial vector bundle $Z\times\mathbb{R}\to Z$, then there exist for $i=0,1$ tubular neighbourhoods $(\tilde{f}_i,\xi)$ of $Z\times 0$ in $Z\times\mathbb{R}$ with the following properties:

- *(i) There exists a neighbourhood U of $Z\times 0$ in $Z\times\mathbb{R}$ such that $\tilde{f}_0|_U=\mathrm{id}_{Z\times\mathbb{R}}|_U$ and $\tilde{f}_1|_U=f|_U$.*
- *(ii) There exists an isotopy of tubular neighbourhoods*

$$\tilde{F}\colon[0,1]\times Z\times\mathbb{R}\to Z\times\mathbb{R},\qquad \tilde{F}(s,z,u)=:\tilde{F}_s(z,u),$$

from $(\tilde{f}_0,\xi)$ to $(\tilde{f}_1,\xi)$ such that $\tilde{F}_1=\tilde{f}_1$.

Moreover, if $f_X(x,i,u)=x$ and $f_{\mathbb{R}}(x,i,u)=u$ for $i=0,1$ and all $(x,u)\in X\times\mathbb{R}$, then $\tilde{F}$ can be chosen to satisfy $\tilde{F}_s(x,t,u)=(x,t,u)$ for all $s\in[0,1]$, where $t\in[0,1]$ is near 0 or 1, and $(x,u)\in X\times\mathbb{R}$ is such that $(x,t,u)\in U$.

Remark B.0.8. As an inspection of the proof shows, Proposition B.0.7 is also valid in the case of a partial tubular neighbourhood $(Z\times(-\delta,\delta),Z\times(0,\infty))\to(Z\times\mathbb{R},Z\times(0,\infty))$, $\delta>0$, of $Z=Z\times 0$ in $Z\times\mathbb{R}$. This is the form in which it will be applied in the proof of Theorem 8.0.1.

Proof. In order to apply Lemma B.0.6, we adapt the proof of [22, Theorem 5.3, page 112].

Consider the function

$$\phi\colon Z\to\mathbb{R},\quad \phi(z)=\lim_{u\to 0}\frac{1}{u}f_{\mathbb{R}}(z,u).$$

Note that ϕ is smooth and $\phi(z)>0$ for all $z\in Z$. (Indeed, as $\phi(x,t)=\lim_{u\to 0}\frac{1}{u}f_{\mathbb{R}}(x,t,u)=\lim_{u\to 0}\frac{1}{u}f_{\mathbb{R}}(x,i,u)=\phi(x,i)$ for t near $i=0,1$ and all $x\in X$, it suffices to show these claims on an open subset $V\subset\operatorname{int}Z=X\times(0,1)$ that is part of a chart $\gamma\colon V\xrightarrow{\cong}V'\subset\mathbb{R}^n$ of Z (where n denotes the dimension of Z). Note that the embedding $g=(g_Z,g_{\mathbb{R}}):=f|\circ(\gamma^{-1}\times\mathrm{id}_{\mathbb{R}})\colon V'\times\mathbb{R}\to Z\times\mathbb{R}$ satisfies $g(v,0)=f(\gamma^{-1}(v),0)=(\gamma^{-1}(v),0)$ for all $v\in V'$. Therefore, for all $z\in V$,

$$\phi(z)=\lim_{u\to 0}\frac{1}{u}f_{\mathbb{R}}(z,u)=\lim_{u\to 0}\frac{g_{\mathbb{R}}(\gamma(z),u)-g_{\mathbb{R}}(\gamma(z),0)}{u-0}=\partial_{n+1}g_{\mathbb{R}}(\gamma(z),0).$$

