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Master's Thesis

Tate's acyclicity and Kiehl's glueing property in sheafy adic Banach rings

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Abstract

On 17 December of 2014 Kiran S. Kedlaya and Ruochan Liu published their article on the foundations of relative p-adic Hodge theory [17] in which they describe a new approach to the area using systematic Witt vector constructions and the work of Huber and Berkovich in nonarchimedean analytic geometry. Our work is based on the second chapter of the article, where Kedlaya and Liu proved that for a sheafy adic Banach ring, the structure sheaf on $\mathrm{Spa}(A, A^+)$ satisfies the Tate sheaf property and the Kiehl glueing property [17, Theorem 2.7.7]. The first property establishes acyclicity for the structure sheaf $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ of the geometric object $\mathrm{Spa}(A, A^+)$, i.e., $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}) = 0$ for $i \neq 0$, while the second ensures a category equivalence between finite projective modules over certain rings and their associated sheaves of modules which are locally free of finite rank. The last theorem is of great importance, since it allow us to establish analogies between the new geometric objects and the classical ones in algebraic geometry, namely the structure sheaf of $X = \mathrm{Spec}(R)$ for a ring R and its corresponding coherent $\mathcal{O}_X$-modules.
The main objective of this thesis is to explain in detail some of the results exposed in the article [17] and enhance some of the definitions, in order to prove the theorem stated above. We assume that the reader has already a mathematical background in algebraic geometry and algebraic number theory. We will divide the presentation in three parts: The first chapter will comprehend the preliminaries of the work stating the classical results from the theories of Huber and Berkovich and some glueing lemmata necessary for the last two parts. Part two will consist of the proof of the Tate sheaf property of the structure sheaf and the last part will be dedicated to the proof of the Kiehl glueing property.

Am 17. Dezember 2014 veröffentlichten S. Kiran Kedlaya und Ruochan Liu ihren Artikel über die Grundlagen der relativen p-adischen Hodgetheorie [17]. In dem Artikel haben die Autoren einen neuen Ansatz, durch systematische Konstruktionen von Wittvektoren, dargelegt und haben die Arbeit von Huber und Berkovich in der nichtarchimedischen analytischen Geometrie beschrieben. Unsere Arbeit basiert auf dem zweiten Kapitel des Artikels, wo Kedlaya und Liu die Tate-Garbe-Eigenschaft und die Kiehl-Aufkleben-Eigenschaft für die Strukturgarbe auf dem Raum $\mathrm{Spa}(A, A^+)$ *eines sheafy adischen Banachringes beweisen [17, Theorem 2.7.7]. Die erste Eigenschaft stellt Azyklizität für die Struckturgarbe*

des geometrischen Objekts sicher, i.e., $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}) = 0$ *für* $i \neq 0$, *während die zweite für eine Kategorieäquivalenz zwischen endlichen projektiven Modulen über bestimmte Ringe und ihrer zugehörigen Garbe von Modulen, die lokal frei endlichen Ranges sind, sorgt. Der letzte Satz ist von großer Bedeutung, da er es uns ermöglicht Analogien zwischen den neuen geometrischen Objekten und den klassischen Objekten der algebraischen Geometrie zu schaffen, spezifisch die Strukturgarbe auf* $X = \mathrm{Spec}(R)$ *für einen Ring* R *und seine entsprechenden kohärenten* $\mathcal{O}_X$*-Module.*
Das Hauptziel dieser Arbeit ist im Detail einige der Ergebnisse in dem Artikel [17] zu erklären und einige der Definitionen zu erweitern, um den oben angegebenen Satz zu beweisen. Wir gehen davon aus, dass der Leser bereits über einen mathematischen Hintergrund in der algebraischen Geometrie und in der algebraischen Zahlenthoerie verfügt. Die Darstellung ist in drei Teile untergliedert: Das erste Kapitel beschreibt die Grundlagen der Arbeit. Die Ergebnisse aus den Theorien von Huber und Berkovich werden angegeben, wie auch einige Verklebungslemmata bewiesen, die notwendig für die beiden letzten Teile sind. Der zweite Teil besteht aus dem Beweis der Tate-Garbe-Eigenschaft der Strukturgarbe und der letzte Teil wird der Kiehl-Aufkleben-Eigenschaft gewidmet.

Contents

1 Introduction

We would like to begin by presenting the theorem we want to prove. After that, we will explain what it says and what the objects of the theorem are.

Theorem 1.0.1. *Let* (A, A^+) *be a sheafy adic Banach ring. Then the structure sheaf on* $\mathrm{Spa}(A, A^+)$ *satisfies the Tate sheaf property and the Kiehl glueing property.*

We will define two different but related geometric spaces associated to a nonarchimedean commutative Banach ring. The first one is the *Gel'fand spectrum* in the sense of Berkovich, while the second one is the *adic spectrum* $\mathrm{Spa}(A, A^+)$ considered by Huber. Consequently, we will proceed by defining a structural presheaf $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ on $\mathrm{Spa}(A, A^+)$ and will call (A, A^+) sheafy if this presheaf is a sheaf. Finally, we will be interested in two particular properties of the structure sheaf. First we will prove the acyclicity of the structure sheaf, where acyclicity means that $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}) = 0$ for $i \neq 0$. The second property is related to a category equivalence between finite projective modules over commutative Banach rings with certain conditions and their associated sheaves of modules which are locally free of finite rank.

2 Part: Preliminaries

2.1 Finite, flat, projective modules and finite étale algebras

We will apply some results from the theory of modules over a ring and do some comparisons with the theory of finite étale algebras over a ring. Specifically we will focus on the case where the ring is a Banach ring.

Before we continue, let us make a convention in order to simplify notation.

Convention 2.1.1. *As in the article [17], we assume all rings to be commutative and unital unless otherwise stated.*

Definition 2.1.2. Let M be a module over a ring R.
We call M *pointwise free*, if $M \otimes_R R_\mathfrak{p}$ is free for each maximal ideal $\mathfrak{p}$ of R.

Definition 2.1.3. Let M be a module over a ring R.
We call M *locally free*, if there exist $f_1, \dots, f_n \in R$ generating the unit ideal such that $M \otimes_R R_f$ is a free module over R_f for $i = 1, \dots, n$.

We will often use the next equivalence for finite projective modules over a ring R.

Theorem 2.1.4. *Let M be a module over a ring R. The following conditions are equivalent:*

(i) M is finitely generated and projective.

(ii) M is a direct summand of a finitely generated free module.

(iii) M is finitely presented and pointwise free.

(iv) M is finitely generated and pointwise free of locally constant rank.

(v) M is finitely generated and locally free.

Proof. See [8, §II.5.2, Théorèm 1]. □

Definition 2.1.5. Let M be a module over a ring R. We say M is *faithfully flat* if M is flat and $M \bigotimes_R N \neq 0$ for every nonzero R-module N. A ring homomorphism $\varphi : R \longrightarrow R'$ is called *faithfully flat* if R', viewed as an R-module via φ, is faithfully flat.

Theorem 2.1.6. *For an R-module M, the following conditions are equivalent:*

(i) M is faithfully flat.

(ii) M is flat and, given a morphism of R-modules $\varphi : N' \longrightarrow N$ such that $\varphi \otimes \mathrm{id}_M : N' \bigotimes_R M \longrightarrow N \bigotimes_R M$ is the zero morphism, then $\varphi = 0$.

(iii) A sequence of R-modules $N' \longrightarrow N \longrightarrow N''$ is exact if and only if the sequence $N' \bigotimes_R M \to N \bigotimes_R M \longrightarrow N'' \bigotimes_R M$ obtained by tensoring over R with M is exact.

(iv) M is flat and, for every maximal ideal $\mathfrak{m} \subset R$, we have $\mathfrak{m}M \neq M$.

Proof. See [5, Proposition 4.2.11]. □

Lemma 2.1.7. *Let $\varphi : R \longrightarrow S$ be a flat ring morphism. Then φ is faithfully flat if and only if for every maximal ideal $\mathfrak{P}$ of R, there exists a maximal ideal $\mathfrak{J}$ of S, such that $\varphi(\mathfrak{J})^{-1} = \mathfrak{P}$.*

Proof. See [8, §I.3.5, Proposition 9]. □

We will now state some results about étale morphisms following [5] in order to characterize the category of finite étale algebras over a ring.

Definition 2.1.8. Let $f : X \longrightarrow S$ be a morphism of schemes.

Then we say that f is *formally étale* if for every affine S-scheme Y and for every closed subscheme $\bar{Y} \subset Y$, given by a quasi-coherent ideal $\mathcal{I} \subset O_Y$ satisfying $\mathcal{I}^2 = 0$, the canonical restriction map

$$\phi : \mathrm{Hom}_S(Y, X) \longrightarrow \mathrm{Hom}_S(\bar{Y}, X)$$

is bijective.

We called f *unramified* at a point $x \in X$ if there exists an open neighborhood $U \subset X$ of x as well as a closed S-immersion $\mathfrak{j} : U \hookrightarrow W \subset \mathbb{A}^n_S$ into a open subscheme W of some affine n-space $\mathbb{A}^n_S$ over S such that:

a.) If $\mathcal{I} \subset \mathcal{O}_W$ is the sheaf of ideals associated to the closed immersion $\mathfrak{j}$,there exist finitely many sections generating $\mathcal{I}$ in a neighborhood of $\mathfrak{j}(x)$.

b.) The differential forms of type dg for sections g of $\mathcal{I}$ where d stands for the exterior differential $d_{\mathbb{A}^n_S/S} : \mathcal{O}_{\mathbb{A}^n_S/S} \longrightarrow \Omega^1_{\mathbb{A}^n_S/S}$, generate $\Omega^1_{\mathbb{A}^n_S/S}$ at $\mathfrak{j}(x)$.

The morphism f is *unramified* if it is unramified at all points of X.

We say that f is *smooth* at a point $x \in X$ (of relative dimension r) if there exists an open neighborhood $U \subset X$ of x as well as a closed S-immersion $\mathfrak{j} : U \hookrightarrow W \subset \mathbb{A}^n_S$ into a open subscheme W of some affine n-space $\mathbb{A}^n_S$ over S such that:

a.) If $\mathcal{I} \subset \mathcal{O}_W$ is the sheaf of ideals associated to the closed immersion $\mathfrak{j}$, there are $n-r$ sections $g_{r+1}, \ldots, g_n$ in $\mathcal{I}$ that generate $\mathcal{I}$ in a neighborhood of $z := \mathfrak{j}(x)$; in particular, we assume $r \leq n$.

b.) The residue classes $dg_{r+1}(z), \ldots, dg_n(z) \in \Omega^1_{\mathbb{A}^n_S/S} \otimes k(z)$ of the differential forms $dg_{r+1}, \ldots, dg_n$ are linearly independent over $k(z)$.

We call f *smooth* on X if it is smooth at all points of X.

The notation $\Omega^1_{\mathbb{A}^n_S/S} \otimes k(z)$, as used above, is an abbreviation for the $k(z)$-vector space

$$\Omega^1_{\mathbb{A}^n_S/S,z} \otimes_{\mathcal{O}_{\mathbb{A}^n_S/S,z}} k(z) \cong \Omega^1_{\mathbb{A}^n_S/S,z}/\mathfrak{m}_z \Omega^1_{\mathbb{A}^n_S/S,z}$$

where $\mathfrak{m}_z \subset \mathcal{O}_{\mathbb{A}^n_S/S,z}$ is the maximal ideal and $k(z) = \mathcal{O}_{\mathbb{A}^n_S/S,z}/\mathfrak{m}_z$ is the residue field of z.

Definition 2.1.9. A morphism of schemes is *étale* if it is locally of finite presentation and formally étale. A morphism of rings is étale if the corresponding morphism of affine schemes is étale.

For a ring R we define the tensor category $\mathbf{F\acute{E}t}(\mathrm{R})$ of finite étale algebras over R, with morphisms being arbitrary morphisms of R-algebras.

It follows from [5, Theorem 8.5.6] and [5, Theorem 8.5.8] that every morphism of schemes $f : X \longrightarrow S$ of locally finite presentation is formally étale if and only if it is smooth and unramified. From here we conclude:

Lemma 2.1.10. *Let $\varphi : A \longrightarrow B$ be a morphism of rings. If φ makes B into a projective A-module of finite type and if the module of relative differential forms of degree* 1 *of B over A is zero, i.e., $\Omega_{B/A} = 0.$, then $B \in$ **FÉt**(A).*

Actually the lemma stated above is reversible and gives a characterization of the finite étale algebras over a ring.

2.2 Seminorms on groups and rings

In order to define Banach rings and eventually spectra of nonarchimedean (commutative) Banach rings it is necessary to consider norms and seminorms on groups and rings. We will introduce the notation as in [17, §2.1].

Definition 2.2.1. Let G be an abelian group. Consider the following conditions on a function $\alpha : G \to [0, +\infty)$.

a.) For all $g, h \in G$, we have $\alpha(g - h) \leq \max\{\alpha(g), \alpha(h)\}$.

b.) We have $\alpha(0) = 0$.

c.) For all $g \in G$, we have $\alpha(g) = 0$ if and only if $g = 0$.

We say that α is a *nonarchimedean seminorm* if it satisfies a.) and b.), and a *nonarchimedean norm* if it satisfies additionally c.).

If α, α' are two seminorms on the same abelian group G, we say α *dominates* α', and write $\alpha \geq \alpha'$ or $\alpha' \leq \alpha$, if there exists $c > 0$ such that $c\alpha(g) \geq \alpha'(g)$ for all $g \in G$. If α and α' dominate each other, we say they are *equivalent*.

Note that conditions a.) and b.) imply

$$\text{For all} \quad g, h \in G, \quad \text{we have} \quad \alpha(g + h) \leq \max\{\alpha(g), \alpha(h)\}$$

since for all $g \in G$, we then have $\alpha(g) = \alpha(-g)$. From here it is clear why we call such a seminorm or norm *nonarchimedean* (compare [6, Part 1, §2]).
Any seminorm α induces a norm on G/H, where $H := \{g \in G : \alpha(g) = 0\}$. That this norm is well defined is easy to see. We will often refer to H as the kernel of α.

Definition 2.2.2. Let G, H be two abelian groups equipped with nonarchimedean seminorms α, β, and let

$$\varphi : G \longrightarrow H$$

be a homomorphism. We say φ is *bounded* if $\alpha \geq \beta \circ \varphi$, and *isometric* if $\alpha = \beta \circ \varphi$.

The *quotient seminorm* induced by α is the seminorm $\bar{\alpha}$ on image(φ) defined by

$$\bar{\alpha}(h) = \inf\{\alpha(g) : g \in G, \varphi(g) = h\}$$

If H is also equipped with a seminorm β, we say φ is *strict* if the two seminorms $\bar{\alpha}$ and β on image(φ) are equivalent; this implies in particular that φ is bounded.

Note that the composition of strict morphisms $g \circ f$ is again strict if f is surjective or g injective.

Definition 2.2.3. For G an abelian group with a nonarchimedean seminorm α, equip the group of Cauchy sequences in G with the seminorm whose value on the sequence $(g_i)_{i \in \mathbb{N}}$ is $\lim_{i \to \infty} \alpha(g_i)$. The quotient by the kernel of this seminorm is the *separated completion* $\widehat{G}$ of G under α. For the unique continuous extension of α to $\widehat{G}$, the homomorphism $G \longrightarrow \widehat{G}$ is isometric, and injective if and only if α is a norm (in which case we call $\widehat{G}$ the *completion* of G).
If G is isomorph to its completion, we say that G is *complete*.

Example 2.2.4. *The two classical examples are $\mathbb{Z}_p$ and $\mathbb{Q}_p$, which are respectively the completions of $\mathbb{Z}$ and $\mathbb{Q}$ for the p-adic norm $|\cdot|_p$.*

Definition 2.2.5. Let A be a ring. Consider the following conditions on a (semi)norm α on the additive group of A.

a'.) For all $g, h \in A$, we have $\alpha(gh) \leq \alpha(g)\alpha(h)$.

b'.) We have a'.), and for all $g \in A$, we have $\alpha(g^2) = \alpha(g)^2$.

c'.) We have b'.), $\alpha(1) = 1$, and for all $g, h \in A$, we have $\alpha(gh) = \alpha(g)\alpha(h)$.

We say α is *submultiplicative* if it satisfies a'.), *power-multiplicative* if it satisfies b'.), and *multiplicative* if it satisfies c'.).

If α is a submultiplicative seminorm and α' is a power-multiplicative seminorm, then α dominates α' if and only if $\alpha'(a) \leq \alpha(a)$ for all $a \in A$. Indeed, if α dominates α',

$$\begin{aligned} \alpha'(a^2) \leq c\alpha(a^2) & \text{ for some } c > 0. \\ \Rightarrow \quad & \alpha'(a)^2 \leq c\alpha(a)^2 \\ \Rightarrow \quad & \frac{\alpha'(a)^2}{c\alpha(a)} \leq \alpha(a) \\ \Rightarrow \quad & \alpha'(a) \leq \alpha(a) \end{aligned}$$

for all $a \in A$ with $\alpha(a) > 0$. The other implication is immediate.
Following this observation, we will often make use of this second property whenever is possible. For example, for any $\beta \in \mathcal{M}(A)$ as in Definition 2.4.1, we get that $\beta(a) \leq |a|$ for all $a \in A$.

Definition 2.2.6. Let A be a ring equipped with a submultiplicative seminorm α. The *spectral seminorm* on A is the power-multiplicative seminorm α_{sp} defined by $\alpha_{sp}(a) = \lim_{s\to\infty}\alpha(a^s)^{1/s}$.

The above limit exists due to a'.) and *Fekete's lemma*, which says that for every subadditive sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ the sequence $\left\{\frac{a_n}{n}\right\}_{n\in\mathbb{N}}$ converges. Note that equivalent choices of α define the same spectral seminorm.

