Deligne-Ribet's work on L-values

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The aim of this series of 3 talks is to report on

I Siegel's result on the *rationality* of the values $L(1-k,\,\varepsilon),\ k \geq 1$, of certain L-functions over totally real number fields to be defined later and

II the *congruences* among those values which have been shown by Deligne and Ribet and which amount to the existence of *p-adic L-functions.*

I. Rationality

In the first lecture we concentrate on topic I. The main idea of Siegel which actually goes back already to Klingen consists of considering certain Eisenstein series the constant term of which are exactly the L-values above. Then he applies the following principle:

Let $f(z)$ be an $SL_2(\mathbb{Z})$-Eisenstein series, $z \in \mathbb{H}$, of weight k with Fourier series

$$f(z) = \sum_{n=0}^{\infty} a_n\, q^n, \quad q = e^{2\pi i z}$$

and assume that $a_1, \cdots, a_k \in \mathbb{Q}$. Then it follows from the following result that also a_0 is rational.

Proposition 1.1 (Siegel)

There exist integers $c_{k,0}, \cdots, c_{k,r}$ with

$$r := r(k) := \dim_{\mathbb{C}} M_k(SL_2(\mathbb{Z})) = \begin{cases} \left[\frac{k}{12}\right] + 1 & \text{if } k \neq 2 \mod 12, \\ \left[\frac{k}{12}\right] & \text{otherwise,} \end{cases}$$

such that

$$c_{k,0}\, a_0 + c_{k,1}\, a_1 + \cdots + c_{k,r}\, a_r = 0$$

and

$$c_{k,0} \neq 0\,.$$

Proof. ([3, §5.2 Cor. 1], [12]) Let X be the compact Riemann surface

$$X = SL_2(\mathbb{Z})\backslash\mathbb{H} \cup \{\infty\} \underset{j}{\cong} \mathbb{P}^1$$

with meromorphic differentials $\Omega^1(X)$ and consider the $\mathbb{C}$-linear map

$$\begin{array}{rcl} M_k(SL_2(\mathbb{Z})) & \overset{\phi}{\hookrightarrow} & \Omega^1(X) \\ f & \mapsto & w(f) := f\,T_k(z)dz = \dfrac{1}{2\pi i}\; f\,T_k(q)\dfrac{dq}{q} \end{array}$$

where

$$T_k(z) := G_{14-k+12(r(k)-1)}\Delta^{-r} = c_{k,r}q^{-r} + \cdots + c_{k,0} + \cdots$$

is a weight $2-k$ modular function, holomorphic on $\mathbb{H}$, with integral Fourier coefficients $c_{k,r} \in \mathbb{Z}$. Here Δ is Ramanujan's Δ-function, a cusp form of weight 12, and G_k is the weight k-Eisensteinseries with constant Fourier coefficient 1. Note that for the weights actually showing up in the T_k's, the q-expansions of the G_i are integral. It is easy to show that the image of ϕ has the basis $\omega_m := j^m dj,\ m = 0, \cdots, r-1$, where $j = \frac{G_4^3}{\Delta} = q^{-1} + \ldots$ is the modular j-function. The constant term of $T_k f$ is obviously

$$c_{k,0}\, a_0 + \cdots + c_{k,r}\, a_r$$

which equals the coefficient of q^{-1} in $\omega(f) = \frac{1}{2\pi i}\, T_k f\, \frac{dq}{q}$. But each $\omega_m = j^m dj = \frac{1}{m+1}\, \frac{dj^{m+1}}{dq}\, dq$ has trivial coefficient in q^{-1}, thus the first claim follows. The fact that $c_{k,0} \neq 0$ is an explicit calculation concerning G_i and Δ, see [3]. □

Remark 1. Siegel's argument in [13] is slightly different: he shows that for the linearly independent modular forms $1, \gamma_1, \cdots, \gamma_{r(k)}$ of weight 0 and k the vectors $1_t, (\gamma_1)_t, \cdots, (\gamma_{r(k)})_t$ of their first $t+1$ Fourier coefficients are still linearly independent (for $t = r(k)+1$). From this fact he is able to express a_0 as a linear combination of $a_1, \cdots, a_{r(k)}$. If $f(z)$ is a weight k Eisenstein series of $\Gamma(N)$ he uses the fact that $f(z)$ satisfies some algebraic relation with coefficients in $SL_2(\mathbb{Z})$-Eisenstein series up to a certain bounded weight ℓ. For the linearly independent modular forms $1, \gamma_1^{k_1}, \cdots, \gamma_{r(k_1)}^{k_1}, \cdots, \gamma_1^{k_d}, \cdots, \gamma_{r(k_d)}^{k_d}$ (γ_i^j of weight j and $k_d = \ell$) he determines an upper bound for t, the minimal integer such that the vectors $(1)_t, \cdots, (\gamma_{r(k_d)}^{k_d})_t$ stay linearly independent. Thereby he again manages to express a_0 as a rational linear combination of the $a_1, \cdots, a_t$ (with bounded denominators).

Since the coefficients $c_{k,j}$ are computable, we obtain explicit numerical formulae for the first Fourier coefficients.

Example 1. The classical Eisenstein series

$$\begin{aligned} \mathbb{E}_k(z) &= \frac{(k-1)!}{2(2\pi i)^k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} (mz+n)^{-k}, \\ &= \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned}$$

with $\sigma_k(n) := \sum_{0<d|n} d^k$ gives

$$\zeta(1-k) = -\frac{2}{c_{k,0}} \sum_{j=1}^{r(k)} \sigma_{k-1}(n) c_{k,j} \in \mathbb{Q}$$

for even integers $k > 2$. Here ζ denotes the Riemann zeta-function.

We now fix some notation for the rest of the talks:

K	totally real number field,
$r = [K:Q]$	
$\mathcal{O} = \mathcal{O}_K$	its ring of integers
$\mathfrak{D} = (\mathcal{O}^*)^{-1}$	different
$d_K = \mathcal{N}(\mathfrak{D})$	discriminant
$\mathcal{N}: K \to \mathbb{Q}$	(or $K \otimes_Q R \to R$ for any $\mathbb{Q}$-algebra R) the norm map.
$\mathfrak{f} \subset \mathcal{O}$	conductor
$\hat{K}$	the ring of finite adeles of K

$\alpha \in K$ is called *totally positive*, $\alpha \gg 0$, if $\sigma(\alpha) > 0$ for all embeddings $\sigma: K \hookrightarrow \mathbb{R}$.

