EXPLICIT RECIPROCITY LAWS IN IWASAWA THEORY
- A SURVEY WITH SOME FOCUS ON THE LUBIN-TATE SETTING -

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ABSTRACT. Starting from Gauß' and Legendre's quadratic reciprocity law we want to sketch how it gave rise to the development of higher and generalized reciprocity laws and even more general explicit reciprocity formulas in Iwasawa theory.

1. INTRODUCTION

The whole article is aimed to serve as an introduction to the different meanings of *explicit reciprocity laws or formulas*. Therefore we confine ourselves here to just give a short guideline through the topic. In section 2 we recall Gauß' reciprocity[1] law and link it to the quadratic Hilbert symbol. We then proceed in section 3 to introduce higher Hilbert symbols in the context of Galois cohomology together with variants in characteristic p or for formal groups. In section 4 we explain some explicit formulas which calculate some of these higher Hilbert symbols. While already here towers of local fields play a crucial role, it will not be before section 6 that we take the full Iwasawa theoretic point of view introduced by Perrin-Riou, Kato (and Coleman), by which we mean the *compatibility of cup-products in Galois cohomology with de Rham duality in p-adic Hodge theory*. As preparation and transition we recall in section 5 the (dual) exponential map of Bloch and Kato and their version of an explicit reciprocity law for the representation $\mathbb{Q}_p(r)$, for $r \geq 1$. Finally, in Iwasawa theory it has become common practice to also call statements like

$$Log_{BK/PR}(res_p(EulerSystem)) = \mathcal{L}_p$$

expressing an p-adic L-functions $\mathcal{L}_p$ as image of some global Euler system under the restriction map res_p to local Galois/Iwasawa cohomology followed by the Bloch-Kato/Perri-Riou (dual) exponential/big logarithm map $Log_{BK/PR}$ as *explicit reciprocity laws* as we will explain in section 7[2]. In the last section 8 we quickly mention the impact of explicit reciprocity laws towards the ε-isomorphism

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[1]We prefer the German spelling "Gauß" of this personal name, although non-German-speakers will be more familiar with "Gauss" perhaps.

[2]David Loeffler pointed out to me that this arguably inaccurate terminology was probably first used in the survey paper [13]: they refer to Kato's work in which he *uses* a reciprocity law for formal groups (in the above mentioned traditional sense of the term) to deduce a formula for the image under the dual Bloch-Kato exponential map of his Euler system class; intentionally or accidentally they may have (mis)applied the name of the intermediate result as the name of the final theorem. Massimo Bertolini responded to me: *My recollection (somewhat vague after 30 years) is that the non-traditional terminology of calling explicit reciprocity laws also the logarithmic images of Euler systems emerged in the early 90's, in connection with Kato's work. Kato used an explicit reciprocity law (in the original sense) for higher dimensional local fields to prove his formula connecting special values of Hasse-Weil L-functions to the logarithms of his classes. After a while our community started to call this type of formula also an explicit reciprocity law. As Henri Darmon points out, one can find a similar pattern in the paper by Coates-Wiles* [29]*, where they used an explicit reciprocity law to establish their main theorem. By taking a quick look at their paper, I have the impression that a terminological distinction between the explicit reciprocity law and the result on special value of L-functions (theorem 29) was maintained.* In any case the link

conjecture.
While we try to look at the general picture we sometimes focus on recent developments regarding the Lubin-Tate setting.

The survey is written mainly from the perspective of reciprocity laws. Nevertheless we aim at pointing out how interwoven the historical development has been with the aspect of special L-values and p-adic L-functions. Thus another option for reading this paper is to start with (or switch directly after section 4 to) section 7.

Needless to say that the presentation involves an increasing gradient of abstraction and complexity according to the topics being discussed, starting from elementary number theory and ending up with current research in this field.

We found the existing surveys [111] on the history and [78] on Kato's explicit reciprocity very helpful when preparing this manuscript, which arose from different colloquium style talks the author had given on this subject.

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2. Gauss' Reciprocity Law and the quadratic Hilbert symbol

In the preface of the wonderful book [77] we found the following nail-on-the-head characterization of the *Quadratic Reciprocity Law*, which we recall shortly in subsection 2.2 below:

Erich Hecke (1923)[56, p.59]:

> Modern number theory dates from the discovery of **the reciprocity law.** By its form it still belongs to the theory of rational numbers, as it can be formulated entirely as a simple relation between rational numbers; however its content points beyond the domain of rational numbers.

Emma Lehmer (1978)[72, p. 467]:

> ... that the famous Legendre law of quadratic reciprocity, of which over 150 proofs are in print, has been generalized over the years [...] to the extent that it has become virtually unrecognizable.

The aim of this article is to is to strengthen the reader's confidence in the first quotation about the importance of Legendre's and Gauss' quadratic reciprocity for the history of algebraic number theory. Most readers will immediately agree with the second statement - one motivation to write this article was also for the author to better understand the link between this quadratic reciprocity law and Perrin Riou's (generalised) reciprocity law. Nevertheless we shall try to uncover the common thread behind

between both, reciprocity laws and special L-values, is so strong - as we try to convince the reader in section 7 - that this usage seems fully justified - over all when combined with David Loeffler's observation in subsection 7.6 below.

the developments which lead mathematicians to call those generalisations again *explicit reciprocity law* or *formula*.

2.1. Legendre symbol. The diophantine equation

$$X^2 + pY = a$$

for $a \in \mathbb{Z}$ and an odd prime p with $(p, a) = 1$ has a solution in $\mathbb{Z}^2$ if and only if

$$X^2 = \overline{a} \in \mathbb{F}_p^\times$$

has a solution in $\mathbb{F}_p$, i.e., if a is a square there. This led Legendre to define the following symbol which later has been extended by Jacobi allowing also non-prime natural numbers in the "denominator":

$$\text{LEGENDRE/JACOBI Symbol} \qquad \left(\frac{a}{p}\right) := \begin{cases} 1, & \overline{a} \in (\mathbb{F}_p^\times)^2; \\ -1, & \text{otherwise.} \end{cases}$$

Another description is given by EULER's criterion

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \mod p.$$

Indeed, since $\mathbb{F}_p^\times$ is cyclic of order $p-1$ the following sequence is exact:

$$0 \longrightarrow (\mathbb{F}_p^\times)^2 \longrightarrow \mathbb{F}_p^\times \xrightarrow{\frac{p-1}{2}} \{1, -1\} \longrightarrow 0, \tag{2.1}$$

where we consider $\mu_2 = \{-1, 1\}$ as subgroup of $\mathbb{F}_p^\times$.

2.2. Gauß' reciprocity law.

The following quadratic reciprocity law had already been discovered by EULER in 1744 and was formulated by LEGENDRE in 1788. It was Gauß who first presented a complete proof of it:

Reciprocity Law I (GAUSS 1801) : Let $l \neq p$ be odd prime numbers. Then we have

$$\left(\frac{l}{p}\right) = (-1)^{\frac{l-1}{2}\frac{p-1}{2}} \left(\frac{p}{l}\right).$$

Supplement: $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$, $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

Slogan:
l is a square modulo p if and only if p is a square modulo l – unless $l \equiv p \equiv 3 \mod 4$, in which case the opposite equivalence holds.

Even more generally the Reciprocity Law holds for odd, pairwise coprime natural numbers m, n instead of l, p using the Jacobi symbol. This formulation of the reciprocity law[3] explains in an obvious way its name, the two "fractions" on the left and right hand side being reciprocal to each other. This literal meaning of "reciprocity" gets lost in all generalisations of it we are going to discuss later on, which is one reason why it is hard to recognise the link to its origin.

Using the multiplicativity of the Legendre symbol in the upper variable and setting $p^* := (-1)^{\frac{p-1}{2}} p$ one obtains the following **equivalent formulation:**

$$\left(\frac{p^*}{l}\right) = \left(\frac{l}{p}\right). \tag{2.2}$$

[3]See [77, Thm. 2.28] for variants.

As mentioned already there are more than 150 proofs in the literature - Gauss himself already had included at least 6 in his Disquisitiones Mathematicae. We would like to sketch at least one proof here, which already is based on certain principles of class field theory. According to the last formulation (2.2) it is natural to consider the quadratic extension $\mathbb{Q}(\sqrt{p^*})$ of $\mathbb{Q}$ which is visibly an abelian extension of $\mathbb{Q}$. By Kronecker's Jugendtraum all abelian extensions of $\mathbb{Q}$ are contained in a cyclotomic extension. By considerations of discriminants it turns out that $\mathbb{Q}(\sqrt{p^*})$ is contained in $\mathbb{Q}(\zeta_p)$, wehre ζ_p denotes a primitive pth root of unity. Now the group $\mathbb{F}_p^\times$ occurs not only as the units of the residue field $\mathbb{F}_p$ at the prime p, but also as the Galois group $G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ of this cyclotomic extension by mapping $a \epsilon \mathbb{F}_p^\times$ to σ_a satisfying $\sigma_a(\zeta_p) = \zeta_p^a$. By definition $\sigma_{\bar{l}}$ is nothing else than the Frobenius automorphism Fr_l at l. This leads to the following reasoning: Firstly, an easy calculation in quadratic extensions shows that $Fr_l(\sqrt{p^*}) = \left(\frac{p^*}{l}\right)\sqrt{p^*}$. Secondly, interpreting the defining sequence (2.1) of the Legendre symbol also in terms of Galois groups attached to the embedding $\mathbb{Q}(\sqrt{p^*}) \subseteq \mathbb{Q}(\zeta_p)$ we have the following commutative diagram with exact rows:[4]

$$\begin{array}{ccccccccc}
 & & & & Fr_l\,\epsilon & & & & \\
0 & \longrightarrow & G(\mathbb{Q}(\zeta_p)/\mathbb{Q}(\sqrt{p^*})) & \longrightarrow & G(\mathbb{Q}(\zeta_p)/\mathbb{Q}) & \longrightarrow & G(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & (\mathbb{F}_p^\times)^2 & \longrightarrow & \mathbb{F}_p^\times & \xrightarrow{\left(\frac{\cdot}{p}\right)} & \{1,-1\} & \longrightarrow & 0. \\
 & & & & \bar{l}\,\epsilon & & & &
\end{array}$$

From these two facts we obtain the equivalences

$$\left(\frac{p^*}{l}\right) = 1 \Leftrightarrow (Fr_l)_{|\mathbb{Q}(\sqrt{p^*})} = \mathrm{id} \Leftrightarrow \bar{l}\,\epsilon\,(\mathbb{F}_p^\times)^2 \Leftrightarrow \left(\frac{l}{p}\right) = 1,$$

which imply the reciprocity law. From the diagram it becomes clear that we can read $\left(\frac{\cdot}{p}\right)$ (or equivalently $\left(\frac{p^*}{\cdot}\right)$) as a homomorphism in $\mathrm{Hom}(G(\mathbb{Q}(\zeta_p)/\mathbb{Q}), \mu_2)$ and for the proof it was crucial to evaluate this homomorphism at the Frobenius automorphisms. Let's keep this in mind for later generalisations.

2.3. **Hilbert symbol.** A central principle in number theory is the **Local-Global-Principle.** Consider the absolute values on $\mathbb{Q}$: Besides the (real) absolute value $|-|_\infty$ there is for each prime p the p-adic norm $|-|_p$ defined as $|p^m \frac{a}{b}|_p := p^{-m}$, if $(p, ab) = 1$. The completions of $\mathbb{Q}$ with respect to these norms show quite different behaviour:

$$\mathbb{Q}_\infty = (\mathbb{Q}, |-|_\infty)^\wedge = \mathbb{R}, \quad \mathbb{Q}_p := (\mathbb{Q}, |-|_p)^\wedge;$$

$$\mathbb{R}^{>0} \xrightarrow{\log} (\mathbb{R}, +), \qquad\qquad \mathbb{Q}_p^\times \xrightarrow{\log_p} \mathbb{Q}_p.$$

For instance, the subset $\mathbb{Z}_p := \{z\,\epsilon\,\mathbb{Q}_p \mid |z|_p \leq 1\}$ forms a subring, the analogous statement for $|\cdot|_\infty$ being obviously false. The elements $v\,\epsilon\,\{p \mid \text{prime}\} \cup \{\infty\}$ are called the places of $\mathbb{Q}$, a place $v \neq \infty$ is called *finite.*

[4] $G(\mathbb{Q}(\sqrt{p^*})/\mathbb{Q})$ is the unique quotient of the cyclic group $G(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ of order 2.

The **Quadratic Hilbert symbol** is now defined *locally* for each place v, respectively each local field $\mathbb{Q}_v$:

$$\left(\frac{m,n}{v}\right) := \begin{cases} 1, & \text{if } mX^2 + nY^2 = Z^2 \text{ has non-trivial solution in } \mathbb{Q}_v; \\ -1, & \text{otherwise.} \end{cases}$$

It defines a symmetric, non-degenerate pairing

$$\left(\frac{\cdot,\cdot}{v}\right): \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 \times \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2 \to \mu_2.$$

By HENSEL'S Lemma the solvability of $X^2 = \overline{a} \in \mathbb{F}_p^\times$ is equivalent to that of $X^2 \equiv a \mod p^n$, for all $n \geq 1$, and in turn to that of $X^2 = a \in \mathbb{Z}_p$. This can be used to easily conclude the identity

$$\left(\frac{p,q}{p}\right) = \left(\frac{q}{p}\right) \tag{2.3}$$

for distinct odd positive primes numbers p, q, see [77, Prop. 2.26] for details and compare with [93, V. Thm. (3.6)] for a different proof.

It is an important insight that in number theory for the full picture one has to consider all places v simultaneously as they are linked among each other, e.g., we have the

Product Formula: $\prod_v |a|_v = 1$ for all $a \in \mathbb{Q}$.

This is a rather trivial consequence of the unique prime factor decomposition of integers. In contrast the following statement is highly non-trivial:

Hilbert's Reciprocity Law II: For $m, n \in \mathbb{Q}^\times$ we have:

$$\prod_{v \text{ all places}} \left(\frac{m,n}{v}\right) = 1. \tag{2.4}$$

Indeed, it turns out that the laws I and II are equivalent; for example the implication "II $\Rightarrow$ I" can be derived as follows:

$$1 = \prod_v \left(\frac{p,q}{v}\right) = \left(\frac{p,q}{2}\right)\left(\frac{p,q}{p}\right)\left(\frac{p,q}{q}\right) = (-1)^{\frac{q-1}{2}\frac{p-1}{2}}\left(\frac{q}{p}\right)\left(\frac{p}{q}\right). \tag{2.5}$$

Here we used the positivity of p, q to see that the contribution at ∞ is one, moreover it is easy to verify that there is no non-trivial contribution from finite places distinct from $2, p, q$. The Hilbert symbol at 2 gives rise to the sign factor while we had seen already in (2.3) how the other two Hilbert symbols specialize to the Legendre symbols. For more details and the opposite implication we refer the interested reader to [77, Thm. 2.28].

3. HIGHER HILBERT SYMBOLS AND VARIANTS

The above proof of the Quadratic Reciprocity Law suggests the importance of studying the group $\mathrm{Hom}(G_\mathbb{Q}, \mu_n)$ for varying natural numbers $n \geq 2$, which leads directly into class field theory and higher symbols as we will see in this section. Note that for a finite extension K and a discrete G_K-module A with trivial G_K-action the group $\mathrm{Hom}_{cts}(G_K, A)$ of continuous group homomorphisms coincides with the first continuous group cohomology $H^1(G_K, A) =: H^1(K, A)$. In Section 2 we were in the lucky situation that $\mu_2 \subseteq \mathbb{Q}$ and hence $\mathrm{Hom}(G_\mathbb{Q}, \mu_2) = H^1(\mathbb{Q}, \mu_2)$. For general n one has either to assume that $\mu_n \subseteq K$ or to work with $H^1(K, A)$.

3.1. **Artin Reciprocity and different concepts of generalisations.** Gauß' Reciprocity Law is the beginning of **class field theory**, which classifies and describes all abelian extensions of global or local fields. The Artinian Reciprocity map can be considered as generalisation of it, for example in the case of (finite) Galois extensions of local fields K/F it comes as a surjective homomorphism

$$(-,K/F):\ GL_1(F)=F^\times \twoheadrightarrow G(K/F)^{ab}$$

whose kernel is the norm subgroup $N_{K/F}K^\times \subseteq F^\times$. Instead of discussing the shape of the global Artin map in general we just mention that in its ideal theoretic version the global Artin symbol $((q),\mathbb{Q}(\zeta_p)/\mathbb{Q})$ attached to the principal ideal (q) for the extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ equals the arithmetic Frobenius automorphism σ_q, which sends ζ_p to ζ_p^q, whence its restriction $((q),\mathbb{Q}(\sqrt{p^*})/\mathbb{Q})$ to the extension $\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}$ equals the Legendre symbol $\left(\frac{p^*}{q}\right)$ in the sense that $((q),\mathbb{Q}(\sqrt{p^*})/\mathbb{Q})\sqrt{p^*}=\left(\frac{p^*}{q}\right)\sqrt{p^*}$, see [77, after Cor. 3.10, Prop. 2.21]. Then Gauß' Reciprocity Law can be seen to be equivalent to the decomposition law of prime ideals in this quadratic extension, see [93, I.Theorem (10.3), Prop. (10.5) and discussion afterwards,VI. Theorem (7.3)].