Definition 2.2.7. Let A, B, C be rings equipped with submultiplicative seminorms α, β, γ, and let $A \longrightarrow B$, $A \longrightarrow C$ be bounded homomorphism. The *product seminorm* on the ring $B \otimes_A C$ is defined by taking $f \in B \otimes_A C$ to the infimum of $\max_i\{\beta(b_i)\gamma(c_i)\}$ over all presentations $f = \sum_i b_i \otimes c_i$. The separated completion of $B \otimes_A C$ for the product seminorm is denoted $B\widehat{\otimes}_A C$ and called the *completed tensor product* of B and C over A.

The product seminorm is not in general a norm. However if A is an analytic field and the rings B, C are Banach rings equipped with submultiplicative norms, then we can assure that the product seminorm is a norm; see [20, Proposition 17.4].

2.3 Banach rings and modules

We will now define the terms Banach ring, Banach algebra and Banach module. We will also define a Tate algebra in order to speak later about rational subspaces and rational localizations, in our new setting as well as in the classical theory of affinoid algebras [6, Part 1, §3].

Definition 2.3.1. As in [17] we will mean by an *analytic field* a field equipped with a nontrivial multiplicative nonarchimedean norm under which it is complete.
A *Banach ring* is a commutative ring R equipped with a submultiplicative norm under which it is complete. We allow the zero ring as a Banach ring, so that the completed tensor product can be defined in the category of Banach rings.
A *Banach algebra* over a Banach ring R is a Banach ring S equipped with the structure of an R-algebra in such a way that the map $R \longrightarrow S$ is bounded.

Lemma 2.3.2. *Let R be a Banach ring.*

(i) For any finite R-module M, the quotient seminorm defined by a surjection $\pi : R^n \longrightarrow M$ of R-modules does not depend, up to equivalence, on the choice of the surjection.

(ii) Let $\psi : R \longrightarrow S$ be a bounded homomorphism of Banach rings. Let M be a finite R-module, let N be a finite S-module, and let $\theta : M \longrightarrow N$ be a R-linear map. Then this map becomes bounded if we equip M and N with the seminorms as described in (i).

Proof. (i) Let us recall the definition of the quotient seminorm $|\cdot|_M$ on M. For $h \in M$,

$$|h|_M = \inf\{|g| : g \in R^n \text{ and } \pi(g) = h\}.$$

Consider a second surjection $\pi' : R^m \longrightarrow M$. Combining π and π' we obtain a third surjection $\pi'' : R^{n+m} \longrightarrow M$. It is enough to check that the quotient seminorms $|\cdot|_M, |\cdot|''_M$ induced by π, π'' are equivalent, as then the same argument will apply with π' and π'' leading to the conclusion that $|\cdot|_M, |\cdot|'_M$ induced by π, π' are equivalent.

Let $e_1, \dots, e_{n+m}$ be the standard basis of R^{n+m}. For $h \in M$,

$$\begin{aligned} |h|''_M =& \inf\{|g''| : g'' \in R^{n+m} \text{ and } \pi''(g'') = h\} \\ \leq& \inf\{|g| : g \in R^n \text{ and } \pi(g) = h\} \\ =& |h|_M, \end{aligned}$$

since the norm on R^{n+m} is compatible with the norm on R^n and lifting an element of M to R^n also gives a lift to R^{n+m}. This proves $|\cdot|''_M \leq |\cdot|_M$.

Now, for $j = n+1, \dots, n+m$ we can write $\pi(e_j) = \sum_{i=1}^{n} A_{ij}\pi(e_i)$ for some $A_{ij} \in R$. For an element $h \in M$ who lifts to $\sum_{i=1}^{n+m} c_i e_i \in R^{n+m}$, we have,

$$\begin{aligned} h &= \pi\Big(\sum_{i=1}^{n+m} c_i e_i\Big) \\ &= \sum_{i=1}^{n} c_i \pi(e_i) + \sum_{j=n+1}^{n+m} c_j \pi(e_j) \\ &= \sum_{i=1}^{n} c_i \pi(e_i) + \sum_{j=n+1}^{n+m} c_j \Big(\sum_{i=1}^{n} A_{ij}\pi(e_i)\Big) \\ &= \sum_{i=1}^{n} \Big(c_i + \sum_{j=n+1}^{n+m} c_j A_{ij}\Big)\pi(e_i) \\ &= \pi\Big(\sum_{i=1}^{n} \Big(c_i + \sum_{j=n+1}^{n+m} c_j A_{ij}\Big) e_i\Big). \end{aligned}$$

So h lifts also to $\sum_{i=1}^{n} \Big(c_i + \sum_{j=n+1}^{n+m} c_j A_{ij}\Big) e_i \in R^n$. Consequently,

$$|\cdot|_M \leq \max\{1, |A|\} |\cdot|''_M$$

This proves the equivalence.

(ii) Choose surjections $\pi_R : R^m \longrightarrow M$, $\pi_S : S^n \longrightarrow N$ of R-modules. We have to verify that $|\cdot|_M \geq |\cdot|_N \circ \theta$.
Define

$$\begin{aligned} p : R^m &\longrightarrow S^n \\ g = (g_i)_{1 \leq i \leq m} &\longmapsto l = (l_j)_{1 \leq j \leq n}, \end{aligned}$$

such that $\pi_S(l) = \theta(\pi_R(g))$, which is well defined due to the fact that θ is a R-linear map.
Let $k_1, \dots, k_m$ be a set of generators of the R-module M. Then $\theta(k_i) = \sum_{1 \leq j \leq n} s_{ij} n_{ij}$ for all $i = 1, \dots, m$, with $s_{ij} \in S$ and generators $n_{ij} \in N$. It follows that for any $g = (g_i)_{1 \leq i \leq m} \in R^m$ the image under p equals $l = (\sum_{i=1}^{m} \psi(g_i) s_{ij})_{1 \leq j \leq n}$,i.e.,

$$g = (g_i)_{1 \leq i \leq m} \longmapsto l = \left(\sum_{i=1}^{m} \psi(g_i) s_{ij}\right)_{1 \leq j \leq n}.$$

For $s := \max_{i,j}\{|s_{ij}|\}$, we get that $|p(g)| \leq cs|g|$ for some constant $c > 0$, since the homomorphism of Banach rings $\psi : R \longrightarrow S$ is bounded. We conclude that $p : R^m \longrightarrow S^n$ is also bounded and makes the following diagram commutative

$$\begin{tikzcd} R^m \arrow[r, "p"] \arrow[d, "\pi_R"'] & S^n \arrow[d, "\pi_S"] \\ M \arrow[r, "\theta"] & N \end{tikzcd}$$

Now, for $h \in M$ we have,

$$\begin{aligned}|\theta(h)|_N &\leq \inf\{|p(g)| : g \in R^m \text{ and } \pi_S(p(g)) = \theta(h)\}\\ &\leq cs \inf\{|g| : g \in R^m \text{ and } \pi_S(p(g)) = \theta(h)\}, \text{ for some } c > 0\\ &= cs \inf\{|g| : g \in R^m \text{ and } \theta(\pi_R(g)) = \theta(h)\}\\ &\leq cs \inf\{|g| : g \in R^m \text{ and } \pi_R(g) = h\}\\ &= cs|h|_M,\end{aligned}$$

which yields the lemma. □

Definition 2.3.3. Let R be a Banach ring. A *Banach module* over R is an R-module M whose additive group is complete for a norm $\|\cdot\|_M$ for which for, some $c > 0$, we have $\|r\mathbf{v}\|_M \leq c\alpha(r)\|\mathbf{v}\|_M$ for all $r \in R, \mathbf{v} \in M$. Here $\alpha(\cdot)$ denotes the norm on R. In particular, any Banach algebra over R is a Banach module over R.

Definition 2.3.4. Let R be a Banach ring with submultiplicative norm α.
An element $z \in R$ is called *topologically nilpotent* if $\lim_{i\to\infty}\alpha(z^i) = 0$.
An element $w \in R$ is called *power-bounded* if the set $\{\alpha(w^n) : n \in \mathbb{N}\} \subset \mathbb{R}_+$ is bounded.

Lemma 2.3.5. *Let R be a Banach ring. Let P be a finite projective R-module. Choose a finite R-module Q and an isomorphism $P \oplus Q \cong R^n$ of R-modules, for n a suitable nonnegative integer. Equip R^n with the supremum norm defined by the canonical basis.*

(i) The subspace norm on P for the inclusion into R^n is equivalent to the quotient norm for the projection from R^n, and gives P the structure of a finite Banach module over R.

(ii) The equivalence class of the norms described in (i.) is independent of the choice of Q and of the presentation $P \oplus Q \cong R^n$.

(iii) The above construction defines a fully faithful functor from the category of finite projective R-modules to the category of finite Banach modules over R whose underlying R-modules are projective (Definition 2.3.6), which is a section of the forgetful functor.

Proof. Consider two copies P', Q' of P, Q respectively. Note that for the presentations

$$(P \oplus Q) \oplus (P' \oplus Q') \cong R^n \oplus R^n, \qquad (P \oplus Q') \oplus (P' \oplus Q) \cong R^n \oplus R^n$$

of $P \oplus Q \oplus P' \oplus Q'$, the two supremum norms $|\cdot|_1, |\cdot|_2$ defined by each presentation respectively are equivalent by Lemma 2.3.2.
The subspace and quotient norms on $P \oplus Q$ induced by $|\cdot|_1$ are identical, and $P \oplus Q$ is complete under these norms. Accordingly, the subspace and quotient norms on $P \oplus Q$ induced by $|\cdot|_2$ are equivalent, and $P \oplus Q$ is complete under these norms. By restricting to P we obtain the subspace and quotient norms induced by the original presentation, so these two are equivalent.
Now, P is the intersection of the closed subspaces $P \oplus Q$ and $P \oplus Q'$ of $P \oplus Q \oplus P' \oplus Q' \cong R^n \oplus R^n$, so P is closed and therefore complete, since $R^n \oplus R^n$ is a Banach space. This proves (i). Parts (ii) and (iii) follow from Lemma 2.3.2, since morphisms in the category of finite Banach R-modules are bounded R-modules homomorphisms. □

Definition 2.3.6. Let R be a Banach ring. A *finite Banach module/algebra* over R is a Banach module/algebra M over R admitting a strict surjection $R^n \longrightarrow M$ of Banach modules over R for some nonnegative integer n (for the supremum norm on R^n defined by the canonical basis).
A morphism between finite Banach modules/algebras over R is a bounded morphism of modules/algebras over R.

Definition 2.3.7. For a Banach ring A and $B \in \mathbf{F\acute{E}t}(A)$, we can view B as a finite Banach module over A by Lemma 2.3.5. The multiplication map $\mu : B \otimes_A B \longrightarrow B$ is then bounded by Lemma 2.3.2 and Lemma 2.3.5, since μ is an additive A-linear map. Consequently, we can find an equivalent norm on B which is submultiplicative (see [20, §17 Proposition 17.4]), and thus consider B as a finite Banach algebra over A.

We end this section by defining the Tate algebra over a ring.

Definition 2.3.8. For $r_1, \dots, r_n > 0$, define the *Tate algebra* over the Banach ring A with submultiplicative seminorm α and radii $r_1, \dots, r_n$ to be the ring

$$A\{T_1/r_1, \dots, T_n/r_n\} := \left\{ f = \sum_I a_I T^I : a_I \in A, \lim_{I \to \infty} \alpha(a_I) r^I = 0 \right\}$$

where $I = (i_1, \dots, i_n)$ runs over n-tuples of nonnegative integers, $T^I = T_1^{i_1}, \dots, T_n^{i_n}$, and $r^I = r_1^{i_1} \dots r_n^{i_n}$. The set $A\{T_1/r_1, \dots, T_n/r_n\}$ is a subring of $A[\![T_1, \dots, T_n]\!]$ complete for the *Gauss norm*

$$\| \sum_I a_I T^I \|_r = \sup_I \{\alpha(a_I) r^I\}$$

which is easily seen to be submultiplicative (resp. power-multiplicative, multiplicative) if the seminorm on A is; see [18, Lemma 1.7]. In case $r_1 = \dots = r_n = 1$, we contract the notation to $A\{T_1, \dots, T_n\}$.
Let $\varphi : A \longrightarrow B$ be a bounded homomorphism of Banach rings. We say φ is *affinoid* if it factors through a strict surjection $\psi : A\{T_1, \dots, T_n\} \longrightarrow B$ for some positive integer n. So we obtain the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "i"'] & B \\
A\{T_1, \dots, T_n\} \arrow[ur, two heads, "\psi\ strict"']
\end{tikzcd}$$

We will also say that B is an *affinoid algebra* over A.
We say that A is *strongly noetherian* if every affinoid algebra over A is noetherian, or equivalently the rings $A\{T_1, \dots, T_n\}$ are noetherian for all $n \geq 0$.

Example 2.3.9. *For A an analytic field, we land in the category of A-affinoid algebras, which present the basic settings in rigid analytic geometry. In particular consider $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$; see [6, Part 1 §3].*

2.4 The Gel'fand spectrum of a Banach ring

Here we introduce the work of Berkovich concerning the definitions and main ideas of the topological space corresponding to a Banach ring; compare [3] and [4].

Throughout the rest of this work, let A be a Banach ring with a norm denoted by $|\cdot|$.

Definition 2.4.1. The *Gel'fand spectrum* $\mathcal{M}(A)$ of A is the set of multiplicative seminorms α on A bounded above by $|\cdot|$ (or equivalently, dominated by $|\cdot|$).
We equip $\mathcal{M}(A)$ with the weakest topology with respect to which all real valued functions on $\mathcal{M}(A)$ of the form $\alpha \mapsto \alpha(a)$, $a \in A$, are continuous. So we may see $\mathcal{M}(A)$ as a closed subspace of the product $\prod_{a\in A}[0, |a|]$; hence $\mathcal{M}(A)$ is compact by Tikhonov's theorem [7, §1.9.5, Théorème 3].
A subbasis of the topology on $\mathcal{M}(A)$ is given by the sets

$$\{\alpha \in \mathcal{M}(A) : \alpha(a) \in I\}$$

for each open interval $I \subseteq \mathbb{R}$ and $a \in A$.
Any bounded homomorphism $\varphi : A \longrightarrow B$ between Banach rings induces a continuous map $\varphi^* : \mathcal{M}(B) \longrightarrow \mathcal{M}(A)$ by $\varphi^*(\beta)(a) := \beta(\varphi(a)), a \in A$.

Lemma 2.4.2. *Let I be a proper ideal in a Banach ring A. Then the closure of I is also a proper ideal. In particular, any maximal ideal in A is closed.*

Proof. Assume the closure of I were A. In that case I would be dense in A. Consider the set:

$$U = \{a \in A : |1 - a| < 1\}.$$

Since U is open in A, there would exist an element $x \in I \cap U$. But then the series $\sum_{i=0}^{\infty}(1 - x)^i$ would converge to an inverse of x, contradicting the assumption that I is proper. □

We present now the first main theorem about the spectrum.

Theorem 2.4.3. *(Berkovich). For A nonzero, $\mathcal{M}(A) \neq \emptyset$.*

Proof. See [3, Theorem 1.2.1]. □

The theorem tells us that if our ring is not trivial, then we can find a multiplicative seminorm on A, which is dominated by the (in general not multiplicative) seminorm $|\cdot|$ of A.

Corollary 2.4.4. *For any proper ideal I of A, there exists $\alpha \in \mathcal{M}(A)$ such that $\alpha(f) = 0$ for all $f \in I$.*

Proof. Let J be the closure of I. By Lemma 2.4.2, A/J is nonzero, so $\mathcal{M}(A/J) \neq \emptyset$ by theorem 2.4.3. Any element of $\mathcal{M}(A/J)$ restricts to an element $\alpha \in \mathcal{M}(A)$ of the desired form. (Compare [3, Corollary 1.2.4]). □

Corollary 2.4.5. *A finite set $f_1, \ldots, f_n$ of elements of A generates the unit ideal if and only if for each $\alpha \in \mathcal{M}(A)$, there exists an index $i \in \{1, \ldots, n\}$ for which $\alpha(f_i) > 0$.*

Proof. Assume the finite set $f_1, \ldots, f_n$ generates the unit ideal in A, then there exists $u_1, \ldots, u_n \in A$ such that $u_1 f_1 + \cdots + u_n f_n = 1$. Since every $\alpha \in \mathcal{M}(A)$ is a multiplicative seminorm, $\max_i\{\alpha(u_i)\alpha(f_i)\} \geq 1$ and so $\alpha(f_i) > 0$ for some index i. Conversely, suppose that $f_1, \ldots, f_n$ generated a nontrivial ideal I; then by Corollary 2.4.4, we can choose $\alpha \in \mathcal{M}(A)$ such that $\alpha(f) = 0$ for all $f \in I$. □

Corollary 2.4.6. *An element $f \in A$ is a unit if and only if $\alpha(f) > 0$ for all $\alpha \in \mathcal{M}(A)$.*

Definition 2.4.7. For $\alpha \in \mathcal{M}(A)$, define the prime ideal $\mathfrak{p}_\alpha := \alpha^{-1}(0)$; then $\alpha \in \mathcal{M}(A)$ induces a multiplicative norm on $A/\mathfrak{p}_\alpha$. The completion of $\mathrm{Frac}(A/\mathfrak{p}_\alpha)$ for the unique multiplicative extension of this norm is called the *residue field* of α, denoted $\mathcal{H}(\alpha)$. The image of the map $\mathcal{M}(A) \longrightarrow \mathrm{Spec}(A)$, $\alpha \mapsto \mathfrak{p}_\alpha$ contains all maximal ideals, by Corollary 2.4.4. The image of an element $a \in A$ in $\mathcal{H}(\alpha)$ will be denoted by $a(\alpha)$.