$\mathbb{H}_K = \{\tau \in K \otimes \mathbb{C} |\ \mathrm{Im}(\tau) \gg \mathcal{O}\} \cong \mathbb{H}_{\mathbb{Q}}^r$

$$\Gamma_{00}(\mathfrak{f}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) | \begin{array}{l} a, d \in 1+\mathfrak{f} \\ b \in \mathfrak{D}^{-1},\ c \in \mathfrak{f}\mathfrak{D} \end{array} \right\}.$$

Let $\mathbb{E}_k(\tau)$, $\tau \in \mathbb{H}_K$, be an $SL_2(\mathcal{O})$-Eisenstein series of weight k. Then its pullback $f = (\Delta^*\mathbb{E}_k)(z)$ along the diagonal

$$\begin{array}{ccc} \mathbb{H}_{\mathbb{Q}} & \overset{\Delta}{\hookrightarrow} & \mathbb{H}_K = \mathbb{H}_{\mathbb{Q}}^r \\ z & \mapsto & (z, \cdots, z) \end{array}$$

is an $SL_2(\mathbb{Z})$-Eisenstein series of weight kr with the same constant Fourier coefficient as $\mathbb{E}_k(\tau)$, thus one may apply Siegel's theorem.

Example 2. For $\mathfrak{a} \subset \mathcal{O}$ consider

$$\mathbb{E}_k(\tau, \mathfrak{a}) = \mathcal{N}(\mathfrak{a})^k \sum_{(\lambda,\mu)\in\mathfrak{a}\times\mathfrak{a}/\mathcal{O}^\times} \mathcal{N}(\lambda\tau + \mu)^{-k}, \text{ for even integer } k > 2,$$

which actually only depends on the class $[\mathfrak{a}] \in Cl_K$; here $\mathcal{N}(\lambda\tau + \mu) = \prod_{\sigma:K\hookrightarrow\mathbb{R}} (\lambda^\sigma \tau_\sigma + \mu^\sigma)$ with $\tau = ((\tau_\sigma)_\sigma)$ and $\mathcal{O}^\times$ acts diagonally on $\mathfrak{a} \times \mathfrak{a}$.

Then $\mathbb{E}_k(\tau) := \sum_{[\mathfrak{a}]\in Cl_K} \mathbb{E}_k(\tau, \mathfrak{a})$ restricted to $\mathbb{H}_{\mathbb{Q}}$ has q-expansion

$$(\Delta^*\mathbb{E}_k)(z) = \zeta_K(k) + \left(\frac{(2\pi i)^k}{(k-1)!}\right)^r d_K^{\frac{2k-1}{2}} \sum_{n=1}^{\infty} \sum_{\substack{0\ll\xi\in\mathfrak{D}^{-1} \\ \mathrm{Tr}(\xi)=n}} \sigma_{k-1}(\xi\mathfrak{D}) q^n$$

where $\zeta_K = \sum_{\mathfrak{b}\subset\mathcal{O}} \mathcal{N}(\mathfrak{b})^{-s}$, $Re(s) > 1$, denotes the Dedekind zeta-function of K and $\sigma_k(\mathfrak{b}) := \sum_{\mathfrak{a}|\mathfrak{b}} \mathcal{N}(\mathfrak{a})^k$.

Thus we obtain Siegel's formula

$$\left(\frac{\Gamma(k)}{(2\pi i)^k}\right)^r \frac{\zeta_K(k)}{d_K^{\frac{2k-1}{2}}} = -\sum_{j=1}^{kd} \frac{c_{kd,j}}{c_{kd,0}} \sum_{\substack{0\ll\xi\in\mathfrak{D}^{-1} \\ \mathrm{Tr}(\xi)=j}} \sigma_{k-1}(\xi\mathfrak{D}) \in \mathbb{Q}.$$

Remark 2. Siegel applies the same technique to (values of) partial ζ-functions, thereby showing the rationality of the L-values $L(1-k, \varepsilon)$ to be defined below.

Let I_0 (resp. $I_0(\mathfrak{f})$) denote the group of fractional ideals of K (prime to $\mathfrak{f}$). Also we write A_0 ($A_0(\mathfrak{f})$) for the submonoid of integral ideals in $\mathcal{O}$ (prime to $\mathfrak{f}$). Then $G_{\mathfrak{f}} := A_0(\mathfrak{f})/\sim_{\mathfrak{f}}$ is the strict ray class group, where $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b} \Longleftrightarrow \mathfrak{a}\mathfrak{b}^{-1} = (\alpha)$ for some $0 \ll \alpha \in 1 + \mathfrak{f}\mathfrak{b}^{-1}$.

For a function $\varepsilon : G_{\mathfrak{f}} \longrightarrow \mathbb{C}$ we set

$$L(s,\varepsilon) := \sum_{\mathfrak{a}\in A_0(\mathfrak{f})} \varepsilon(\mathfrak{a})\mathfrak{N}\mathfrak{a}^{-s} = \sum_{[\mathfrak{a}]=a\in G_{\mathfrak{f}}} \varepsilon(a)\zeta(\mathfrak{a},\mathfrak{f},s)$$

where

$$\zeta(\mathfrak{a},\mathfrak{f},s) = \sum_{\mathfrak{b}\in A_0(\mathfrak{f}),\mathfrak{b}\sim_{\mathfrak{f}}\mathfrak{a}} \mathcal{N}\mathfrak{b}^{-s}\,.$$