This shows that ϕ is smooth on V. Next, we show that $\phi(z)>0$ for all $z\in V$. It follows from $f_{\mathbb{R}}(Z\times(0,\infty))\subset(0,\infty)$ that $\phi(z)\geq 0$ for all $z\in Z$. Suppose that $\phi(z)=0$ for some $z\in V$. Then, the above calculation implies that $\partial_{n+1}g_{\mathbb{R}}(\gamma(z),0)=0$. Moreover, for $1\leq i\leq n$,

$$\partial_i g_{\mathbb{R}}(\gamma(z),0)=\lim_{r\to 0}\frac{g_{\mathbb{R}}(\gamma(z)+re_i,0)-g_{\mathbb{R}}(\gamma(z),0)}{r-0}=\lim_{r\to 0}\frac{0-0}{r-0}=0.$$

(Here, e_i denotes the i-th standard unit vector in $\mathbb{R}^n$.) Hence, the gradient of $g_{\mathbb{R}}$ vanishes at $(\gamma(z), 0)$, which is a contradiction to the fact that g is an embedding. Thus, $\phi(z) \neq 0$ for all $z \in V$.) The tubular neighbourhood of $Z = 0 \times Z$ in $\mathbb{R} \times Z$ given by

$$\Phi\colon Z \times \mathbb{R} \to Z \times \mathbb{R}, \qquad \Phi(z, u) = (z, \phi(z) \cdot u),$$

is called *fiber derivative* in the proof of [22, Theorem 5.3, p. 112].

Since $s \cdot \phi(z) + 1 - s > 0$ for all $z \in Z$, the smooth map

$$F^{(1)}\colon [0,1] \times Z \times \mathbb{R} \to Z \times \mathbb{R}, \qquad F^{(1)}_s(z, u) = (z, (s \cdot \phi(z) + 1 - s) \cdot u),$$

defines for every $s \in [0, 1]$ a tubular neighbourhood $(F^{(1)}_s\colon Z \times \mathbb{R} \to Z \times \mathbb{R}, \xi)$ of $Z \times 0$ in $Z \times \mathbb{R}$. Moreover, we claim that

$$F^{(2)}\colon [0,1] \times Z \times \mathbb{R} \to Z \times \mathbb{R}, \qquad F^{(2)}_s(z, u) = \begin{cases} \Phi(z, u), & s = 0, \\ (f_Z(z, su), \frac{1}{s} f_{\mathbb{R}}(z, su)), & 0 < s \leq 1, \end{cases}$$

is a smooth map such that $(F^{(2)}_s\colon Z \times \mathbb{R} \to Z \times \mathbb{R}, \xi)$ is for every $s \in [0, 1]$ a tubular neighbourhood of $Z \times 0$ in $Z \times \mathbb{R}$. Once smoothness of $F^{(2)}$ is checked, the claim of the proposition follows directly from the application of Lemma B.0.6 (note that X is compact) to the smooth map

$$F\colon [0,1] \times Z \times \mathbb{R} \to Z \times \mathbb{R}, \qquad F_s = \begin{cases} F^{(1)}_{\tau(2s)}, & s \in [0, 1/2), \\ F^{(2)}_{\tau(2s-1)}, & s \in [1/2, 1], \end{cases}$$

where $\tau\colon [0,1] \to [0,1]$ is a chosen smooth map such that $\tau(t) = i$ for t near $i = 0, 1$. (Note that, by construction, F is a smooth map such that $F_0 = \mathrm{id}_{Z \times \mathbb{R}}$, $F_1 = f$, and $F_s\colon Z \times \mathbb{R} \to Z \times \mathbb{R}$ is a tubular neighbourhood of $Z \times 0$ in $Z \times \mathbb{R}$ for all $s \in [0, 1]$.)

Obviously, $F^{(2)}_s$ is for every $s \in [0, 1]$ a tubular neighbourhood of $Z = 0 \times Z$ in $\mathbb{R} \times Z$. (Indeed, note that $F^{(2)}_0 = \Phi$, and for every $s \in (0, 1]$,

$$F^{(2)}_s = (\mathrm{id}_Z \times \mathrm{mult}_{1/s}) \circ f \circ (\mathrm{id}_Z \times \mathrm{mult}_s).)$$

It remains to check that $F^{(2)}$ is smooth. (This corresponds to the proof of smoothness of the homotopy H in the proof of [22, Theorem 5.3, page 112].)