The homomorphism

$$\hat{} : A \longrightarrow B := \prod_{\alpha \in \mathcal{M}(A)} \mathcal{H}(\alpha),$$

which sends $a \in A$ to the element $\hat{a} = (a(\alpha))_{\alpha \in \mathcal{M}(A)}$ is called the *Gel'fand tranform.*

Note that the induced map $\mathcal{M}(B) \longrightarrow \mathcal{M}(A)$ is surjective.

From this, we could also deduce the compactness of $\mathcal{M}(A)$ using the following result:

Lemma 2.4.8. *Let $\{K_i\}_{i \in I}$ be a family of valuation fields. Then the spectrum $\mathcal{M}(A)$ of the Banach ring $A = \prod_{i \in I} K_i$ is homeomorphic to the Stone-Čech compactification of the (discrete) set I.*

Proof. [3, Proposition 1.2.3]. □

2.5 The adic spectrum of an adic Banach ring

We continue by introducing another type of topological space corresponding to a Banach ring, following [17] and the work of Huber; [14], [15] and [16].

As in [17] we will begin by stating a basic result in general topology.

Lemma 2.5.1. *Let $f : X \longrightarrow Y$ be a continuous map of a quasicompact topological space to a Hausdorff topological space. Then f is closed.*

Proof. See [7, §1.9.4, Corollaire 2]. □

From now on we will impose some conditions on the Banach ring A. From this section onwards we will only consider Banach rings which are Banach algebras over an analytic field K.

Let us consider an important consequence of our assumption:

Let A be a Banach ring. If we may view A as a Banach algebra over some analytic field, then there exists a topologically nilpotent unit $z \in A$ such that $\alpha_{sp}(z)\alpha_{sp}(z^{-1}) = 1$. We will refer to any such z as a *uniform unit* in A. Note that for any $\beta \in \mathcal{M}(A)$,

$$1 = \beta(z)\beta(z^{-1}) \leq \alpha_{sp}(z)\alpha_{sp}(z^{-1}) = 1$$

and so $\beta(z) = \alpha_{sp}(z)$.

Consequently from now on, we only consider Banach rings containing a topologically nilpotent unit.

To justify the existence of a uniform unit z in a nonzero Banach algebra A over an analytic field K, consider an element $z \in K$, such that $|z|_K \in (0,1)$. Then $|z| \leq c|z|_K$ for some $c > 0$, where $|\cdot|_K$ denotes the norm in K. Since $|\cdot|_K$ is multiplicative, we have $|z^n|_K = |z|_K^n$ for any $n \in \mathbb{N}$ and $|z|_K^n \to 0$, when $n \to \infty$. It follows that $|z^n| \to 0$, when $n \to \infty$, which proves that z is a topologically nilpotent element of A.
Now, by Theorem 2.4.3, there exists $\beta \in \mathcal{M}(A)$. For this β we get:

$$\begin{aligned}
\alpha_{sp}(z)\alpha_{sp}(z^{-1}) &= \lim_{s\to\infty} |z^s|^{\frac{1}{s}} \lim_{r\to\infty} |(z^{-1})^r|^{\frac{1}{r}} \\
&\geq \lim_{s\to\infty} \beta(z^s)^{\frac{1}{s}} \lim_{r\to\infty} \beta((z^{-1})^r)^{\frac{1}{r}} \\
&= \lim_{s\to\infty} \beta(z) \lim_{r\to\infty} \beta(z^{-1}) \\
&= \beta(z)\beta(z^{-1}) \\
&= \beta(1) \\
&= 1.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\alpha_{sp}(z)\alpha_{sp}(z^{-1}) &= \lim_{s\to\infty} |z^s|^{\frac{1}{s}} \lim_{r\to\infty} |(z^{-1})^r|^{\frac{1}{r}} \\
&\leq \lim_{s\to\infty} |(z^s)|_K^{\frac{1}{s}} \lim_{r\to\infty} |(z^{-1})^r|_K^{\frac{1}{r}} \\
&= \lim_{s\to\infty} |z|_K \lim_{r\to\infty} |z^{-1}|_K \\
&= |z|_K|z^{-1}|_K \\
&= |1|_K \\
&= 1.
\end{aligned}$$

We conclude that $\alpha_{sp}(z)\alpha_{sp}(z^{-1}) = 1$.

Definition 2.5.2. For A a Banach ring equipped with a submultiplicative norm, let A° denote the subring of power-bounded elements of A.
An *adic Banach ring* is a pair (A, A^+) in which A is a Banach ring (which as mentioned above is now a Banach algebra over some analytic field) and A^+ is a subring of A° which is open and integrally closed in A. These conditions ensure that every topologically nilpotent element of A must belong to A^+.

A *morphism* of adic Banach rings $(A, A^+) \longrightarrow (B, B^+)$ is a bounded homomorphism $\varphi : A \longrightarrow B$ of Banach rings such that $\varphi(A^+) \subseteq B^+$. With this definition, the correspondence $A \longmapsto (A, A^\circ)$ defines a functor from the category of Banach rings to the category of adic Banach rings.

For $(A, A^+) \longrightarrow (B, B^+)$, $(A, A^+) \longrightarrow (C, C^+)$ two morphisms of adic Banach rings, their coproduct in the category of adic Banach rings will be denoted by $(B, B^+) \otimes_{(A,A^+)} (C, C^+)$. It consists of (D, D^+) where $D = B\hat{\otimes}_A C$ and D^+ is the completion of the integral closure of $B^+ \otimes_{A^+} C^+$ in D.

Definition 2.5.3. Let Γ be a totally ordered abelian group, and let Γ_0 be a pointed monoid $\Gamma \cup \{0\}$ ordered so that $0 < \gamma$ for all $\gamma \in \Gamma$. A *semivaluation* on a ring A with values in Γ is a function $v : A \longrightarrow \Gamma_0$ satisfying the following properties.

a.) For all $a, b \in A$, we have $v(a - b) \leq \max\{v(a), v(b)\}$.

b.) For all $a, b \in A$, we have $v(ab) = v(a)v(b)$.

c.) We have $v(0) = 0$ and $v(1) = 1$.

If morover $v^{-1}(\{0\}) = \{0\}$, we say that v is a *valuation*.

As an example consider $\Gamma = \mathbb{R}_{\geq 0}$, then a (semi)valuation is the same as a multiplicative (semi)norm.

For A a Banach ring, we declare two semivaluations on A, which possibly are defined into different ordered groups, to be *equivalent* if they define the same order relation on A. It is clear that this defines an equivalence relation and that the equivalence classes form a set. We denote this set by $\mathrm{Spv}(A)$. We will identify each equivalence class in $\mathrm{Spv}(A)$ with a particular representative in an arbitrary but fixed manner.

A semivaluation v on A is *continuous* if for every nonzero γ in the value group of v (i.e., the subgroup of Γ generated by the nonzero images of v) there is a neighborhood U of 0 in A such that $v(u) < \gamma$ for all $u \in U$.

The *adic spectrum* of (A, A^+) is the subset $\mathrm{Spa}(A, A^+)$ of $\mathrm{Spv}(A)$ consisting of the equivalence classes of continuous semivaluations on A bounded by 1 on A^+. Since A^+ is integrally closed in A, we have the following equality

$$A^+ = \{x \in A : v(x) \leq 1 \text{ with } v \in \mathrm{Spa}(A, A^+)\ \}.$$

See [15, Proposition 1.6] for more details.

We equip $X := \mathrm{Spa}(A, A^+)$ with the topology generated by the sets of the form

$$\{v \in \mathrm{Spv}(A, A^+) : v(a) \leq v(b) \neq 0\} \qquad (a, b \in A).$$

A *rational subspace* of $\mathrm{Spa}(A, A^+)$ is a set of the form:

$$X\left(\frac{f_1}{g}, \dots, \frac{f_n}{g}\right) = \{v \in \mathrm{Spa}(A, A^+) : v(f_i) \leq v(g) \neq 0,\ (i = 1, \dots, n)\}.$$

for some $f_1, \ldots, f_n, g \in A$ generating the unit ideal. Note that we would get the same definition if we only require that $f_1, \ldots, f_n$ generate the unit ideal, since then it would be harmless to append g as an extra generator.
Note that any morphism $\psi : (A, A^+) \longrightarrow (B, B^+)$ of adic Banach rings induces a continuous map

$$\psi^* : \mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+),$$

under which the inverse image of any rational subspace is again a rational subspace.

One of the most important theorems of the work of Huber about adic spectra is the following result, which can be seen as an analogue to the compactness of the Gel'fand spectrum.

Theorem 2.5.4. *(Huber). The space* $\mathrm{Spa}(A, A^+)$ *is quasicompact and the rational subspaces form a topological basis consisting of quasicompact open subsets.*

Proof. See [14, Theorem 3.5(i,ii)]. □

We will now relate this construction to the Gel'fand spectrum.

Definition 2.5.5. There is a natural map $\mathcal{M}(A) \longrightarrow \mathrm{Spa}(A, A^+)$ taking each $\alpha \in \mathcal{M}(A)$ to the equivalence class of α as a semivaluation. Note that this map is not continuous.
Now suppose that A contains a uniform unit z; then there is a map

$$j : \mathrm{Spa}(A, A^+) \longrightarrow \mathcal{M}(A)$$

defined as follows. Given a semivaluation $v \in \mathrm{Spa}(A, A^+)$, define the multiplicative seminorm $\alpha = \alpha(v) \in \mathcal{M}(A)$ by the formula

$$\alpha(x) = \inf\{\alpha_{sp}(z)^{r/s} : r \in \mathbb{Z}, s \in \mathbb{Z}_{>0}, v(z^r) < v(x^s)\}.$$

The composition $\mathcal{M}(A) \longrightarrow \mathrm{Spa}(A, A^+) \longrightarrow \mathcal{M}(A)$ is the identity. In particular, the map $\mathcal{M}(A) \longrightarrow \mathrm{Spa}(A, A^+)$ is injective, and by Theorem 2.4.3, $\mathrm{Spa}(A, A^+) \neq \emptyset$ whenever $A \neq 0$.

Definition 2.5.6. We define a *rational subspace* of $\mathcal{M}(A)$ as the intersection of $\mathcal{M}(A)$ with a rational subspace of $\mathrm{Spa}(A, A^+)$. For a rational subspace U of $\mathrm{Spa}(A, A^+)$ as in Definition 2.5.3, the corresponding rational subspace of $\mathcal{M}(A)$ is

$$\{\alpha \in \mathcal{M}(A) : \alpha(f_i) \leq \alpha(g) \text{ for } i = 1, \ldots, n\}$$

and the image of U in $\mathcal{M}(A)$ is equal to the intersection $U \cap \mathcal{M}(A)$. As a corollary, we see that every nonempty rational subspace of $\mathrm{Spa}(A, A^+)$ meets $\mathcal{M}(A)$, so $\mathcal{M}(A)$ is dense in $\mathrm{Spa}(A, A^+)$.
Rational subspaces of $\mathcal{M}(A)$ are closed, not open; as a result, not every rational subspace containing some $\alpha \in \mathcal{M}(A)$ is a neighborhood of α.However, those which are neighborhoods form a neighborhood basis of a in $\mathcal{M}(A)$; we say that such rational subspaces *encircle* a.

Lemma 2.5.7. *Given a rational subspace* $U = X(\frac{f_1}{g}, \dots, \frac{f_n}{g})$ *of* $\mathrm{Spa}(A, A^+)$*, let* $\overline{U}$ *be the image of* U *under the projection* $j : \mathrm{Spa}(A, A^+) \longrightarrow \mathcal{M}(A)$ *. Then* $\overline{U}$ *is compact in* $\mathcal{M}(A)$ *and* $c := \inf\{\alpha(g) : \alpha \in \overline{U}\} > 0$.

Proof. Since U is quasicompact by Theorem 2.5.4 and j is continuous by the discussion above, $\overline{U}$ is a closed subset of a compact space, due to the topology of $\mathcal{M}(A)$ and Lemma 2.5.1. It follows that $\overline{U}$ is compact.
For the second assertion notice that we have $\alpha(g) > 0$ for all $\alpha \in \overline{U}$, so by compactness

$$c = \inf\{\alpha(g) : \alpha \in \overline{U}\} > 0.$$

□

Keeping the same notation as above, consider

$$0 < \epsilon < \min\left\{\frac{1}{\max\{|u_1|, \dots, |u_n|, |u|\}}, c\right\},$$

where $f_1 u_1 + \cdots + f_n u_n + gu = 1$.

Any $f_1', \cdots, f_n', g' \in A$ satisfying $|f_i - f_i'| < \epsilon, |g - g'| < \epsilon$ generate the unit ideal, since

$$|f_1' + \cdots + f_n' + g' - 1| \le \epsilon \max\{\{|u_1|, \cdots, |u_n|, |u|\} < 1$$

and $f_1' + \cdots + f_n' + g' \in A$ would have an inverse.

For $f_1', \cdots, f_n', g'$ we get that $v(f_i - f_i'), v(g - g') < v(g)$, for any $v \in U$, since assuming the contrary leads to $v(f_i - f_i')^s, v(g - g')^s \ge v(g)^s$ for any $s \in \mathbb{Z}_{>0}$, implying $j(v)(f_i - f_i'), j(v)(g - g') \ge j(v)(g)$, which produces the contradiction

$$\begin{aligned} j(v)(g) &> \epsilon \\ &> |f_i - f_i'|, |g - g'| \\ &\ge j(v)(f_i - f_i'), j(v)(g - g') \\ &\ge j(v)(g). \end{aligned}$$

Let $v \in U$. Then, since $v(g - g') < v(g)$,

$$v(g') = v(g' - g + g) = \max\{v(g - g'), v(g)\} = v(g),$$

so $v(g) = v(g')$. This proves that

$$U \subseteq U' := \{v \in \mathrm{Spa}(A, A^+) : v(f_i') \le v(g') \text{ for } i = 1, \dots, n\}$$

For the other inclusion, notice that

$$1 = v(1) = v(f_1' u_1' + \cdots + f_n' u_n' + g' u') \le \max\{v(u_1'), \cdots, v(u_n'), v(u')\} v(g'),$$

for every $v \in U'$. Thus $0 < v(g')$. As explained above, we obtain $v(f_i - f_i'), v(g - g') < v(g') = v(g)$, for every $v \in U'$. We conclude that $U = U'$.

Consequently, one may drop the condition $v(g) \neq 0$ for the definition of a rational subspace.

Definition 2.5.8. An *adic field* is an adic Banach ring (K, K^+) in which K is an analytic field and K^+ is a valuation ring in K (i.e., a subring containing either x or $1/x$ for each $x \in K^\times$). The space $\mathrm{Spa}(K, K^+)$ is not in general a point; however, the valuation corresponding to K^+ defines the unique closed point of $\mathrm{Spa}(K, K^+)$.
Given $v \in \mathrm{Spa}(A, A^+)$, let $(\mathcal{H}(v), \mathcal{H}(v)^+)$ be the adic field with $\mathcal{H}(v) = \mathcal{H}(\alpha(v))$ and $\mathcal{H}(v)^+$ equal to the valuation ring of the continuous multiplicative extension of v to $\mathcal{H}(\alpha(v))$. By construction, there is a canonical morphism $(A, A^+) \longrightarrow (\mathcal{H}(v), \mathcal{H}(v)^+)$ such that the induced map $\mathrm{Spa}(\mathcal{H}(v), \mathcal{H}(v)^+) \longrightarrow \mathrm{Spa}(A, A^+)$ maps the unique closed point of $(\mathcal{H}(v), \mathcal{H}(v)^+)$ to v.

Definition 2.5.9. Let U be a quasicompact open subset of $\mathrm{Spa}(A, A^+)$. We say that U is an *affinoid subdomain* of $\mathrm{Spa}(A, A^+)$ if there exists an affinoid homomorphism $\varphi : (A, A^+) \longrightarrow (B, B^+)$ which is initial among morphisms $\psi : (A, A^+) \longrightarrow (C, C^+)$ of adic Banach rings for which $\psi^*(\mathrm{Spa}(C, C^+)) \subseteq U$. We refer to the representing morphism $(A, A^+) \longrightarrow (B, B^+)$ as an affinoid localization.
Every rational subspace U is an affinoid subdomain and the map $\mathrm{Spa}(B, B^+) \cong U$ is a homeomorphism, as we will see in the Lemma 2.5.10 below. We thus refer to U also as a *rational subdomain* and the corresponding affinoid localization also as a rational localization.

Lemma 2.5.10. *Let U be a rational subspace of* $\mathrm{Spa}(A, A^+)$ *as in Definition 2.5.3.*

(i) The subspace U is an affinoid subdomain represented by $\varphi : (A, A^+) \longrightarrow (B, B^+)$, where B is the quotient of $A\{T_1, \dots, T_n\}$ for the closure of the ideal $(gT_1 - f_1, \dots, gT_n - f_n)$, equipped with the quotient norm, and B^+ is the completion of the integral closure of the image of $A^+[T_1, \dots, T_n]$ in B.

(ii) The map $\varphi^* : \mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+)$ induces a homeomorphism $\mathrm{Spa}(B, B^+) \cong U$. More precisely, the rational subspaces of $\mathrm{Spa}(B, B^+)$ correspond to the rational subspaces of $\mathrm{Spa}(A, A^+)$ contained in U.

Proof. We will guide our proof following [15] and the notes of Professor T. Wedhorn on adic spaces.