Then, setting $\mathfrak{b} := \mathfrak{a}\mathfrak{D}^{-1}\mathfrak{f}^{-1}$, for $k \geq 1$ Siegel constructs a Hilbert modular Eisenstein series $\mathbb{E}_k(\mathfrak{a},\mathfrak{f})(\tau)$ whose pullback $(\Delta^*\mathbb{E}_k(\mathfrak{a},\mathfrak{f}))(z)$ $(=\varphi(z)$ in [13, (98)]) is an elliptic modular form of weight kr

$$\left(\frac{\Gamma(k)}{(2\pi i)}^{k}\right)^{r} d_K^{\frac{2k-1}{2}}\mathcal{N}(\mathfrak{f})^{k-1}\mathcal{N}(\mathfrak{b})^{k}\left(\sum_{\substack{(x,\lambda)\in K^2/\mathcal{O}_+^{\times}\\ x\in\mathfrak{a}\mathfrak{D}^{-1},\lambda\in\mathfrak{b}}} e^{2\pi i \mathrm{Tr}\lambda}\mathcal{N}(xz+\lambda)^{-k}\mathcal{N}|xz+\lambda|^{-s}\right)_{|s=0},$$

where $\mathcal{O}_+^{\times} := \{\alpha\in\mathcal{O}^{\times}|\alpha = 1 \mod \mathfrak{f},\alpha \gg 0\}$, and has Fourier coefficients (loc. cit., (100), (101) and the last displayed formula on page 19)

$$\begin{aligned} a_0 &= \zeta(\mathfrak{a},\mathfrak{f},1-k)\ (k>2,\ \text{if } K=\mathbb{Q},\ \mathfrak{f}=1),\\ a_n &\in \mathbb{Q}. \end{aligned}$$

Thus the rationality of the partial zeta values follows. Moreover, if $\varepsilon : G_{\mathfrak{f}} \to \mathbb{Q}$ takes values in the rational numbers, we obtain

Theorem I. $L(1-k,\varepsilon)\in\mathbb{Q}$ *for all* $k\geq 1$ $(K\neq\mathbb{Q})$.

Furthermore, we can extend the definiton of $L(s,\varepsilon)$ to all functions $\varepsilon : G_{\mathfrak{f}} \to V$ with values in a $\mathbb{Q}$-vectorspace V, e.g. $\mathbb{Q}_p$, by setting

$$L(s,\varepsilon) := \sum_{a\in G_{\mathfrak{f}}} \zeta(\mathfrak{a},\mathfrak{f},s)\varepsilon(a)\in V\,.$$

II. Congruences

In a second step we are now going to discuss the type of congruences which Deligne-Ribet prove in order to show the existence of p-adic L-functions. To this end let

$$G = \varprojlim_{n} \; G_{\mathfrak{f}p^n}$$

be the strict ray class group of conductor $\mathfrak{f}p^\infty$, which by class field theory corresponds to the maximal abelian extension of K, which is unramified at all finite places v not dividing $\mathfrak{f}p$ (infinite places are allowed to ramify!). Let

$$\mathcal{N} : G \to \mathbb{Z}_p^\times$$

be the unique continuous character which is compatible with the usual norm map $\mathcal{N} : A_0(\mathfrak{f}p) \to \mathbb{Z}_p$ and the natural map $A_0(\mathfrak{f}p) \to G$. Note that $\mathcal{N}$ is actually nothing else than the p-cyclotomic character of K. For $\varepsilon : G_\mathfrak{f} \to \mathbb{Q}_p$ and $c \in G$ we set

$$\Delta_c(1-k, \varepsilon) := L(1-k, \varepsilon) - \mathcal{N}c^k L(1-k, \varepsilon_c) \in \mathbb{Q}_p$$

for $k \geq 1$, where $\varepsilon_c(g) := \varepsilon(\bar{c}g)$ if $\bar{c}$ denotes the image of c under the canonical projection $G \twoheadrightarrow G_f$. Note that for a character ε this is equal to

$$\left(1 - \mathcal{N}c^k \; \varepsilon(\bar{c})\right) \; L(1-k, \, \varepsilon)\,.$$

For a family $(\varepsilon_k : G_\mathfrak{f} \to \mathbb{Q}_p)_{k \geq 1}$ (almost all zero) we define a map φ on $A_0(\mathfrak{f})$ via

$$\varphi(\mathfrak{a}) = \sum_{k \geq 1} \; \varepsilon_k(\mathfrak{a}) \mathcal{N}\mathfrak{a}^{k-1} \in \mathbb{Q}_p\,.$$

Theorem II. *If $\varphi(\mathfrak{a}) \in \mathbb{Z}_p$ for all $\mathfrak{a} \in A_0(\mathfrak{f})$, then*

$$\Delta := \sum_{k \geq 1} \; \Delta_c(1-k, \; \varepsilon_k) \in \mathbb{Z}_p \quad \textit{for all} \quad c \in G\,.$$

These kinds of congruences were (axiomatically) first introduced by Coates [1]. In the case $K = \mathbb{Q}$ they are nothing else than the "generalized Kummer congruences" as considered by Mazur and others and they can be rather easily proved from the fact that over $\mathbb{Q}$ the values $L(1-k, \varepsilon)$ for $k \geq 1$ can be expressed in terms of Bernoulli polynomials; see [2, §1]. Thus, from now on we shall concentrate on the case $K \neq \mathbb{Q}$.

For the purpose of p-adic L-functions we now assume that all primes lying above p divide the conductor $\mathfrak{f} \subset \mathcal{O}$. Note that a locally constant function $\varepsilon : G \to \mathbb{Q}_p$ factorizes over $G_{\mathfrak{f}'}$ for some $\mathfrak{f}|\mathfrak{f}'|(\mathfrak{f}p)^n$, $n \geq 1$, which allows us to define $\Delta_c(1-k,\varepsilon)$ (independently of the choice of $\mathfrak{f}'$, as $A_0(\mathfrak{f}')=A_0(\mathfrak{f})$ for any such $\mathfrak{f}'$). Thus we obtain a distribution for any $k\geq 1,\ c\in G$,

$$\begin{aligned}\mu_{c,k}:\{\varepsilon:G\to\mathbb{Q}_p \text{ locally constant}\} &\longrightarrow \mathbb{Q}_p\\ \varepsilon &\longmapsto \Delta_c(1-k,\varepsilon).\end{aligned}$$

Setting $\varepsilon_k:=\varepsilon:G\to\mathbb{Z}_p$ (locally constant) and $\varepsilon_i:=0$ for $i\neq k$, it follows from Theorem II that $\mu_{c,k}(\varepsilon)\in\mathbb{Z}_p$, i.e., the distribution $\mu_{c,k}$ is bounded or in other words:

Theorem III. *$\mu_{c,k}$ is a measure on G with values in $\mathbb{Z}_p$. Moreover, $\mu_{c,k}=\mathcal{N}^{k-1}\cdot\mu_{c,1}$.*