A *non-abelian generalisation* of the Artin reciprocity is the **local (classical or p-adic) Langlands program** relating roughly speaking certain n-dimensional representations of the absolute Galois group G_F of F to certain representations of the reductive algebraic groups GL_n. For a recent survey on reciprocity laws and Galois representations, see [115].

In this article we focus rather on **generalisations of the quadratic Hilbert symbol**. To this aim we fix a natural number $n\geq 2$, a finite extension $F/\mathbb{Q}_p$ containing μ_n, and we denote by L/F the maximal abelian extension of exponent n of F, i.e., such that $nG(L/F)=1$. The *Kummer sequence*

$$1\longrightarrow \mu_n \longrightarrow \bar{F}^\times \xrightarrow{\ .n\ } \bar{F}^\times \longrightarrow 1$$

induces by going over to the attached long exact sequence in Galois cohomology a canonical isomorphism

$$F^\times/(F^\times)^n \xrightarrow{\delta} H^1(F,\mu_n)\cong \mathrm{Hom}(G(L/F),\mu_n),$$

while the local Artin map induces an isomorphism

$$F^\times/(F^\times)^n \xrightarrow{(-,L/F)} G(L/F)$$

upon noting that the norm group $N_{L/F}L^\times$ of L coincides with $(F^\times)^n$. Then the **nth Hilbert symbol** $\left(\frac{,}{F}\right)_n$ is defined by the commutativity of the upper half of the following diagram, in which the first line is the Pontrjagin duality pairing, i.e., given by evaluating a group homomorphism from the right at an element from the left:

(3.6)
$$\begin{array}{ccccc}
G(L/F) & \times\mathrm{Hom}(G(L/F),\mu_n) & \xrightarrow{Pontrjagin} & \mu_n \\
\Big\uparrow{\scriptstyle (-,L/F)} & \Big\uparrow{\scriptstyle \delta} & & \Big\| \\
F^\times/(F^\times)^n \quad \times & F^\times/(F^\times)^n & \xrightarrow{\left(\frac{,}{F}\right)_n} & \mu_n \\
{\scriptstyle -\delta}\Big\downarrow & {\scriptstyle \delta}\Big\downarrow & & {\scriptstyle \cong}\Big\uparrow{\scriptstyle \mathrm{inv}\otimes\mu_n} \\
H^1(F,\mu_n) \quad \times & H^1(F,\mu_n) & \xrightarrow{\cup} & H^2(F,\mu_n^{\otimes 2})\cong H^2(F,\mu_n)\otimes\mu_n .
\end{array}$$

Unravelling this definition one obtains that

$$\left(\frac{a,b}{F}\right)_n=\frac{(a,L/K)(\sqrt[n]{b})}{\sqrt[n]{b}}$$

for $a,b\in F^\times$, see [93, V Satz (3.1)]. By [94, Cor. (7.2.13)] also the lower part of the diagram commutes, which establishes a relation between the Hilbert symbol and the cup product pairing for

$\mathbb{G}_m$ respectively the Galois representation $\mathbb{Q}_p(1)$ given by the cyclotomic character. Moreover, for $F = \mathbb{Q}_p$, $n = 2$ we recover our quadratic Hilbert symbol $\left(\frac{\,,\,}{\mathbb{Q}_p}\right)_2 = \left(\frac{\,,\,}{p}\right)$ as follows from [93, V. Satz (3.6)]. In other words, we encounter two ways of generalisation of the quadratic Hilbert symbol:

1. The nth Hilbert symbol, which will turn out to be related directly to the search of reciprocity laws concerning higher power residues, i.e., the congruences $x^n \equiv a \mod p$, with $n > 2$.
2. Tate's local cup-product paring for different Galois representations. This topic will govern the Iwasawa theoretic development as discussed in section 6.

For now we will introduce some variants of the higher Hilbert symbol.

3.2. **Schmid-Witt Residue Formula.** For a variant with $\mathrm{char}(F) = p$ we consider the field $F = \mathbb{F}_p((Z))$. Replacing the Kummer sequence by the *Artin-Schreier* sequence

$$0 \longrightarrow \mathbb{F}_p \longrightarrow F^{\mathrm{sep}} \xrightarrow{\wp} F^{\mathrm{sep}} \longrightarrow 0$$

with $\wp(x) := (Fr - \mathrm{id})(x) = x^p - x$ induces a pairing

$$(\ ,\ ] : F^\times/(F^\times)^p \times F/\wp(F) \to \mathbb{F}_p$$

by the same recipe. It can be explicitly calculated by the **Schmid-Witt Residue Formula**

$$(a, b] = \mathrm{Res}\left(b\frac{da}{a}\right)$$

with $\mathrm{Res}((\sum_i c_i Z^i)dZ) := c_{-1}$.

For $n > 1$ there is similarly the Artin-Schreier-Witt pairing

$$[\ ,\ )_F : W_n(F) \times F^\times \longrightarrow W_n(\mathbb{F}_p),$$

involving Witt vectors $W_n(F)$ of length n, for which the Schmid-Witt formula gives again an explicit description, see [102, §7] for a version for *ramified* Witt vectors, which will be crucial for the proof of Theorem 12 below. There is the classical article by Witt [117]. In the case of local fields of characteristic p, Parshin [95] develops the higher class field theory using the generalized Witt pairing.

3.3. **Lubin-Tate formal groups.** Another variant of the Hilbert symbol can be formulated for (Lubin-Tate) formal groups $\mathcal{F}$ over a finite extension F of $\mathbb{Q}_p$ attached to the prime π (and similarly for p-divisible groups). The sequence

$$0 \longrightarrow \mathcal{F}[\pi^n] \longrightarrow \mathcal{F}(\mathfrak{m}_{\bar{F}}) \xrightarrow{[\pi^n]} \mathcal{F}(\mathfrak{m}_{\bar{F}}) \longrightarrow 0$$

induces a pairing

$$(\ ,\ )_{\mathcal{F},n} : F^\times \times \mathcal{F}(\mathfrak{m}_F) \to \mathcal{F}[\pi^n].$$

We shall say more about the Lubin-Tate setting just before Theorem 4.

So the culminating question is:

How can one compute the Hilbert symbol $\left(\frac{\,,\,}{F}\right)_n$ (and its above variants) explicitly?

4. Explicit Formulas

4.1. **Tame case.** In the *tame case*, i.e., $p \nmid n$, the Hilbert symbol has a simple description:

Let q be the cardinality of $O_F/\pi_F O_F$ and consider the unique decomposition

$$O_F^\times \cong \mu_{q-1} \times (1+\pi_F O_F), \quad u \mapsto \omega(u)\langle u\rangle$$

with $\omega(u) \,\epsilon\, \mu_{q-1}$ and $\langle u\rangle \,\epsilon\, 1+\pi_F O_F$; let v_F be the normalized valuation, i.e., $v_F(\pi_F) = 1$.

For $a, b \,\epsilon\, F^\times$ such that $\alpha = v_F(a)$, $\beta = v_F(b)$ we have by [93, V Satz (3.4)]:

$$\left(\frac{a,b}{F}\right)_n = \omega\left((-1)^{\alpha\beta}\frac{b^\alpha}{a^\beta}\right)^{\frac{q-1}{n}}.$$

In particular, for $a = \pi_F$ and $u \,\epsilon\, O_F^\times$ we obtain that $\left(\frac{\pi_F, u}{F}\right)_n = \omega(u)^{\frac{q-1}{n}}$ is the root of unity $\zeta \,\epsilon\, \mu_n$ determined by $\zeta \equiv u^{\frac{q-1}{n}} \mod \pi_F O_F$, i.e., we have (see V.Satz (3.5) in (loc. cit.)):

$$\left(\frac{\pi_F, u}{F}\right)_n = 1 \iff u \text{ is an } n\text{th power} \quad \mod \pi_F O_F.$$

In other words the **nth power residue symbol**

$$\left(\frac{u}{\pi_F}\right)_n := \left(\frac{\pi_F, u}{F}\right)_n \text{ for } u \,\epsilon\, O_F^\times \tag{4.7}$$

generalises the Legendre symbol $\left(\frac{a}{p}\right)$ which is recovered by specializing to $F = \mathbb{Q}_p$ and $n = 2$. See VI §8 in [93] for an extension to the definition of a global nth power residue symbol over arbitrary number fields K (containing μ_n) and the proof of a general reciprocity law (Theorem (8.3)) for it as a consequence of the abstract formalism of class field theory, i.e., Artin reciprocity. Indeed, one has again a product formula for $a, b \,\epsilon\, K^\times$

$$\prod_v \left(\frac{a,b}{K_v}\right) = 1,$$

which results from the product formula $\prod_v (a, L_v/K_v) = 1$ of (local) Artin maps for (principal idèles) $a \,\epsilon\, K^\times$. Similar as in (2.5) upon using (4.7) one obtains for $a, b \,\epsilon\, K^\times$ prime to each other and to n the general formula

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^{-1} = \prod_{v \mid n\infty} \left(\frac{a,b}{K_v}\right).$$

Theorem VI.(8.4) in (loc. cit.) or (2.5) explains how Gauß' reciprocity law is a special case. In other words: The long quest in number theory for similar laws for nth power residue symbols lead to the discovery of the Artin reciprocity law, which in turn eventually led to a full explanation of the general reciprocity law of higher power residue symbols - except for the explicit calculation of the nth Hilbert symbol in the case of (wild) ramification, i.e., for $v \mid n$, which we are going to discuss in the next subsection.

4.2. **Wild case: Kummer, Artin-Hasse, Iwasawa, Wiles, Brückner, ...** Therefore henceforth we consider for odd p the other extreme

$$n = p^k,$$

i.e., we replace n by p^n and take $F := K_n := \mathbb{Q}_p(\zeta_{p^n})$ where ζ_{p^n} is a primitive p^nth root of unity. Then $\pi_n := \zeta_{p^n} - 1$ forms a prime element for the valuation ring of integers $o_{K_n} = \{|z|_p \le 1\}$. We also introduce the trace map $Tr := Tr_{K_n/\mathbb{Q}_p} := \sum_{\sigma \,\epsilon\, G(K_n/\mathbb{Q}_p)} \sigma$ for this extension $F/\mathbb{Q}_p$.

It was Kummer [70] who found the following description of the Hilbert symbol for $n = 1$, in which differential logarithms of units show up: For $\beta \,\epsilon\, 1 + \pi_1 o_{K_1}$ he writes $\beta = f(\pi_1)$ for some $f \,\epsilon\, \mathbb{Z}_p[[Z]]$

and performs calculus with regard to the variable π_1. so, formally $d\log b$ amounts to the logarithmic derivative $d\log f(Z) = \frac{f'(Z)}{f(Z)}$ evaluated at $Z = \pi_1$. In the same spirit the residue Res_{π_1} will be taken with respect to π_1.

Theorem 1 (Kummer 1858). *For $\alpha, \beta \in 1 + \pi_1 o_{K_1}$ we have:*

$$(\alpha, \beta)_p = \zeta_p^{\mathrm{Res}_{\pi_1}(\frac{\log \alpha \cdot d\log \beta}{\pi_1^p})} = \zeta_p^{\frac{1}{p}Tr(\zeta_p \log \alpha \cdot d\log \beta)}.$$

As Franz Lemmermeyer [58] points outs that "Kummer worked out the arithmetic of cyclotomic extensions guided by his desire to find the higher reciprocity laws; notions such as unique factorization into ideal numbers, the ideal class group, units, the Stickelberger relation, Hilbert 90, norm residues and Kummer extensions owe their existence to his work on reciprocity laws. His work on Fermat's Last Theorem is connected to the class number formula and the "plus" class number, and a meticulous investigation of units, in particular Kummer's Lemma, as well as the tools needed for proving it, his differential logarithms, which much later were generalized by Coates and Wiles." In his talk during the memorial conference of John Coates in Cambridge 2023 Andrew Wiles pointed out that indeed, Kummers work and Iwasawa's interpretation of it was crucial for both the development of his reciprocity law in the Lubin-Tate setting (Theorem 4 below) as well as the link to special-L values ((7.33),(7.40)) and p-adic L-functions ((7.34), (7.41)), which we will discuss in section 7.

Only many years later Artin and Hasse [5] obtained a result for arbitrary n.

Theorem 2 (Artin-Hasse 1928). *For $\beta \in 1 + \pi_n o_{K_n}$ we have:*

$$(\zeta_{p^n}, \beta)_{p^n} = \zeta_{p^n}^{\frac{1}{p^n}Tr(\log \beta)};$$
$$(\beta, \pi_n)_{p^n} = \zeta_{p^n}^{\frac{1}{p^n}Tr(\frac{\zeta_{p^n}}{\pi_n}\log \beta)}.$$

Iwasawa [60] contributed the following version, which specializes to Kummer's version for $n = 1$.

Theorem 3 (Iwasawa 1968). *For $\beta = (\beta_k) \in \varprojlim_{k,Norm} K_k^\times$, $g_\beta \in \mathbb{Z}_p[[Z]]$ with $g_\beta(\pi_n) = \beta_n$ and $\alpha \in 1 + \pi_n o_{K_n}$ we have:*

$$(\beta_n, \alpha)_{p^n} = \zeta_{p^n}^{\frac{1}{p^n}Tr(\log(\alpha)\cdot D\log \beta)}$$

with invariant logarithmic derivation

$$D\log \beta = \left((1+Z)\frac{g'_\beta(Z)}{g_\beta(Z)} \right)_{|Z=\pi_n}.$$

This result should also be compared with Coleman's more complete formula in [32]. Due to our interest towards Iwasawa theory we concentrate here on the above versions, but we would like to also mention: Helmut Brückner [23] discovered in 1964 an explicit description by a residue formula over not necessarily cyclotomic fields, see [93, V. Theorem (3.7)]. Another treatment has been given by Guy Henniart [57]. See also the work of Sergei Vostokov et. al. [111, 112, 113].

Wiles [116] generalized Iwasawa's result to the **Lubin-Tate setting:** we fix a finite extension $L/\mathbb{Q}_p$ with prime element $\pi \in o_L$ as well as a Lubin-Tate formal group $\mathcal{F} = \mathcal{F}_\pi$ over L attached to π. We write $[a]_\mathcal{F}(Z) \in o_L[[Z]]$, $a \in o_L$, for the power series giving the o_L-action on it and $\log_\mathcal{F}$ for the logarithm of the formal group. The π_L^n-division points generate a tower of Galois extensions $L_n = L(\mathcal{F}[\pi_L^n])$ of L the union of which we denote by L_∞ with Galois group Γ_L. We let $T := T_\pi\mathcal{F} = \varprojlim_n \mathcal{F}[\pi^n] = o_L\eta$ denote its Tate-module with o_L-generator $\eta = (\eta_n)$ and on which the Galois action is described by the Lubin-Tate character $\chi_{LT}: G_L \longrightarrow o_L^\times$. Furthermore, we write ∂_{inv} for the invariant derivation with respect to $\mathcal{F}$ and $t_L := \log_{\mathcal{F}}(Z)$, i.e., $\partial_{\mathrm{inv}}(f) = g_{\mathcal{F}}^{-1} f'$, where $g_{\mathcal{F}}$ is the formal derivative of $\log_{\mathcal{F}}$. As before let $Tr = Tr_{L_n/L}$ denote the corresponding trace map. For $L = \mathbb{Q}_p$, $\pi = p$, $\mathcal{F} = \hat{\mathbb{G}}_m$ this specializes to the above cyclotomic setting:

Lubin-Tate		**cyclotomic**
$L/\mathbb{Q}_p$	degree d extension of	$\mathbb{Q}_p$
o_L	integers	$\mathbb{Z}_p$
$\pi_L \in o_L$	prime element	p
$q = \lvert o_L/\pi_L o_L\rvert$	cardinality of residue field	$p = \lvert\mathbb{Z}_p/p\mathbb{Z}_p\rvert$
$\mathcal{F} = \mathcal{F}_\pi$	Lubin-Tate formal group/L attached to $\pi = \pi_L$	$\hat{\mathbb{G}}_m$
$[a](Z) \in o_L[[Z]]$	giving o_L-action, $a \in o_L$	$(Z+1)^a - 1$
$T_\pi \mathcal{F} = \varprojlim_n \mathcal{F}[\pi^n]$	Tate-module, G_L-action by	$\mathbb{Z}_p(1)$,
$\chi_{LT}: G_L \longrightarrow o_L^\times$	Lubin-Tate character	χ_{cyc}
Ω	period of Cartier dual	
φ_L	$f(Z) \mapsto f([\pi_L](Z))$	$\varphi_{\mathbb{Q}_p}$
ψ_L	(almost)left inverse of φ_L	$\psi_{\mathbb{Q}_p}$

Then Wiles obtained the following generalisation.