Let (A, A^+) be an adic Banach ring, $g \in A$ and let $T = \{f_1, \dots, f_n\} \subseteq A$ be a finite subset such that TA is open in A. Then there exists on $A_g := A[\frac{1}{g}]$ a (unique) nonarchimedean ring topology, defined by T, making it into a topological ring

$$A_g = A\Big(\frac{T}{g}\Big) = A\Big(\frac{f_1, \dots, f_n}{g}\Big),$$

such that $\{f_i g^{-1} : f_i \in T\}$ is power-bounded in $A(\frac{T}{g})$ and such that the canonical homomorphism $\varphi : A \longrightarrow A(\frac{T}{g})$ satisfy the following universal property. If B is a nonarchimedean topological ring and $\psi : A \longrightarrow B$ is a continuous homomorphism such that $\psi(g)$ is invertible in B and such that the set $\{\psi(f_i)\psi(g)^{-1} : f_i \in T\}$ is power-bounded in B, then there exists a unique continuous ring homomorphism $\alpha : A(\frac{T}{g}) \longrightarrow B$ with $\psi = \alpha \circ \varphi$.

The topology defined by T on A_g is the coarsest and finest topology such that the subgroups

$$E_i := \{f_i g^{-1} : f_i \in T\}$$

of $(A, +)$ build a fundamental system of neighborhoods of 0.
Let A_g^+ be the integral closure of $A^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ in A_g. Then A_g^+ is a ring of integral elements in $A(\frac{T}{g})$. The completion of $(A(\frac{T}{g}), A_g^+)$ is denoted by $(A\langle\frac{T}{g}\rangle, A^+\langle\frac{T}{g}\rangle^+)$. Hence, $(A\langle\frac{T}{g}\rangle, A^+\langle\frac{T}{g}\rangle^+)$ is an adic Banach ring.
In a similar way, the canonical continuous homomorphism of adic Banach rings

$$\rho : (A, A^+) \longrightarrow \Big(A\Big\langle\frac{T}{g}\Big\rangle, A\Big\langle\frac{T}{g}\Big\rangle^+\Big)$$

is universal for continuous homomorphisms of adic Banach rings $\psi : A \longrightarrow B$, with $\psi(g) \in B^\times$ and $\frac{\psi(f_i)}{\psi(g)} \in B^+$ for all $f_i \in T$.
Since

$$A\{T_1, \dots, T_n\}/\overline{(gT_1 - f_1, \dots, gT_n - f_n)}$$

fulfills this universal property, it follows that

$$A\{T_1, \dots, T_n\}/\overline{(gT_1 - f_1, \dots, gT_n - f_n)} \cong A\Big\langle\frac{T}{g}\Big\rangle.$$

Therefore, it would be enough to show the Lemma for $(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$.
Note that in our present setting, to be a continuous homomorphism of adic Banach rings, is equivalent to be a bounded homomorphism of adic Banach rings.

(i) The definition of $(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$ shows that for all $v \in \operatorname{Spa}(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$ one has $v(\rho(f_i)) \leq v(\rho(g))$ for all $i = 1, \dots, n$. This means that

$$\rho^* : \operatorname{Spa}\Big(A\Big\langle\frac{T}{g}\Big\rangle, A\Big\langle\frac{T}{g}\Big\rangle^+\Big) \longrightarrow \operatorname{Spa}(A, A^+)$$

factors through U. Now, consider a continuous homomorphism from (A, A^+) to an adic Banach ring (B, B^+), $\psi : (A, A^+) \longrightarrow (B, B^+)$ such that ψ^* factors through U.
As ψ^* factors through U, we have $w(\psi(f_i)) \leq w(\psi(g)) \neq 0$ for all $w \in \operatorname{Spa}(B, B^+)$ and for all $i = 1, \dots, n$. This implies that $\psi(g) \in B^\times$ by Corollary 2.4.6. Moreover, for all $w \in \operatorname{Spa}(B, B^+)$ we have $w(\frac{\psi(f_i)}{\psi(g)}) \leq 1$ for all $i = 1, \dots, n$. This implies that $\frac{\psi(f_i)}{\psi(g)} \in B^+$ for all $i = 1, \dots, n$. Thus, we can apply the universal property of

$$\rho : (A, A^+) \longrightarrow \Big(A\Big\langle\frac{T}{g}\Big\rangle, A\Big\langle\frac{T}{g}\Big\rangle^+\Big)$$

and conclude that there exists a unique continuous homomorphism of Banach rings $\beta : (A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+) \longrightarrow (B, B^+)$, such that $\beta \circ \rho = \psi$.

(ii) We saw that the map

$$\rho^* : \operatorname{Spa}\Big(A\Big\langle\frac{T}{g}\Big\rangle, A\Big\langle\frac{T}{g}\Big\rangle^+\Big) \longrightarrow \operatorname{Spa}(A, A^+)$$

factors through U, so by definition, ρ^* gives us a continuous map from $\mathrm{Spa}(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$ to U. To see that the map is bijective, choose any $v \in U$. Again, by the universal property of $\rho : (A, A^+) \longrightarrow (A\langle\frac{T}{g}\rangle, \langle\frac{T}{g}\rangle^+)$, the map $(A, A^+) \longrightarrow (\mathcal{H}(v), \mathcal{H}(v)^+)$ factors uniquely through a unique continuous homomorphism of adic Banach rings $\beta : (A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+) \longrightarrow (\mathcal{H}(v), \mathcal{H}(v)^+)$. The closed point of $(\mathcal{H}(v), \mathcal{H}(v)^+)$ maps to the unique point of $\mathrm{Spa}(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$ in the preimage of v.
To see that the induced homomorphism

$$j' : \mathrm{Spa}\Big(A\Big\langle\frac{T}{g}\Big\rangle, A\Big\langle\frac{T}{g}\Big\rangle^+\Big) \longrightarrow U$$

is a homeomorphism, it suffices to check the final assertion, i.e., that any rational subspace of $\mathrm{Spa}(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$ is also a rational subspace of $\mathrm{Spa}(A, A^+)$.
Let $V = X(\frac{h_1}{r}, \dots, \frac{h_m}{r})$ be a rational subspace of $(A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$, for $h_1, \dots, h_m, r \in A(\frac{T}{g})$ by Lemma 2.5.7. Multiplying $h_1, \dots, h_m, r$ with a suitable power of g we may assume that all these elements lie in the image of $\rho : (A, A^+) \longrightarrow (A\langle\frac{T}{g}\rangle, A\langle\frac{T}{g}\rangle^+)$, say $r = \rho(q)$ and $\{h_1, \dots, h_m\} = \rho(H)$ for some $q \in A$ and some finite subset H of A. As V is quasi-compact, then $j'(V)$ is quasi-compact. Now, every $v \in j'(V)$ is of the form $v \circ \rho$ for some $v \in V$. Thus by definition $v(q) \neq 0$ for all $v \in j'(V)$. Consider the following rational subspaces:

$$X_k := \{v \in \mathrm{Spa}(A, A^+) : v(f_i)^k \leq v(q) \neq 0 \text{ for all } i = 1, \dots, n\},$$

for each $k \in \mathbb{N}$.
Then $j'(V) \subseteq \bigcup_k X_k$, which implies that $j'(V) \subseteq X_l$ for some $l \in \mathbb{N}$. It follows that for the rational subspace W of $\mathrm{Spa}(A, A^+)$ generated by $H \cup \{f_1^l, \dots, f_n^l\} \cup \{q\}$, we get $j'(V) = U \cap W$. Hence, $j'(V)$ is a rational subspace of $\mathrm{Spa}(A, A^+)$. □

Remark 2.5.11. In the category of affinoid K-algebras, where K is an analytic field, we can characterize the intersection of two affinoid subdomains in the following way. Let $\tau_1 : R \longrightarrow A_1$ and $\tau_2 : R \longrightarrow A_2$ be homomorphisms of affinoid K-algebras. Then the complete tensor product $A_1 \hat{\otimes}_R A_2$ is an affinoid K-algebra as well by [6, Theorem 6 Appendix B]. Accordingly, the category of affinoid K-algebras admits amalgamated sums and the completed tensor product of two affinoid (resp. rational) localizations is again such a localization. By [6, §3.3, Proposition 13] such a localization correspond to the intersection of the affinoid (resp. rational) subdomains.
We have a similar result for rational subspaces of $\mathrm{Spa}(A, A^+)$:
Consider two rational subspaces U_1, U_2 of $\mathrm{Spa}(A, A^+)$. Let $T_1 := \{f_1, \dots, f_n, g\}$ and $T_2 := \{f_1', \dots, f_n', g'\}$ be generating sets for U_1 and U_2 respectively. Then the intersection of U_1 and U_2 is equal to the rational subspace generated by $T := \{t_1 t_2 : t_i \in T_i\}$, i.e.,

$$\mathrm{Spa}\Big(A\Big\langle\frac{T_1}{g}\Big\rangle, A\Big\langle\frac{T_1}{g}\Big\rangle^+\Big) \cap \mathrm{Spa}\Big(A\Big\langle\frac{T_2}{g'}\Big\rangle, A\Big\langle\frac{T_2}{g'}\Big\rangle^+\Big) = \mathrm{Spa}\Big(A\Big\langle\frac{T}{gg'}\Big\rangle, A\Big\langle\frac{T}{gg'}\Big\rangle^+\Big)$$

It follows, that the intersection of two rational subspaces correspond to the completed tensor product of its rational localizations, since it fulfills the universal property of $(A\langle\frac{T}{gg'}\rangle, A\langle\frac{T}{gg'}\rangle^+)$.

In order to establish analogies to the theory of algebraic geometry and in order to comprehend fully our *Theorem 1.0.1*, we will need to define a structure sheaf on adic Banach rings. At this point it is important to mention that the structure sheaf that we will define, following [17, §2], differs from the structure sheaf on an affine scheme, since it will be defined in the category of complete topological rings. For example, for an affine scheme $X = \mathrm{Spec}(A)$ with A a Banach ring and $\mathcal{O}_X$ its classical structure sheaf, we know that $\Gamma(D(f), \mathcal{O}_X) \cong A_f$, where $D(f)$ is an element of the base of the topology of X and $f \in A$. For the structure sheaf we will defined, the global section of an element of the basis of the topology (a rational subspace) will be the completion $\widehat{A_f}$ for A_f for some $f \in A$, which in general differs from A_f; see Definition 2.5.13 and compare to Lemma 2.5.10.

Definition 2.5.12. By a *rational covering* (resp. *affinoid covering*) of $\mathrm{Spa}(A, A^+)$, we will mean either a finite colletion $\{U_i\}_i$ of rational (resp. affinoid) subdomains of $\mathrm{Spa}(A, A^+)$ forming a set-theoretic covering, or the corresponding collection $\{\mathrm{Spa}(A, A^+) \longrightarrow \mathrm{Spa}(B, B^+)\}_i$ of a rational (resp. affinoid) localizations, depending on the context. Note that a rational covering of $\mathrm{Spa}(A, A^+)$ induces a set-theoretic covering of $\mathcal{M}(A)$ by rational subspaces, but not conversely in general. However, for a finite collection of rational subspaces $\{U_i \cap \mathcal{M}(A)\}_{i \in I}$ whose relative interiors cover $\mathcal{M}(A)$, we may consider the inverse image under the projection j of those interiors, which will be equal to U_i in $\mathrm{Spa}(A, A^+)$, for all $i \in I$. Thus, we can induce a rational covering of $\mathrm{Spa}(A, A^+)$, since $\mathcal{M}(A)$ is dense in $\mathrm{Spa}(A, A^+)$; we call such a covering a *strong rational covering* of $\mathrm{Spa}(A, A^+)$ (or of $\mathcal{M}(A)$).

Definition 2.5.13. Define the *structure presheaf* $\mathcal{O}$ on $\mathrm{Spa}(A, A^+)$ as the functor taking each open subset U to the inverse limit of B over all rational localizations $(A, A^+) \longrightarrow (B, B^+)$ for which $\mathrm{Spa}(B, B^+) \subseteq U$ (i.e., $\mathcal{O} : U \mapsto \varprojlim_{\mathrm{Spa}(B,B^+) \subseteq U} B$). In particular, for any rational localization $(A, A^+) \longrightarrow (B, B^+)$, we have

$$\Gamma(\mathrm{Spa}(B, B^+), \mathcal{O}) = B,$$

since $\mathrm{Spa}(B, B^+)$ is final in the index category.

We say that (A, A^+) is *sheafy* if the structure presheaf is a sheaf on $\mathrm{Spa}(A, A^+)$; we will prove below (Lemma 3.2.1) that an equivalent condition is that for any rational localization $(A, A^+) \longrightarrow (B, B^+)$ and for any rational covering $\mathfrak{B} := (U_i)_{i \in I}$ of $\mathrm{Spa}(B, B^+)$, the map $B \longrightarrow \check{H}^0(\mathrm{Spa}(B, B^+), \mathcal{O}; \mathfrak{B})$ is an isomorphism. In this case, $(\mathrm{Spa}(A, A^+), \mathcal{O})$ is a locally ringed space.

Let us mention another important and interesting Theorem due to Huber.

Theorem 2.5.14. *(Huber). Let (A, A^+) be an adic Banach ring such that A is strongly noetherian. Then (A, A^+) is sheafy.*

Proof. See [15, Theorem 2.2]. □

We will end the first part of our work presenting a key argument that will allow us to prove the Tate and Kiehl properties of the structure sheaf on $\mathrm{Spa}(A, A^+)$. The argument is based on the theory of rigid geometry for affinoid algebras (see [6, Part I, §4]), which consists in reducing certain questions about coverings to coverings of a simple form.

Definition 2.5.15. For $f_1, \ldots, f_n \in A$ generating the unit ideal, the *standard rational covering* of $\mathrm{Spa}(A, A^+)$ generated by $f_1, \ldots, f_n$ is the covering by rational subspaces

$$U_i = X\left(\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}\right) = \{v \in \mathrm{Spa}(A, A^+) \ : v(f_j) \le v(f_i), \text{ for } j = 1, \ldots, n\},$$

with $i = 1, \ldots, n$.

For $f_1, \ldots, f_n \in A$ arbitrary and $v \in \mathrm{Spa}(A, A^+)$ we either have $v(f_i) < 1$, $v(f_i) > 1$ or $v(f_i) = 1$, for every $i = 1, \ldots, n$. This motivates the definition of the *standard Laurent covering* generated by $f_1, \ldots, f_n$, which is the covering given by the rational subspaces

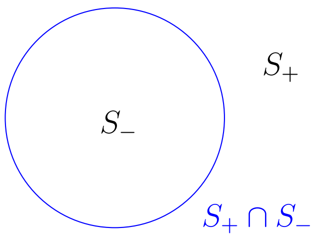
$$S_e = \bigcap_{i=1}^{n} S_{i,e_i} = \bigcap_{i=1}^{n} X\left(f_i^{e_i}\right) \qquad (e = (e_1, \ldots, e_n) \in \{-, +\}^n),$$

where

$$S_{i,-} = X\left(f_i^{+}\right) = X\left(\frac{f_i}{1}\right) = \{v \in \mathrm{Spa}(A, A^+) : v(f_i) \le 1\},$$
$$S_{i,+} = X\left(f_i^{-}\right) = X\left(\frac{1}{f_i}\right) = \{v \in \mathrm{Spa}(A, A^+) : v(f_i) \ge 1\}.$$

A standard Laurent covering with $n = 1$ is also called a *simple Laurent covering.*

We can think of a simple Laurent covering as a separation of $\mathrm{Spa}(A, A^+)$ through a ball of radius one between the elements inside the ball, the ones outside the ball and the ones on the border of the ball:



Definition 2.5.16. Consider two coverings $\mathfrak{A} = (U_i)_{i \in I}$ and $\mathfrak{B} = (V_j)_{j \in J}$ of a topological space X. Then $\mathfrak{B}$ is called a *refinement* of $\mathfrak{A}$ if there exists a map $\tau : J \longrightarrow I$ such that $V_j \subset U_{\tau(j)}$ for all $j \in \tau$.

Lemma 2.5.17. *The following statements hold.*

(i) Any rational covering can be refined by a standard rational covering.

(ii) For any standard rational covering $\mathfrak{A}$ of $X = \mathrm{Spa}(A, A^+)$, there exists a standard Laurent covering $\mathfrak{B}$ of X such that for each $V = \mathrm{Spa}(B, B^+) \in \mathfrak{B}$, the restriction of $\mathfrak{A}$ to V (omitting empty intersections) is a standard rational covering generated by units in B.

(iii) Any standard rational covering generated by units can be refined by a standard Laurent covering generated by units.

Proof. (i) Consider a rational covering given by $(U_i)_{1\le i\le n}$ with

$$U_i = X\Big(\frac{f_1^{(i)}}{f_0^{(i)}}, \ldots, \frac{f_{r_i}^{(i)}}{f_0^{(i)}}\Big)$$

Now, consider the set I of all tuples $(v_1,\ldots,v_n)\in\mathbb{N}^n$ with $0\le v_i\le r_i$ and set

$$f_{v_1,\ldots,v_n} := \prod_{i=1}^{n} f_{v_i}^{(i)}$$

for such tuples. Writing I' for the set of all $(v_1,\ldots,v_n)\in I$ such that $v_i=0$ for at least one i, then the functions

$$f_{v_1,\ldots,v_n}, \qquad (v_1,\ldots,v_n)\in I'$$

form a standard rational covering $\mathfrak{B}$.
Indeed, suppose that the functions $f_{v_1,\ldots,v_n}$, $(v_1,\ldots,v_n)\in I'$, do not generate the unit ideal in A, then by Corollary 2.4.5, there exists $\alpha\in\mathcal{M}(A)$ with $\alpha(f_{v_1,\ldots,v_n})=0$ for all $f_{v_1,\ldots,v_n}$, $(v_1,\ldots,v_n)\in I'$. Since the $(U_i)_{1\le i\le n}$ form a covering, there exists U_j such that $\alpha\in U_j$. Then $\alpha(f_0^{(j)})>0$ and $\alpha(\prod_{i\ne j} f_{v_i}^{(i)})=0$ for $0\le v_i\le r_i$. This contradicts the fact that for each i, the functions $f_0^{(i)},\ldots,f_{r_i}^{(i)}$ generate the unit ideal of A, since for each i there exists $0\le v_i\le r_i$ such that $\alpha(f_{v_i}^{(i)})>0$, due to Corollary 2.4.5, and then it would be impossible to have $\prod_{i\ne j}\alpha(f_{v_i}^{(i)})=\alpha(\prod_{i\ne j} f_{v_i}^{(i)})=0$. Thus $\mathfrak{B}$ is well defined.

