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GAUS AG: Arithmetic of critical p-adic L-functions

AG Venjakob

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ORGANISATION: We meet every Thursday at 11 (c.t.) in SR8 Mathematikon, Heidelberg. Online participation via zoom:
`https://us06web.zoom.us/j/81273398429?pwd=IMzCuKcWZfF5LUNFMj5ZJ6HEUNZUdA.1`.
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In this seminar, we study the 2023 monograph [BBb], where Benois-Büyükboduk construct and investigate, from an arithmetic perspective, p-adic L-functions on the eigencurve that are attached to so-called *θ-critical* modular forms. The latter is a class of modular forms which includes for example those coming from CM elliptic curves.

Historically, constructions of p-adic L-functions for sufficiently well-behaved modular forms have been known for several decades, with the θ-critical forms remaining the last unsolved case for some time. In 2012, Bellaïche completed the picture with an analytic (i.e. working in Betti cohomology) construction of p-adic L-functions for θ-critical modular forms. These feature very interesting behaviour unparalleled in the non-θ-critical case, like the *"exceptional zero phenomenon"* or the presence of a *secondary p-adic L-function*.

In [BBb], the authors reconstruct critical p-adic L-functions in an arithmetic way (i.e. using étale cohomology), and investigate the arithmetic and Iwasawa-theoretic aspects of the above-mentioned phenomena appearing in the θ-critical scenario. One example of the novel techniques developed by Benois-Büyükboduk is the principle of *"eigenspace transition via differentiation"*, to control the complicated p-adic Hodge theory of the Galois representation attached to a θ-critical form. Another central new concept is that of *"infinitesimal thickening"* of *Selmer complexes, fundamental lines* etc., which is introduced to remedy the somewhat degenerate behaviour of the usual Selmer complex in the θ-critical setting, allowing for instance the formulation of appropriate versions of the Iwasawa Main Conjecture in that setting.

With their new methods, Benois-Büyükboduk prove a BSD-like leading term formula for their thick p-adic L-function in the analytic rank-one case, which will be treated in the final talk of the seminar.

Talk 1: Overview – Max Witzelsperger (18.4.)

Give an introduction to the work [BBb], and provide some motivation and context. In particular, briefly introduce modular forms and their complex L-functions; modular symbols should also be mentioned. Review the more classical analytic approach to construct p-adic L-functions from [Bel]. For background, [W] might be useful.

Explain the different, arithmetic method in [BBb], and give a rough overview of the main goals of the seminar, that is, the results of §2.4.4 and Theorem 5.29 of [BBb].

Talk 2: The eigencurve – Anna Blanco Cabanillas (25.4.)

Briefly review Tate's notions of affinoid algebras and rigid spaces. As a reference, cf. [Bo, Ch. 3 and 5], but don't go into detail on the definitions. Define and study the weight space $\mathcal{W} = \mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ in some detail, following §§6.3.1, 6.3.3 and 6.3.5 in [Ei].

Explain the construction of the Coleman-Mazur eigencurve in [BBa, §5.2]: First describe the general formalism of constructing eigenvarieties (the "eigenvariety machine"), cf. [Ei, §3.6]. Then go on to introduce the Banach spaces of modular symbols from [BBa, §5.1] to feed the machine. Describe how newforms correspond to classical points on the eigencurve, using the interpretation of points [Ei, Theorems 7.3.1 and 3.7.1] and the one-dimensionality of Hecke eigenspaces [Bel, Theorem 1].

Talk 3: Analytic construction of p-adic L-functions on the eigencurve – ??? (2.5.)

Explain refinements of newforms [Bel, §1.3] (called p-stabilizations in [BBb]) and give an overview over the various notions of criticality for modular forms (last paragraph in the introduction of [Bel] above §1.4). It is ok to restrict attention to cusp forms throughout. State Prop. 2.11 and Def. 2.12 ibid and prove as much as possible, in particular the relationship with the geometry of the eigencurve, but avoid mentioning overconvergent modular forms.

Introduce the Mellin tranform in the context of distributions-valued modular symbols [Bel, §3.3]. Having gathered all necessary tools, and assuming [Bel, Thm. 1] as a black box, present Bellaïche's construction of the 2-variable p-adic L-function of a refined cuspidal newform (§4.3 ibid). This is reviewed in a streamlined way in [BBb, §§2.3.3-2.3.4]; also explain the infinitesimal thickening from §2.3.5 ibid. State its interpolation behaviour on classical points ([Bel, Prop. 4.14] and [BBb, Equ. (2.31)]). Stress the difference between the "classical" scenario and the more complicated θ-critical case. Prove Theorem 3 [Bel, §4.3.3] and cover secondary p-adic L-functions (§4.4 ibid).

Talk 4: Large exponential maps à la Perrin-Riou – Marlon Kocher (16.5.)