Theorem 4 (Wiles 1978). *For* $\beta = (\beta_k) \in \varprojlim_{k,Norm} L_k^\times$, $g_\beta \in o_L[[Z]]$ *with* $g_\beta(\eta_n) = \beta_n$ *and* $\alpha \in \mathcal{F}(\eta_n o_{L_n})$ *we have:*

$$(\beta_n, \alpha)_{\mathcal{F},n} = [\frac{1}{\pi^n} Tr\,(\log_{\mathcal{F}}(\alpha) D \log g_\beta(\eta_n))]_{\mathcal{F}}(\eta_n)$$

with invariant logarithmic derivation $D \log g_\beta = \frac{1}{\log'_{\mathcal{F}}} \frac{g'_\beta}{g_\beta}$.

While Coates and Wiles [28, 30, 29] were working on extending Iwasawa theoretic methods to CM-elliptic curves (as will be later discussed in section 7), Coleman systematized their methods. In particular, he discovered what one now calls Coleman power series, which have the property to interpolate a full norm-compatible system of local units for all layers of the Lubin-Tate tower simultaneously. We recall the injective ring endomorphism

$$\begin{aligned} \varphi_L : o_L[[Z]] &\longrightarrow o_L[[Z]] \\ f(Z) &\longmapsto f([\pi_L](Z)) . \end{aligned}$$

Moreover, there is a unique multiplicative map $\mathscr{N}: o_L[[Z]] \longrightarrow o_L[[Z]]$ such that

$$\varphi_L \circ \mathscr{N}(f)(Z) = \prod_{a \in LT_1} f(a +_{\mathcal{F}} Z) \qquad \text{for any } f \in o_L[[Z]]$$

([31] Thm. 11).

Theorem 5 (Coleman)**.** *For any norm-coherent sequence $u = (u_n)_n \in \varprojlim_n L_n^\times$ there is a unique Laurent series $g_{u,\eta} \in (o_L((Z))^\times)^{\mathscr{N}=1}$ such that $g_{u,\eta}(\eta_n) = u_n$ for any $n \geq 1$. This defines a multiplicative isomorphism*

$$\varprojlim_n L_n^\times \xrightarrow{\cong} (o_L((Z))^\times)^{\mathscr{N}=1}$$
$$u \longmapsto g_{u,\eta}\ .$$

We now reformulate Iwasawa's theorem 3 in the way of [64, §1.1]. Via the isomorphism $\mathbb{Z}/p^m\mathbb{Z} \cong \mu_{p^m}$, $\bar{1} \mapsto \zeta_{p^m}$, we interpret the p^mth Hilbert symbol as a pairing

$$K_m^\times \times K_m^\times \to \mathbb{Z}/\mathfrak{p}^m\mathbb{Z}.$$

Fixing $m \geq 0$ in the first argument and taking inverse limits in the second argument (with respect to norm maps) and in the target gives pairings

$$K_m^\times \times \varprojlim_n K_n^\times \to \mathbb{Z}_p, \quad \text{and } (-,-) : (K_m^\times) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \times \varprojlim_n K_n^\times \to \mathbb{Q}_p. \tag{4.8}$$

Hence the non-degenerateness of the trace pairing implies the unique existence of a map $\lambda_m : \varprojlim_n K_n^\times \to K_m$ making the following diagram commutative

$$\begin{array}{ccccc} (K_m^\times) \otimes_{\mathbb{Z}} \mathbb{Q} & \times & \varprojlim_n K_n^\times & \xrightarrow{(-,-)} & \mathbb{Q}_p \\ \uparrow{\scriptstyle \exp} & & {\scriptstyle \lambda_m}\downarrow & & \downarrow \\ K_m & \times & K_m & \xrightarrow{Tr_{K_m/\mathbb{Q}_p}} & \mathbb{Q}_p, \end{array} \tag{4.9}$$

where $\exp(a)$ is defined as $\exp(p^k a) \otimes p^{-k}$ for k sufficiently large. More explicitly, λ_m is the composite

$$\varprojlim_n K_n^\times \xrightarrow{\text{Hilbert symbol}} \mathrm{Hom}_{cts}(K_m^\times, \mathbb{Q}_p) \xrightarrow{\exp^*} \mathrm{Hom}_{cts}(K_m, \mathbb{Q}_p) \underset{\cong}{\xleftarrow{Tr}} K_m. \tag{4.10}$$

Then the statement of Theorem 3 means that

$$\lambda_m(\beta) = \frac{1}{p^m}(D \log g_\beta)_{|Z=\pi_m}. \tag{4.11}$$

In §1.2 of (loc. cit.) Kato explains a relationship between cyclotomic units and some values of partial Riemann zeta functions, which gives a first hint that explicit reciprocity laws are closely related to Euler systems and special L-values or even p-adic L-functions - a connection we shall resume in section 7. A similar reformulation is of course also possible for Wiles's theorem with $\lambda_m : \varprojlim_n L_n^\times \to L_m$ given by $\frac{1}{\pi_L^m}(D \log g_-)_{|Z=\eta_m}$. Note that λ_m is nothing else than (part of) the Coates-Wiles homomorphism $\psi^1_{CW,m}$ in sections 5 and 7.

We close this section by giving comments to the literature and mentioning that there are lots of further formulas by various mathematicians - we are very grateful to Denis Benois for guidance in this regard:

CLASSICAL RECIPROCITY LAWS: "Classical" results can be divided in two groups: Artin-Hasse type formulas and Shavarevich-Brückner-Vostokov type formulas. The foundational papers for these approaches are the paper of Artin-Hasse [5] and the paper of Shafarevich [114], respectively; see also Kneser [68]. The link between these formulas can be understood in the framework of syntomic cohomology (see below).

The following three important papers added new techniques to the development:
Sen's formula in [104] is more general than the formulas of Artin-Hasse- Iwasawa. More importantly, his approach uses the computation of the Galois cohomology of the field $\mathbb{C}_p$. In some sense, this is the first application of methods of p-adic Hodge theory to explicit reciprocity laws.
In [61] Kato first uses syntomic cohomology to prove an explicit reciprocity law. In some sense, the explicit reciprocity law of Bloch-Kato (to be discussed in the next section) and other explicit reciprocity laws proved by Kato take their origin in this paper. This paper gives a conceptual explanation why Hilbert symbols can be computed using differential forms. See also the appendix of [55] written by Kurihara in which he shows that both types of explicit formulas for the Hilbert symbol (Artin-Hasse-Iwasawa type and Shafarevich-Brückner-Vostokov type formulas) can be recovered by syntomic techniques.
Fontaine's appendix of [97] is the first paper where the relationship between the characteristic 0 and characteristic p cases is explicitly established. For the further development of this point, see the papers of Abrashkin [3] and [2].

EXPLICIT RECIPROCITY LAWS FOR FORMAL GROUPS: The paper [107] by Tavares Ribeiro generalizes the result of Abrashkin (the condition that μ_{p^n} is contained in the ground field is removed): The approach is different from Abrashkin's and introduces (φ, Γ)-modules in the false Tate curve extensions.

To sum up, the most general formulas for formal groups *over one-dimensional local fields* are obtained in the paper of Tavares Ribeiro (for Brückner-Vostokov formulas, only formal groups over unramified local fields can be covered by this type of formulas) and Benois' paper [7] (for Artin-Hasse type formulas, the ground field is arbitrary, but there are restrictions on the valuation of the second argument). Kolyvagin's formula in [69] is really explicit only in the case of Lubin-Tate groups and coincides in that case with Wiles' explicit reciprocity law. In the case of general formal group his formula depends on some non explicit Galois invariants. These invariants can be explicitly computed using p-adic periods accoding to [7].
The paper of Flórez [48] concerns Lubin-Tate formal groups *over higher local fields*. The case of an arbitrary formal group in the higher dimensional case was studied by Fukaya [53]. Her result is a direct generalization of [7]. For the Hilbert symbol in the higher dimensional field case see also the article [1] by Abrashkin and Jenni.
In characteristic p, concerning *Formal Drinfeld Modules*, recently there has been remarkable progress by Eddine [46, 45]. Her results are more general that the formulas of Bars and Longhi [6]. She follows the approach of Kolyvagin but applies it in the characteristic p case, where the theory of p-adic periods is not available.

We recommend to consult the monograph [47] as well as the surveys [111], [6] and the literature listed therein.

5. BLOCH-KATO'S RECIPROCITY LAW

The explicit calculations of the higher symbols in section 4 culminated in the interpretation (4.11) based on the reformulation of the pairing as given in (4.10). This can also be rewritten as the statement that the class in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = \mathrm{Hom}_\Gamma(H^{ab}_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$ associated with such λ_m below is the image of 1 under a transition map $\partial^1 : \mathbb{Q}_p \to H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ arising from p-adic Hodge theory. This is the starting point for the variant of Bloch and Kato.

Bloch and Kato [21] found a way to generalize the exponential map $\exp_{\mathcal{F}}$ of a formal group (as it shows up in Kato's reformulation (4.11) of Iwasawa's and Wiles's explicit reciprocity law) to de Rham Galois representations. Building on this they were able to generalize the explicit reciprocity for $\mathbb{Q}_p(1)$ (corresponding to the multiplicative formal group $\mathcal{F} = \hat{\mathbb{G}}_m$) to $\mathbb{Q}_p(r)$.

Let K be a finite extension of $\mathbb{Q}_p$. For a continuous representation of G_K on a finite dimensional $\mathbb{Q}_p$-vector space V we write as usual[5]

$$D_{dR,K}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K} \supseteq D^0_{dR,K}(V) := (B^+_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K} \quad \text{and}$$
$$D_{cris,K}(V) := (B_{max,\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} V)^{G_K} .$$

The quotient $tan_K(V) := D_{dR,K}(V)/D^0_{dR,K}(V)$ is called the tangent space of V.

Henceforth we assume that V is de Rham. Then the usual Bloch-Kato exponential map $\exp_{K,V} : tan_K(V) \to H^1(K,V)$ can be defined as follows. Apply the tensor functor $- \otimes_{\mathbb{Q}_p} V$ to the exact sequence

$$0 \to \mathbb{Q}_p \to B^{\phi_p=1}_{max,\mathbb{Q}_p} \to B_{dR}/B^+_{dR} \to 0 \tag{5.12}$$

and take the (first) connecting homomorphism in the associated G_K-cohomology sequence. Furthermore, the dual exponential map $exp^*_{K,V}$ is defined by the commutativity of the following diagram

$$\begin{array}{ccc} H^1(K,V) & \xrightarrow{\exp^*_{K,V}} & D^0_{dR,K}(V) \\ \cong\downarrow & & \downarrow\cong \\ H^1(K,V^*(1))^* & \xrightarrow{(\exp_{K,V^*(1)})^*} & (D_{dR,K}(V^*(1))/D^0_{dR,K}(V^*(1)))^*, \end{array} \tag{5.13}$$

where the left, resp. right, vertical isomorphism comes from local Tate duality, resp. from the perfect pairing

$$D_{dR,K}(V) \times D_{dR,K}(\mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))) \longrightarrow D_{dR,K}(\mathbb{Q}_p(1)) \cong K, \tag{5.14}$$

in which the $D^0_{dR,K}$-subspaces are orthogonal to each other. Note that the isomorphism $K \cong D_{dR,K}(\mathbb{Q}_p(1))$ sends a to $at^{-1}_{\mathbb{Q}_p} \otimes \eta^{cyc}$. Also, $(-)^*$ here means the $\mathbb{Q}_p$-dual, $t_{\mathbb{Q}_p}$ is Fontaine's period satisfying $g(t_{\mathbb{Q}_p}) = \chi_{cyc}(g)t_{\mathbb{Q}_p}$ similarly as the basis η^{cyc} of $\mathbb{Q}_p(1)$.

For a formal group $\mathcal{F}$ with p-adic Tate-module T and $V := T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ it is shown in [22, Example 3.10.1] that the following diagram commutes:

$$\begin{array}{ccc} tan(\mathcal{F}) & \xrightarrow{\exp_{\mathcal{F}}} & \mathcal{F}(o_K) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \cong\downarrow & & \downarrow Kummer \\ tan_K(V) & \xrightarrow{\exp_{K,V}} & H^1(K,V). \end{array} \tag{5.15}$$

In particular, $\exp_{K,\mathbb{Q}_p(1)}$ is induced by the classical p-adic exponential map $\exp$ while $\exp^*_{K,\mathbb{Q}_p(1)}$ is the trivial map, because $tan_K(\mathbb{Q}_p) = 0$.

We can now interpret the map λ_m in (4.10) as

(5.16)

$$\varprojlim_n K_n^\times \xrightarrow{\text{Kummer map}} H^1_{Iw}(\mathbb{Z}_p(1)) \xrightarrow{cor} \mathrm{H}^1(K_m, \mathbb{Q}_p(1)) \xrightarrow{\exp^*_{BK,\mathbb{Q}_p(1)}} D^0_{dR,K_m}(\mathbb{Q}_p(1)) \xleftarrow{\cong} K_m$$

[5] We refer the reader to [50] for the foundations in p-adic Hodge theory.

and it is this shape which will be suitable for generalizations. In particular, it should be compared with the diagonal map in Corollary 7 below specialized to $\mathcal{F} = \hat{\mathbb{G}}_m$.

Recall from (4.11) that the classical reciprocity law just states, that λ_m is given by the first Coates-Wiles homomorphism. We will see later how the general Coates-Wiles homomorphisms arose in the study of p-adic L-functions in 7.1.2. Bloch and Kato interpreted them as classes in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(r))$ for $r \geq 1$ and reformulated for $r = 1$ the classical reciprocity law by saying that this class arises as the image of 1 under a certain transition map $\partial^r : \mathbb{Q}_p \to H^1(\mathbb{Q}_p, \mathbb{Q}_p(r))$ arising from p-adic Hodge theory. We shall sketch here its generalisation in [102, §8] to the Lubin-Tate setting as introduced before Theorem 4.