Proof. It suffices to show for $k\geq 1$ and $n\geq 1$ that

$$\int \varepsilon\mathcal{N}^{k-1}d\mu_{c,1}\equiv\Delta_c(1-k,\varepsilon) \bmod p^n\,\mathbb{Z}_p$$

for each locally constant $\varepsilon:G\to\mathbb{Z}_p$. For this, let $\eta:G\to\mathbb{Z}_p$ be a locally constant function such that

$$\eta\equiv\mathcal{N}^{k-1}\text{mod } p^n.$$

Taking $\varepsilon_1:=\varepsilon_\eta,\varepsilon_k:=-\varepsilon$, and $\varepsilon_i:=0$ for $i\neq 1,k$, the function $\varphi=\varepsilon\eta-\varepsilon\mathcal{N}^{k-1}$ takes values in $p^n\mathbb{Z}_p$, and Theorem II implies:

$$\Delta_c(0,\varepsilon\eta)-\Delta_c(1-k,\varepsilon)\in p^n\mathbb{Z}_p,$$

which gives

$$\Delta_c(1-k,\varepsilon)\equiv\Delta_c(0,\varepsilon\eta)=\int\varepsilon\eta d\mu_{c,1}\equiv\int\varepsilon\mathcal{N}^{k-1}d\mu_{c,1}\,\mathrm{mod}\,p^n\mathbb{Z}_p.$$

□

Setting $\lambda_c:=\mathcal{N}^{-1}\mu_{c,1}\in\mathbb{Z}_p[[G]]$ for $c\in G$ we obtain

$$\int\mathcal{N}^k\varepsilon d\lambda_c=\Delta_c(1-k,\varepsilon).$$

Then $\lambda:=\frac{1}{1-c}\lambda_c$ in the total quotient ring of $\mathbb{Z}_p[[G]]$ is a *pseudo-measure* à la Serre [11] for $1-c$ a non-zero divisor in $\mathbb{Z}_p[[G]]$:

$$(1-c')\lambda=\lambda_{c'}\in\mathbb{Z}_p[[G]]\,\text{for all}\,c'\in G.$$

Also, it can be deduced that for $k\geq 1$

$$\int \mathcal{N}^k \varepsilon d\lambda = L(1-k,\varepsilon).$$

Indeed, if ε belongs to the set of continuous characters X_G of G (with values in $\mathbb{C}_p^\times$) we calculate

$$\int \mathcal{N}^k \varepsilon d\lambda = \frac{\int \mathcal{N}^k \varepsilon d\lambda_c}{1-\mathcal{N}c^k\varepsilon(c)} = \frac{L(1-k,\varepsilon)-\mathcal{N}c^k L(1-k,\varepsilon_c)}{1-\mathcal{N}c^k\varepsilon(c)} = L(1-k,\varepsilon),$$

where we choose a c which does not lie in the kernel of $\mathcal{N}^k\varepsilon$.
Setting

$$\begin{array}{rrcl} \mathcal{L}=\mathcal{L}_{\mathfrak{f}}: & X_G\setminus\{\mathbb{1}\} & \longrightarrow & \mathbb{C}_p^\times \\ & \varphi & \longmapsto & \int \varphi d\lambda \end{array}$$

we get the p-adic zeta-function of K as

$$\zeta_{K,p}(s) = \mathcal{L}(<\chi_K>^{1-s}),\ s\in\mathbb{Z}_p\setminus\{1\},$$

satisfying

$$\begin{array}{lll} \zeta_{K,p}(1-k) & = \mathcal{L}(\omega_K^{-k}\chi_K^k) & = L_{\mathfrak{f}}(1-k,\omega_K^{-k}) \\ & & = \zeta_{K,\mathfrak{f}}(1-k) \qquad \text{if } \ k|\#\omega_K. \end{array}$$

Here χ_K denotes the p-cyclotomic character, ω_K the Teichmüller character of K and $<>$ the projection from $\mathbb{Z}_p^\times$ to $1+p\mathbb{Z}_p$. More generally, the p-adic L-function for $\phi\in X_G$ arises as

$$L_{p,K}(s,\phi) := \mathcal{L}(\phi <\chi_K>^{1-s})$$

with

$$L_{p,K}(1-k,\phi) = L_{\mathfrak{f}}(1-k,\phi\omega_K^{-k}),$$

where $L_{\mathfrak{f}}\big(s,\phi\,\omega_K^{-k}\big)$ is the Artin L-function attached to $\phi\,\omega_K^{-k}$ without the Euler factors of the places dividing $\mathfrak{f}$.

Consider the following decomposition and projection

$$\alpha: G\cong A\times\Gamma \twoheadrightarrow \Gamma \xrightarrow{\sim} \mathbb{Z}_p,$$

$\alpha(\gamma)=1$ for a fixed topological generator γ of Γ, where $A = ker(<>)$ where we simply write $<>$ also for the composite $<>\circ\mathcal{N}$. For the convenience of the reader we recall the following well-known

Proposition 2.1([10, (4.8)-(4.10)]) Let $\varepsilon : G \to \mathbb{Z}_p[\varepsilon]^\times$ be an (even) character of finite order.

(i) Then, for each c, there exist unique $H'_{\varepsilon,c}, G'_{\varepsilon,c} \in \mathbb{Z}_p[\varepsilon][[X]]$ such that

$$\begin{array}{ll} \int <x>^{1-s}\varepsilon(x)d\lambda_c(x) & = G'_{\varepsilon,c}(<\gamma>^{1-s}-1), \\ 1-\varepsilon(c)<c>^{1-s} & = H'_{\varepsilon,c}(<\gamma>^{1-s}-1). \end{array}$$

(ii) If $\varepsilon_{|A} \not\equiv 1\!\!1$, then

$$L_{p,K}(s,\varepsilon) = F_\varepsilon\left(<\gamma>^{1-s}-1\right)$$

for some $F_\varepsilon \in \mathbb{Z}_p[\varepsilon][[X]]$. Clearly, $F_\varepsilon = \frac{G'_{\varepsilon,c}}{H'_{\varepsilon,c}}$ for all c such that $H'_{\varepsilon,c} \neq 0$ even though $H'_{\varepsilon,c}$ need not be a unit.