If t is near $i = 0, 1$, then we have for all $(x, u) \in X \times \mathbb{R}$ that

$$\begin{aligned} &F^{(2)}(0, x, t, u) = \Phi(x, t, u) = (x, t, \phi(x, i) \cdot u), \\ &F^{(2)}(s, x, t, u) = (f_Z(x, t, su), \frac{1}{s} f_{\mathbb{R}}(x, t, su)) = (f_X(x, i, su), t, \frac{1}{s} f_{\mathbb{R}}(x, i, su)), \qquad s \in (0, 1]. \end{aligned}$$

Hence, it suffices to show that $F^{(2)}|_{[0,1] \times V \times \mathbb{R}}$ is smooth for any open subset $V \subset X \times (0, 1)$ that is part of a chart $\gamma\colon V \xrightarrow{\cong} V' \subset \mathbb{R}^n$ of Z. As above, we define the embedding $g := f| \circ (\gamma^{-1} \times \mathrm{id}_{\mathbb{R}})\colon V' \times \mathbb{R} \to Z \times \mathbb{R}$. Using $g_{\mathbb{R}}(v, 0) = 0$ for all $v \in V'$, Taylor's formular implies that for all $(v, u) \in V' \times \mathbb{R}$,

$$g_{\mathbb{R}}(v, u) = \partial_{n+1} g_{\mathbb{R}}(v, 0) \cdot u + \sigma(v, u) \cdot u,$$

for some smooth function $\sigma\colon V' \times \mathbb{R} \to \mathbb{R}$ such that $\sigma(v,0)=0$ for all $v \in V'$. Then

$$\overline{F}^{(2)}\colon \mathbb{R}\times V \times \mathbb{R} \to Z \times \mathbb{R}, \qquad \overline{F}^{(2)}_s(z,u) = (f_Z(z,su), (\phi(z)+\sigma(\gamma(z),su))\cdot u),$$

is a smooth function that satisfies $\overline{F}^{(2)}|_{[0,1]\times V\times\mathbb{R}} = F^{(2)}|_{[0,1]\times V\times\mathbb{R}}$. In fact, for all $(z,u) \in Z\times\mathbb{R}$,

$$\overline{F}^{(2)}_0(z,u) = (f_Z(z,0), (\phi(z)+\sigma(\gamma(z),0))\cdot u) = (z, \phi(z)\cdot u) = F^{(2)}_0(z,u).$$

Moreover, for $s \in (0,1]$ and all $(z,u)\in Z\times\mathbb{R}$, the equation

$$\frac{1}{s} f_{\mathbb{R}}(z,su) = \frac{1}{s} g_{\mathbb{R}}(\gamma(z),su) = \partial_{n+1} g_{\mathbb{R}}(\gamma(z),0)\cdot u + \sigma(\gamma(z),su)\cdot u = (\phi(z)+\sigma(\gamma(z),su))\cdot u$$

implies that

$$\overline{F}^{(2)}_s(z,u) = (f_Z(z,su), (\phi(z)+\sigma(\gamma(z),su))\cdot u) = (f_Z(z,su), \frac{1}{s} f_{\mathbb{R}}(z,su)) = F^{(2)}_s(z,u).$$

All in all, this shows that $F^{(2)}$ is smooth.