We define the Coates-Wiles homomorphisms in this context for $r \geq 1$ and $m \geq 0$ by

$$\begin{aligned} \psi^r_{CW,m} :\ \mathbb{U}(L_\infty) := \varprojlim_n o^\times_{L_n} &\longrightarrow L_m \\ u &\longmapsto \frac{1}{r!\pi_L^{rm}} \left(\partial_{\mathrm{inv}}^{r-1} \frac{\partial_{\mathrm{inv}} g_{u,\eta}}{g_{u,\eta}} \right)_{|Z=\eta_m}, \end{aligned}$$

where $g_{u,\eta}$ is the Coleman power series from Theorem 5; for $m > 0$ one can extend the domain of definition to $\varprojlim_n L_n^\times$ while for $m = 0$ one cannot evaluate at $\eta_0 = 0$. Then the map

$$\begin{aligned} \Psi^r_{CW,m} :\ \mathbb{U}(L_\infty) &\longrightarrow L_m t_L^r \\ u &\longmapsto \psi^r_{CW,m}(u) t^r_L \end{aligned}$$

is G_L-equivariant (it depends on the choice of η). In the following we abbreviate $\psi^r_{CW} := \psi^r_{CW,0}$ and $\Psi^r_{CW} := \Psi^r_{CW,0}$. One might think about these maps in terms of the formal identity

$$\log g_{u,\eta}(Z) = \sum_r \psi^r_{CW}(u) t^r_L = \sum_r \Psi^r_{CW}(u) \qquad \text{in } L[[t_L]] \subseteq B_{dR} \tag{5.17}$$

or the identity

$$d \log g_{u,\eta}(Z) = \frac{d g_{u,\eta}(Z)}{g_{u,\eta}(Z)} = \sum_{r \geq 1} r \psi^r_{CW}(u) t_L^{r-1} dt_L \ . \tag{5.18}$$

From class field theory we have an exact sequence

$$0 \longrightarrow \mathbb{U}(L_\infty) \xrightarrow{rec} H_L^{ab} \longrightarrow \hat{\mathbb{Z}} \longrightarrow 0$$

of Γ_L-modules, where the latter one has trivial action and H_L^{ab} denotes the maximal abelian quotient of the absolute Galois group $H_L = G(\bar{L}/L_\infty)$ of L_∞ with Γ_L-action induced by inner conjugation within G_L. Therefore, for $r \geq 1$, one has isomorphisms [6]

$$\begin{aligned} H^1(L, L(\chi^r_{LT})) &\cong H^1(L_\infty, L(\chi^r_{LT}))^{\Gamma_L} \\ &= \mathrm{Hom}_{\Gamma_L}(H_L^{ab}, L(\chi^r_{LT})) \\ &= \mathrm{Hom}_{\Gamma_L}(\mathbb{U}(L_\infty), L(\chi^r_{LT})). \end{aligned}$$

Since Ψ^r_{CW} belongs to the latter group, we can interpret it as class in $H^1(L, L(\chi^r_{LT}))$. On the other hand the exact sequence

$$0 \longrightarrow L t^r_L \longrightarrow Fil^r B^+_{max,L} \xrightarrow{\pi_L^{-r}\phi_q - 1} B^+_{max,L} \longrightarrow 0 \tag{5.19}$$

[6] Note that $H^i(\Gamma_L, L(\chi^r_{LT})) = 0$ for $i, r \geq 1$!

from [103, Lem. 8.2 (i)] induces, for any $r \geq 1$, the connecting homomorphism in continuous Galois cohomology

$$L = (B^+_{max,L})^{G_L} \xrightarrow{\partial^r} H^1(L, Lt_L^r)$$

and one can show [102, Prop. 8.5] that upon identifying $Lt_L^r \cong L(\chi_{LT}^r)$ we have a commutative diagram

$$\begin{array}{ccc} L = tan_L(L(\chi_{LT}^r)) & \xrightarrow{\quad 1-\pi^{-r} \quad} & L \\ & {\scriptstyle \exp_{L,L(\chi_{LT}^r)}} \searrow \qquad \swarrow {\scriptstyle \partial^r} & \\ & H^1(L, L(\chi_{LT}^r)). & \end{array}$$

Theorem 6 (Bloch-Kato explicit reciprocity law [22, Thm. 2.1],[102, Thm. 8.6])**.** *For all $a \in L$ and $r \geq 1$, we have the identities*

$$\partial^r(a) = ar\Psi^r_{CW}$$

and

$$\exp_{L,L(\chi_{LT}^r)}(a) = (1 - \pi_L^{-r})ar\Psi^r_{CW}.$$

interpreted as maps on $\mathbb{U}(L_\infty)$.

Strictly speaking their explicit reciprocity law is Theorem 2.6 in (loc. cit.), which calculates the cup-product

$$H^1(K, \mathbb{Z}/p^n\mathbb{Z}(r)) \times H^1(K(\mu_{p^n}), \mathbb{Z}/p^n\mathbb{Z}(1)) \to H^2(K(\mu_{p^n}), \mathbb{Z}/p^n\mathbb{Z}(r+1))$$

via differential forms using Coleman power series and which immediately implies the result stated here.

The following immediate consequence [102, Cor. 8.7] is again derived by considering duals and setting $\mathbf{d}_r := t_L^r t_{\mathbb{Q}_p}^{-1} \otimes (\eta^{\otimes -r} \otimes \eta^{cyc})$, where η^{cyc} is a generator of the cyclotomic Tate module $\mathbb{Z}_p(1)$, and $t_{\mathbb{Q}_p} := \log_{\mathbb{G}_m}([\iota(\eta^{cyc}) + 1] - 1)$.

Corollary 7 (A special case of Kato's explicit reciprocity law)**.** *For $r \geq 1$ the diagram*

$$\begin{array}{ccc} \varprojlim_n o_{L_n}^\times \otimes_{\mathbb{Z}} T^{\otimes -r} & & \\ \downarrow {\scriptstyle -\kappa \otimes \mathrm{id}} & \searrow {\scriptstyle ''(1-\pi_L^{-r})r\psi^r_{CW}(-)\mathbf{d}_r''} & \\ H^1_{Iw}(L_\infty/L, T^{\otimes -r}(1)) & & \\ \downarrow {\scriptstyle \mathrm{cores}} & & \\ H^1(L, T^{\otimes -r}(1)) & \xrightarrow{\quad \exp^* \quad} & D^0_{dR,L}(V^{\otimes -r}(1)) = L\mathbf{d}_r, \end{array}$$

commutes, i.e., the diagonal map sends $u \otimes a\eta^{\otimes -r}$ to

$$a(1 - \pi_L^{-r})r\psi^r_{CW}(u)\mathbf{d}_r = a\frac{1 - \pi_L^{-r}}{(r-1)!}\partial^r_{\mathrm{inv}} \log g_{u,\eta}(Z)_{|Z=0}\mathbf{d}_r\ .$$

We recommend also to compare with [44] and to consult [99] for a survey on Bloch's and Kato's explicit reciprocity law. In that whole volume *The Bloch-Kato Conjecture for the Riemann Zeta Function* also the meaning for the Tamagawa number conjecture for the motive $\mathbb{Z}(r)$ is discussed in detail. The general case of Kato's explicit reciprocity law in this setting can be found in [62], it has been generalized by Tsuji in [108]. As Laurent Berger pointed out his result is one of the rare ones which are not restricted to the L-analytic setting in the sense of subsection 6.5 below and it would be most desirable (but difficult) generalizing Theorem 5.3 in (loc. cit.) to higher dimensional representations.

6. Perrin-Riou's Reciprocity Law

Perrin-Riou's Reciprocity Law stands for an explicit computation of *Tate's local cup-product pairing* or *Iwasawa cohomology pairing* by means of the big exponential map $\Omega_{V^*(1)}$ (and/or the regulator map $\mathcal{L}_V$) for a crystalline[7] representation V of $G_{\mathbb{Q}_p}$ *in terms of easier to handle dualities from p-adic Hodge-theory.* So this is a generalization of the interpretation of the Hilbert symbol as cup-product as in (3.6), which justifies the use of "reciprocity law".

6.1. **Iwasawa cohomology and big exponential map.** In order to explain Perrin Riou's point of view we first have to introduce **Iwasawa cohomology** attached to the Galois tower of number fields

$$K \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots \subseteq K_\infty = \bigcup K_n$$

with Galois group $\Gamma := G(K_\infty/K)$ a p-adic Lie Group. We write $\Lambda(\Gamma) = \mathbb{Z}_p[[\Gamma]]$ for the Iwasawa algebra, i.e., completed group algebra. For a Galois stable lattice $T \subseteq V$ we define its Iwasawa cohomology groups as

$$H^i_{Iw}(T) := \varprojlim_n H^i(K_n, T) \cong H^i(K, \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)$$

which are finitely generated modules over $\Lambda(\Gamma)$ and important protagonists in the Iwasawa Main Conjectures together with its rational versions $H^i_{Iw}(V) := H^i_{Iw}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, see [92], [54].

In [96, 97, 98] Perrin-Riou introduced the (big) exponential map for a crystalline representation V of $G_{\mathbb{Q}_p}$ satisfying $Fil^{-h} D_{cris}(V) = D_{cris}(V)$ for some integer $h \geq 1$ (such h always exists):[8]

$$\Omega_{V,h} : \ D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V) \to D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V).$$

Here $D(\Gamma, \mathbb{Q}_p)$ denotes the distribution algebra of locally ($\mathbb{Q}_p$)-analytic distributions of Γ with coefficients in $\mathbb{Q}_p$. Note that all the twists $V(j)$ can be obtained as Galois equivariant quotients of $D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} V$. The big exponential map is uniquely characterized by the fact that it interpolates the Bloch-Kato exponential maps, the construction of which is recalled in section 5,

$$\exp_{K_n, V(j)} : \ D_{cris, K_n}(V(j)) \subseteq D_{dR, K_n}(V(j)) \to H^1(K_n, V(j))$$

for all cyclotomic twists $V(j)$ of V, $j \in \mathbb{Z}$, and all integers $n \geq 0$ and which satisfy a twist-behaviour with respect to the cyclotomic character.[9] Later Berger [10, Thm: II.13] found an easier direct construction based on use of (φ, Γ)- and Wach modules. Up to tensoring $\Omega_{V,h}$ with the field of fractions of $D(\Gamma, \mathbb{Q}_p)$ one can extend the definition to all $h \in \mathbb{Z}$.

---

[7]E.g. representations attached to abelian varietes over $\mathbb{Q}_p$ with *good* reduction. We refer the reader to [50] for the foundations in p-adic Hodge theory.

[8]Strictly speaking $\Omega_{V,h}$ is only defined on some submodule $\left(D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V)\right)^{\Delta=0}$, see e.g. [10, Def. II.12]. Moreover, in general, its image is only well-defined in $D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V)/V^{H_{\mathbb{Q}_p}}$. Up to replacing V by an appropriate twist $V(j)$ one can always avoid these restrictions!

[9]$Tw_{V(j)} \circ \Omega_{V(j),h} \circ D = -\Omega_{V(j+1),h+1}$ in [97, Thm. 3.2.3] or [10, Rem. II.15] without the sign.

6.2. **Different formulations of Perrin-Riou's explicit reciprocity law.** In section 3 we saw that one can interpret reciprocity laws via duality. This principle can be lifted to the level of Iwasawa cohomology extended to modules over $D(\Gamma, \mathbb{Q}_p)$:

Perrin-Riou formulates her explicit reciprocity law [97, Conj. 3.6.4] as the compatibility of Iwasawa duality with the (base change of the) crystalline (or de Rham) duality via her big exponential map as stated in (the lower part of) diagram (6.20) below. As indicated in the upper part of the diagram, the Iwasawa duality pairing[10] $\{,\}_{Iw}$ interpolates the cup-product $\langle,\rangle_{Tate}$ of usual Galois cohomology à la Tate. In this sense, one can use the crystalline pairing $[,]_{D_{cris}}$ in order to calculate explicitly both the Iwasawa pairing and then the cup-product, hence her name *explicit reciprocity law* is in full accordance with the previous use in section 3. Perrin-Riou was able to prove this conjecture for $\mathbb{Q}_p(r)$, $r \in \mathbb{Z}$, more generally it was shown independently by COLMEZ 1998 [34], BENOIS 1998 [8], KATO-KURIHARA-TSUJI (UNPUBLISHED)

Theorem 8 (First formulation)**.** *Let V be crystalline. Then, for all integers h, j we have a commutative diagram*

$$
\begin{array}{ccccc}
H^1(K_n, V^*(1-j)) & \times & H^1(K_n, V(j)) & \xrightarrow{\langle,\rangle_{Tate}} & H^2(K_n, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \\
\uparrow{\scriptstyle pr_n} & & \uparrow{\scriptstyle pr_n} & & \uparrow{\scriptstyle (-1)^j \int \chi_{trivial} -} \\
D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V) & \xrightarrow{\{,\}_{Iw}} & D(\Gamma, \mathbb{Q}_p) \\
\uparrow{\scriptstyle \Omega_{V^*(1),1-h}} & & \uparrow{\scriptstyle \Omega_{V,h}} & & \| \\
D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V) & \xrightarrow{[,]_{D_{cris}}} & D(\Gamma, \mathbb{Q}_p).
\end{array}
\tag{6.20}
$$

Colmez [34, Thm. 9] proves the explicit reciprocity law in a different, but equivalent version, our *second formulation*: He shows that **the big exponential map not only interpolates the Bloch-Kato exponential maps themselves, but simultaneously the dual Bloch-Kato exponential maps**, see also [98, §4.2.1, especially (4.2.1)]. The equivalence is explained in [98, §4.2]. We shall only state these kind of formulae later in the Lubin-Tate setting, see Theorem 17 below: those formulae specialize to those on p. 123 in Berger's article: [10, II.16] uses the same approach for a very elegant proof based on such interpolation formulae in Thm. II.10 of (loc. cit.) calculated via the theory of (φ, Γ)- and Wach modules again.

Noting that $\Omega_{V,h}$ possesses an (almost) inverse $\mathcal{L}_{V,h}$, called *regulator map,* we obtain a *third formulation* as follows.

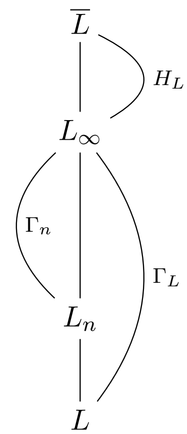
Theorem 9 (Third formulation: adjunction of big exponential- and regulator map)**.** *The regulator map is adjoint to the big exponential map:*

$$
\begin{array}{ccccc}
D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_\Lambda H^1_{Iw}(V) & \xrightarrow{\{,\}_{Iw}} & D(\Gamma, \mathbb{Q}_p) \\
\uparrow{\scriptstyle \Omega_{V^*(1),1-h}} & & {\scriptstyle \mathcal{L}_{V,h}}\downarrow & & \| \\
D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V^*(1)) & \times & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V) & \xrightarrow{[,]_{D_{cris}}} & D(\Gamma, \mathbb{Q}_p).
\end{array}
$$

6.3. **Iwasawa cohomology via (φ, Γ)-modules.** Fontaine [49] observed that Iwasawa cohomology can be expressed in terms of (φ, Γ)-modules.

[10] This can either be defined by a limit process of Tate duality running up the tower K_n as in [97] or by a residue pairing as in [65, §4.2] or [103, §4.5.4].

We recall his equivalence of categories directly in the Lubin-Tate setting, as introduced before Theorem 4, see [102] for more details:
$L_n = L(\mathcal{F}[\pi_L^n])$ is the extension of L generated by the π_L^n-torsion points of $\mathcal{F}$,
$L_\infty := \bigcup_n L_n$,
$\Gamma_L := \mathrm{Gal}(L_\infty/L)$ and $\Gamma_n := \mathrm{Gal}(L_\infty/L_n)$;
the Lubin-Tate character χ_{LT} induces isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$ and $Lie(\Gamma_L) \xrightarrow{\cong} L$, and $\nabla \in Lie(\Gamma_L)$ is the preimage of 1. Moreover, H_L denotes the kernel of χ_{LT}.



To this aim we write $\mathbf{A}_L := \widehat{o_L[[Z]][\frac{1}{Z}]}$ for the p-adic completion of $o_L[[Z]][\frac{1}{Z}]$. This ring comes with commuting actions by $\varphi = [\pi_L]^*$ and Γ_L via $[\chi_{LT}(\gamma)]^*$. Moreover, we have the (almost) left inverse operator ψ_L satisfying $\psi_L \circ \varphi_L = \frac{q}{\pi_L}$, with $q := \#(o_L/\pi_L o_L)$.

Then, $\Phi\Gamma(\mathbf{A}_L)$, the category of (φ, Γ)-modules, consists of finitely generated $\mathbf{A}_L$-modules M with commuting (semi-linear) actions by some endomorphism φ_M and by Γ_L. M is called étale, if the linearized map

$$\varphi_M^{lin}: \mathbf{A}_L \otimes_{\mathbf{A}_L, \varphi_L} M \xrightarrow{\cong} M$$
$$f \otimes m \longmapsto f\varphi_M(m)$$

is an isomorphism. We write $\Phi\Gamma^{et}(\mathbf{A}_L)$ for the full subcategory of étale objects, to which e.g. the module $\Omega^1 := \mathbf{A}_L dZ$ of differential forms belongs. By $d: \mathbf{A}_L \to \Omega^1$ we denote the canonical map sending f to $df = \frac{d}{dZ} f dZ$. The reader is invited to recall from the table just before Theorem 4 how the Lubin-Tate setting specializes to the cyclotomic one.

Let $\mathrm{Rep}_{o_L}(G_L)$ denote the abelian category of finitely generated o_L-modules equipped with a continuous linear G_L-action. The following result is established in [67] Thm. 1.6., where also the ring $\mathbf{A}$ is introduced.

Theorem 10. *The functors*

$$V \longmapsto D_{LT}(V) := (\mathbf{A} \otimes_{o_L} V)^{H_L} \qquad \text{and} \qquad M \longmapsto (\mathbf{A} \otimes_{\mathbf{A}_L} M)^{\phi_q \otimes \varphi_M = 1}$$

are exact quasi-inverse equivalences of categories between $\mathrm{Rep}_{o_L}(G_L)$ *and* $\Phi\Gamma(\mathbf{A}_L)$.