(iii) If $\varepsilon_{|A} \not\equiv 1\!\!1$ and $\theta : G \to \mathbb{C}_p^\times$ is a character of finite order such that $\theta_{|A} = 1\!\!1$ is trivial, then

$$F_{\theta\varepsilon}(X) = F_\varepsilon(\zeta(1+X)-1) \ \text{ with } \ \zeta := \theta(\gamma) \in \mu_{p^\infty}.$$

Remark 3. Setting

$$H_\epsilon = \begin{cases} -H'_{\epsilon,\gamma}, & \text{if } \varepsilon_{|A} \equiv 1\!\!1, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$G_\epsilon = \begin{cases} -G_{\epsilon,\gamma}, & \text{if } \varepsilon_{|A} \equiv 1\!\!1, \\ F_\epsilon, & \text{otherwise,} \end{cases}$$

we obtain the unique power series in [14] such that

$$L_{p,K}(s,\varepsilon) = \frac{G_\varepsilon(<\gamma>^{1-s}-1)}{H_\varepsilon(<\gamma>^{1-s}-1)}.$$

Proof. (sketch) Note that for any $\psi \in X_G$ there exists a character ψ_A which is trivial on Γ and such that

$$\psi(g) = \psi_A(g)\ \psi(\gamma)^{\alpha(g)}$$

for all $g \in G$.

We will use the following well-known

Fact: Let μ be a measure on G. Then

$$\int \psi d\mu = F(\psi(\gamma) - 1)$$

where $F = F_{\psi_{A,\mu}} := \sum\limits_{n \geq 0} a_n X^n \in \mathbb{Z}_p[\psi][[X]]$ with

$$a_n = \int\limits_G \psi_A(x) \binom{\alpha(x)}{n} d\mu(x) \in \mathbb{Z}_p[\psi].$$

Taking $\psi =< >^{1-s}$ and $\mu = \varepsilon\lambda_c$ we have

$$\int < >^{1-s} \varepsilon d\lambda_c = F(< \gamma >^{1-s} -1).$$

Setting $G'_{\varepsilon,c} := F = F_{< >^{1-s}{}_A, \varepsilon\lambda_c}$ and $H'_{\varepsilon,c} := 1 - \varepsilon(c)(X+1)^{\alpha}$, if $< c >=< \gamma >^{\alpha}$ for $\alpha \in \mathbb{Z}_p$, we obtain (i).

(ii) By (4.3) in (loc. cit.) there exist a measure (not just pseudo-measure) μ' on G such that

$$L_{p,K}(s, \epsilon) = \int < >^{1-s} \varepsilon d\mu'$$

for all ϵ which are non-trivial on A. Now apply the above fact to the measure $\mu = \varepsilon\mu'$ and $\psi =< >^{1-s}$.

(iii) By definition we have

$$\begin{aligned} F_{\theta\varepsilon}(< \gamma >^{1-s} -1) &= \int < x >^{1-s} \theta\varepsilon d\mu' \\ &= F_{(< >^{1-s}\theta)_A, \varepsilon\mu'}(< \gamma >^{1-s} \theta(\gamma) - 1) \end{aligned}$$

by the above fact. But

$$F_{\varepsilon} = F_{< >_A^{1-s}, \varepsilon\mu'} = F_{(< >^{1-s}\theta)_A, \varepsilon\mu'}$$

because $\theta_A = 1\!\!1$ and the claim follows. $\square$

For the proof of Theorem II we need a q-expansion principle which we shall state now, but explain in more detail later.

For simplicity we assume $\mathfrak{f} = N$, a positive integer, i.e.,

$$\Gamma_{00}(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) | \begin{matrix} a, d \in 1 + N\mathcal{O} \\ b \in \mathfrak{D}^{-1}, c \in N\mathfrak{D} \end{matrix} \right\}$$

Let $F : \mathbb{H}_K \to \mathbb{C}$ be a Hilbert modular form of weight k with Fourier expansion

$$F_1 = a_0 + \sum_{\mu \in \mathcal{O},\, \mu \gg 0} a_\mu, \quad q^\mu = e^{2\pi i Tr(\mu\tau)}$$

at 1 and

$$F_\alpha$$

at other cusps α. Actually, the cusps we are interested in are all induced from finite ideles $\alpha \in \hat{K}^\times$ [2, (5.3)], which in turn induce elements $c(\alpha)$ in G (called $j(\alpha)$ just below [2, (2.22)]).

Perhaps the hardest part in [2] is the proof of the geometric irreducibility of the fibres of moduli space of Hilbert-Blumenthal Abelian Varieties (HBAV), which implies the following

q-expansion-principle (Rapoport, Deligne-Ribet)

I If the q-expansion coefficients of F at one cusp are all rational, then this is true for all cusps.

Now let $F_k, k \geq 0$, be forms of weight k, almost all zero, such that $F_{k,1} \in$ "$\mathbb{Q}[[q]]$" and set

$$S(\alpha) := \sum_{k \geq 0} \mathcal{N}\alpha_p^{-k} F_{k,\alpha} \in \mathbb{Q}_p[[q]],$$

which corresponds (only) to a p-adic modular form, as it mixes up different weights! Here α_p denotes the p-component of $\alpha \in \hat{K}^\times$.

II If $S(\alpha)$ has coefficients in $\mathbb{Z}_p$ for one α, then for all α.

III Assume that all non-constant coefficients of $S(\alpha)$ lie in $\mathbb{Z}_p$. Then this holds also for $S(\beta)$ at any other cusp β and

$$a_0(S(\alpha)) - a_0(S(\beta) \in \mathbb{Z}_p.$$

Now the idea of the proof of Theorem II consists of finding suitable Hilbert modular forms F_{k,ε_k} such that for all cusps α its Fourier expansion are essentially of the form

$$(F_{k,\varepsilon_k})_\alpha = \mathcal{N}c(\alpha)^k \mathcal{N}\alpha_p{}^k \Big\{ L(1-k, \varepsilon_{k,c(\alpha)}) + \sum_{\mu \gg 0} \left(\sum_{?} \varepsilon_{k,c(\alpha)} \mathcal{N}^{k-1}(\mu?) \right) q^\mu \Big\},$$