The goal of this talk is to define the large exponential map for rigid-analytic families of (φ, Γ)-modules [BBb, §1.1.8], which is the first ingredient towards an arithmetic alternative to the approach from Talk 3. Start by briefly reviewing the basics on (φ, Γ)-modules over the Robba ring, particularly in families, see [BBa, §3.1]; the standard reference for background is [KPX]. Also introduce Iwasawa cohomology both on the side of representations and of (φ, Γ)-modules [BBb, §1.1.2].

Go on to construct the large exponential map for crystalline (φ, Γ)-modules [BBb, §1.1.4] and explain its relationship with the usual Bloch-Kato exponential map (Thm. 1.2 ibid). Extend it to families [BBb, §1.1.8].

If time permits, compare the above construction with Perrin-Riou's classical large exponential map for crystalline representations [BBb, §1.1.5].

Talk 5: Triangulations and abstract construction of p-adic L-functions – Marvin Schneider (23.5.)

Define trianguline (φ, Γ)-modules [KPX, Def. 6.3.1] and discuss Theorem 6.3.9 ibid (ideally, also sketch the proof). You can assume that X is smooth and that the rank d is 2 in order to keep notations simpler. Discuss [KPX, Prop. 6.4.5], which entails one of the main difficulties of the θ-critical theory.

Then go on to present the abstract construction of the "Perrin-Riou-style" p-adic L-function; see [BBa, §3.3 - §4.2] or [BBb, §2.2.1 - §2.2.3]. In particular, work with a general map of affinoids $\mathcal{X} \to \mathcal{W}$ of the type considered in [BBb, §2.2.1.1], but indicate that our situation of interest (the weight map on the eigencurve) fits into this framework; in particular, the Galois representation $V_{\mathcal{X}}$ and the cohomology class $\mathbf{z}$ will be given a concrete meaning in Talks 7 and 8.

Note that in [BBb, §2.2.3], the θ-critical scenario is assumed, while [BBa, §4.2] treats the complementary case. We focus on the former, more difficult one, but if time permits, compare the two cases. In [BBb], prove Prop. 2.12 and the subsequent Cor. 2.13, the "extreme exceptional zero phenomenon".

Talk 6: Secondary p-adic L-functions – Max Witzelsperger (6.6.)

Continuing to work in the abstract setting of the previous talk, introduce the secondary p-adic L-function [BBb, §2.2.5], a phenomenon unique to the θ-critical case. The goal of the talk is to prove Theorem 2.16 ibid, which should be motivated as the analogue of Bellaïche's interpolation result on secondary p-adic L-functions; see [BBb, Prop. 2.19], or the paragraph "Interpolation" on p. 11 in [Bel].

To establish the above Theorem, present the content of [BBb, §2.1.2], and prove the "eigenspace transition principle" (Prop. 2.6), which is the key ingredient for Thm. 2.16, because it allows to circumvent the problem of non-saturation of the global triangulation at θ-critical points (cf. the previous talk). Develop as much as needed from §2.1.1.

Talk 7: Big Galois representations – Rustam Steingart (13.6.)

The goal of this talk is to construct the rigid-analytic family of Galois representations $V_{\mathcal{X}}$ to which we want to apply the methods from the previous talks (§6.4.3 in [BBa]).

Introduce the modular curves and their étale sheaves (§6.1, particularily 6.1.4 in [BBa]). Discuss [BBa, Def. 6.4] in some detail. State and prove Prop. 6.5 ibid, after establishing the necessary preliminaries from §5.2. Finally, motivate and define the representations $V_{\mathcal{X}}$, $V'_{\mathcal{X}}$ and their canonical pairing (§§6.4.1 and 6.4.3 in [BBa]), and cover as much of §6.4.4 as possible.

Talk 8: Arithmetic construction of p-adic L-functions on the eigencurve – ??? (20.6.)

Introduce the big Beilinson-Kato element on the eigencurve (Def. 6.7 and 6.14 in [BBa]). Start by giving a little background, sketching Def. 6.1 in [BBa] of Kato's $\mathbb{BK}$-element (also, cf. [Kat, §§1-2]), but don't go into detail on the construction. Point out the significance of this cohomology class by stating [BBb, Prop. 2.20] without proof. For background, cf. Ex. 13.3 and Thm. 16.6 in [Kat], and also [BBa, Thm. 7.3(iv)].

Now define the arithmetic θ-critical two-variable p-adic L-function [BBb, Def. 2.22]. Deduce Theorem 2.23 from the results of the previous talks. Finally, prove Theorem 2.24 of [BBb], the comparison with the critical p-adic L-function from Talk 3; for this, recall the crucial "general position" condition (GP) from the previous talk (introduced in §2.1.1.4 in [BBb]).

For the rest of this program, all references are made with respect to the article [BBb].

Talk 9: Iwasawa theoretic invariants – Otmar Venjakob (27.6.)

The goal of this talk is to construct the Iwasawa theoretic $\mathscr{L}$-invariant $\mathscr{L}_{\mathrm{Iw}}^{\mathrm{cr}}$ which appears in Prop. 2.27, and to prove said proposition. This comparison result between the improved p-adic L-function at the critical-slope eigenvalue and the p-adic L-function at the slope-0 eigenvalue has no analogue in the analytic setting and further motivates the arithmetic approach of Benois-Büyükboduk.