Herr observed how to express Galois cohomology in terms of (φ, Γ)-modules, see [71] for a generalisation in the Lubin-Tate setting. The following results, generalising Fontaine's observation from the cyclotomic to the general Lubin-Tate setting, expresses Iwasawa cohomology, where the twist by $\tau := \chi_{LT}^{-1}\chi_{cyc}$ is a new phenomenon in the general Lubin-Tate stting (disappearing obviously in the cyclotomic case).

Theorem 11 ([102, Thm. 5.13])**.** *Let V in $\mathrm{Rep}_{o_L}(G_L)$. Then, with $\psi = \psi_{D_{LT}(V(\tau^{-1}))}$, we have a short exact sequence*

$$(6.21) \quad 0 \longrightarrow H^1_{Iw}(L_\infty/L, V) \longrightarrow D_{LT}(V(\tau^{-1})) \xrightarrow{\psi - 1} D_{LT}(V(\tau - 1)) \longrightarrow H^2_{Iw}(L_\infty/L, V) \longrightarrow 0,$$

which is functorial in V.

Now the above mentioned version of the Schmid-Witt formula 3.2 for ramified Witt vectors (of finite length) gives an explicit determination of elements in Iwasawa cohomology (in the Lubin-Tate setting) as images under the Kummer map

$$\kappa : \varprojlim_{n,m} L_n^\times / L_n^{\times p^m} \xrightarrow{\cong} H^1_{Iw}(L_\infty/L, \mathbb{Z}_p(1)) ,$$

by means of (φ, Γ_L)-modules. We write $T^* = o_L\eta^*$ for the dual of the Tate module $T := \varprojlim_n \mathcal{F}[\pi_L^n] = o_L\eta$ of $\mathcal{F}$ and we consider the map

$$\nabla : (\varprojlim_n L_n^\times) \otimes_{\mathbb{Z}} T^* \longrightarrow \mathbf{A}_L^{\psi=1}, u \otimes a\eta^* \longmapsto a \frac{\partial_{\mathrm{inv}}(g_{u,\eta})}{g_{u,\eta}}(\iota_{LT}(\eta)) .$$

Theorem 12 (*cycl:* BENOIS[8, Prop. 2.1.5], COLMEZ[36, Thm. 7.4.1],[26, Prop. V.3.2 (iii)] | *LT:* SCHNEIDER-VENJAKOB[102, Thm. 6.2])**.** *The diagram*

$$\begin{array}{ccc} (\varprojlim_n L_n^\times) \otimes_{\mathbb{Z}} T^* & \xrightarrow{\kappa \otimes T^*} & H^1_{Iw}(L_\infty/L, o_L(\tau)) \\ & \searrow^{\nabla} & \cong \downarrow Exp^* \\ & & \mathbf{A}_L^{\psi=1} = D_{LT}(o_L)^{\psi=1} \end{array}$$

is commutative. Here Exp^* denotes the isomorphism between Iwasawa cohomology and the $\psi = 1$ part of the attached (φ, Γ)-module, which is originally due to Fontaine and which has been constructed in the Lubin-Tate setting in [102, Thm. 5.13].

Colmez [36, §7.4] calls this already an *explicit reciprocity law*, probably as the determination of the Kummer map is achieved by using the cup-product pairing, at least in [102, §6]. Moreover, the latter is calculated by the explicit Schmid-Witt formula as had been noticed already by Fontaine in §4.4 of his letter to Perrin Riou [97]. Finally, as Colmez explains in Remark following [36, Thm. 7.4.1] Exp^* produces - up to the Amice-transform and restricting (pseudo)measures from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$ - the Kubota-Leopoldt zeta function from the system of cyclotomic units - as we discuss in detail in section 7.[11] See also [26, Thm. IV.2.1, Prop. IV.3.1 (ii)], [34, Thm. 9 (iii)] for an explicit reciprocity involving Exp^* for general de Rham representations interpolating dual Bloch-Kato exponential maps for the twists $V(k)$ of V :

$$p^{-n}\varphi^{-n}(Exp^*(\mu)) = \sum_{k \,\epsilon\, \mathbb{Z}} \exp^*_{K_n, V(-k)} \left(\int_{\Gamma_{K_n}} \chi_{cyc}(x)^{-k}\mu(x) \right)$$

for $\mu \,\epsilon\, H^1_{Iw}(K, V)$ and n big enough. As remarked in [33, §5, Remark (ii)] the term $CW_{V,k,n}(\mu)$ corresponding to $\exp^* \left(\int_{\Gamma_{K_n}} \chi_{cyc}(x)^{-k}\mu(x) \right)$ in the sum can be defined directly from $Exp^*(\mu)$ without any reference to $\exp^*$ and the maps $\mu \mapsto CW_{V,k,n}(\mu)$ are generalizations of the Coates-Wiles homomorphisms from sections 5 and 7. The meaning of the above formula is that they are related to Bloch-Kato's dual exponential maps, which is an explicit reciprocity law in the sense of Corollary 7. Compare also with formulae (5.17) and (5.18) combined with the formula in Theorem 12, which also indicates the relation of Exp^* with Coleman's isomorphism Col in (7.35) and (7.45).

[11] $\nabla(\mathbf{c}(a,b)) = D\log g_{\mathbf{c}(a,b)} = D\log f(T)$ with the notation in (7.31). The restriction of measures from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$ corresponds to applying the operator $1 - \varphi = 1 - \varphi\psi$ given that we start with a power series fixed under ψ. Therefore, Colmez' interpretation is compatible with that of subsection 7.1.2.

6.4. **Colmez' abstract reciprocity law.** In [38] Colmez formulates a reciprocity law purely in terms of (φ, Γ)-modules as an abstraction of Theorems 8,9 above. To this end he replaces the representation V by $D = D(V) \in \Phi\Gamma(\mathbf{A}_{\mathbb{Q}_p})$, $H^1_{Iw}(V)$ by $D^{\Psi=1}$ (which maps via $1-\varphi$ to $D^{\psi=0}$, which is the same as $D \boxtimes \mathbb{Z}_p^\times$ in his terminology), $\mathcal{L}_V$ (respectively the inverse of $\Omega_{V^*(1)}$) by $1-\varphi$. Furthermore, he defines $\mathbf{A}_{\mathbb{Q}_p}(\Gamma) := \widehat{\Lambda(\Gamma)[\frac{1}{\gamma-1}]}$ as p-adic completion and constructs a further pairing $\{,\}_{conv}$ by generalising the convolution product of distributions to (φ, Γ)-modules. Then his abstract reciprocity law has the following shape.

Theorem 13 (**Colmez' abstract reciprocity law**)**.** *The canonical pairing*

$$\check{D} := Hom_{\mathbf{A}_{\mathbb{Q}_p}}(D, \Omega^1) \times D \to \Omega^1$$

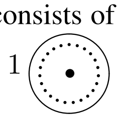
induces a commutative diagram

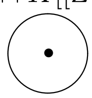
$$\begin{array}{ccccccc} \check{D}^{\psi=0} & \times & D^{\psi=0} & \xrightarrow{\{,\}_{Iw}} & \mathbf{A}_{\mathbb{Q}_p}(\Gamma) & \xrightarrow{\mathfrak{M}} & \mathbf{A}_{\mathbb{Q}_p}^{\psi=0} \\ \downarrow{\scriptstyle \sigma_{-1}\iota_*} & & \| & & & & \downarrow{\scriptstyle d} \\ \check{D}^{\psi=0} & \times & D^{\psi=0} & & \xrightarrow{\{,\}_{conv}} & & (\Omega^1)^{\psi=0}. \end{array}$$

Here, ι_* denotes a certain involution, $\sigma_1 \in \Gamma$ corresponds to -1 under the cyclotomic character and $\mathfrak{M}$ denotes the Mellin transform.

This compatibility has been used by Colmez to study locally algebraic vectors in order to compare the p-adic with the classical local Langlands correspondence (for $GL_2(\mathbb{Q}_p)$).

This abstract version in the cyclotomic setting served as a role model for an abstract reciprocity law in the Lubin-Tate setting as we will describe now. As there is no good integral theory in that case so far, we will only obtain a version over the Robba ring.

6.5. **Abstract Reciprocity formula in the Lubin-Tate setting.** We start recalling the definition of the Robba ring and (φ_L, Γ_L)-modules in the context of Lubin-Tate extensions over the base field L. Let $L \subseteq K$ be a complete field containing the period $\Omega \in K$. The Robba ring $\mathcal{R} := \mathcal{R}_K \subseteq K[[Z, Z^{-1}]]$ with coefficients in K consists of those analytic functions which converge on some annulus $r \leq |Z| < 1$ for some $0 < r < 1$ 

while $\mathcal{R}^+ := \mathcal{R}_K^+ := \mathcal{R} \cap K[[Z]]$ consists of those analytic functions which converge on the open unit disk $\mathbb{B} := \{|Z| < 1\}$. 

On $\mathcal{R}_K, \mathcal{R}_K^+$ the elements $\gamma \in \Gamma_L$ and φ_L act via $Z \mapsto [\chi_{LT}(\gamma)](Z)$ and $[\pi_L](Z)$, respectively. Moreover, there exists an (almost) left inverse operator ψ satisfying $\psi \circ \varphi_L = \frac{q}{\pi_L}$.

We write $Rep_L(G_L)$ for the category of finite dimensional L-vector spaces with continuous, linear G_L-action and $Rep_L^{an}(G_L)$ for the full subcategory consisting of *analytic* representations V, i.e., $\mathbb{C}_p \otimes_{\sigma,L} V$ is the trivial semilinear $\mathbb{C}_p$-representation $\mathbb{C}_p^{\dim_L V}$ for all $\iota \neq \sigma : L \to \mathbb{C}_p$. Furthermore, $\Phi\Gamma^{an}(\mathcal{R}_L)$ denotes the category of (φ_L, Γ_L)-modules[12] over $\mathcal{R}_L$ with L-**linear** $Lie(\Gamma_L)$-action (the latter derived action is always $\mathbb{Q}_p$-linear!).

[12] From now on we assume them to be free of finite rank over $\mathcal{R}_L$.

Theorem 14 (KISIN-REN/FONTAINE, BERGER)**.** *There is an equivalence of categories*

$$Rep_L^{an}(G_L) \longleftrightarrow \Phi\Gamma^{an,ét}(\mathcal{R}_L)$$
$$W \mapsto D^{\dagger}_{rig}(W).$$

Note that without the superscripts ^{an} the statement is false for $L \neq \mathbb{Q}_p$, because the theorem of overconvergence by Cherbonnier and Colmez [25] does not hold, see [52] for a counterexample.

Now let V be an L-linear continuous representation of G_L such that $V^*(1)$ (and hence $V(\tau^{-1})$) is L-analytic and crystalline. Then $M := D^{\dagger}_{\mathrm{rig}}(V(\tau^{-1}))$ and $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}}) \cong D^{\dagger}_{\mathrm{rig}}(V^*(1))$ belong to $\Phi\Gamma^{an}(\mathcal{R})$ over the Robba ring $\mathcal{R}$. Building on BERGER's comparison isomorphism

$$\mathrm{comp}_{\check{M}} : \mathcal{R}[\frac{1}{t_{LT}}] \otimes_{\mathcal{R}} \check{M} \xrightarrow{\cong} \mathcal{R}[\frac{1}{t_{LT}}] \otimes_L D_{cris,L}(V^*(1))$$

we prove the following abstract reciprocity law.

Theorem 15 (SCHNEIDER-VENJAKOB[103, (140)])**.** *The following diagram*

$$\begin{array}{ccccccc}
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \xrightarrow{\{,\}'_{M,Iw}} & \mathcal{R}(\Gamma_L) & \xrightarrow[\cong]{\mathfrak{M}} & \mathcal{R}_L^{\psi=0} \\
\| & & \| & & & & \downarrow d \\
\check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \longrightarrow & & & (\Omega^1_{\mathcal{R}})^{\psi=0} \\
\uparrow \mathrm{comp}_{\check{M}} \downarrow & & \uparrow \mathrm{comp}_M \downarrow & & & & \uparrow \mathrm{comp}_{\Omega^1} \downarrow \\
\mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V^*(1)) & \times & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(V(\tau^{-1})) & & \xrightarrow{[,]_{D_{cris,L}(V(\tau^{-1}))}} & & \mathcal{R}^{\psi_L=0} \otimes_L D_{cris,L}(L(\chi_{LT}))
\end{array}$$

commutes up to inverting $t_{LT} := \log_{LT}$. Here, the broken arrows indicate that we have only an isomorphism after inverting t_{LT}.

Actually, this theorem is a consequence of Serre duality on certain rigid analytic varieties comparing additive and multiplicative residuum maps. More precisely, we show the commutativity of

$$\begin{array}{ccc}
\mathcal{R}_L(\Gamma_L) := \mathcal{R}_L(\mathfrak{X}^{\times}) & \xrightarrow[\cong]{\cdot d\log_{\mathfrak{X}^{\times}}} & \Omega^1_{\mathfrak{X}^{\times}} \\
 & & \downarrow \mathrm{res}_{\mathfrak{X}^{\times}} \\
(-)(\mathrm{ev}_1 d\log_{\mathfrak{X}}) \downarrow \cong & & L \\
 & & \uparrow \mathrm{res}_{\mathfrak{X}} \\
(\Omega^1_{\mathfrak{X}})^{\psi=0} & \xrightarrow{\mathrm{ev}_{-1}\cdot} & \Omega^1_{\mathfrak{X}},
\end{array}$$

with SCHNEIDER-TEITELBAUM's character varieties $\mathfrak{X}, \mathfrak{X}^{\times}$ for the groups $o_L, o_L^{\times}$ using their Fourier theory and Lubin-Tate isomorphism $\mathfrak{X} \cong \mathbb{B}$ over $\mathbb{C}_p$. See [103, Thm. 4.5.12] for details.

From the above abstract version one derives now a reciprocity formula in the Lubin-Tate setting analogous to Theorem 9. We write $D(\Gamma_L, \mathbb{C}_p)$ for the algebra of locally L-analytic distributions. Recall that already Fourquaux [51], who initiated the investigation of Perrin-Riou's approach for Lubin-Tate extensions in his thesis in 2005, had achieved a generalization of Colmez' construction of the *Perrin-Riou logarithm*. In the following result enters instead the big exponential map, he has constructed together with Berger.

Theorem 16. *If* $\mathrm{Fil}^{-1}D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ *and* $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$*, then the following diagram commutes:*

$$\begin{array}{ccccc}
D^{\dagger}_{\mathrm{rig}}(V^*(1))^{\psi_L=\frac{q}{\pi_L}} & \times & D_{LT}(V(\tau^{-1}))^{\psi_L=1} & \xrightarrow{\{,\}_{Iw}} & D(\Gamma_L,\mathbb{C}_p) \\
\uparrow{\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \mathcal{L}^0_V}\downarrow & & \| \\
D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V^*(1)) & \times & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V(\tau^{-1})) & \xrightarrow{[,]} & D(\Gamma_L,\mathbb{C}_p),
\end{array}$$

where $\Omega_{V^*(1),1}$ *denotes Berger's and Fourquaux' big exponential map* [11] *while the regulator map* $\mathcal{L}^0_V$ *is defined in* [103, §5.1].