Finally, suppose that $f_X(x,i,u)=x$ and $f_{\mathbb{R}}(x,i,u)=u$ for $i=0,1$ and all $(x,u)\in X\times\mathbb{R}$. Let $t\in[0,1]$ be near 0 or 1, and let $(x,u)\in X\times\mathbb{R}$ such that $(x,t,u)\in U$. Since the desired $\tilde{F}$ is constructed from $F^{(2)}$ by means of Lemma B.0.6, we have $\tilde{F}_s|_U = F_s|_U$ for all $s\in[0,1]$ by property (ii) of Lemma B.0.6. Hence, to prove the claim $\tilde{F}_s(x,t,u)=(x,t,u)$, it suffices to show that $F_s(x,t,u)=(x,t,u)$. For this purpose, we distinguish between the following three cases for $s\in[0,1]$:

- $s\in[0,1/2)$. Then,

$$F_s(x,t,u) = F^{(1)}_{\tau(2s)}(x,t,u) = (x,t,(\tau(2s)\cdot\phi(x,t)+1-\tau(2s))\cdot u) = (x,t,u).$$

  Here, we have used that $\phi(x,t) = \lim_{u\to 0}\frac{1}{u} f_{\mathbb{R}}(x,t,u) = 1$, where $f_{\mathbb{R}}(x,t,u) = f_{\mathbb{R}}(x,i,u) = u$ since t is near $i=0,1$.

- $s=1/2$. Then, using again that $\phi(x,t)=1$ since t is near $i=0,1$,

$$F_s(x,t,u) = F^{(2)}_{\tau(2s-1)}(x,t,u) = F^{(2)}_0(x,t,u) = \Phi(x,t,u) = (x,t,\phi(x,t)\cdot u) = (x,t,u).$$

- $s\in(1/2,1]$. Then, using $f_X(x,t,\tau(2s-1)u) = f_X(x,i,\tau(2s-1)u)$, $f_{[0,1]}(x,t,\tau(2s-1)u) = t$ and $f_{\mathbb{R}}(x,t,\tau(2s-1)u) = f_{\mathbb{R}}(x,i,\tau(2s-1)u)$ for t near $i=0,1$,

$$\begin{aligned}
F_s(x,t,u) &= F^{(2)}_{\tau(2s-1)}(x,t,u)\\
&= (f_Z(x,t,\tau(2s-1)u), \frac{1}{\tau(2s-1)} f_{\mathbb{R}}(x,t,\tau(2s-1)u))\\
&= (f_X(x,t,\tau(2s-1)u), f_{[0,1]}(x,t,\tau(2s-1)u), \frac{1}{\tau(2s-1)} f_{\mathbb{R}}(x,t,\tau(2s-1)u))\\
&= (f_X(x,i,\tau(2s-1)u), t, \frac{1}{\tau(2s-1)} f_{\mathbb{R}}(x,i,\tau(2s-1)u))\\
&= (x,t,\frac{1}{\tau(2s-1)}\cdot\tau(2s-1)u) = (x,t,u).
\end{aligned}$$

□

Appendix C

Some Morse Theory

The purpose of the present section is to present some fundamental material on Morse theory that will serve as a convenient reference for Part III.

Following the presentation in [41], we will make use of the notion of *Morse functions* (see [41, Definition 2.3, page 8]) on *smooth manifold triads* (see [41, Definition 1.3, page 2]).

The following result is a careful reformulation of [41, Theorem 7.8, page 97]:

Lemma C.0.1. *Let* (W, V_0, V_1) *be a smooth manifold triad of dimension* $n := \dim W \geq 6$ *. Given an integer* $2 \leq r < n-2$ *and a self-indexing (see [41, Definition 4.9, p. 44]) Morse function*

$$f\colon (W, V_0, V_1) \to ([-1/2, n+1/2], -1/2, n+1/2),$$

the following statements hold (where all occurring homology groups are taken with integer coefficients):

(a) If $H_r(W, V_0) = H_{r+1}(W, V_0) = 0$ *and* f *has no critical points of index different from* r *and* $r+1$ *, then* f *has an equal number of critical points of index* r *and* $r+1$ *.*

(b) Suppose that W *,* V_0 *and* V_1 *are all simply connected. If* $H_r(W, V_0) = 0$ *and* f *has no critical points of index* $< r$ *, then the critical points of* f *of index* r *can be cancelled against an equal number of critical points of* f *of index* $r+1$ *. More precisely, there exists a self-indexing Morse function* $g\colon (W, V_0, V_1) \to ([-1/2, n+1/2], -1/2, n+1/2)$ *that agrees with* f *outside a given neighbouhood of* $f^{-1}([r, r+1])$ *and has no critical points of index* $< r+1$ *.*

Proof. We reproduce the proof of [41, Theorem 7.8, page 97] with the necessary modifications.