Lets see that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$, or that $\mathfrak{B}$ refines $\mathfrak{A}$.
Consider a tuple $(v_1,\ldots,v_n)\in I'$ and look at the set

$$X_{v_1,\ldots,v_n} := X\Big(\frac{f_{\mu_1,\ldots,\mu_n}}{f_{v_1,\ldots,v_n}} : (\mu_1,\ldots,\mu_n)\in I'\Big)\in\mathfrak{B}$$

where, for example $v_n=0$. We will show that $X_{v_1,\ldots,v_n}\subset U_n$. Choose $v\in X_{v_1,\ldots,v_n}$ and an index μ_n, $0\le\mu_n\le r_n$.
We have to proof that

$$v(f_{\mu_n}^{(n)})\le v(f_0^{(n)})=v(f_{v_n}^{(n)}).$$

Again, since the $(U_i)_{1\le i\le n}$ form a covering, there exists U_j such that $v\in U_j$. If $j=n$, then there is nothing to be proved. Assume thus that j is different from n, say $j=1$. It follows $v(f_{v_1}^{(1)})\le v(f_0^{(1)})$, for $0\le v_1\le r_1$ and

$$v\Big(\prod_{i=1}^{n-1} f_{v_i}^{(i)}\Big)v(f_{\mu_n}^{(n)})\le v(f_0^{(1)})v\Big(\prod_{i=2}^{n-1} f_{v_i}^{(i)}\Big)v(f_{\mu_n}^{(n)})\le v\Big(\prod_{i=1}^{n} f_{v_i}^{(i)}\Big) \tag{1}$$

as the tuple $(0,v_2,\ldots,v_{n-1},\mu_n)$ belongs to I'.
Suppose that

$$v(f_{\mu_n}^{(n)}) > v(f_0^{(n)})=v(f_{v_n}^{(n)}).$$

Then

$$v(f_{v_n}^{(n)})v\Big(\prod_{i=1}^{n-1} f_{v_i}^{(i)}\Big) < v(f_{\mu_n}^{(n)})v\Big(\prod_{i=1}^{n-1} f_{v_i}^{(i)}\Big),$$

which contradicts (1). Hence, $v(f_{\mu_n}^{(n)}) \leq v(f_0^{(n)}) = v(f_{v_n}^{(n)})$.
We conclude that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$.

(ii) Let $\mathfrak{A}$ be a standard rational covering consisting of $(U_i)_{0\leq i\leq n}$, with

$$U_i = X\Big(\frac{f_0}{f_i}, \dots, \frac{f_r}{f_i}\Big).$$

Since $f_1 \dots, f_n$ generate the unit ideal of A, we have that for each $\alpha \in \mathcal{M}(A)$ there exists $0 \leq i \leq r$ with $\alpha(f_i) > 0$, due to Corollary 2.4.5.
Now consider the projection $j : \mathrm{Spa}(A, A^+) \longrightarrow \mathcal{M}(A)$. As previously discussed j is continuous and following the notation of Lemma 2.5.7 we conclude that $\overline{U_i} =: j(U_i)$ is compact in $\mathcal{M}(A)$, and

$$c_{(i)} := \inf\{\alpha(f_i) : \alpha \in \overline{U_i}\} > 0.$$

For each $\alpha \in \mathcal{M}(A)$, $\alpha \in U_l$ for some $0 \leq l \leq n$, since $\mathfrak{A}$ is a covering. It follows, that $\alpha \in \overline{U_l}$ and as we have just seen the quantity $c_{(l)} = \inf\{\alpha(f_l) : \alpha \in \overline{U_l}\} > 0$. We conclude then,

$$c := \inf\left\{\max_{0\leq i\leq r} \alpha(f_i) : \alpha \in \mathcal{M}(A)\right\} > 0.$$

Now we use the fact that our ring contains a topologically nilpotent unit z. Following Definition 2.3.4, there exists $k \in \mathbb{N}$ such that $|z^k| < c$. For this choice of k, we have for all $\alpha \in \mathcal{M}(A)$:

$$1 < \frac{\left\{\max\limits_{0\leq i\leq r} \alpha(f_i)\right\}}{|z^k|} \leq \frac{\left\{\max\limits_{0\leq i\leq r} \alpha(f_i)\right\}}{\alpha(z^k)} = \left\{\max_{0\leq i\leq r} \alpha\Big(\frac{f_i}{z^k}\Big)\right\},$$

since $\alpha(z^k) \leq |z^k|$.
We conclude that for the generators $g_0 := f_0 z^{-k}, \dots, g_r := f_r z^{-k}$, the quantity

$$d := \inf\left\{\max_{0\leq i\leq r} \alpha(g_i) : \alpha \in \mathcal{M}(A)\right\}$$

fulfills: $d > 1$.
Note that $g_0, \dots, g_r$ generate the same standard rational covering $\mathfrak{A}$, since we just multiply the generators of $\mathfrak{A}$ by a constant element z^{-k}.

We claim that the standard Laurent covering $\mathfrak{B}$ defined by $g_0, \dots, g_r$ has the desired property. To justify this, consider a set $V = X\Big(g_0^{e_0}, \dots, g_r^{e_r}\Big) \cong \mathrm{Spa}(B, B^+) \in \mathfrak{B}$, where $e_0, \dots, e_r \in \{-, +\}$ and $\varphi : (A, A^+) \longrightarrow (B, B^+)$ is the rational localization associated to V. We may assume that $e_0 = \dots = e_s = +$ and that $e_{s+1} = \dots = e_r = -$, for some $0 \leq s \leq r$.
Let $v \in V$. Then for any g_i with $i = 0, \dots, s$, we have that

$$v(g_i) < \max_{i=s+1,\dots,r} \{v(g_i)\},$$

since assuming the contrary gives rise to the contradiction

$$\begin{aligned}\max_{i=s+1,\dots,r}\{j_B(v)(g_i)\} &> 1\\ &\geq |g_i|_B\\ &\geq j_B(v)(g_i)\\ &\geq \max_{i=s+1,\dots,r}\{j_B(v)(g_i)\},\end{aligned}$$

where $|\cdot|_B$ denotes the norm in B and $j_B : \mathrm{Spa}(B, B+) \longrightarrow \mathcal{M}(B)$ is the projection in B. Remember that $j_B|_A \in \mathcal{M}(A)$, which justifies the first inequality. It follows that

$$U_i \cap V = \emptyset \quad \text{ for } i = 0, \dots, s,$$

since

$$\max_{i=0,\dots,s}\{v(g_i)\} < \max_{i=s+1,\dots,r}\{v(g_i)\},$$

and in particular

$$\max_{i=0,\dots,r}\{v(g_i)\} = \max_{i=s+1,\dots,r}\{v(g_i)\}.$$

And so $\mathfrak{A}|_V$ is a standard rational covering of $\mathrm{Spa}(B, B^+)$ generated by the units $g_{s+1}, \dots, g_r$. To support this claim, notice that for all $\alpha \in \mathcal{M}(B) \hookrightarrow \mathrm{Spa}(B, B^+) \cong V$, $\alpha(g_i) \geq 1 > 0$ for $s+1 \leq i \leq r$. So by Corollary 2.4.5., each g_i with $s+1 \leq i \leq r$ is a unit in B.

(iii) Consider the standard rational covering $\mathfrak{A} = (U_k)_{0 \leq j \leq r}$ generated by the units $f_1, \dots, f_r$. Let $\mathfrak{B}$ be the Laurent covering of X generated by all products

$$f_i f_j^{-1} \qquad 1 \leq i < j \leq r.$$

We claim that $\mathfrak{B}$ refines $\mathfrak{A}$.
To verify this consider a $V \in \mathfrak{B}$. Given elements $i, j \in S := \{1, \dots, r\}$, we write $i << j$ if $v(f_i) \leq v(f_j)$ for all $v \in V$. The relation ”$<<$” is transitive, since if $i << j$ and $j << r$, then $v(f_i) \leq v(f_j)$ and $v(f_j) \leq v(f_r)$ for all $v \in V$ and the order relation ”$\leq$” is transitive. The relation ”$<<$” is also total, since for $v \in V$ we have

$$\begin{gathered} v(f_i f_j^{-1}) \leq 1 \text{ or } v(f_i f_j^{-1}) \geq 1\\ \Longleftrightarrow\\ v(f_i) \leq v(f_j) \text{ or } v(f_j) \leq v(f_i).\end{gathered}$$

It is known that every total transitive order on a finite set has a maximal element, so let $i_s \in S$ be the maximal element of the relation ”$<<$”. It follows that $v(f_i) \leq v(f_{i_s})$ for all $v \in V$ and so

$$V \subset X\Big(\frac{f_0}{f_{i_s}}, \dots, \frac{f_r}{f_{i_s}}\Big).$$

In this way $\mathfrak{B}$ is a refinement of $\mathfrak{A}$. $\square$

Using the above Lemma, we establish the following criterion for rational coverings.

Lemma 2.5.18. *Let $\mathcal{P}$ be a property of rational coverings of rational subdomains of* $\mathrm{Spa}(A, A^+)$ *satisfying the following conditions.*

a.) The property $\mathcal{P}$ is local: if it holds for a refinement of a given covering, it also holds for the original covering.

b.) The property $\mathcal{P}$ is transitive: if it holds for a covering $\{(B, B^+) \longrightarrow (C_i, C_i^+)\}_i$ and for some coverings $\{(C_i, C_i^+) \longrightarrow (D_{ij}, D_{ij}^+)\}_j$ for each i, then it holds for the composite covering $\{(B, B^+) \longrightarrow (D_{ij}, D_{ij}^+)\}_{i,j}$.

c.) The property $\mathcal{P}$ holds for any simple Laurent covering.

Then the property $\mathcal{P}$ holds for any rational covering of any rational subdomain of $\mathrm{Spa}(A, A^+)$.

Proof. We will divide the demonstration into four observations:

Let $U \cong \mathrm{Spa}(B, B^+)$ be a rational subdomain of $\mathrm{Spa}(A, A^+)$.

(i) We may deduce $\mathcal{P}$ for any *standard Laurent covering generated by units in B.*

We argue by induction. Let $\mathfrak{B}$ be a standard Laurent covering of $U = \mathrm{Spa}(B, B^+)$. In the case $\mathfrak{B}$ is defined by a unit $f_1 \in B$, then by condition c.) property $\mathcal{P}$ holds for $\mathfrak{B}$. Say then that the property $\mathcal{P}$ holds for the case when $\mathfrak{B} = (S_e)_{e \in \{-,+\}^{n-1}}$ is defined by $f_1, \ldots, f_{n-1}$, all units in B. Consider the simple Laurent covering $\mathfrak{A}$ defined by a unit $f_n \in B$. For each $S_e \in \mathfrak{B}$,

$$S_e = X\left(f_1^{e_1}, \ldots, f_{n-1}^{e_{n-1}}\right), \quad e_1, \ldots, e_{n-1} \in \{+, -\},$$

the covering $\mathfrak{A}|_{S_e} = (U_{e,j})_{0 \leq j \leq 1}$, with

$$U_{e,j} = X\left(f_1^{e_1}, \ldots, f_{n-1}^{e_{n-1}}, f_n^j\right) \in \mathfrak{A}|_{S_e}, \quad j \in \{+, -\},$$

fulfills the property $\mathcal{P}$ by condition c.)
Then we can apply b.) and conclude that the composition

$$\{(B, B^+) \longrightarrow (C_e, C_e^+) \longrightarrow (D_{e,j}, D_{e,j}^+)\}_{e,j},$$

where $(C_e, C_e^+) \cong S_e$ and $(D_{e,j}, D_{e,j}^+) \cong U_{e,j}$, satisfy the property $\mathcal{P}$. But this composition is precisely the standard Laurent covering generated by the units $f_1, \ldots, f_n$. We conclude that any standard Laurent covering is a composition of simple Laurent coverings, thus we can apply b.) and c.) to deduce (i).

(ii) We may deduce $\mathcal{P}$ for any *standard rational covering generated by units.*

Apply Lemma 2.5.17(iii), to refine the covering by a standard Laurent covering generated by units, then invoke a.) and (i).

(iii) We may deduce $\mathcal{P}$ for any *standard rational covering.*

Given a standard rational covering $\{(B, B^+) \longrightarrow (C_i, C_i^+)\}_i$, use Lemma 2.5.17(ii) to obtain a standard Laurent covering $\{(B, B^+) \longrightarrow (D_j, D_j^+)\}_j$ such that for each j, the covering

$$\{(D_j, D_j^+) \longrightarrow (C_i, C_i^+)\widehat{\otimes}_{(B,B^+)}(D_j, D_j^+)\}_i$$

is a standard rational covering generated by units in D_j. Note that

$$(C_i, C_i^+)\widehat{\otimes}_{(B,B^+)}(D_j, D_j^+)$$

correspond to the intersection of $\mathrm{Spa}(D_j, D_j^+)$ and $\mathrm{Spa}(C_i, C_i^+)$ by Remark 2.5.11.
We may thus deduce $\mathcal{P}$ for the covering

$$\{(B, B^+) \longrightarrow (C_i, C_i^+)\widehat{\otimes}_{(B,B^+)}(D_j, D_j^+)\}_{i,j}$$

by invoking (ii) and b.), which is a refinement of $\{(B, B^+) \longrightarrow (C_i, C_i^+)\}_i$. Invoking a.) follows that any standard rational covering fulfills $\mathcal{P}$.

(iv) We may deduce $\mathcal{P}$ for any *rational covering* by applying Lemma 2.5.17(i) to refine the covering by a standard rational covering and then invoke (iii) and a.). This proofs our claim. $\square$

3 Part: Tate sheaf property

In this section we will prove that for a sheafy adic Banach ring (A, A^+) the structural sheaf on $\mathrm{Spa}(A, A^+)$ satisfies acyclicity for every rational localization $(A, A^+) \to (B, B^+)$, where $U \cong \mathrm{Spa}(B, B^+)$.
We will use Lemma 2.5.18 in order to verify acyclicity of sheaves of rings. First for simple Laurent coverings, and then for any rational covering, by proving that acyclicity fulfills the conditions of the Lemma. We begin with some important results in the theory of Čech cohomology.

3.1 Čech cohomology

Here we will discuss the overall setting of Čech cohomology that will be required in this work. We will give some basic definitions, as well as settle some notation for the Čech complexes. Our objective for this part will be to proof some results relating Čech cohomology and Grothendieck cohomology, as it will be the conclusion of the comparison Theorem for Čech cohomology and its Corollary, see Theorem 3.1.9 and Corollary 3.1.10.

Let X be a topological space, and let $\mathfrak{A} = (U_i)_{i\in I}$ be an open covering of X. Fix, once and for all, a well-ordering of the index set I. For any finite set of indices $i_0, \dots, i_q \in I$

we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_q}$ by $U_{i_0 \dots i_q}$.
Now let $\mathcal{F}$ be a presheaf of abelian groups on X. We define a complex $C^\bullet(\mathfrak{A}, \mathcal{F})$ of abelian groups as follows. For each $q \geq 0$, let

$$C^q(\mathfrak{A}, \mathcal{F}) = \prod_{i_0 < \cdots < i_q} \mathcal{F}(U_{i_0 \dots i_q}).$$

Thus an element $\alpha \in C^q(\mathfrak{A}, \mathcal{F})$ is determined by giving an element

$$\alpha_{i_0 \dots i_q} \in \mathcal{F}(U_{i_0 \dots i_q})$$

for each $(q+1)$-tuple $i_0 < \cdots < i_q$ of elements of I. We define the co-boundary map $d^q : C^q \longrightarrow C^{q+1}$ by setting

$$(d^q \alpha)_{i_0 \dots i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k \alpha_{i_0, \dots, \hat{i}_k, \dots i_{q+1}} |_{U_{i_0 \cdots i_{q+1}}}.$$

Here the notation $\hat{i}_k$ means to omit i_k. Since $\alpha_{i_0, \cdots \hat{i}_k, \dots i_{q+1}}$ is an element of $\mathcal{F}(U_{i_0, \cdots \hat{i}_k, \dots i_{q+1}})$, we restrict to $U_{i_0 \dots i_{q+1}}$ to get an element of $\mathcal{F}(U_{i_0 \dots i_{q+1}})$. One checks easily that $d^q \circ d^{q-1} = 0$, so we have indeed defined a complex of abelian groups.

Definition 3.1.1. Let X be a topological space and let $\mathfrak{A}$ be an open covering of X. For any presheaf of abelian groups $\mathcal{F}$ on X, we define the qth *Čech cohomology group of $\mathcal{F}$, with respect to the covering* $\mathfrak{A}$, to be

$$\check{H}^q(X, \mathcal{F}; \mathfrak{A}) = h^q(C^\bullet(\mathfrak{A}, \mathcal{F})),$$

the qth cohomology object of the complex $C^\bullet(\mathfrak{A}, \mathcal{F})$.

Remark 3.1.2. If $\alpha \in C^q(\mathfrak{A}, \mathcal{F})$, it is sometimes convenient to have the symbol $\alpha_{i_0 \dots i_q}$ defined for *all* $(q+1)$-tuples of elements of I, as for examples in double complexes, see below. If there is a repeated index in the set $\{i_0, \dots, i_q\}$, we define $\alpha_{i_0 \dots i_q} = 0$. If the indices are all distinct, we define $\alpha_{i_0 \dots i_q} = (-1)^\delta \alpha_{\delta(i_0) \dots \delta(i_q)}$, where δ is the permutation for which $\delta(i_0) < \cdots < \delta(i_q)$. With these conventions, one can check that the formula given above for $d^q \alpha$ remains correct for any $(q+2)$-tuple $i_0, \dots, i_{q+1}$ of elements of I.