We write $\mathrm{Ev}_{W,n} : \mathcal{R}^+_L \otimes_L D_{cris,L}(W) \to L_n \otimes_L D_{cris,L}(W)$ for the composite $\partial_{D_{cris,L}(W)} \circ \varphi_q^{-n}$ from the introduction of [11], which actually sends $f(Z)\otimes d$ to $f(\eta_n)\otimes \varphi_L^{-n}(d)$. By abuse of notation we also use $\mathrm{Ev}_{W,0}$ for the analogous map $\mathcal{R}^+_K \otimes_L D_{cris,L}(W) \to K \otimes_L D_{cris,L}(W)$. For $x \in D(\Gamma_L,K)\otimes_L D_{cris,L}(W)$ we denote by $x(\chi_{LT}^j)$ the image under the map $D(\Gamma_L,K)\otimes_L D_{cris,L}(W) \to K\otimes_L D_{cris,L}(W)$, $\lambda\otimes d \mapsto \lambda(\chi_{LT}^j)\otimes d$. With this notation Berger's and Fourquaux' interpolation property reads as follows:

Theorem 17 (Berger-Fourquaux [11, Thm. 3.5.3])**.** *Let* W *be* L*-analytic and* $h \geq 1$ *such that* $\mathrm{Fil}^{-h}D_{cris,L}(W) = D_{cris,L}(W)$*. For any* $f \in \left((\mathcal{R}^+)^{\psi=0}\otimes_L D_{cris,L}(W)\right)^{\Delta=0}$ *and* $y \in (\mathcal{R}^+ \otimes_L D_{cris,L}(W))^{\psi=\frac{q}{\pi_L}}$ *with* $f = (1-\varphi_L)y$ *we have: If* $h+j \geq 1$*, then*

$$\begin{aligned}
&h^1_{L_n,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\Omega_{W,h}(f))) = \\
&(-1)^{h+j-1}(h+j-1)! \begin{cases} \exp_{L_n,W(\chi_{LT}^j)}\left(q^{-n}\mathrm{Ev}_{W(\chi_{LT}^j),n}(\partial_{\mathrm{inv}}^{-j}y\otimes e_j)\right) & \text{if } n \geq 1; \\ \exp_{L,W(\chi_{LT}^j)}\left((1-q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi_{LT}^j),0}(\partial_{\mathrm{inv}}^{-j}y\otimes e_j)\right), & \text{if } n = 0. \end{cases}
\end{aligned} \tag{6.22}$$

If $h+j \leq 0$*, then*

$$\begin{aligned}
&\exp^*_{L_n,W(\chi_{LT}^j)}\left(h^1_{L_n,W(\chi_{LT}^j)}(tw_{\chi_{LT}^j}(\Omega_{W,h}(f)))\right) = \\
&\frac{1}{(-h-j)!}\begin{cases} q^{-n}\mathrm{Ev}_{W(\chi_{LT}^j),n}(\partial_{\mathrm{inv}}^{-j}y\otimes e_j) & \text{if } n \geq 1; \\ (1-q^{-1}\varphi_L^{-1})\mathrm{Ev}_{W(\chi_{LT}^j),0}(\partial_{\mathrm{inv}}^{-j}y\otimes e_j), & \text{if } n = 0. \end{cases}
\end{aligned} \tag{6.23}$$

The reciprocity law Theorem 16 then delivers the interpolation property of the regulator map

$$\mathbf{L}_V : H^1_{Iw}(L_\infty/L,T) \to D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(V(\tau^{-1})),$$

which generalizes [83, Thm. A.2.3] and [84, Thm. B.5] from the cyclotomic case (See §5.2.4 of [103, Thm. 6.4] for the missing notation).

Theorem 18 ([103, Thm. 6.4])**.** *Assume that* $V^*(1)$ *is* L*-analytic with* $\mathrm{Fil}^{-1}D_{cris,L}(V^*(1)) = D_{cris,L}(V^*(1))$ *and* $D_{cris,L}(V^*(1))^{\varphi_L=\pi_L^{-1}} = D_{cris,L}(V^*(1))^{\varphi_L=1} = 0$*. Then we have for* $j \geq 0$

$$\begin{aligned}
\Omega^j\mathbf{L}_V(y)(\chi_{LT}^j) &= j!\left((1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\right)\otimes e_j \\
&= j!(1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\left(\widetilde{\exp}^*_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\otimes e_j\right)
\end{aligned}$$

and for $j \leq -1$:

$$\Omega^j \mathbf{L}_V(y)(\chi_{LT}^j) = \frac{(-1)^j}{(-1-j)!}\Big((1-\pi_L^{-1}\varphi_L^{-1})^{-1}(1-\frac{\pi_L}{q}\varphi_L)\widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\Big)\otimes e_j$$
$$= \frac{(-1)^j}{(-1-j)!}(1-\pi_L^{-1-j}\varphi_L^{-1})^{-1}(1-\frac{\pi_L^{j+1}}{q}\varphi_L)\Big(\widetilde{\log}_{L,V(\chi_{LT}^{-j}),\mathrm{id}}(y_{\chi_{LT}^{-j}})\otimes e_j\Big),$$

if the operators $1-\pi_L^{-1-j}\varphi_L^{-1}, 1-\frac{\pi_L^{j+1}}{q}\varphi_L$ *or equivalently* $1-\pi_L^{-1}\varphi_L^{-1}, 1-\frac{\pi_L}{q}\varphi_L$ *are invertible on* $D_{cris,L}(V(\tau^{-1}))$ *and* $D_{cris,L}(V(\tau^{-1}\chi_{LT}^j))$*, respectively.*

We leave it to the interested reader to check that the combination (of parts of) Theorem 17 and 18 implies again the reciprocity law 16 by the principle of p-adic interpolation.

6.6. **Nakamura's Explicit Reciprocity Law.** In the cyclotomic setting Nakamura [91] extended the definition of the (dual) Bloch-Kato exponential map to all (de Rham) (φ,Γ)-modules over the Robba ring, i.e., they need not be étale. In the rank one case he established an explicit reciprocity law in the style of Theorem 18. Part of his work has been carried over to the Lubin-Tate situation in [89], from where we cite the following result. We define a regulator map

$$\mathbf{L}_{\mathcal{R}(\delta)}:\ H^1_{Iw,+}(\mathcal{R}(\delta)) := (\mathcal{R}_L^+(\delta))^{\Psi=1} \xrightarrow{1-\varphi} (\mathcal{R}_L^+(\delta))^{\Psi=0} \xleftarrow[\cong]{\mathfrak{M}_\delta\circ\sigma_{-1}} D(\Gamma_L),$$

where $\mathfrak{M}_\delta$ denotes again a certain Mellin-transform. Moreover, we recall that the Amice transform is the map

$$A_-:\ D(o_L,K)\to\mathcal{R}_K^+,$$

sending a distribution μ to

$$A_\mu(Z) = \int_{o_L}\eta(x,Z)\mu(x)$$

with $\eta(x,Z) := \exp\left(\Omega x \log_{LT}(Z)\right)\ \epsilon\ 1+Zo_{\widehat{L_\infty}}[[Z]]$.

Theorem 19 (Explicit reciprocity formula [91, 4B1],[89, Prop. 9.13/17])**.** *Let* $\delta=\delta_{lc}x^k$ *be de Rham. For* $k\leq 0$*, the following diagram is commutative:*
(6.24)

$$\begin{array}{ccc}
H^1_{Iw,+}(\mathcal{R}(\delta)) & \xrightarrow{\mathbf{L}_{\mathcal{R}(\delta)}} & D(\Gamma_L)\\
\Big\downarrow{\scriptstyle x\mapsto[(0,C_{Tr}(\mathfrak{z}_n)^{-1}x)]} & & \Big\downarrow{\scriptstyle pr_{\Gamma_n}}\\
H^1_{\Psi,\mathfrak{z}_n}(\mathcal{R}(\delta)) \xrightarrow{\exp^{*,(n)}} D_{dR}^{(n)}(\mathcal{R}(\delta))\cong L_n\otimes_L D_{dR}(\mathcal{R}(\delta)) & \xleftarrow{\Sigma} & K[\Gamma_L/U],
\end{array}$$

i.e., a class $[A_\mu \mathbf{e}_\delta]\ \epsilon\ H^1_{\Psi,\mathfrak{z}_n}(\mathcal{R}(\delta))^{\Gamma_L}$*, is mapped under* $\exp^*$ *to*

$$\mathfrak{C}(\delta)\int_{o_L^\times}\delta(x)^{-1}\mu(x)\frac{1}{t_{LT}^k}\mathbf{e}_\delta,$$

where $\mathfrak{C}(\delta)$ defines a certain ε-constant defined in (loc. cit.). For $k \geq 1$, the following diagram is commutative:

(6.25)
$$\begin{array}{ccc}
H^1_{Iw,+}(\mathcal{R}(\delta)) & \xrightarrow{\quad\mathbf{L}_{\mathcal{R}(\delta)}\quad} & D(\Gamma_L) \\
\Big\downarrow{\scriptstyle x\mapsto[(0,C_{Tr}(\mathfrak{z}_n)^{-1}x)]} & & \Big\downarrow{\scriptstyle pr_{\Gamma_n}} \\
H^1_{\Psi,\mathfrak{z}_n}(\mathcal{R}(\delta)) \xleftarrow{\exp^{(n)}} D^{(n)}_{dR}(\mathcal{R}(\delta)) \cong L_n \otimes D_{dR}(\mathcal{R}(\delta)) & \xleftarrow{\Sigma'} & K[\Gamma_L/U],
\end{array}$$

i.e., a class $[A_\mu \mathbf{e}_\delta] \in H^1_{\Psi,\mathfrak{z}_n}(\mathcal{R}(\delta))^{\Gamma_L}$, is mapped under $\exp^{-1}_{\mathcal{R}(\delta)}$ to

$$\mathfrak{C}'(\delta) \int_{o_L^\times} \delta(x)^{-1}\mu(x)\frac{1}{t^k_{LT}}\mathbf{e}_\delta,$$

where $\mathfrak{C}'(\delta)$ defines a certain ε-constant defined in (loc. cit.).

7. Regulator maps and Euler systems

Let us begin this quasi-final section with two quotations, which in our opinion set the stage very well.

Coates-Wiles [29, Introduction]:

> The above result on cyclotomic fields [the link between the divisibility of $(2\pi i)^{-k}\zeta(k)$ by p for even integers k with $1 < k < p-1$ and the non-triviality of the χ^k_{cyc}-th eigenspace of the $G(\mathbb{Q}_p(\mu_p)/\mathbb{Q}_p)$-modules U^1/C of 1-units modulo the closure of cyclotomic units] was probably known, in essence, to Kummer. However, we have been influenced by the important, and somewhat neglected, paper ...[[59]] of Iwasawa, which establishes a deeper result. Accordingly, the present paper has been written in the spirit of Iwasawa's work. In particular, we study explicit reciprocity laws in certain fields of division points on E, and introduce analogues of the mysterious maps ψ_n, used by Iwasawa in the cyclotomic case. However, since this paper was first written, a number of people have pointed out to us that these explicit reciprocity laws can be avoided in the proof of Theorem 1 [the main result concerning BSD in that article]. While this is true ..., we have retained a discussion of them for two reasons. Firstly, we believe that the explicit reciprocity laws form the correct conceptual framework for discussing Kummer's p-adic logarithmic derivatives (which are essential for the proof of Theorem 1). Secondly, we suspect that these laws will play a vital role in future attempts to prove the finiteness of the Tate-Šafarevič group of E when $L(E/F, 1) \neq 0$.

Perrin-Riou (1999)[98, Introduction]:

> Le développement de ces lois [explicites de réciprocité] s'est fait en parallèle et en liaison avec le développement de la théorie d'Iwasawa locale; dans le cas classique, il s'agit de l'étude du comportement des unités locales sur la $\mathbb{Z}_p^\times$-extension cyclotomique K_∞ à l'aide de l'application exponentielle (Iwasawa, Coates-Wiles, Coleman).

7.1. **Cyclotomic untis.**

7.1.1. *Special L-values.* The development of explicit reciprocity laws (especially from Kummer's to Iwasawa's version) happened simultaneously with the discovery of the description of special ζ-values through special units. Already in Kummer's version 1 it was crucial to form "(logarithmic) derivatives" from such units, which means to express such units in terms of power series, which allow a differential calculus, and doing the manipulation that way. Inspired by Iwasawa this was immanent in the early work of Coates-Wiles already and then systematized by Coleman's invention of the power series attached to certain local units and named after him, see 5.

In the easiest case to begin with, viz of the Riemann zeta-function, we have the well-known formulae of the Bernoulli numbers $\mathcal{B}_n$, $n \geq 0$,

$$\frac{1}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\mathcal{B}_n}{n!} t^{n-1} \tag{7.26}$$

and its relation to the ζ-values

$$\zeta(1-k) = -\frac{\mathcal{B}_k}{k} \quad (k = 2, 4, 6, \ldots). \tag{7.27}$$

Setting

$$F(z) = \frac{e^{-\frac{a}{2}z} - e^{\frac{a}{2}z}}{e^{-\frac{b}{2}z} - e^{\frac{b}{2}z}}$$

we obtain for its logarithmic derivative

$$g(z) := \frac{d}{dz} \log F(z) \tag{7.28}$$

$$= \frac{b}{2}\left(\frac{1}{e^{-bz}-1} - \frac{1}{e^{bz}-1}\right) - \frac{a}{2}\left(\frac{1}{e^{-az}-1} - \frac{1}{e^{az}-1}\right) \tag{7.29}$$

$$= \sum_{k=2 \text{ (even)}}^{\infty} \frac{\mathcal{B}_k z^{k-1}}{k!}(a^k - b^k) \tag{7.30}$$

where we used (7.26) for the last equation. Using (7.27) we thus obtain

$$\left(\frac{d}{dz}\right)^{k-1} g(z)_{|z=0} = (b^k - a^k)\zeta(1-k) \quad (k = 2, 4, 6, \ldots). \tag{7.31}$$

Using the transformation

$$F(z) = f(T) = \frac{(1+T)^{-\frac{a}{2}} - (1+T)^{\frac{a}{2}}}{(1+T)^{-\frac{b}{2}} - (1+T)^{\frac{b}{2}}} \quad \text{with } T = e^z - 1$$

the differentiation $\frac{d}{dz}$ corresponds to $D := (T+1)\frac{d}{dT}$ and we may rewrite (7.31) as

$$\left(D^{k-1} D \log f(T)\right)_{|T=0} = (b^k - a^k)\zeta(1-k) \quad (k = 2, 4, 6, \ldots). \tag{7.32}$$

For $\pi_n = \xi_n - 1$ where $\xi = (\xi_n)_n \, \epsilon \, \mathbb{Z}_p(1)$ is a generator, i.e., a norm-compatible system of p^nth roots of unity, the elements $f(\pi_n) =: c_n(a,b)$ are so called *cyclotomic units* whose importance with respect to special ζ-values were discovered already by Kummer: the values of the Riemann zeta function at the odd negative integers arise as the higher logarithmic derivatives (seeing the uniformizing element as the variable) of them. They form a norm-compatible system $\mathbf{c}(a,b) = (c_n(a,b))$ in the inverse limit $\mathbb{U}(K_\infty) := \varprojlim_n o_{K_n}^\times$ of local units with respect to the norm maps, where $K_n := \mathbb{Q}_p(\xi_n)$ runs through the p-cyclotomic tower. In fact, $f(T) = g_{\mathbf{c}(a,b)}(T)$ is the Coleman power series attached to $\mathbf{c}(a,b)$ with regard to ξ.

This observation led Coates and Wiles to the general definition of the higher logarithmic derivative homomorphism for each $k \geq 1$

$$\delta_k : \mathbb{U}(K_\infty) \to \mathbb{Z}_p$$
$$\mathbf{u} \mapsto \left(D^{k-1} D \log g_{\mathbf{u}}(T) \right)_{|T=0}.$$

In this terminology (7.32) now becomes

$$\delta_k(\mathbf{c}(a,b)) = (b^k - a^k)\zeta(1-k) \quad (k = 2,4,6,\ldots) \tag{7.33}$$

relating *cyclotomic units* to ζ-values.

7.1.2. *p-adic L-functions.* We will now extend this relation to the Kubota-Leopoldt p-adic ζ-function. More precisely, for $k \geq 1$, there is a commutative diagram ([27, 3.5.2]

$$\begin{array}{ccc} \mathbb{U}(K_\infty) & \xrightarrow{Col} & \mathbb{Z}_p[[\Gamma]] \\ \downarrow{\scriptstyle \delta_k} & & \downarrow{\scriptstyle \chi^k_{cyc}} \\ \mathbb{Z}_p & \xrightarrow{1-p^{k-1}} & \mathbb{Z}_p \end{array}$$

with Col being the composite of

$$\mathbb{U}(K_\infty) \to \mathbb{Z}_p[[T]]^{\psi=0}, \mathbf{u} \mapsto \frac{1}{p} \log \left(\frac{g_{\mathbf{u}}^p}{\varphi(g_{\mathbf{u}})} \right),$$

with the inverse of the Mellin transform

$$\mathfrak{M} : \ \mathbb{Z}_p[[\Gamma]] \xrightarrow{\cong} \mathbb{Z}_p[[T]]^{\psi=0}, \lambda \mapsto \lambda \cdot (1+T) \quad \text{with } \gamma \cdot (1+T) = (1+T)^{\chi_{cyc}(\gamma)},$$

and where the integration map $\chi^k_{cyc} : \mathbb{Z}_p[[\Gamma]] \to \mathbb{Z}_p$ sends γ to $\chi^k_{cyc}(\gamma)$. We shall also write $\lambda(\chi) := \chi(\lambda)$. We obtain that $Col(\mathbf{c}(a,b))$ is the measure satisfying

$$Col(\mathbf{c}(a,b))(\chi^k_{cyc}) = (b^k - a^k)(1-p^{k-1})\zeta(1-k) \quad (k = 2,4,6,\ldots),$$

i.e., the Kubota-Leopoldt p-adic ζ-function is nothing else than

$$\frac{Col(\mathbf{c}(a,b))}{\sigma_b - \sigma_a} \tag{7.34}$$

where $\chi_{cyc}(\sigma_a) = a$.