It is well-known that there exists a chain complex of free abelian groups (or $\mathbb{Z}$-modules)

$$C_{n-2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{r+1} \xrightarrow{\partial} C_r \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2$$

such that the rank of C_λ is given by the number of critical points of f of index λ (see [41, page 89] and [41, Section 3, page 36]), and $H_*(C_*) \cong H_*(W, V_0)$ (see [41, Theorem 7.4, page 90]).

(a) Since every critical point of f has index r or $r+1$, we have $C_\lambda = 0$ for $\lambda \notin \{r, r+1\}$. As $H_r(W, V_0) = H_{r+1}(W, V_0) = 0$, the above chain complex reduces to an isomorphism $\partial\colon C_{r+1} \xrightarrow{\cong} C_r$. In particular, these two free abelian groups have the same rank, which implies that f has an equal number of critical points of index r and $r+1$.

(b) Since f has no critical points of index $< r$, we have $C_\lambda = 0$ for $\lambda < r$. Let $z_1, \dots, z_k$ be a basis of the free abelian group C_r. (In particular, k denotes the number of critical points of f of index r.) It follows from $H_r(W, V_0) = 0$ that the above chain complex is exact at C_r. Hence, there exist $b_1, \dots, b_k \in C_{r+1}$ such that $\partial(b_i) = z_i$ for all $i = 1, \dots, k$. Let $y_1, \dots, y_l$ be a basis for the kernel of $\partial\colon C_{r+1} \to C_r$. Then $y_1, \dots, y_l, b_1, \dots, b_k$ constitutes a basis for C_{r+1}. Since $2 \le r < r+1 \le n-2$ and W is connected, we can use the basis theorem [41, Theorem 7.6, page 92] to find a Morse function h and a gradient-like vector field ξ such that the left-hand discs of the critical points of f of index r represent the basis $\{z_1, \dots, z_k\}$ of C_r and the left-hand discs of the critical points of f of index $r+1$ represent the basis $y_1, \dots, y_l, b_1, \dots, b_k$ of C_{r+1}. Finally, the process described in the proof of [41, Theorem 7.8, page 97ff] allows us to eliminate the pairs (z_i, b_i) successively for $i = 1, \dots, k$. □

Lemma C.0.2. *Let (W, V_0, V_1) be a smooth manifold triad of dimension $n := \dim W \ge 2$ that admits a Morse function $(W, V_0, V_1) \to ([0,1], 0, 1)$ without critical points of index < 2. If V_0 is simply connected, then W is simply connected as well.*

Proof. By Morse theory there is a sequence of cobordisms

$$W_1 := V_0 \times [0,1] \subset W_2 \subset \cdots \subset W_k := W,$$

such that W_{i+1} is for $i = 1, \dots, k-1$ homotopy equivalent to W_i with a $(\nu_i + 1)$-ball attached along an embedded ν_i-sphere ($\nu_i \ge 1$). Hence, if W_i is simply connected, then W_{i+1} is simply connected as well by the Seifert van Kampen theorem. (Note that if $\nu_i = 1$ for some i, then the embedded 1-sphere is not simply connected, but the 2-ball and W_i are.) Thus, it follows inductively that W is simply connected.

□

The proof of $(i) \Rightarrow (ii)$ in the following Lemma is an adaption of the proof of Smale's h-cobordism theorem in [41, Theorem 9.1, page 107].