Definition 3.1.3. Let X be a topological space and $\mathcal{F}$ a presheaf of abelian groups. Then, for any open covering $\mathfrak{A} = (U_i)_{i \in I}$ of X, one defines an augmentation homomorphism

$$\begin{gathered} \varepsilon : \mathcal{F}(X) \longrightarrow C^0(\mathfrak{A}, \mathcal{F}) \\ f \mapsto (f|_{U_i}) \end{gathered}$$

mapping $\mathcal{F}(X)$ into $\ker d^0$. The map ε is used to construct the so-called *augmented Čech complex* $C^\bullet_{aug}(\mathfrak{A}, \mathcal{F})$, which is given by $C^q_{aug}(\mathfrak{A}, \mathcal{F}) := C^q(\mathfrak{A}, \mathcal{F})$ for $q \neq -1$ and $C^{-1}_{aug}(\mathfrak{A}, \mathcal{F}) := \mathcal{F}(X)$ with co-boundary homomorphism d^q_{aug} as in $C^q(\mathfrak{A}, \mathcal{F})$ except that $d^{-1}_{aug} := \varepsilon$. The associated cohomology object $h^q_{aug}(C^\bullet_{aug}(\mathfrak{A}, \mathcal{F}))$ of the complex $C^\bullet_{aug}(\mathfrak{A}, \mathcal{F})$ coincides with the Čech cohomology group $\check{H}^q(X, \mathcal{F}; \mathfrak{A})$, for $q \geq 1$.
If all cohomology objects $h^q_{aug}(C^\bullet_{aug}(\mathfrak{A}, \mathcal{F}))$ vanish, i.e., if the sequence

$$0 \to \mathcal{F}(X) \xrightarrow{\varepsilon} C^0(\mathfrak{A}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathfrak{A}, \mathcal{F}) \to \dots$$

is exact, the covering $\mathfrak{A}$ is called *$\mathcal{F}$-acyclic.* This condition is equivalent to the fact that $\check{H}^q(X, \mathcal{F}; \mathfrak{A}) = 0$ for all $q \neq 0$ and that ε induces a bijection $\mathcal{F} \xrightarrow{\sim} \check{H}^0(X, \mathcal{F}; \mathfrak{A})$. To give another characterization of acyclicity, we consider the trivial covering $\mathfrak{A}_0 = X$ of X. This covering is $\mathcal{F}$-acyclic by [5, Corollary 7.6.2], since the augmentation is the identity map in this case. Therefore, an arbitrary open covering $\mathfrak{A}$ is $\mathcal{F}$-acyclic if and only if all homomorphisms

$$\varrho^q(\mathfrak{A}, \mathfrak{A}_0) : \check{H}^q(X, \mathcal{F}; \mathfrak{A}_0) \longrightarrow \check{H}^q(X, \mathcal{F}; \mathfrak{A}) \tag{2}$$

are bijective. Namely, we have $\check{H}^q(X, \mathcal{F}; \mathfrak{A}) = 0$ for all $q \geqslant 1$. Furthermore, if $\tau^\bullet : C^\bullet(\mathfrak{A}_0, \mathcal{F}) \longrightarrow C^\bullet(\mathfrak{A}, \mathcal{F})$ is the complex homomorphism associated to the refinement $\mathfrak{A}$ of $\mathfrak{A}_0$, then $\tau^0 : C^0(\mathfrak{A}_0, \mathcal{F}) \longrightarrow C^0(\mathfrak{A}, \mathcal{F})$ coincides with the augmentation $\varepsilon : \mathcal{F}(X) \longrightarrow C^0(\mathfrak{A}, \mathcal{F})$.

For what it is to come we follow [1, §8.1]. We would like to see in more detail the relation between Čech complexes associated to refinements of a given covering of a topological space X. In turn, this will help us proof the Comparison Theorem for Čech cohomology.

Consider two open coverings $\mathfrak{A} = (U_i)_{i \in I}$ and $\mathfrak{B} = (V_j)_{j \in J}$ of a topological space X and assume that $\mathfrak{B}$ is a refinement of $\mathfrak{A}$, i.e., that there exists a map $\tau : J \longrightarrow I$ satisfying $V_j \subset U_{\tau(j)}$, for all $j \in J$. Any such map τ induces a homomorphism

$$\tau^q : C^q(\mathfrak{A}, \mathcal{F}) \longrightarrow C^q(\mathfrak{B}, \mathcal{F}),$$

where $\mathcal{F}$ is a presheaf of abelian groups on X and $f \in C^q(\mathfrak{A}, \mathcal{F})$ is mapped onto the element $\tau^q(f)$ with components

$$(\tau^q(f))_{j_0 \ldots j_q} := f_{\tau(j_0) \ldots \tau(j_q)}|_{V_{j_0 \ldots j_q}}.$$

The maps τ^q constitute a homomorphism of complexes

$$\tau^\bullet : C^\bullet(\mathfrak{A}, \mathcal{F}) \longrightarrow C^\bullet(\mathfrak{B}, \mathcal{F}).$$

Although the map $\tau : J \longrightarrow I$ is not uniquely determined by the coverings $\mathfrak{A}$ and $\mathfrak{B}$, we can show that the induced maps

$$\check{H}^q(\tau^\bullet) : \check{H}^q(X, \mathcal{F}; \mathfrak{A}) \longrightarrow \check{H}^q(X, \mathcal{F}; \mathfrak{B})$$

are independent of τ. Namely, let $\tau' : J \longrightarrow I$ be a second map satisfying $V_j \subset U_{\tau'(j)}$ for all $j \in J$. Then one verifies that the homomorphisms

$$l^q : C^q(\mathfrak{A}, \mathcal{F}) \longrightarrow C^{q-1}(\mathfrak{B}, \mathcal{F})$$

given by

$$(l^q(f))_{j_0 \ldots j_{q-1}} = \sum_{k=0}^{q-1} (-1)^k f_{\tau(j_0) \ldots \tau(j_k) \tau'(j_k) \ldots \tau'(j_{q-1})}|_{V_{j_0 \ldots j_{q-1}}}$$

define a homotopy between $\tau^\bullet$ and $\tau'^\bullet$; thus, the maps $\check{H}^q(\tau^\bullet)$ and $\check{H}^q(\tau'^\bullet)$ must coincide for all q. We will use the notation $\varrho^q(\mathfrak{B}, \mathfrak{A})$ instead of $\check{H}^q(\tau^\bullet)$ or $\check{H}^q(\tau'^\bullet)$, as in [1, §8.1]. Note that $\varrho^q(\mathfrak{A}, \mathfrak{A})$ =id and that $\varrho^q(\mathfrak{W}, \mathfrak{A}) = \varrho^q(\mathfrak{W}, \mathfrak{B}) \circ \varrho^q(\mathfrak{B}, \mathfrak{A})$ if $\mathfrak{W}$ is a refinement of $\mathfrak{B}$ and $\mathfrak{B}$ is a refinement of $\mathfrak{A}$. In particular, we have

Lemma 3.1.4. *Assume that the coverings $\mathfrak{A}$ and $\mathfrak{B}$ are refinements of each other. Then $\varrho^q(\mathfrak{B},\mathfrak{A}) : \check{H}^q(X,\mathcal{F};\mathfrak{A}) \longrightarrow \check{H}^q(X,\mathcal{F};\mathfrak{B})$ is bijective and its inverse is $\varrho^q(\mathfrak{A},\mathfrak{B})$ for all q.*

Using the previous definitions for $\mathcal{F}$-acyclicity and remembering that X is an open subset of itself, one derives

Lemma 3.1.5. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be open coverings of X which are refinements of each other. Then, $\mathfrak{A}$ is $\mathcal{F}$-acyclic if and only if $\mathfrak{B}$ is $\mathcal{F}$-acyclic.*

Now we center our attention into proving the Comparison Theorem for Čech cohomology. In order to do this, some knowledge in the cohomology of double complexes is needed, especially the results that are linked to argumentations involving spectral sequences; see [21, § 0.2.3].

Let R be a commutative ring. A *double complex* $K^{\bullet\bullet}$ of R-modules consists of a collection of R-modules $K^{p,q}, p, q \in \mathbb{Z}$, and of R-module homomorphisms

$$\begin{aligned} {'d}^{p,q} &: K^{p,q} \longrightarrow K^{p+1,q} \qquad \text{and} \\ {''d}^{p,q} &: K^{p,q} \longrightarrow K^{p,q+1}, \end{aligned}$$

satisfying

$$\begin{aligned} &{'d}^{p+1,q} \circ {'d}^{p,q} = 0, \\ &{''d}^{p,q+1} \circ {''d}^{p,q} = 0, \quad \text{and} \\ &{''d}^{p+1,q} \circ {'d}^{p,q} +' d^{p,q+1} \circ {''d}^{p,q} = 0 \end{aligned}$$

for all $p, q \in \mathbb{Z}$. Thus, we may interpret $K^{\bullet\bullet}$ as a diagram

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ \cdots \arrow[r] & K^{p,q} \arrow[r, "{''d^{p,q}}"] \arrow[d, "{'d^{p,q}}"] & K^{p,q+1} \arrow[r] \arrow[d, "{'d^{p,q+1}}"] & \cdots \\ \cdots \arrow[r] & K^{p+1,q} \arrow[r, "{''d^{p+1,q}}"] \arrow[d] & K^{p+1,q+1} \arrow[r] \arrow[d] & \cdots \\ & \vdots & \vdots & \end{tikzcd}$$

with co-boundary homomorphisms ${'d}^{p,q}$ (for a fixed q) as well as ${''d}^{p,q}$ (for p fixed) such that all squares are anticommutative.
It follows in particular, that for a fixed $q \in \mathbb{Z}$, the modules $K^{p,q}$ with co-boundary homomorphisms ${'d}^{p,q}$ constitute a single complex. This complex is called the q-th column of $K^{\bullet\bullet}$, and it will be denoted by ${'K}^{\bullet q}$. Analogously, one defines the p-th row ${''K}^{p\bullet}$ of $K^{\bullet\bullet}$ by using the maps ${''d}^{p,q}$ as co-boundary maps.

There is a third way of deriving a single complex from $K^{\bullet\bullet}$. For $r \in \mathbb{Z}$, set $K^r := \bigoplus_{p+q=r} K^{p,q}$ and define homomorphisms $d^r : K^r \longrightarrow K^{r+1}$ by $d^r|K^{p,q} :={}'d^{p,q}+{}''d^{p,q}$ for all pairs (p,q) such that $p+q=r$. It is easily seen that $d^{r+1} \circ d^r = 0$ for all r; hence the modules K^r and the maps d^r constitute a complex $K^\bullet$ which is called the *single complex associated to* $K^{\bullet\bullet}$. The cohomology objects of $K^\bullet$ are also referred to as the cohomology objects of $K^{\bullet\bullet}$, i.e., we set

$$h^r(K^{\bullet\bullet}) := h^r(K^\bullet), \qquad r \in \mathbb{Z}.$$

In the following we are only interested in double complexes $K^{\bullet\bullet}$ which vanish at negative integers. Thus, we always assume that $K^{p,q} = 0$ for $p < 0$ or $q < 0$.

Lemma 3.1.6. *Let $K^{\bullet\bullet}$ be a double complex and consider the homomorphism of complexes ${}'\pi : K^\bullet \longrightarrow {}'K^{\bullet 0}$ induced by the natural projections $K^r = \bigoplus_{p+q=r} K^{p,q} \longrightarrow K^{r,0}$.*
Then, if $h^p({}'K^{\bullet q}) = 0$ for all $p \geq 0$ and for all $q > 0$, the maps $h^r({}'\pi) : h^r(K^\bullet) \longrightarrow h^r({}'K^{\bullet 0})$ are bijective for all r.

Proof. For $i \geq 0$, we consider the subcomplex $K_i^\bullet$ of $K^\bullet$ which is defined by

$$K_i^r := \bigoplus_{\substack{p+q=r \\ q \geq i}} K^{p,q}.$$

Note that $K_0^\bullet = K^\bullet$. Furthermore, there are natural isomorphisms $K_i^r/K_{i+1}^r \xrightarrow{\sim} K^{r-i,i}$ which constitute an isomorphism of complexes $K_i^\bullet/K_{i+1}^\bullet \xrightarrow{\sim} {}'K^{\bullet i}$ of degree $-i$ (i.e., one obtains an isomorphism in the usual sense if the indices of all modules in ${}'K^{\bullet i}$ are enlarged by i). Thus we have $h^r(K_i^\bullet/K_{i+1}^\bullet) \cong h^{r-i}({}'K^{\bullet i}) = 0$ for $i > 0$ and for all r by our assumption. Looking at the long cohomology sequence corresponding to

$$0 \longrightarrow K_i^\bullet/K_{i+1}^\bullet \longrightarrow K^\bullet/K_{i+1}^\bullet \longrightarrow K^\bullet/K_i^\bullet \longrightarrow 0$$

we get bijections $h^r(K^\bullet/K_{i+1}^\bullet) \xrightarrow{\sim} h^r(K^\bullet/K_i^\bullet)$ for $i > 0$, and, by induction

$$h^r(K^\bullet/K_i^\bullet) \xrightarrow{\sim} h^r(K^\bullet/K_1^\bullet) = h^r({}'K^{\bullet 0})$$

for $i > 0$. Since $h^r(K^\bullet)$ is canonically isomorphic to $h^r(K^\bullet/K_i^\bullet)$ for $i \geq r+2$, the assertion of the lemma follows.

□

We will use the previous lemma to derive a key argument for the proof of the Comparison Theorem for Čech cohomology.

Lemma 3.1.7. *Let $K^{\bullet\bullet}$ be a double complex, $K^\bullet$ the associated single complex. Denote by $K''^\bullet$ the subcomplex*

$$K''^q := ker\ {}'d^{0,q}.$$

Then, if $h^p({}'K^{\bullet q}) = 0$ for all $p > 0$ and for all q, the inclusion $K''^\bullet \hookrightarrow K^\bullet$ induces bijections

$$h^r(K''^\bullet) \xrightarrow{\sim} h^r(K^\bullet),$$

for all r.

Proof. Considering the long cohomology sequence corresponding to the short exact sequence

$$0 \longrightarrow K''^\bullet \longrightarrow K^\bullet \longrightarrow K^\bullet/K''^\bullet \longrightarrow 0,$$

we have only to show that $h^r(K^\bullet/K''^\bullet) = 0$ for all r. For this purpose we introduce the double complex $L^{\bullet\bullet}$ defined by

$$L^{p,q} := \begin{cases} K^{p,q} & \text{if} \quad p \neq 0 \\ K^{0,q}/K''^q & \text{if} \quad p = 0. \end{cases}$$

with co-boundary maps being induced by $K^{\bullet\bullet}$. Then, by our construction,

$$h^0('L^{\bullet q}) = 0$$

for all q. From the definition of $L^{\bullet\bullet}$ and the assumption on $h^p('K^{\bullet q})$, we have

$$h^p('L^{\bullet q}) = h^p('K^{\bullet q}) = 0$$

for all $p > 0$ and for all q. Since $K^\bullet/K''^\bullet$ is the single complex associated to $L^{\bullet\bullet}$, we get from Lemma 3.1.6

$$h^r(K^\bullet/K''^\bullet) = h^r('L^{\bullet 0}) = 0$$

for all r.