Col is the prototype of a regulator map! Indeed, the *Perrin-Riou regulator map* for crystalline representations V defined by Lei-Loeffler-Zerbes in [74, 73]

$$H^1_{Iw}(V) \xrightarrow{\mathcal{L}_{V,1}} D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(V)$$

fits into the following commutative diagram, in case $V = \mathbb{Q}_p(1)$,

$$\begin{array}{ccc} \mathbb{U}(K_\infty) & \xrightarrow{\qquad Col \qquad} & \mathbb{Z}_p[[\Gamma]] \\ {\scriptstyle Kummer}\downarrow & & \downarrow{\scriptstyle incl \otimes e_1} \\ H^1_{Iw}(\mathbb{Q}_p(1)) & \xrightarrow{\ell_0^{-1}\mathcal{L}_{\mathbb{Q}_p(1),1}} & D(\Gamma, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_{cris}(\mathbb{Q}_p(1)) \end{array} \tag{7.35}$$

which is a immediate consequence of Theorem 12 and the definitions.[13] Here ℓ_0 denotes the element in the locally $\mathbb{Q}_p$-analytic distribution algebra $D(\Gamma, \mathbb{Q}_p)$ of Γ induced by the element $\nabla \ \epsilon \ Lie(\Gamma)$

[13] Compare with the diagram in [83, Appendix C] and with [110, (4) and p. 2402]

according to [101, §2.3], see also [83, Def. 3.3.3]. Moreover, $e_1 = t^{-1} \otimes t_1$ denotes the natural $\mathbb{Q}_p$-basis of $D_{cris}(\mathbb{Q}_p(1))$ with $t \in B_{cris}$ Fontaine's p-adic period and t_1 a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1)$.

7.2. **Elliptic units I.**

7.2.1. *Special L-values.* Now we consider the setting of [30] and let K be an imaginary quadratic field with class number 1, and $\mathcal{O}$ the ring of integers of K. Let E be an elliptic curve defined over K, with complex multiplication by $\mathcal{O}$, and let ψ be the Grössencharacter of E over K. Choose $p > 3$ to be a rational prime, at which E has good reduction at all primes of K above p, and which splits in K, say into $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. We put $\pi = \psi(\mathfrak{p})$, so that π is a generator of $\mathfrak{p}$. Now, as E has complex multiplication by $\mathcal{O}$, we can also view π as an endomorphism of E. For each $n \geq 1$, , let E_{π^n} be the kernel of the endomorphism π^n of E. We set $F_n = K(E_{\pi^n})$ and $F_\infty = \bigcup_n F_n$. Let U_n be the local principal units, i.e., which are congruent $1 \mod \mathfrak{p}_n$, of the completion of F_n at the uniqe prime $\mathfrak{p}_n$ above $\mathfrak{p}$.

Let $\mathcal{L}$ be the period lattice of the Weierstrass $\wp$-function associated with a Weierstrass model for E. Since K has class number 1, there exists $\Omega \in \mathcal{L}$, such that $\mathcal{L} = \Omega\mathcal{O}$. For each integer $k \geq 1$, let $L(\overline{\psi}^k, s)$ be the complex Hecke L-function of $\overline{\psi}^k$. It has been shown by Hurwitz, Birch and Swinnerton-Dyer, and Damerell that the numbers $\Omega^{-k}L(\overline{\psi}^k, k),\ k = 1, 2, \dots$, belong to K. We can therefore view these numbers as lying not only in the complex field, but also in the completion $K_\mathfrak{p}$ of K at the non-archimedean prime $\mathfrak{p}$.

Let $\kappa : \Gamma \to \mathbb{Z}_p^\times$ be the canonical character giving the action of $\Gamma = G(F_\infty/K)$ on the Tate module $T_\pi E$. Write $\mathfrak{f}$ for the conductor of ψ and fix a generator $f \in K$ of it. For each integer $k \geq 1$, write

$$\mu_k = 12(-1)^{k-1}(k-1)! \left(\frac{\Omega}{f}\right)^{-k}.$$

We consider pairs $\mathfrak{s} = (\{\mathfrak{a}_j \mid j \in J\}, \{n_j \mid j \in J\})$ of sets consisting of integral ideals $\mathfrak{a}_j$ prime to $6\mathfrak{f}\mathfrak{p}$ and integers n_j, respectively, over an arbitrary finite index set J satisfying $\sum_J n_j(N\mathfrak{a}_j - 1) = 0$ for the absolute norms $N\mathfrak{a}_j$. Attached to such $\mathfrak{s}$ comes

$$\Theta(z, \mathfrak{s}) := \prod_J \Theta(z, \mathfrak{a}_j)^{n_j},$$

where $\Theta(z, \mathfrak{a})$ is an elliptic function for the lattice $\mathcal{L}$ defined in (loc. cit.), and for each integer $k \geq 0$

$$h_k(\mathfrak{s}) := \sum_J n_j(N\mathfrak{a}_j - \psi^k(\mathfrak{a}_J)).$$

Moreover,

$$\Lambda(z, \mathfrak{s}) := \prod_J \Lambda(z, \mathfrak{a}_j)^{n_j}$$

with

$$\Lambda(z, \mathfrak{a}) := \prod_{\mathfrak{b} \in B} \Theta(z + \psi(\mathfrak{b})\frac{\Omega}{f}, \mathfrak{a}),$$

where B denotes a set of representatives, which are integral and prime to $\mathfrak{f}$, of the ray class group of K modulo $\mathfrak{f}$.

The Bernoulli numbers $\mathcal{B}_n$ from our previous example are now replaced by the Eisenstein numbers $\mathcal{C}_k(\mathfrak{a})$, i.e., the coefficients of the Eisenstein series attached to the logarithmic derivative

$$g(z) := \frac{d}{dz} \log \Lambda(z, \mathfrak{s}) = \sum_{n=1}^{\infty} \mathcal{C}_k(\mathfrak{s}) z^{n-1} \tag{7.36}$$

and its relation to the L-values

$$\mathcal{C}_k(\mathfrak{s}) = 12(-1)^{k-1}\left(\frac{\Omega}{f}\right)^{-k} h_k(\mathfrak{s})L(\overline{\psi}^k, k) \quad (k \geq 1). \tag{7.37}$$

We thus obtain

$$\left(\frac{d}{dz}\right)^{k-1} g(z)_{|z=0} = (k-1)!\mathcal{C}_k(\mathfrak{s}) = \mu_k h_k(\mathfrak{s})L(\overline{\psi}^k, k) \quad (k \geq 1). \tag{7.38}$$

Using the transformation

$$\Lambda(z, \mathfrak{s}) = R(T, \mathfrak{s}) \quad \text{with } T = \exp_{\hat{E}}(z) \text{ and } z = \log_{\hat{E}}(T)$$

the exponential and logarithm, respectively, map of the formal Lubin-Tate group $\hat{E}$, the differentiation $\frac{d}{dz}$ corresponds to $D := \frac{1}{\log'_{\hat{E}}}\frac{d}{dT}$ and we may rewrite (7.38) as

$$\left(D^{k-1} D \log R(T, \mathfrak{s})\right)_{|T=0} = \mu_k h_k(\mathfrak{s})L(\overline{\psi}^k, k) \quad (k \geq 1). \tag{7.39}$$

For $u = (u_n)_n \in T_\pi \hat{E}$ a basis as $\mathbb{Z}_p$-module and $\tau_n \in \mathbb{C}/\mathcal{L}$ with

$$\exp_{\hat{E}}(\tau_n) = -2\frac{\wp(\tau_n)}{\wp'(\tau_n)} = -2\frac{x(\tau_n)}{y(\tau_n)} = u_n,$$

the elements $\Lambda(\tau_n, \mathfrak{s}) =: e_n(\mathfrak{s})$ are so-called *elliptic units* (due to Robert (1973)), whose importance with respect to special L-values is the following: the values of the Hecke L-function at the positive integers arise as the higher logarithmic derivatives (seeing the uniformizing element as the variable) of them. They form a norm-compatible system $\mathbf{e}(\mathfrak{s}) = (e_n(\mathfrak{s}))$ of global units in F_n inducing also norm-compatible elements in the inverse limit $\mathbb{U}(L_\infty) := \varprojlim_n U_n$ of principal local units of L_n with respect to the norm maps, where $L_n := (F_n)_{\mathfrak{p}_n} = \mathbb{Q}_p(\hat{E}[\pi_n])$ runs through the Lubin-Tate tower attached to $\hat{E}$. In fact, $R(T, \mathfrak{s}) = g_{\mathbf{e}(\mathfrak{s})}(T)$ is the Coleman power series with coefficients in $\mathbb{Z}_p$ attached to $\mathbf{e}(\mathfrak{s})$ with regard to u :

$$R(u_n, \mathfrak{s}) = \Lambda(\tau_n, \mathfrak{s})$$

by [30, Theorem 5 and (16)].

Again Coates and Wiles considered a higher logarithmic derivative homomorphism for each $k \geq 1$:

$$\delta_k : \mathbb{U}(L_\infty) \to \mathbb{Z}_p,$$
$$\mathbf{u} \mapsto \left(D^{k-1} D \log g_{\mathbf{u}}(T)\right)_{|T=0}.$$

In this terminology (7.39) now becomes

$$\delta_k(\mathbf{e}(\mathfrak{s})) = \mu_k h_k(\mathfrak{s})L(\overline{\psi}^k, k) \quad (k \geq 1) \tag{7.40}$$

relating *elliptic units* to Hecke L-values, see [30, (21)].

This relation between elliptic units and special L-values was important in the works of Coates and Wiles ([29]) on Birch and Swinnerton-Dyer conjectures, see also [43].

7.2.2. *p-adic L-functions.* We will now extend this relation to the p-adic L-function. We fix an isomorphism

$$\eta : \hat{\mathbb{G}}_m \to \hat{E}; \quad T = \eta(S) = \Omega_p S + \cdots \,\epsilon\, \widehat{\mathbb{Z}_p^{nr}}[[S]].$$

Then, for $k \geq 1$, there is a commutative diagram ([43, I§3.5 (11)]

$$\begin{array}{ccc} \mathbb{U}(L_\infty) & \xrightarrow{Col} & \widehat{\mathbb{Z}_p^{nr}}[[\Gamma]] \\ \downarrow \delta_k & & \downarrow \kappa^k \\ \mathbb{Z}_p & \xrightarrow{\Omega_p^k(1-\frac{\pi^k}{p})} & \widehat{\mathbb{Z}_p^{nr}} \end{array}$$

with Col being now the composite of

$$\mathbb{U}(L_\infty) \to \mathbb{Z}_p[[T]]^{\psi_{\hat{E}}=0}, \mathbf{u} \mapsto \frac{1}{p}\log\left(\frac{g_{\mathbf{u}}^p}{\varphi_{\hat{E}}(g_{\mathbf{u}})}\right),$$

$$\mathbb{Z}_p[[T]]^{\psi_{\hat{E}}=0} \to \widehat{\mathbb{Z}_p^{nr}}[[S]]^{\psi_{\hat{\mathbb{G}}_m}=0}, T \mapsto \eta(S),$$

and the inverse of the Mellin transform

$$\mathfrak{M} : \widehat{\mathbb{Z}_p^{nr}}[[\Gamma]] \xrightarrow{\cong} \widehat{\mathbb{Z}_p^{nr}}[[S]]^{\psi_{\hat{\mathbb{G}}_m}=0}, \lambda \mapsto \lambda \cdot (1+T) \quad \text{with } \gamma \cdot (1+T) = (1+T)^{\kappa(\gamma)},$$

and where the integration map $\kappa^k : \widehat{\mathbb{Z}_p^{nr}}[[\Gamma]] \to \widehat{\mathbb{Z}_p^{nr}}$ sends γ to $\kappa^k(\gamma)$. We shall also write $\lambda(\kappa^k) := \kappa^k(\lambda)$.

We obtain that $Col(\mathbf{e}(\mathfrak{s}))$ is the measure satisfying

$$Col(\mathbf{e}(\mathfrak{s}))(\kappa^k) = \Omega_p^k(1 - \frac{\psi(\mathfrak{p})^k}{N\mathfrak{p}})\mu_k h_k(\mathfrak{s}) L(\overline{\psi}^k, k) \quad (k \geq 1),$$

i.e., the p-adic L-function is nothing else than

$$\frac{Col(\mathbf{e}(\mathfrak{s}))}{h(\mathfrak{s})} \tag{7.41}$$

where $h(\mathfrak{s})$ is the measure which interpolates the factor $h_k(\mathfrak{s})$ (its existence follows from the discussion before [30, Thm. 18]).

7.3. **Elliptic Units II.** Now we consider again the general Lubin-Tate situation as in section 6.3.

Setting as before $\mathbb{U} := \mathbb{U}(L_\infty) := \varprojlim_n o_{L_n}^\times$ with transition maps given by the norm we are looking for a map

$$\mathcal{L} : \mathbb{U} \otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L, \mathbb{C}_p) \otimes_L D_{cris,L}(L(\tau))$$

such that

$$\frac{\Omega^r}{r!} \frac{1 - \pi_L^{-r}}{1 - \frac{\pi_L^r}{q}} \mathcal{L}(u \otimes a\eta^*)(\chi_{LT}^r) \otimes (t_{LT}^{r-1} \otimes \eta^{\otimes -r+1}) = CW(u \otimes a\eta^{\otimes -r}) \tag{7.42}$$

for all $r \geq 1, u \,\epsilon\, \mathbb{U}, a \,\epsilon\, o_L$, where CW denotes the diagonal map in Corollary 7. We set $\mathcal{L} = \mathfrak{L} \otimes \mathbf{d}_1$ with $\mathfrak{L}$ given as follows

$$\mathfrak{L} : \mathbb{U} \otimes T_\pi^* \xrightarrow{\nabla} o_L[[\omega_{LT}]]^{\psi_L=1} \xrightarrow{(1-\frac{\pi_L}{q}\varphi)} \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \xrightarrow{\log_{LT}\cdot} \mathcal{O}_{\mathbb{C}_p}(\mathbf{B})^{\psi_L=0} \xrightarrow{\mathfrak{M}^{-1}} D(\Gamma_L, \mathbb{C}_p),$$

where the map ∇ has been defined before Theorem 12 as the homomorphism

$$\begin{aligned} \nabla : \mathbb{U} \otimes_{\mathbb{Z}} T^* &\longrightarrow o_L[[\omega_{LT}]]^{\psi=1} \\ u \otimes a\eta^* &\longmapsto a\frac{\partial_{\mathrm{inv}}(g_{u,\eta})}{g_{u,\eta}}(\omega_{LT}) \,. \end{aligned}$$

Note that due to the multiplication by $\log_{LT}$ the maps $\mathcal{L},\ \mathfrak{L}$ are not Γ_L-equivariant. In [103, §5.1.1] it is shown that indeed $\mathcal{L}$ satisfies (7.42) and that

$$\text{(7.43)}\quad \mathbb{U}\otimes_{\mathbb{Z}} T_\pi^* \xrightarrow{-\kappa\otimes T_\pi^*} H^1_{Iw}(L_\infty/L, o_L(\tau)) \xrightarrow{\mathcal{L}_{L(\tau\chi_{LT})}\otimes o_L(\chi_{LT}^{-1})\otimes t_{LT}} D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau))$$

coincides with

$$\mathcal{L}:\ \mathbb{U}\otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau)).$$

We set $\mathfrak{l}_i := t_{LT}\partial_{\mathrm{inv}} - i$, $\partial_{\mathrm{inv}} = \frac{d}{dt_{LT}}$ and by abuse of notation we also write $\mathfrak{l}_i = \nabla_{\mathrm{Lie}} - i$ for the corresponding element in $D(\Gamma_L, K)$.