for $k \geq 1$ ($F_0 = 0$). Then, by the assumption of Theorem II all non-constant coefficients belong to $\mathbb{Z}_p$. Thus the q-expansion principle III implies that

$$\begin{aligned} \Delta = \sum_{k\geq 1} \Delta_{c(\alpha)}(1-k, \varepsilon_k) &= \sum_{k\geq 1} \big(L(1-k, \varepsilon_k) - \mathcal{N}c(\alpha)^k L(1-k, \varepsilon_{k,c(\alpha)}) \big) \\ &= a_0(S(1)) - a_0(S(\alpha)) \in \mathbb{Z}_p \end{aligned}$$

and Theorem II follows. □

The needed Eisenstein series F_{k,ε_k} for arbitrary locally constant functions ε (these are needed for the distribution property) are similar to those studied by Siegel (see page 5), but while for rationality it sufficed for Siegel to realise the special L-values as constant terms of elliptic modular forms, for the purpose of congruences Deligne-Ribet had to find Hilbert modular forms with the corresponding L-values as constant terms; they are constructed in [2, §6] in an adelic setting! Also we should note that for $k = 1$ such $F_{1,\varepsilon}$ apparently does not exist and Deligne and Ribet have to circumvent this problem, which leads to some technical problems which we do not want to discuss here.

The rest of this survey is devoted to explain the q-expansion principle in a little more detail.

II.1. **The Moduli space.** References for this paragraph concerning the moduli problem of classifying abelian varieties with real multiplication are [9], [8, §1], [2, §4].

Recall that a Hilbert-Blumenthal abelian variety (HBAV) relative to $\mathcal{O}$ over a base S is an abelian scheme X/S, furnished with a homomorphism $m : \mathcal{O} \hookrightarrow \mathrm{End}(X)$ making its Lie algebra $Lie(X/S)$ into a locally free $\mathcal{O} \otimes \mathcal{O}_S$ -module of rank 1, in particular the relative dimension of X over S equals $r = [K : \mathbb{Q}]$. Fixing a polarization module $\mathfrak{c} \subset K$, and considering for simplicity only the case $S = \mathrm{Spec}(R)$ for any ring R, a $\mathfrak{c}$-polarization of X/S is a (positive) isomorphism

$$\lambda : \mathcal{P}(X) := \mathrm{Hom}_{\mathcal{O}}(X, X^t) \xrightarrow{\sim} \mathfrak{c}$$

of $\mathcal{O}$-modules, where X^t denotes the dual abelian variety of X.

Finally, for $N \geq 1$ a $\Gamma_{00}(N)$-structure of X is an $\mathcal{O}$-linear immersion $\mathcal{O} \otimes \mu_{N/S} \hookrightarrow X$. Now consider the moduli problem, given by the functor:

$$\begin{array}{rccl} F(\mathfrak{c}, N): & \text{Schemes} & \longrightarrow & \text{Sets} \\ & S & \longmapsto & \left\{ \begin{array}{l} \text{isomorphism classes of triples} \\ (X, \lambda, i) \text{ with } X \text{ a HBAV } /S, \\ \lambda \text{ a } \mathfrak{c}-\text{polarization}, \\ i \text{ a } \Gamma_{00}(N)\text{-structure} \end{array} \right\}. \end{array}$$

The moduli stack $\mathfrak{M}(\mathfrak{c}, N)$ defined by this functor is an algebraic stack, smooth of relative dimension r over $\mathbb{Z}$ and is *represented* by an algebraic space, also denoted $\mathfrak{M}(\mathfrak{c}, N)$, smooth of relative dimension r over $\mathbb{Z}$ for $N \geq 4$. In particular, there is a unique universal $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(N)$-structure

$$\begin{array}{ccc} \varphi^* \mathcal{X} & \longrightarrow & \mathcal{X} := (X_{\text{univ}}, \lambda_{\text{univ}}, i_{\text{univ}}) \\ \downarrow & & \downarrow \pi \\ S & \xrightarrow{\varphi} & \mathfrak{M}(\mathfrak{c}, N) \end{array}$$

and all elements in $F(\mathfrak{c}, N)(S)$ arise as pullback $\varphi^* \mathcal{X}$ for some $\varphi \in \text{Mor}(S, \mathfrak{M}(\mathfrak{c}, N))$. In particular, for a field L, the L-valued points

$$\mathfrak{M}(\mathfrak{c}, N)(L)$$

consists of $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(N)$-structure defined over L.

II.2. **Hilbert-modular forms (HMF).** Assume $N \geq 4$ and denote by $\underline{\omega}$ the sheaf $\pi_*(\Omega^1_{X_{\text{univ}}/\mathfrak{M}})$ on $\mathfrak{M} := \mathfrak{M}(\mathfrak{c}, N)$ and let

$$\underline{\omega}(k) := \bigotimes_{\sigma} \underline{\omega}(\sigma)^k$$

be $\underline{\omega}$ extended by the character $\mathcal{N}^k$, which is an invertible sheaf over $\mathfrak{M}_R := \mathfrak{M} \otimes R$, on which $(\mathcal{O} \otimes R)^\times$ acts by $\mathcal{N}^k$, see [5, §1.3] for details. Here we fix a ground ring R and for every R-algebra R' and as before $\mathcal{N}$ denotes the norm character $\mathcal{O} \otimes R' \to R'$. Then the $\mathfrak{c}$-HMFs of weight k on $\Gamma_{00}(N)$ defined over R are by definition the global sections

$$\mathcal{M}_k(\mathfrak{c}, N, R) := H^0\left(\mathfrak{M}(\mathfrak{c}, N)_R, \underline{\omega}(k)\right).$$

This can also be interpreted (and this makes even sense for $N \geq 1$) as a rule F which assigns to a quadruple

$$(X, \lambda, \omega, i)$$

over R' an element $F(X, \lambda, \omega, i) \in R'$, where

1. (X, λ) is a $\mathfrak{c}$-polarized HBAV over an R-algebra R',
2. ω is a nowhere vanishing differential on X and
3. i is a $\Gamma_{00}(N)$-structure

subject to the following conditions

(i) $F(X, \lambda, \omega, i)$ depends only on the R'-isomorphism-class of (X, λ, ω, i),
(ii) Formation of $F(X, \lambda, \omega, i)$ is compatible with any extension of scalars $R'' \to R'$ of R-algebras,
(iii) $F(X, \lambda, a^{-1}\omega, i) = \mathcal{N}(a)^k F(X, \lambda, \omega, i)$ for all $a \in (\mathcal{O} \otimes R')^\times$.