Lemma C.0.3. *Fix integers $m \ge 8$ and $k \in \{1, \dots, \lfloor \frac{m}{2} \rfloor\}$. If (W, Σ_0, Σ_1) is a smooth manifold triad of dimension $m = \dim W$ such that Σ_0 and Σ_1 are homotopy spheres, then the folowing statements are equivalent:*

- (i) *W is $(k-1)$-connected.*
- (ii) *(W, Σ_0, Σ_1) admits a Morse function with only critical points of index $\{k, \dots, m-k\}$.*

Proof. (All homology groups in the present proof are taken with integer coefficients.)

$(i) \Rightarrow (ii)$. In preparation of the construction of the desired Morse function, we first show that

$$(*) \qquad H_i(W, \Sigma_0) = 0 = H_i(W, \Sigma_1) \qquad \text{for } i = 0, 1, \dots, k-1.$$

In fact, consider the following portion of the long exact sequence of reduced homology groups for the pair (W, Σ_0) (see [20, page 118]):

$$\tilde{H}_i(\Sigma_0) \to \tilde{H}_i(W) \to \tilde{H}_i(W, \Sigma_0) \to \tilde{H}_{i-1}(\Sigma_0).$$

As the occuring reduced homology groups of the homotopy sphere Σ_0 vanish, we conclude that

$$H_i(W, \Sigma_0) = \tilde{H}_i(W, \Sigma_0) \cong \tilde{H}_i(W) \qquad \text{for } i = 0, 1, \ldots, k-1.$$

Analogously, $H_i(W, \Sigma_1) \cong \tilde{H}_i(W)$ for $i = 0, 1, \ldots, k-1$. As W is $(k-1)$-connected by (i), the Hurewicz theorem implies that $\tilde{H}_i(W) = 0$ for $i = 0, 1, \ldots, k-1$. This proves $(*)$.

By [41, Theorem 4.8, page 44], there exists a *self-indexing* Morse function

$$f\colon (W, \Sigma_0, \Sigma_1) \to ([-\frac{1}{2}, m + \frac{1}{2}], -\frac{1}{2}, m + \frac{1}{2})$$

(see [41, Definition 4.9, page 44]), i.e. any critical point p of f has index $f(p)$. In order to construct the desired Morse function, we will use $(*)$ to simplify f via a finite sequence of eliminations of critical points of successive indices. Since $H_0(W, \Sigma_0) = 0$, it follows from [41, Theorem 8.1, page 100] that all critical points of f of index 0 can be cancelled against an equal number of critical points of f of index 1. Moreover, for $k > 1$, as W and Σ_0 are simply connected, all critical points of f of index 1 can be traded for an equal number of critical points of f of index 3 by the same theorem. Thus, we may assume that f has no critical points of index 0 and 1. Next, if $k > 2$, then we repeatedly use Lemma C.0.1(b) and $H_i(W, \Sigma_0) = 0$ for $i = 2, \ldots, k-1$ to cancel all critical points of f of index i against an equal number of critical points of f of index $i+1$. Consequently, we may assume that f has no critical points of index $0, 1, \ldots, k-1$. Replacing f by $m - f$, we obtain a self-indexing Morse function on the triad (W, Σ_1, Σ_0) with no critical points of index $m-k+1, \ldots, m$. By the same arguments as for f, we can now use $(*)$ to eliminate the critical points of $m - f$ of index $0, 1, \ldots, k-1$. This results in a Morse function f with only critical points of index k.

$(ii) \Rightarrow (i)$. As in the proof of $(i) \Rightarrow (ii)$ one can show that $H_i(W, \Sigma_0) \cong \tilde{H}_i(W)$ for $i = 0, \ldots, k-1$. Hence, the beginning of the proof of Lemma C.0.1 and (ii) imply that $\tilde{H}_i(W) = 0$ for $i = 0, \ldots, k-1$. Finally, if $k \geq 2$, then W is simply connected by Lemma C.0.2. ☐

Lemma C.0.4. *Consider a Morse function $f\colon M \to \mathbb{R}$ without critical points of index 1 and $n-1$ on a connected closed smooth manifold M^n of dimension $n \geq 2$. Then the following statements hold:*

(a) M is orientable.