If one defines $\mathcal{L}_{L(\tau)}$ [14] as a twist of $\mathcal{L}_{L(\tau\chi_{LT})}$ by requiring the commutativity of the following diagram:

$$\begin{array}{ccc} H^1_{Iw}(L_\infty/L, o_L(\tau)) & \xrightarrow{\mathcal{L}_{L(\tau)}} & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau)) \\ \downarrow{\cong} & & \downarrow{\frac{\nabla_{\mathrm{Lie}} Tw_{\chi^{-1}}}{\Omega}\otimes t_{LT}^{-1}} \\ H^1_{Iw}(L_\infty/L, o_L(\tau\chi_{LT}))\otimes_{o_L} o_L(\chi_{LT}^{-1}) & \xrightarrow{\mathcal{L}_{L(\tau\chi_{LT})}\otimes o_L(\chi_{LT}^{-1})} & D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau\chi_{LT}))\otimes_L L(\chi_{LT}^{-1}), \end{array}$$

then

$$\mathcal{L}:\ \mathbb{U}\otimes_{\mathbb{Z}} T_\pi^* \to D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau))$$

also coincides with
(7.44)

$$\mathbb{U}\otimes_{\mathbb{Z}} T_\pi^* \xrightarrow{-\kappa\otimes T_\pi^*} H^1_{Iw}(L_\infty/L, o_L(\tau)) \xrightarrow{(\frac{1}{\Omega}\nabla_{\mathrm{Lie}} Tw_{\chi^{-1}}\otimes \mathrm{id})\circ\mathcal{L}_{L(\tau)}} D(\Gamma_L,\mathbb{C}_p)\otimes_L D_{cris,L}(L(\tau)).$$

Example 7.1. We refer the interested reader to §5 of [100] for an example of a CM-elliptic curve E with supersingular reduction at p in which they attach to a norm-compatible sequence of elliptic units $e(\mathfrak{a})$ (in the notation of [43, II 4.9]) a distribution $\mu(\mathfrak{a})\ \epsilon\ D(\Gamma_L, K)$ in [100, Prop. 5.2] satisfying a certain interpolation property with respect to the values of the attached (partial) Hecke-L-function. Without going into any detail concerning their setting and instead referring the reader to the notation in (loc. cit.) we just want to point out that up to twisting this distribution is the image of $\kappa(e(\mathfrak{a}))\otimes\eta^{-1}$ under the regulator map $\mathcal{L}_{L(\tau)}$:

$$\mathcal{L}_{L(\tau)}(\kappa(e(\mathfrak{a}))\otimes\eta^{-1}) = \Omega Tw_{\chi_{LT}}(\mu(\mathfrak{a}))\otimes \mathbf{d}_1.$$

Here, $L = \mathbf{K}_p = \mathbf{F}_\wp$ (in their notation) is the unique unramified extension of $\mathbb{Q}_p$ of degree 2, $\pi_L = p$, $q = p^2$, and the Lubin-Tate formal group is $\hat{E}_\wp$ while $K = \widehat{L_\infty}$.

Indeed, we have a commutative diagram

$$\text{(7.45)}\qquad \begin{array}{ccc} \mathbb{U} & \xrightarrow{Col} & D(\Gamma_L, K) \\ \downarrow{-\kappa(-)\otimes\eta^{-1}} & & \downarrow{\Omega Tw_{\chi_{LT}}\otimes\mathbf{d}_1} \\ H^1_{Iw}(L_\infty/L, o_L(\tau)) & \xrightarrow{\mathcal{L}_{L(\tau)}} & D(\Gamma_L, K)\otimes_L D_{cris,L}(L(\tau)), \end{array}$$

[14] Since the representation $L(\tau)$ does not satisfy the conditions for the definition of the regulator map at the beginning of chapter 5.1 in (loc. cit.) while $L(\tau\chi_{LT})$ does.

where the Coleman map Col is given as the composite in the upper line of the following commutative diagram

$$
\begin{array}{ccccccccc}
\mathbb{U} & \xrightarrow{\log g_-} & \mathcal{O}^{\psi_L=\frac{1}{\pi_L}} & \xrightarrow{1-\frac{\varphi_L}{p^2}} & \mathcal{O}^{\psi_L=0} & = & \mathcal{O}^{\psi_L=0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K) \\
\downarrow & & \downarrow{\scriptstyle \partial_{\mathrm{inv}}} & & \downarrow{\scriptstyle \partial_{\mathrm{inv}}} & & \downarrow{\scriptstyle \mathfrak{l}_0} & & \downarrow{\scriptstyle \nabla_{\mathrm{Lie}}} \\
\mathbb{U} \otimes T_p^* & \xrightarrow{\nabla} & \mathcal{O}^{\psi_L=1} & \xrightarrow{1-\frac{\pi_L}{q}\varphi_L} & \mathcal{O}^{\psi_L=0} & \xrightarrow{\log_{LT}\cdot} & \mathcal{O}^{\psi_L=0} & \xrightarrow{\mathfrak{M}^{-1}} & D(\Gamma_L, K),
\end{array}
\tag{7.46}
$$

in which the second line is just $\mathfrak{L}$. Then the commutativity of (7.45) follows by comparing (7.46) with (7.44). Finally, $Col(e(\mathfrak{a})) = \mu(\mathfrak{a})(= \mathfrak{M}^{-1}(g_{\mathfrak{a}}(Z))$ in their notation) holds by construction in (loc. cit.) upon noting that on $\mathcal{O}^{\psi_L=\frac{1}{\pi_L}}$ the operator $1 - \frac{\pi}{p^2}\varphi_L \circ \psi_L$, which is used implicitly to define $g_{\mathfrak{a}}(Z)(= (1 - \frac{\pi}{p^2}\varphi_L \circ \psi_L) \log Q_{\mathfrak{a}}(Z))$, equals $1 - \frac{\varphi_L}{p^2}$.

Example 7.2. Another application of an L-analytic regulator map is about to show up in the ongoing PhD project of Muhammad Manji [90] supervised by David Loeffler. Just like the Perrin-Riou regulator map above, one hopes that the L-analytic regulator map has global applications to Iwasawa theory, in particular that it can see p-adic L-functions which lie in the L-analytic distribution algebra. The below example takes again $L/\mathbb{Q}_p$ unramified quadratic, where one already sees rich new behaviour and encounters a lot of obstacles. He is considering a p-adic Galois representation V associated to an ordinary automorphic representation Π defined over the unitary group $GU(2, 1)$ with respect to the imaginary quadratic field $\mathbf{K}$, assuming p is inert in $\mathbf{K}$. As in the previous example we consider $L = \mathbf{K}_p$, $\pi_L = p$, $q = p^2$.

Consider the 2-variable Iwasawa algebra $\Lambda = \mathcal{O}_L[\![\mathcal{O}_L^\times]\!]$ and the L-analytic distribution algebra $\Lambda_\infty^L = D(\mathcal{O}_L^\times, K) \cong \Gamma(\mathscr{W}^L, \mathcal{O}_{\mathscr{W}^L})$, where $\mathscr{W}^L$ is the locus of L-analytic characters in $\mathrm{Spf}(\Lambda)^{\mathrm{rig}}$. This is the ring Manji expects the p-adic L-function for V to lie in, although it has not been constructed yet. For the lack of such a construction he takes $L_p^*(V) := \mathcal{L}_S(c^\Pi)$ where S is a 1-dimensional subquotient of V and c^Π is the Loeffler–Skinner–Zerbes Euler system for Π constructed in [82]. He must impose the condition that $S^*(1)$ is L-analytic. The explicit reciprocity law in this case would then be to show that $L_p^*(V)$ is really "the" p-adic L-function $L_p(V)$ for V. Moreover, one hopes that such $L_p(V)$ would be bounded, but the notion of boundedness in Λ_∞^L is not so well understood, see [4] regarding this unsolved problem.

There are some hitches in the theory, for example Iwasawa cohomology base changed to Λ_∞^L no longer compares with $\psi = 1$ invariants of the associated L-analytic (ϕ, Γ)-module when $L \neq \mathbb{Q}_p$, see [105, 106]. With some adjustments Manji can define a local condition at p which he uses to define a Selmer group $\widetilde{H}^i(V) \subset H^i_{Iw}(\mathbf{K}_{\Sigma_p}/\mathbf{K}, V) \otimes \Lambda_\infty^L$ where $\mathbf{K}_{\Sigma_p}$ denotes the maximal abelian extension of $\mathbf{K}$ unramified away from p. From this he is about to obtain a statement towards a main conjecture:

Theorem 20 ([90])**.** *Suppose V satisfies some technical conditions and that $L_p^* \neq 0$. Then,*

$$\mathrm{char}_{\Lambda_\infty^L}\left(\widetilde{H}^2(V)\right) \Big| \left(L_p^*(V)\right).$$

The full statement of an Iwasawa main conjecture in this context would be the following:

Conjecture. *Under (possibly relaxed) hypothesis,*

$$\mathrm{char}_{\Lambda_\infty^L}\left(\widetilde{H}^2(V)\right) = \left(L_p(V)\right).$$

7.4. **Kato's explicit reciprocity law.** Let us begin this subsection with a quotations, which in our opinion sets the stage very well.

KATO [61, Introduction], [64]:

> The explicit reciprocity law is classically a mysterious relation between Hilbert symbols and differential forms.
> The classical explicit reciprocity [....] is concerned with the explicit description of this homomorphism [λ_m (4.11)] by using differential forms.

In (one of the main applications of) [64] Kato replaces the tower K_n of number fields by a tower of (open) modular curves Y_n and he forms the inverse limit $\varprojlim_n K_2(Y_n)$ with respect to the norm maps instead of $\varprojlim_n K_1(K_n)$. Then he defines maps λ_m, which look like the following when specialized to this application

$$\lambda_m : \varprojlim_n K_2(Y_n) \to \varprojlim_n H^1(Y(m), \mathbb{Z}/p^n\mathbb{Z}(1)) \to H^1(\mathbb{Q}_p, V) \xrightarrow{\exp^*_{\mathbb{Q}_p,V}} M_2(X_m) \otimes \mathbb{Q}_p$$

with $V = H^1(Y(m)_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)(1)$ and $M_2(X_m)$ being the space of weight 2 modular forms on the compact modular curve X_m. These maps generalize (5.16), (4.10).

The explicit reciprocity laws Theorem 4.3.1/4 (and 6.1.9) in (loc. cit.) express these maps λ_m in terms of differential forms. They contain Theorem 4, Theorem 6 and Corollary 7 as special cases.

The Euler system in this case is given by Beilinson elements in K_2 of modular cuves, which are analytically related to $\lim_{s\to 0} s^{-1}L(f,s)$ for elliptic cusp forms f of weight two via regulator maps while via the maps λ_m to the values $L(f, k-1)$.

See [78] for a more detailed explanation.

7.5. **Euler systems and localisation at $\ell = p$.** In this subsection we sketch the general philosophy. Morally, Iwasawa cohomology groups are the groups where the local parts at p of Euler systems of a global p-adic representation V live. Perrin-Riou's regulator map $\mathcal{L}_V$ (often also called big logarithm map Log_V) sends them p-adic L-functions considered as measures or distributions.

In a diagram:

$$\begin{array}{ccccc} H^1_{Iw}(\mathbb{Q}, V) & \xrightarrow{loc_p} & H^1_{Iw}(\mathbb{Q}_p, V) & \xrightarrow{Log_V} & \text{Measures/Distributions} \\ p\text{-adic Euler systems} \mapsto & & & & p\text{-adic } L\text{-functions}\ . \end{array}$$

See [35, 37] for further references on this subject and the survey [13] for a further discussion and more examples of Euler systems.

Recently there have been new examples by Loeffler-Zerbes et. al. [66, 76, 75, 79, 82, 81, 80, 88, 87, 85, 86, 24] and Bertolini-Darmon et. al. [40, 42, 41, 39, 15, 18, 12, 16, 17, 20, 19] on generalized Kato classes and progress on the BSD-conjecture.

7.6. **Heegner points and localisation at $\ell \neq p$.** There is yet another class of statements which are referred to as "reciprocity laws" as was pointed out to us by David Loeffler: these are the reciprocity laws in the theory of level-raising congruences for Heegner points going back to Bertolini and Darmon's paper [14], and more recently back in fashion again with the work of Yifeng Liu et al. These "reciprocity laws" describe the localisation loc_ℓ of p-adic Euler systems at primes $\ell \neq p$, in contrast to the reciprocity laws for Euler systems described in subsection 7.5, which are about describing the localisation at $\ell = p$.

Without going into too much details the setting in (loc. cit.) is as follows: K denotes an imaginary quadratic field and K_∞ its anti-cyclotomic $\mathbb{Z}_p$-extension, i.e., the unique $\mathbb{Z}_p$-extension of K such that $c\gamma c = \gamma^{-1}$ for all $\gamma \in G_\infty = G(K_\infty/K) \cong \mathbb{Z}_p$ for complex conjugation c. Then $\Lambda = \mathbb{Z}_p[[G_\infty]]$

denotes the corresponding Iwasawa algebra. Attached to certain ordinary eigenforms f (on a certain finite graph $\mathcal{T}/\Gamma$ attached to a certain definite quaternion algebra B) they construct an element $\mathcal{L}_f \in \Lambda$ such that $L_p(f, K) := \mathcal{L}_f \mathcal{L}_f^{\iota}$ - with the involution ι on Λ sending γ to its inverse - denotes the anti-cyclotomic p-adic Rankin-L-function attached to f. Using Heegner points the authors construct elements $\kappa(\ell)$ in global Iwasawa cohomology groups $H^1_{Iw}(K_\infty, T_f/p^nT_f)$ indexed by so called n-admissible primes ℓ attached to f (necessarily prime to p), where $T_f \subseteq V_f$ denotes a Galois stable $\mathbb{Z}_p$-lattice in the Galois representation V_f attached to f. They also consider semi-local Iwasawa cohomology groups together with a decomposition of Λ-modules

$$H^1_{Iw}(K_{\infty,\ell}, T_f/p^nT_f) \cong H^1_{Iw,fin}(K_{\infty,\ell}, T_f/p^nT_f) \oplus H^1_{Iw,sing}(K_{\infty,\ell}, T_f/p^nT_f)$$

together with the projections v_ℓ and ∂_ℓ onto the first and second factor, respectively. Combined with the localisation map $loc_\ell : \ H^1_{Iw}(K_\infty, T_f/p^nT_f) \to H^1_{Iw}(K_{\infty,\ell}, T_f/p^nT_f)$, their first explicit reciprocity law then reads as follows:

Theorem 7.3 ([14, Thm. 4.1])**.** If ℓ denotes an n-admissible prime, then

$$\partial_\ell(loc_\ell(\kappa(\ell))) = \mathcal{L}_f \mod p^n$$

holds in $H^1_{Iw,sing}(K_{\infty,\ell}, T_f/p^nT_f) \cong \Lambda/p^n\Lambda$ (up to multiplication by elements of $\mathbb{Z}_p^\times$ and G_∞).

Instead of stating also their second explicit reciprocity law, Thm. 4.2 in (loc. cit.), we just state the following consequence from both, which is Cor. 4.3 in (loc. cit.):

Corollary 7.4. For all pairs of n-admissible primes ℓ_1, ℓ_2 attached to f, the equality

$$v_{\ell_1}(loc_{\ell_1}(\kappa(\ell_2)) = v_{\ell_2}(loc_{\ell_2}(\kappa(\ell_1))$$

holds in $H^1_{Iw,fin}(K_{\infty,\ell}, T_f/p^nT_f) \cong \Lambda/p^n\Lambda$ (up to multiplication by elements of $\mathbb{Z}_p^\times$ and G_∞).

Note that in contrast to most other generalisations these level-raising reciprocity laws do have some symmetry concerning the primes ℓ_1 and ℓ_2, which reminds one of the original quadratic reciprocity law.

We wonder, moreover, whether the authors' view of explicit reciprocity laws, expressed in their comment [*Both theorems are instances of explicit reciprocity laws relating these explicit cohomology classes to special values of L-functions ...*] near the beginning of section 4.1 in (loc. cit.), might offer valuable guidance in answering the historical question of how the name "explicit reciprocity law" got attached to statements like those in subsection 7.5: once the terminology was established for theorems describing localisations of Euler systems at $\ell \neq p$, the jump to also using the term in the $\ell = p$ case seems quite natural.

8. ε-ISOMORPHISMS

A survey of this kind would be incomplete if we would not at least mention that Perrin Riou's explicit reciprocity law has an immediate impact towards the ε-isomorphisms à la Kato [62], [63] or Fukaya-Kato [54]. Needless to say that such kind of application has been fascinating the author for a long time. Unfortunately, a thorough discussion of this topic is out of the scope of this article. For a survey on these local Tamagawa Number Conjectures, which can be considered as a form of local Iwasawa main conjectures (while the global TNC corresponds to global Iwasawa main conjectures), see [109], [110]. Here we just want to point out that the formulae of type 17 are used by Benois and Berger [9] (strictly speaking they need and use the formulas from [8] providing information on the integral level), the formulae of type 18 in [83], the formulae 19 by Nakamura [91] (in the cyclotomic case) and in [89] (in the Lubin-Tate setting) in order to show different cases of this conjecture. The proof of Kato's rank one case [110] is very much based on the map Col from section 7.1.2.

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