II.3. **q-expansions.** Let α be in $\hat{K}^\times$, $\mathfrak{a} = (\alpha)$ the (fractional) ideal generated by it and $\mathfrak{b} := \mathfrak{a}\mathfrak{c}^{-1}$. Then there is an HBAV-analogue of the Tate-elliptic curve: more precisely, there exist a certain Laurent series ring $\mathbb{Z}((q)) := \mathbb{Z}((q^\mu, \mu \in \mathfrak{a}\mathfrak{b}))$ and a HBAV $\mathrm{Tate}_\alpha(q)$ defined over $\mathbb{Z}((q))$ together with a canonical polarization λ_{can}, a differential $\omega_\mathfrak{a}$ and a Γ_{00}-structure i_α. Then the value

$$F(\mathrm{Tate}_\alpha(q), \lambda_{can}, \omega_\alpha, i_\alpha) \in R \otimes \mathbb{Z}((q)) \subseteq R((q))$$

has by definition of the latter ring a q-expansion

$$F_\alpha = \sum_{\mu \in \mathfrak{a}\mathfrak{b}} a_\mu(F_\mathfrak{a}) q^\mu$$

at the "cusp α" (actually there are some choices involved which for simplicity we do not want to mention here).

In the case $K \neq \mathbb{Q}$, one knows that

$$a_\mu(F_\alpha) = 0 \text{ unless } \mu = 0 \text{ or } \mu \gg 0.$$

Theorem IV (Ribet)**.** *The geometric fibres of $\mathfrak{M}(\mathfrak{c}, N)$ over $Spec(\mathbb{Z})$ are all geometrically irreducible.*

From the standard geometric argument[1] in [9], see also [6, thm. 1.6.1] or [7] one obtains the following

Corollary (q-expansion principle I)

Fix $N \geq 1,\ k \geq 0$ and R.

(i) Then for all cusps α as above the map

$$\begin{array}{rl} \mathcal{M}_k(\mathfrak{c}, N, R) & \hookrightarrow R((q)) \\ F & \longmapsto F_\alpha \end{array}$$

is injective.

(ii) Let $R \subset R'$ and let F be in $\mathcal{M}_k(\mathfrak{c}, N, R')$. If the coefficients of F_α are all in R, then F arises actually from a unique HMF defined already over R, i.e., the sequence

$$0 \to \mathcal{M}_k(\mathfrak{c}, N, R) \to \mathcal{M}_k(\mathfrak{c}, N, R') \to R'((q))/R((q))$$

is exact.

Remark 4. For $K \neq \mathbb{Q}$ the modular forms $\mathcal{M}_k(\mathfrak{c}, N, \mathbb{C})$ over $\mathbb{C}$ can also be identified with the set of holomorphic functions $F : \mathbb{H}_K \to \mathbb{C}$ such that

$$F_{|M}(\tau) = \mathcal{N}(\gamma\tau + \delta)^{-k} F\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) = F(\tau)$$

for all

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{00}(N)\,.$$

To this end one defines

$$F(\tau) := F\left(\frac{K \otimes \mathbb{C}}{2\pi i(\mathfrak{a}^{-1}\mathfrak{D}^{-1} + \mathfrak{a}\mathfrak{c}^{-1}\tau)}, \lambda_{can}, \omega_{can}, i_{can}\right)$$

for $\omega_{can} = dz$ and certain canonical choices λ_{can}, i_{can} (cf. [2] before (5.7), compare also with [4, §4.13]).

---

[1] Roughly speaking the idea is as follows: One has to show that the morphism Spec $(\mathbb{Z}((q))) \to \mathfrak{M}(\mathfrak{c}, N)$ corresponding to the point $\mathrm{Tate}_\alpha(q)$ of $\mathfrak{M}$ has the following topological property: its image contains the generic point (of each fibre). Then the vanishing of the modular form F in $\mathrm{Tate}_\alpha(q)$ implies the vanishing of F in each of these generic points. By the smoothness and irreducibility (which imply the absence of embedded components) this implies the vanishing of F everywhere.

By the translation-invariance with respect to elements of $(\mathfrak{a}\mathfrak{b}\mathfrak{D})^{-1}$ thus F admits a Fourier-expansion

$$F_\alpha(\tau) = \sum_{\substack{\mu\in\mathfrak{a}\mathfrak{b} \\ \mu=0 \\ \text{or} \\ \mu\gg 0}} c_\mu(F_\alpha) q^\mu \quad \left(q^\mu := e^{2\pi i\, tr(\tau\cdot\mu)}\right) \quad \text{at the cusp } \alpha.$$

By a GAGA-like result one has

$$c_\mu(F_\alpha) = a_\mu(F_\alpha),$$

where a_μ is the Fourier-coefficient defined earlier.

II.4. ***p*-adic modular forms.** Let R be a p-adic ring, i.e.,

$$R = \varprojlim_n R/p^n R.$$

Then the R-module $V(N,R)$ of p-adic modular forms on $\Gamma_{00}(N)$ over R (in the sense of Katz) consists of functions f on isomorphism classes of triples (X,λ,i) such that (X,λ) is a $\mathfrak{c}$-polarized HBAV over a p-adic R-algebra R' and i is a $\Gamma_{00}(Np^\infty)$-structure, i.e., a compatible system of $\Gamma_{00}(Np^n)$-structures, $n \geq 0$, subject to the condition that f is compatible with scalar extensions $R' \to R''$ of p-adic R-algebras.