Introduce the various Selmer groups attached to the Galois representation of a θ-critical form (§3.2) and cover the entire §3.2.3, which closes with the proof of Prop. 2.27. Explain as much of the material from §3.1 as is necessary to formulate and prove the results of §3.2.3. Formulate Conjecture G about the non-vanishing of this invariant; perhaps also mention the refined Conjecture H, and Cor. 3.21 as a positive result in that direction.

Talk 10: Punctual and thick Iwasawa Main Conjectures – Annie Littler (4.7.)

Introduce the notion of a Selmer complex (§1.2.2) and define the Selmer complexes attached to V_f from 3.3.1.1. Define fundamental lines and the modules of algebraic p-adic L-functions (Def. 3.30), and discuss Prop. 3.31. State the ordinary and oritical Main Conjectures (3.32 and 3.33), and discuss Prop. 3.34 about their equivalence, given the non-vanishing of the Iwasawa invariant $\mathscr{L}_{\mathrm{Iw}}^{\mathrm{cr}}$.

In the second, main part of the talk, introduce the Iwasawa theory of the infinitesimal deformation V_k of V_f: Cover §4.1.1, and define the thick Selmer complexes and fundamental lines (§4.1.2.5 and 4.1.2.6). Then prove Prop. 4.3, which allows you to formulate the Main Conjecture 4.5 for the infinitesimal thickening of the module of algebraic p-adic L-functions (Def. 4.4). Show that the thick Main Conjecture implies the punctual one (Prop. 4.6).

Talk 11: Thick Selmer groups and Bloch-Kato Selmer groups – ??? (11.7.)

Consider the punctual resp. thick Selmer complexes $\mathbf{R\Gamma}({}^cV,{}^c\mathbf{D})$ resp. $\mathbf{R\Gamma}({}^c\widetilde{V},{}^c\widetilde{\mathbf{D}})$ that are investigated in §5.2.2.1 resp. §5.2.4. Start by establishing indications for the degenerate behaviour of the former (Prop. 5.5 and Cor. 5.8); cf. §1.3.1 for the necessary background on p-adic heights afforded by cyclotomic variation.

Concerning the latter, work through §5.2.4. The goal is to prove the crucial Prop. 5.13 and the subsequent Cor. 5.14, which relates the thick Selmer group to the Bloch-Kato Selmer group. The latter object is expected to encode behaviour of the complex L-function.

Talk 12: Galois cohomology at the central critical twist – ??? (18.7.)

This talk covers the content of §§5.3.1 and 5.3.2, laying the foundation for the last talk. Introduce the Selmer complex $\mathbf{R\Gamma}({}^cV_{\mathcal{X}},{}^c\mathbf{D}_{\mathcal{X}})$, where $V_{\mathcal{X}}$ denotes the big Galois representation on a sufficiently small neighbourhood of a θ-critical form f_α, $\mathbf{D}_{\mathcal{X}}$ its triangulation, and ${}^c(-)$ denotes the central critical twist (Def. 5.17).

The talk is devoted to the investigation of H^1 of this complex. Show that it is a free $\mathcal{O}_{\mathcal{X}}$-module (Lem. 5.18), and isomorphic to the Greenberg-style Selmer group (Prop. 5.20). Then restrict to the case that the analytic rank of f at the centre of the critical range is equal to one (§5.3.2). The main result of the talk is Thm. 5.22, which says that H^1 has rank one in this case, and also provides a generator in terms of the Beilinson-Kato element (cf. Talk 8).

Talk 13: The leading term formula – ??? (25.7.)

Introduce the $\mathcal{O}_{\mathcal{X}}$-adic height pairing on the H^1 of the Selmer complexes from the last talk (§5.3.3). Construct the central twist $\mathrm{EXP}_{{}^c\mathbf{D}}$ of the large exponential map (§5.3.4.1), and the corresponding map LOG_η (Def. 5.26) and prove Prop. 5.27.

Define the regulator $R_{\mathcal{X}}$ (§5.3.5.1) and show Prop. 5.28; in order to carry out the proof, first explain the key Thm. 1.15, which we treat as a black box. Finally, prove the $\mathcal{O}_{\mathcal{X}}$-adic leading term formula at the central critical point in the analytic rank-one case (Thm. 5.29).

References

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[BBb] Denis Benois and Kâzım Büyükboduk, 2024, *"Arithmetic of critical p-adic L-functions"*, arxiv preprint, `https://arxiv.org/abs/2403.16076`

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[Ei] Joël Bellaïche, 2021, *"The eigenbook"*, Birkhäuser

[Bo] Siegfried Bosch, 2014, *"Lectures on Formal and Rigid Geometry"*, Lecture Notes in Mathematics 2105

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[W] Chris Williams, *"An Introduction to p-adic L-Functions II: Modular Forms"*, `https://warwick.ac.uk/fac/sci/maths/people/staff/cwilliams/lecturenotes/lecture_notes_part_ii.pdf`