□

As usual let X be a topological space. We want to define a double Čech complex $C^{\bullet\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$ depending on two open coverings $\mathfrak{A} = (U_i)_{i\in I}$ and $\mathfrak{B} = (V_j)_{j\in J}$ of X. For $p, q \geq 0$, we set (using the same notations as before)

$$C^{p,q}(\mathfrak{A}, \mathfrak{B}) := \prod_{\substack{(i_0 \dots i_p)\in I^{p+1} \\ (j_0 \dots j_q)\in J^{q+1}}} \mathcal{F}(U_{i_0\dots i_p} \cap V_{j_0\dots j_q})$$

and define homomorphisms

$$'d^{p,q} : C^{p,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) \longrightarrow C^{p+1,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) \quad \text{and}$$

$$''d^{p,q} : C^{p,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) \longrightarrow C^{p,q+1}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$$

where, for any $f \in C^{p,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$, the $(i_0, \dots, i_{p+1}, j_0, \dots, j_q)$-component of $'d^{p,q}(f)$ is given by

$$\sum_{k=0}^{p+1} (-1)^{k+q} f_{i_0\dots\hat{i}_k\dots i_{p+1}, j_0\dots j_q}|_{U_{i_0\dots i_{p+1}} \cap V_{j_0\dots j_q}}$$

and the $(i_0, \ldots, i_p, j_0, \ldots, j_{q+1})$-component of $''d^{p,q}(f)$ is given by

$$\sum_{l=0}^{q+1}(-1)^l f_{i_0\ldots i_p, j_0\ldots\hat{j}_l\ldots j_q}|_{U_{i_0\cdots i_{p+1}}\cap V_{j_0\cdots j_{q+1}}}.$$

It is easy to see that the objects $C^{p,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$ together with the maps $'d^{p,q}$ and $''d^{p,q}$ constitute a double complex $C^{\bullet\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$. From this double complex, one can derive single complexes as outlined before. The q-th column and the p-th row of $C^{\bullet\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$ are described as follows:

$$'C^{\bullet q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) = \prod_{(j_0,\ldots,j_q)\in J^{q+1}} C^{\bullet}_{(-1)^q}(\mathfrak{A}|_{V_{j_0\cdots j_q}}, \mathcal{F}) \quad \text{and}$$

$$''C^{p\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} C^{\bullet}(\mathfrak{B}|_{U_{i_0\cdots i_p}}, \mathcal{F})$$

Here the product of complexes is understood in the obvious way. Furthermore, $C^{\bullet}_{(-1)^q}$ is the complex obtained from $C^{\bullet}$ by multiplying the co-boundary maps with $(-1)^q$, and for any open $V \subseteq X$, the covering $\mathfrak{A}|_V := \{U_i \cap V\}_{i\in I}$ is the restriction of $\mathfrak{A}$ to V (likewise for $\mathfrak{B}$). Also, on the right-hand sides, $\mathcal{F}$ must be interpreted as its restriction to $V_{j_0\ldots j_q}$ or $U_{i_0\ldots i_p}$.
The augmentations

$$\mathcal{F}(V_{j_0\ldots j_q}) \longrightarrow C^0(\mathfrak{A}|_{V_{j_0\cdots j_q}}, \mathcal{F})$$

induce homomorphisms

$$C^q(\mathfrak{B}, \mathcal{F}) \longrightarrow \ker {'d^{0,q}} \subset C^{0,q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}),$$

which, in turn, can be interpreted as a homomorphism

$$i'' : C^{\bullet}(\mathfrak{B}, \mathcal{F}) \longrightarrow C^{\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$$

into the single complex associated to $C^{\bullet\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$. Furthermore, i'' maps $C^{\bullet}(\mathfrak{B}, \mathcal{F})$ into the subcomplex $C''^{\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$ of $C^{\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$ which is given by

$$C''^{q}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) = \ker {'d^{0,q}}.$$

Now assuming that the covering $\mathfrak{A}|_{V_{j_0\cdots j_q}}$ is $\mathcal{F}$-acyclic for all indices $j_0, \ldots, j_q \in J$ and for all q, we see that i'' maps $C^{\bullet}(\mathfrak{B}, \mathcal{F})$ isomorphically onto $C''^{\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$. In addition, it follows from our description of the q-th row that

$$h^p({'C^{\bullet q}}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})) = 0$$

for all $p > 0$ and for all q. Thus, Lemma 3.1.7 can be applied and we get

Lemma 3.1.8. *If the covering $\mathfrak{A}|_{V_{j_0\cdots j_q}}$ is $\mathcal{F}$-acyclic for all indices $j_0, \ldots, j_q \in J$ and for all q, then the homomorphisms*

$$i'' : C^{\bullet}(\mathfrak{B}, \mathcal{F}) \longrightarrow C^{\bullet}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$$

induces bijections

$$h^r(i'') : \check{H}^r(X, \mathcal{F}; \mathfrak{B}) \xrightarrow{\sim} h^r({'C^{\bullet}}(\mathfrak{A}, \mathfrak{B}; \mathcal{F}))$$

Of course, there is an analogue of Lemma 3.1.8 for the homomorphism

$$i' : C^\bullet(\mathfrak{A}, \mathcal{F}) \longrightarrow C^\bullet(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$$

which is derived from the augmentations

$$\mathcal{F}(U_{i_0 \dots i_p}) \longrightarrow C^0(\mathfrak{B}|_{U_{i_0 \dots i_p}}, \mathcal{F}).$$

Thus, we obtain the following result:

Theorem 3.1.9 (Comparison Theorem)**.** *Assume that all coverings $\mathfrak{A}|_{V_{j_0 \cdots j_q}}$ and $\mathfrak{B}|_{U_{i_0 \cdots i_p}}$ are $\mathcal{F}$-acyclic. Then one gets bijections*

$$\check{H}^r(X, \mathcal{F}; \mathfrak{A}) \xrightarrow[\sim]{h^r(i')} h^r('C^\bullet(\mathfrak{A}, \mathfrak{B}; \mathcal{F})) \xleftarrow[\sim]{h^r(i'')} \check{H}^r(X, \mathcal{F}; \mathfrak{B})$$

and a commutative diagram

$$\begin{tikzcd}
\check{H}^r(X, \mathcal{F}; \mathfrak{A}) \arrow[r, "h^r(i')", "\sim"'] & h^r('C^\bullet(\mathfrak{A}, \mathfrak{B}; \mathcal{F})) & \check{H}^r(X, \mathcal{F}; \mathfrak{B}) \arrow[l, "h^r(i'')"', "\sim"] \\
\check{H}^r(X, \mathcal{F}; \mathfrak{A}_0) \arrow[u, "\varrho^r(\mathfrak{A}, \mathfrak{A}_0)"] \arrow[urr, "\varrho^r(\mathfrak{B}, \mathfrak{A}_0)"'] & &
\end{tikzcd}$$

for all r. In particular, $\mathfrak{A}$ is $\mathcal{F}$-acyclic if and only if $\mathfrak{B}$ is $\mathcal{F}$-acyclic.

Proof. The acyclicity statement follows from Lemma 3.1.8 if one realizes that there is a canonical augmentation

$$\mathcal{F}(X) \longrightarrow C^0(\mathfrak{A}, \mathfrak{B}; \mathcal{F}) = C^{0,0}(\mathfrak{A}, \mathfrak{B}; \mathcal{F})$$

which is compatible with the augmentations $\mathcal{F}(X) \longrightarrow C^0(\mathfrak{A}, \mathcal{F})$ and $\mathcal{F}(X) \longrightarrow C^0(\mathfrak{B}, \mathcal{F})$ via i' and i''. Indeed, the criterion of $\mathcal{F}$-acyclicity used in (2) and the commutativity of the diagram yield the result. □

Corollary 3.1.10. *Assume $\mathfrak{B} = (V_j)_{j \in J}$ is a refinement of $\mathfrak{A} = (U_i)_{i \in I}$ and that $\mathfrak{B}|_{U_{i_0, \dots, i_p}}$ is $\mathcal{F}$-acyclic for all $i_0, \dots, i_p \in I$ and for all p. Then, the covering $\mathfrak{A}$ is $\mathcal{F}$-acyclic if and only if $\mathfrak{B}$ is $\mathcal{F}$-acyclic.*

Proof. We only have to show that all coverings $\mathfrak{A}|_{V_{j_0, \dots, j_q}}$ are $\mathcal{F}$-acyclic. However, this follows form Lemma 3.1.5, since $\mathfrak{A}|_{V_{j_0, \dots, j_q}}$ and the trivial covering of $V_{j_0 \dots j_q}$ are refinements of each other.

□

We will end this section by giving a standard argument that will allow us to attain acyclicity in general cohomology by acyclicity in Čech cohomology, for any rational covering of a rational subspace of $\mathrm{Spa}(A, A^+)$. For this part some basic knowledge in cohomology theory is required. See [12, §III.1] or [21].

Lemma 3.1.11. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category of presheaves of $\mathcal{O}_X$-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}_X$-modules. For each point $x \in X$, the stalk $\mathcal{F}_x$ is defined and it is an $\mathcal{O}_{X,x}$-module. By [12, Proposition 2.1A], there exists an injection $\mathcal{F}_x \longrightarrow I_x$, where I_x is an injective $\mathcal{O}_{X,x}$-module. Consider the inclusion

$$j : \{x\} \longrightarrow X$$

for each $x \in X$ and the sheaf

$$\mathcal{J} := \prod_{x \in X} j_*(I_x).$$

Here we consider I_x as a sheaf on the one-point space $\{x\}$, and j_* is the direct image functor; see [12, II, § 1].
For any presheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules, we have

$$Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J}) = \prod_{x \in X} Hom_{\mathcal{O}_X}(\mathcal{G}, j_*(I_x)).$$

Now, for each point $x \in X$

$$Hom_{\mathcal{O}_X}(\mathcal{G}, j_*(I_x)) \cong Hom_{\mathcal{O}_{X.x}}(\mathcal{G}_x, I_x)$$

which is easily verified.
So we obtain an injection

$$\mathcal{F} \hookrightarrow \mathcal{J},$$

induced by all the local maps $\mathcal{F}_x \longrightarrow I_x$.
Furthermore $Hom_{\mathcal{O}_X}(\cdot, \mathcal{J}) = \prod_{x \in X} Hom_{\mathcal{O}_{X,x}}(\cdot, I_x) \circ F$ is exact, since I_x is injective and the stalk functor $F : \mathcal{G} \mapsto \mathcal{G}_x$ is exact.
We conclude that $\mathcal{J}$ is an injective presheaf of $\mathcal{O}_X$-modules. □

Corollary 3.1.12. *The category of abelian presheaves on a topological space X has enough injectives.*

Proof. Let $\mathcal{O}_X$ be the constant sheaf of rings $\underline{\mathbb{Z}}$ as in [12, II, § 2]. Then $(X, \mathcal{O}_X)$ is a ringed space, and the category of abelian presheaves on X coincides with the category of presheaves of $\mathcal{O}_X$-modules. □

Lemma 3.1.13. *Let X be a ringed space. Let $\mathfrak{A} = (U_i)_{i \in I}$ be a covering of X. Let $\mathcal{I}$ be an injective object in the category of abelian presheaves on X. Then*

$$H^q(X, \mathcal{I}) = \check{H}^q(X, \mathcal{I}; \mathfrak{A}) = 0, \textit{ for } q > 0.$$

Proof. For the Grothendieck cohomology see [12, §III.1, Theorem 1.1A].

For the Čech cohomology consider an injective presheaf $\mathcal{I}$ of $\mathcal{O}_X$-modules. In this case the functor $Hom_{\mathcal{O}_X}(\cdot, \mathcal{I})$ is exact.
Denote $j_{i_o \cdots i_p} : U_{i_o \cdots i_p} \longrightarrow X$ the inclusion of $U_{i_o \cdots i_p}$ into X. Consider the complex $K(\mathfrak{A})_\bullet$ of presheaves of $\mathcal{O}_X$-modules

$$\cdots \longrightarrow \bigoplus_{i_0, i_1, i_2} (j_{i_0 i_1 i_2})_{p!} \mathcal{O}_{U_{i_0 i_1 i_2}} \longrightarrow \bigoplus_{i_0, i_1} (j_{i_0 i_1})_{p!} \mathcal{O}_{U_{i_0 i_1}} \longrightarrow \bigoplus_{i_0} (j_{i_0})_{p!} \mathcal{O}_{U_{i_0}} \longrightarrow 0$$

where $(j_{i_0\cdots i_p})_{p!}\mathcal{O}_{U_{i_0\cdots i_p}}$ denotes the extended sheaf by zero as in [12, II, Exercise 1.19], where the last nonzero term is place in degree 0 and the map

$$(j_{i_0\cdots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0\cdots i_{p+1}}} \longrightarrow (j_{i_0\cdots \hat{i_j}\cdots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0\cdots \hat{i_j}\cdots i_{p+1}}}$$

is given by $(-1)^j$ times the canonical map.
Then

$$Hom_{\mathcal{O}_X}((j_{i_0\cdots i_{p+1}})_{p!}\mathcal{O}_{U_{i_0\cdots i_p}}, \mathcal{F}) \cong Hom_{\mathcal{O}_X}(\mathcal{O}_X|_{U_{i_0\cdots i_p}}, \mathcal{F}|_{U_{i_0\cdots i_p}}) \cong \mathcal{F}(U_{i_0\cdots i_p}).$$

for any abelian presheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules. Consequently, $Hom_{\mathcal{O}_X}(K(\mathfrak{A})_\bullet, \mathcal{I}) \cong C^\bullet(\mathfrak{A}, \mathcal{I})$.
Since $Hom_{\mathcal{O}_X}(\cdot, \mathcal{I})$ is exact, we conclude that

$$h^q(Hom_{\mathcal{O}_X}(K(\mathfrak{A})_\bullet, \mathcal{I})) \cong h^q(C^\bullet(\mathfrak{A}, \mathcal{I})) = 0 \quad \text{for} \quad q > 0.$$

Thus, $\check{H}^q(X, \mathcal{I}; \mathfrak{A}) = 0$, for $q > 0$. □

Lemma 3.1.14. *Let $U \cong \operatorname{Spa}(B, B^+)$ be a rational subspace of $\operatorname{Spa}(A, A^+)$ and $\mathcal{F}$ an abelian presheaf on U. Assume that for any rational covering $\mathfrak{B} = (U_i)_{i\in I}$, $\check{H}^q(U, \mathcal{F}; \mathfrak{B}) = 0$ for all $q > 0$.*

Then $H^q(U, \mathcal{F}) = 0$ for all $q > 0$.

Proof. Since the category of abelian presheaves has enough injectives, we can choose an embedding $\mathcal{F} \longrightarrow \mathcal{I}$ into an injective abelian presheaf $\mathcal{I}$. By Lemma 3.1.13,

$$\check{H}^q(U, \mathcal{I}; \mathfrak{B}) = 0 \text{ for } q > 0.$$

Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

It follows that for any $U_i \in \mathfrak{B}$ the sequence

$$0 \longrightarrow \mathcal{F}(U_i) \longrightarrow \mathcal{I}(U_i) \longrightarrow \mathcal{Q}(U_i) \longrightarrow 0 \qquad (1)$$

is exact. In particular we obtain a short exact sequence of Čech complexes

$$0 \longrightarrow C^\bullet(\mathfrak{B}, \mathcal{F}) \longrightarrow C^\bullet(\mathfrak{B}, \mathcal{I}) \longrightarrow C^\bullet(\mathfrak{B}, \mathcal{Q}) \longrightarrow 0.$$

Looking at the long exact sequence of Čech cohomology groups we conclude that for the abelian presheaf $\mathcal{Q}$,

$$\check{H}^q(U, \mathcal{Q}; \mathfrak{B}) = 0 \text{ for all } q > 0.$$

Now consider the long exact cohomology sequence

$$\begin{array}{ccccc} 0 \longrightarrow H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \mathcal{I}) & \longrightarrow & H^0(U, \mathcal{Q}) \\ H^1(U, \mathcal{F}) & \longrightarrow & H^1(U, \mathcal{I}) & \longrightarrow & H^1(U, \mathcal{Q}) \\ \cdots & & \cdots & & \cdots \end{array}$$

By Lemma 3.1.13, $H^q(U,\mathcal{I}) = 0$ for $q > 0$. We conclude that $H^q(U,\mathcal{F}) \cong H^{q-1}(U,\mathcal{Q})$ for $q > 1$ and from (1), we obtain $H^1(U,\mathcal{F}) = 0$, since $H^0(U,\mathcal{I}) \longrightarrow H^0(U,\mathcal{Q})$ is surjective.
We argue now by induction on $q > 0$. The case $q = 1$ is done.
Suppose $H^{q-1}(U,\mathcal{F}) = 0$ for all abelian presheaves on U with $\check{H}^q(U,\mathcal{F};\mathfrak{B}) = 0$ for all $q > 0$.
Then $\mathcal{Q}$ is such a presheaf and by the conclusion made above

$$0 = H^{q-1}(U,\mathcal{Q}) \cong H^q(U,\mathcal{F}).$$

It follows that $H^q(U,\mathcal{F}) = 0$ for all $q > 0$. □

3.2 Acyclic sheaves

From here on we will follow the main ideas of the classical Tate's acyclicity Theorem in rigid analytic geometry; see [1, §8.2]. As mentioned before our objective is to proof acyclicity for simple Laurent coverings, and then use Lemma 2.5.18 to obtain acyclicity for every rational covering.

Lemma 3.2.1. *Let $\mathcal{F}$ be a presheaf of abelian groups on $\mathrm{Spa}(A,A^+)$. Suppose that for every rational subdomain $U = \mathrm{Spa}(B,B^+)$ of $\mathrm{Spa}(A,A^+)$ and every simple Laurent covering $\{V_1,V_2\}$ of U, we have*

$$\check{H}^0(U,\mathcal{F};\{V_1,V_2\}) = \mathcal{F}(U), \quad \text{resp.}$$
$$\check{H}^i(U,\mathcal{F};\{V_1,V_2\}) = \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

Then for every rational subdomain U of $\mathrm{Spa}(A,A^+)$ and every rational covering $\mathfrak{B}$ of U,

$$H^0(U,\mathcal{F}) = \check{H}^0(U,\mathcal{F};\mathfrak{B}) = \mathcal{F}(U), \quad \text{resp.}$$
$$H^i(U,\mathcal{F}) = \check{H}^i(U,\mathcal{F};\mathfrak{B}) = \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. We will divide the proof into several steps in order to derive the two conclusions of the Lemma. In first part we will proof that the property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U,\mathcal{F};\mathfrak{B})$ is injective" for a rational covering $\mathfrak{B}$ of U satisfies a.), b.) and c.) of Lemma 2.5.18.

Throughout this argument, let U be an arbitrary rational subdomain of $\mathrm{Spa}(A,A^+)$, with $U \cong \mathrm{Spa}(B,B^+)$ and $\mathfrak{B}$ be a rational covering of U.

(1) The property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U,\mathcal{F};\mathfrak{B})$ is injective" for a rational covering $\mathfrak{B}$ of U is local:

Let $\mathfrak{S} = (V_j)$ be a refinement of $\mathfrak{B} = (U_i)$ such that

$$\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U,\mathcal{F};\mathfrak{S})$$

is injective.
Since $\mathfrak{S}$ is a refinement of $\mathfrak{B}$ we obtain a morphism

$$\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(V_{i,j}) \quad \text{for} \quad V_{i,j} \subset U_i.$$

Consequently, there is a morphism between the two Čech complexes:

$$\tau^\bullet : C^\bullet(U, \mathcal{F}; \mathfrak{B}) \longrightarrow C^\bullet(U, \mathcal{F}; \mathfrak{S}).$$

Thus we get morphisms $\psi^i : \check{H}^i(U, \mathcal{F}; \mathfrak{B}) \longrightarrow \check{H}^i(U, \mathcal{F}; \mathfrak{S})$ for all $i \geq 0$.
Consider the canonical morphism

$$\varphi_1 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B}).$$

Then $\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$ factors through $\psi^0 \circ \varphi_1$, as shown in the following commutative diagram:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \\ \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[ur, "\psi^0"'] & \end{tikzcd}$$

Since φ_2 is injective, $\psi^0 \circ \varphi_1$ must also be injective. We conclude that φ_1 is injective.