Note first that there is no concept of weights anymore. Secondly, if p is nilpotent in R, any R-algebra R' is automatically p-adic, thus there is a natural identification

$$\mathcal{M}_0(\mathfrak{c}, Np^\infty, R/p^n R) \cong V(N, R/p^n R)$$

of weight 0 $\mathfrak{c}$-HMF with p-adic modular forms, both over R/p^nR. Thus we obtain the following interpretation for $N \geq 4$. Let $\mathfrak{M}_R^{p\text{-adic}}$ be the formal scheme $\{\mathfrak{M}(\mathfrak{c}, Np^\infty)_{R/p^nR}\}_{n\in\mathbb{N}}$ with structure sheaf $\mathcal{O}_{\mathfrak{M}^{p\text{-adic}}}$. Then the global sections

$$\begin{aligned} H^0(\mathfrak{M}_R^{p\text{-adic}}, \mathcal{O}_{\mathfrak{M}^{p\text{-adic}}}) &\cong \varprojlim_n H^0(\mathfrak{M}(\mathfrak{c}, Np^\infty)_{R/p^nR}, \mathcal{O}_{\mathfrak{M}^{p\text{-adic}}}) \\ &\cong \varprojlim_n \mathcal{M}_0(\mathfrak{c}, Np^\infty, R/p^nR) \\ &\cong \varprojlim_n V(N, R/p^n) \\ &\cong V(N,R) \end{aligned}$$

are nothing else than the p-adic modular forms over R.

Now we concentrate on the case $R = \mathbb{Z}_p$. To a finite idele $\alpha \in \hat{K}^\times$ one can also attach a *p-adic cusp* [2, (5.11)] and we have the q-expansion-maps

$$\begin{array}{rccl} \phi_\alpha : & V(N, \mathbb{Z}_p) & \longrightarrow & \mathbb{Z}_p[[q]] \\ & f & \longmapsto & f(\mathrm{Tate}_\alpha(q), \lambda_{can}, i_\alpha) \end{array}$$

and $\phi_\alpha \otimes \mathbb{Q}_p : V(N, \mathbb{Q}_p) := V(N, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[q]] \subseteq \mathbb{Q}_p[[q]]$.

Theorem 2.4

(i) ϕ_α *is injective.*
(ii) *coker* (ϕ_α) *does not have any p-torsion.*
(iii) *The sequence*

$$0 \to V(N, \mathbb{Z}_p) \to V(N, \mathbb{Q}_p) \stackrel{\phi_\alpha \otimes \mathbb{Q}_p}{\longrightarrow} \mathbb{Q}_p[[q]]/\mathbb{Z}_p[[q]]$$

is exact, i.e., a p-adic modular form $f \in V(N, \mathbb{Q}_p)$ *all of whose q-expansion coefficients at one p-adic cusp* α *belong to* $\mathbb{Z}_p$ *is already defined over* $\mathbb{Z}_p$.

Sketch of proof:
Since ϕ_α is the inverse limit of the injective maps (by the Corollary of Ribet's theorem)

$$V(N, \mathbb{Z}_p/p^n\mathbb{Z}_p) = \mathcal{M}_0(\mathfrak{c}, Np^\infty, \mathbb{Z}_p/p^n) \hookrightarrow \mathbb{Z}_p/p^n\mathbb{Z}_p[[q]]$$

it is clearly injective itself. Furthermore with $\mathfrak{M}(\mathfrak{c}, Np^\infty)/\mathbb{Z}$ also $\mathfrak{M}^{p\text{-adic}}_{\mathbb{Z}_p}/\mathbb{Z}_p$ is flat, because flatness is stable under base change, here by $(\mathbb{Z} \to)\mathbb{Z}_{(p)} \to \mathbb{Z}_p$. Hence we have the exact sequence

$$0 \to \mathcal{O}_{\mathfrak{M}^{p\text{-adic}}} \stackrel{p}{\to} \mathcal{O}_{\mathfrak{M}^{p\text{-adic}}} \to \mathcal{O}_{\mathfrak{M}^{p\text{-adic}}}/p \to 0.$$

Taking global sections gives

$$(1) \qquad 0 \to V(N, \mathbb{Z}_p) \stackrel{p}{\to} V(N, \mathbb{Z}_p) \to V(N, \mathbb{Z}_p/p\mathbb{Z}_p).$$

Now let $f(q) \in \mathbb{Z}_p[[q]]$ be a power series in q such that $pf(q) = g(q)$ for some $g \in V(N, \mathbb{Z}_p)$. Then the image of g in $V(N, \mathbb{Z}_p/p\mathbb{Z}_p)$ is zero by (i). From (1) we see that there is a unique $f \in V(N, \mathbb{Z}_p)$ such that $g = pf$ and with q-expansion $f(q)$. The claim (ii) follows. Finally, tensoring the exact sequence

$$0 \to V(\mathbb{Z}_p) \stackrel{\phi_\alpha}{\to} \mathbb{Z}_p[[q]] \to \text{ coker } \phi_\alpha \to 0$$

with $\mathbb{Q}_p/\mathbb{Z}_p$ claim (iii) follows from (ii). $\square$

Finally we remark that there is a canonical map

$$\mathcal{M}_k(\mathfrak{c}, N, R) \overset{\pi_k}{\to} V(N, R)$$

for any $k \geq 0$, induced by sending (X, λ, i) to $(X, \lambda, \omega(i), i)$ where $\omega(i)$ is a certain differential attached to i. Now we are able to prove the q-expansion principle III. In [2, (5.15)] it is explained that

$$S(\alpha) = \sum_{k\geq 1} \mathcal{N}\alpha_p{}^{-k} F_{k,\alpha} = \left(\sum_{k\geq 1} \pi_k F_k \right)_\alpha ,$$

where the last q-expansion is with respect to the p-adic cusp attached to α, while $F_{k,\alpha}$ denotes the expansion at the usual cusp. Setting F for $\sum_{k\geq 1} \pi_k F_k$, we have by the assumption of III that the q-expansion at the p-adic cusp α of the p-adic modular form $(F - t)$ has coefficients in $\mathbb{Z}_p$:

$$(F - t)_\alpha \in \mathbb{Z}_p[[q]],$$

where $t = a_0(F_\alpha)$ denotes the constant term at α, considered as weight zero HMF or constant p-adic modular form, respectively. Thus $F - t$ belongs to $V(N, \mathbb{Z}_p)$ by (iii) of the last theorem. Hence the q-expansion of $F - t$ at any other cusp β, say, also has coefficients in $\mathbb{Z}_p$. In particular, its constant term

$$\begin{aligned} a_0\left((F - t)_\beta\right) &= a_0(F_\beta) - a_0(t_\beta) \\ &= a_0(F_\beta) - t \\ &= a_0(F_\beta) - a_0(F_\alpha) \end{aligned}$$

belongs to $\mathbb{Z}_p$.

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