(b) f has precisely one critical point of index 0 and precisely one critical point of index n.

(c) All nonempty fibers $f^{-1}(t)$, $t \in \mathbb{R}$, are connected.

(d) f restricts to an open map $M \to f(M)$.

(e) M is simply connected.

Proof. (a). Since $C_{n-1} = 0$ yields $H_{n-1}(M) = 0$, M is orientable by [20, Corollary 3.28, p. 238]. Alternatively, use (e).

(b). (All homology groups in this proof are understood with integer coefficients.) It is well-known that there exists a chain complex of free abelian groups (or $\mathbb{Z}$-modules)

$$C_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{r+1} \xrightarrow{\partial} C_r \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

such that the rank of C_λ is given by the number of critical points of f of index λ (see [41, page 89] and [41, Section 3, page 36]) and $H_*(C_*) \cong H_*(M)$ since $\partial M = \emptyset$ (see [41, Theorem

7.4, page 90]). Observe that $H_0(C_*) = C_0$ because $C_1 = 0$ by assumption. On the other hand, $H_0(C_*) \cong \mathbb{Z}$ because M is connected. Therefore, $C_0 \cong \mathbb{Z}$, which shows that f has precisely one critical point of index 0. Application of the same argument to the Morse function $-f$ (which has no critical points of index 1 as well) shows that f has precisely one critical point of index n.

(c). The claim follows for non-singular fibers of f from the fact that by [44, Proposition 4.19(iii), p. 56] the effect of a p-surgery on a closed smooth manifold of dimension $d \geq p+2$ does not affect its homology groups in dimensions strictly below $\min(p, d-p-1)$. In our case, $d = n-1$ and $p \in \{1, \dots, n-3\}$. As far as singular fibers of f are concerned, recall that the fiber $f^{-1}(t)$ of a singular value $t \in \mathbb{R}$ of f of index λ is homeomorphic to the cone the inclusion $S^{\lambda-1} \times S^{n-\lambda-1} \hookrightarrow S^{\lambda-1} \times D^{n-\lambda} \overset{\alpha}{\to} f^{-1}(t')$, where α is a tubular neighbourhood of the left-hand sphere in a nearby regular fiber $f^{-1}(t')$, $t' < t$. If more than one critical point lies on the same level t, then one cones off the corresponding inclusions separately.

(d). It suffices to cover M by open subsets such that f restricts on each of these subsets U to an open map $U \to f(M)$. One of these open subsets can obviously be taken to be the complement of the critical points of f in M. It remains to consider the image of sufficiently small open neighbourhoods of the origin under the the local normal form of a non-degenerate critical point of index $\lambda \in \{0, \dots, n\}$,

$$\mu^n_\lambda \colon \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda} \to \mathbb{R}, \qquad (x, y) \mapsto -|x|^2 + |y|^2.$$

We distinguish between the following two cases:

- If $\lambda \in \{1, \dots, n-1\}$, then μ^n_λ maps an open r-ball centered at the origin onto the open subset $(-r^2, r^2) \subset \mathbb{R}$.
- For $\lambda \in \{0, n\}$ let c_λ denote the unique critical point of f of index λ (see part (b)), and set $a_\lambda := f(c_\lambda)$. Since $f(M) = [a_0, a_n]$, it suffices to observe that μ^n_0 maps an open r-ball centered at the origin onto the open subset $[0, r^2) \subset [0, \infty)$ and μ^n_n maps an open r-ball centered at the origin onto the open subset $(-r^2, 0] \subset (-\infty, 0]$.

(e). This follows from Lemma C.0.2. □

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