(2) The property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$ is injective" for a rational covering $\mathfrak{B}$ of U is transitive:

For each $U_i \in \mathfrak{B}$, let $\mathfrak{S}_i = (V_{ij})$ be a covering of U_i, such that

$$\varphi_1 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \text{ and } \varphi_i : \mathcal{F}(U_i) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}_i)$$

are injective.

Consider the covering $\mathfrak{S} := \bigcup_i \mathfrak{S}_\mathfrak{i}$ of U and the canonical morphism

$$\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}).$$

For $f \in \mathcal{F}(U)$, we have that $\varphi_2(f) = (f|_{V_{ij}})_{ij} = (f|_{U_i|_{V_{ij}}})_{ij} = \psi^0 \circ \varphi_1(f)$, since $\mathcal{F}$ is a presheaf. It follows that the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \\ \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[ur, "\psi^0"'] & \end{tikzcd}$$

commutes. Here the morphism $\psi^0 : \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$ is induced by the morphism of Čech complexes

$$\tau^\bullet : C^\bullet(U, \mathcal{F}; \mathfrak{B}) \longrightarrow C^\bullet(U, \mathcal{F}; \mathfrak{S}),$$

which exists since $\mathfrak{S}$ is a refinement of $\mathfrak{B}$.
Now, consider the morphism

$$\tau^0 : \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(V_{ij})$$

given by $(\tau^0(f))_{ij} = f_i|_{V_{ij}}$, where $f := (f_i)_{i\in I}$ and $V_{ij} \in \mathfrak{S}_i$. Since

$$\varphi_i : \mathcal{F}(U_i) \longrightarrow \check{H}^0(U_i, \mathcal{F}; \mathfrak{S}_\mathrm{i})$$

is injective for all $U_i \in \mathfrak{B}$, it follows that τ^0 is also injective. Looking at the following commutative diagram,

$$\begin{tikzcd} \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[r, hook] \arrow[d, "\psi^0"] & \prod_i \mathcal{F}(U_i) \arrow[d, hook, "\tau^0"] \\ \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \arrow[r, hook] & \prod_{i,j} \mathcal{F}(V_{ij}) \end{tikzcd}$$

we conclude that ψ^0 is injective. Hence, $\varphi_2 = \psi^0 \circ \varphi_1$ must also be injective.

(3) The property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \{V_1, V_2\})$ is injective" holds for simple Laurent coverings $\{V_1, V_2\}$ of U.

By hypothesis $\check{H}^0(U, \mathcal{F}; \{V_1, V_2\}) = \mathcal{F}(U)$.

It follows by Lemma 2.5.18 that for any rational covering $\mathfrak{B}$ of U the canonical morphism $\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$ is injective.

Now we are ready to proof the first statement of the Lemma, which we will do by proving that the property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$" is bijective" for a rational covering $\mathfrak{B}$ of U satisfies a.), b.) and c.) of Lemma 2.5.18.

a.) The property "$\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$ is bijective" for a rational covering $\mathfrak{B}$ of U is local:

Let $\mathfrak{S} = (V_j)$ be a refinement of $\mathfrak{B} = (U_i)$ given by the function $\lambda : j \longmapsto i$, such that

$$\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$$

is bijective.
As mentioned above, there exists a morphism

$$\psi^0 : \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$$

and the canonical morphism

$$\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$$

factors through $\psi^0 \circ \varphi_1$, as shown in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \\
\check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[ur, "\psi^0"'] &
\end{tikzcd}$$

Since φ_2 is bijective, ψ^0 is surjective.
We will show that ψ^0 is also injective, thus establishing an isomorphism between $\check{H}^0(U, \mathcal{F}; \mathfrak{B})$ and $\check{H}^0(U, \mathcal{F}; \mathfrak{S})$. Then we can deduce the surjectivity of φ_1 from the surjectivity of φ_2.

Consider the following commutative diagram:

$$\begin{tikzcd}
\check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[r, "\psi^0"] \arrow[d] \arrow[dr, "\gamma"] & \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \arrow[d] \\
\prod_i \mathcal{F}(U_i) \arrow[r, "\tau^0"] & \prod_j \mathcal{F}(V_j)
\end{tikzcd}$$

where $\tau^0((f_i)_i) := (f_{\lambda(j)}|_{V_j})$. Let us prove that γ is injective, which implies the injectivity of ψ^0.

For any $U_i \in \mathfrak{B}$ we have a rational covering $\mathfrak{S}|_{U_i} = (U_i \cap V_j)$, with $V_j \in \mathfrak{S}$. We get a canonical morphism for each U_i

$$\begin{aligned}
\beta_i : \mathcal{F}(U_i) &\longrightarrow \check{H}^0(\mathcal{F}, U_i, \mathfrak{S}|_{U_i}) \\
f_i &\longmapsto (f_i|_{U_i \cap V_j})_j
\end{aligned}$$

which is injective as mentioned before.

Given $f := ((f_i))_i \in \check{H}^0(U_i, \mathcal{F}; \mathfrak{B})$ such that $\gamma(f) = 0$, then

$$\begin{aligned}
f_i|_{U_i \cap U_j} &= f_j|_{U_i \cap U_j} \quad \text{and} \\
f_{\lambda(j)}|_{V_j} &= 0,
\end{aligned}$$

for all i and j.
For a fixed index i, we obtain

$$\begin{aligned}
(\beta_i(f_i))_j &= f_i|_{U_i \cap V_j} \\
&= (f_i|_{U_i \cap U_{\lambda(j)}})|_{U_i \cap V_j}, \quad \text{since } \mathcal{F} \text{ is a presheaf} \\
&= (f_{\lambda(j)}|_{U_i \cap U_{\lambda(j)}})|_{U_i \cap V_j} \\
&= (f_{\lambda(j)}|_{V_j})|_{U_i \cap V_j}, \quad \text{since } \mathcal{F} \text{ is a presheaf} \\
&= 0.
\end{aligned}$$

We conclude that $f_i = 0$ by the injectivity of β_i. Consequently, $f = 0$ after considering all indices i. Hence, γ is injective and a.) is proven.

b.) The property that $\mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$ is bijective, is transitive:

For each $U_i \in \mathfrak{B}$, let $\mathfrak{S}_i = (V_{ij})$ be a covering of U_i, such that

$$\varphi_1 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \text{ and } \varphi_i : \mathcal{F}(U_i) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}_i)$$

are bijective.

Consider the covering $\mathfrak{S} := \bigcup_i \mathfrak{S}_i$ of U. As in the injective case, there exists a canonical morphism

$$\varphi_2 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S})$$

and an injective morphism

$$\psi^0 : \check{H}^0(U, \mathcal{F}; \mathfrak{B}) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}),$$

induced by the morphism of Čech complexes

$$\tau^\bullet : C^\bullet(U, \mathcal{F}; \mathfrak{B}) \longrightarrow C^\bullet(U, \mathcal{F}; \mathfrak{S}),$$

which exists since $\mathfrak{S}$ is a refinement of $\mathfrak{B}$, given by the function $\lambda : IJ \longrightarrow I, ij \longmapsto i$. In this way we obtain again a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & \check{H}^0(U, \mathcal{F}; \mathfrak{S}) \\
\check{H}^0(U, \mathcal{F}; \mathfrak{B}) \arrow[ur, "\psi^0"'] &
\end{tikzcd}$$

such that φ_1 is bijective and ψ^0, φ_2 are injective.
Given $g := (g_{ij}) \in \check{H}^0(U, \mathcal{F}; \mathfrak{S})$, we have that $g_{i_0j_0}|_{V_{i_0j_0} \cap V_{i_1j_1}} = g_{i_1j_1}|_{V_{i_0j_0} \cap V_{i_1j_1}}$ for all $V_{ij} \in \mathfrak{S}$. In particular, $g_{i_0j_0}|_{V_{i_0j_0} \cap V_{i_1j_1}} = g_{i_1j_1}|_{V_{i_0j_0} \cap V_{i_1j_1}}$ for all $V_{ij} \in \mathfrak{S}_i$. It follows that for any fixed i, say i_0, the components i_0j of g are elements of $\check{H}^0(U, \mathcal{F}; \mathfrak{S}_{i_0})$, i.e.,

$$(g_{i_0j})_j \in \check{H}^0(U, \mathcal{F}; \mathfrak{S}_{i_0}).$$

Since

$$\varphi_{i_0} : \mathcal{F}(U_{i_0}) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}_{i_0})$$

is bijective, there exists $f_{i_0} \in \mathcal{F}(U_{i_0})$ such that $\varphi_{i_0}(f_{i_0}) = (f_{i_0}|_{V_{i_0j}})_j = (g_{i_0j})_j$.
Varying the index i_0, we obtain a morphism

$$\tau'^0 : \prod_i \mathcal{F}(U_i) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{S}),$$

given by $(\tau'^0(f))_{ij} = f_i|_{V_{ij}} = g_{ij}$, where $f := (f_i)_{i \in I}$.

For each $\mathcal{F}(U_{i_0i_1})$ consider the morphism

$$\epsilon_{i_0i_1} : \mathcal{F}(U_{i_0i_1}) \longrightarrow \prod_{\substack{\lambda(j_0)=i_0 \\ \lambda(j_1)=i_1}} \mathcal{F}(V_{j_0j_1}).$$

Which arise by considering $\mathfrak{H}_{i_0 i_1} := \{V_{j_0 j_1}\}_{\lambda(j_0)=i_0, \lambda(j_1)=i_1}$ as a covering of $U_{i_0 i_1}$. As we previously proved, the canonical morphism

$$\beta : \mathcal{F}(U_{i_0 i_1}) \longrightarrow \check{H}^0(U_{i_0 i_1}, \mathcal{F}; \mathfrak{H}_{i_0 i_1})$$

is injective and so

$$\epsilon_{i_0 i_1} : \mathcal{F}(U_{i_0 i_1}) \longrightarrow \prod_{\substack{\lambda(j_0)=i_0 \\ \lambda(j_1)=i_1}} \mathcal{F}(V_{j_0 j_1}),$$

is injective for each $U_{i_0 i_1}$.
Consider now a part of the two augmented Čech complexes given by

$$\begin{array}{ccccccc}
0 \longrightarrow \mathcal{F}(U) & \xrightarrow{\epsilon} & \prod_i \mathcal{F}(U_i) & \xrightarrow{d^0} & \prod_{i_0<i_1} \mathcal{F}(U_{i_0 i_1}) \\
 & & \downarrow{\scriptstyle \tau^0} & & \downarrow{\scriptstyle \tau^1} \\
0 \longrightarrow \mathcal{F}(U) & \xrightarrow{\epsilon} & \prod_{i,j} \mathcal{F}(V_{ij}) & \xrightarrow{d'^0} & \prod_{i_0 j_0 < i_1 j_1} \mathcal{F}(V_{i_0 j_0 i_1 j_1})
\end{array}$$

As $d'^0 \circ \tau^0(f) = \tau^1 \circ d^0(f) = 0$, it follows that

$$\tau^1 \left((f_{i_1} - f_{i_0}|_{U_{i_0 i_1}})_{i_0 i_1} \right) = 0,$$

where $(f_{i_1} - f_{i_0}|_{U_{i_0 i_1}})_{i_0 i_1} \in \prod_{i_0<i_1} \mathcal{F}(U_{i_0 i_1})$. In particular, for any $f_{i_1} - f_{i_0}|_{U_{i_0 i_1}} \in \mathcal{F}(U_{i_0 i_1})$ we have

$$\epsilon_{i_0 i_1} \left(f_{i_1} - f_{i_0}|_{U_{i_0 i_1}} \right) = 0.$$

Thus, $f_{i_1} - f_{i_0}|_{U_{i_0 i_1}} = 0$ for each $i_0 i_1$. This implies that $f = (f_i)_{i \in I} \in \check{H}^0(U, \mathcal{F}; \mathfrak{B})$. Since

$$\varphi_1 : \mathcal{F}(U) \longrightarrow \check{H}^0(U, \mathcal{F}; \mathfrak{B})$$

is bijective, there exists $x \in \mathcal{F}(U)$ such that $\varphi_2(x) = g$. This proves that φ_2 is surjective. Consequently, φ_2 is bijective.

c.) By hypothesis $\check{H}^0(U, \mathcal{F}; \{V_1, V_2\}) = \mathcal{F}(U)$.
By the definition of Grothendieck cohomology, we get that $H^0(U, \mathcal{F}) \cong \mathcal{F}(U)$. This proofs the first part of the Lemma.

Assume now that

$$\check{H}^i(U, \mathcal{F}; \{V_1, V_2\}) = \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

Define the *property* $\mathcal{P}$ for any rational covering $\mathfrak{S}$ of U:=

> *Given another covering* $\mathfrak{A}= (U_i)_{i \in I}$ *of* U, $\mathfrak{S}|_{U_{i_0,\dots,i_p}}$ *is* $\mathcal{F}$*-acyclic for all* $i_0, \dots, i_p \in I$ *and for all* p*, where* $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap U_{i_p}$.

We will proof that the *property* $\mathcal{P}$ satisfies a.), b.) and c.) of Lemma 2.5.18.

a.) Let $\mathfrak{B}'$ be a refinement of $\mathfrak{B} = (U_i)_{i\in I}$ with the *property* $\mathcal{P}$. Since $\mathfrak{B}$ is a covering of U and $\mathfrak{B}'$ fulfills the *property* $\mathcal{P}$, it follows that $\mathfrak{B}'|_{U_{i_0,\dots,i_p}}$ is $\mathcal{F}$-acyclic for all $i_0,\dots,i_p \in I$ and for all p. Now, consider any covering $\mathfrak{S} = (V_j)_{j\in J}$ of U. It suffices to proof that $\mathfrak{B}|_{V_j}$ is $\mathcal{F}$-acyclic in order to show that $\mathfrak{B}$ fulfills the *property* $\mathcal{P}$. The covering $\mathfrak{B}'|_{V_j}$ is a refinement of $\mathfrak{B}|_{V_j}$ as a covering of V_j and $\mathfrak{B}'|_{V_j}$ is $\mathcal{F}$-acyclic, since $\mathfrak{B}'$ fulfills the *property* $\mathcal{P}$. For the covering $\mathfrak{B} \cup \{V_j\}$ of U, we have as above that

$$\mathfrak{B}'|_{U_{i_0\dots i_p}\cap V_j} = \mathfrak{B}'|_{V_j}|_{U_{i_0\dots i_p}}$$

is $\mathcal{F}$-acyclic for all $i_0,\dots,i_p \in I$ and for all p. Thus we can apply Corollary 3.1.10 and conclude that $\mathfrak{B}|_{V_j}$ is $\mathcal{F}$-acyclic. This proofs a.)

b.) Let $\mathfrak{B} = (U_i)_{i\in I}$ be a covering of U with the *property* $\mathcal{P}$, and for each U_i let $\mathfrak{B}'_i = (W_{ik})_{k\in K}$ be a covering of U_i with the *property* $\mathcal{P}$. We want to proof that $\mathfrak{B}' = \bigcup_i \mathfrak{B}'_i$ fulfills the *property* $\mathcal{P}$. To see this, consider any covering $\mathfrak{S} = (V_j)_{j\in J}$ of U. It suffices to proof that $\mathfrak{B}'|_{V_j}$ is $\mathcal{F}$-acyclic in order to show that $\mathfrak{B}'$ fulfills the *property* $\mathcal{P}$.
We would like to apply Corollary 3.1.10 to the coverings $\mathfrak{B}|_{V_j}$ and $\mathfrak{B}'|_{V_j}$ of V_j, since the covering $\mathfrak{B}'|_{V_j}$ is a refinement of $\mathfrak{B}|_{V_j}$ as a covering of V_j. For the purpose of satisfying the conditions of the Corollary 3.1.10, it would suffice to proof that $\mathfrak{B}'|_{U_j}$ is $\mathcal{F}$-acyclic for some $U_j \in \mathfrak{B}$.
Consider the covering $\mathfrak{B}'_j = (W_{jk})_{k\in K}$ of U_j, which is a refinement of

$$\mathfrak{B}'|_{U_j} = (U_j \cap W_{ik})_{i\in I, k\in K}$$

as a covering of U_j. Since $\mathfrak{B}'_j$ fulfills the *property* $\mathcal{P}$, then

$$\mathfrak{B}'_j|_{U_j \cap W_{i_0k_0\dots i_pk_p}}$$

is $\mathcal{F}$-acyclic for all $i_0k_0,\dots,i_pk_p$ and for all p. Thus, we can apply Corollary 3.1.10 to the coverings $\mathfrak{B}'_j$ and $\mathfrak{B}'|_{U_j}$ and conclude that $\mathfrak{B}'|_{U_j}$ is $\mathcal{F}$-acyclic if and only if $\mathfrak{B}'_j$ is $\mathcal{F}$-acyclic. Regarding the trivial cover $\mathfrak{A}_0 := \{U_j\}$ as a cover of U_j, it follows that $\mathfrak{B}'_j$ is $\mathcal{F}$-acyclic by the *property* $\mathcal{P}$. Consequently, $\mathfrak{B}'|_{U_j}$ is $\mathcal{F}$-acyclic.
Now, the conditions of the Corollary 3.1.10 are met and $\mathfrak{B}'|_{V_j}$ is $\mathcal{F}$-acyclic if and only if $\mathfrak{B}|_{V_j}$ is $\mathcal{F}$-acyclic. Since $\mathfrak{B}$ fulfills the *property* $\mathcal{P}$, it follows that $\mathfrak{B}|_{V_j}$ is $\mathcal{F}$-acyclic. Hence, $\mathfrak{B}'|_{V_j}$ is $\mathcal{F}$-acyclic. This proofs b.)

c.) Let $\{V_1, V_2\}$ be a simple Laurent covering of U. Consider another covering $\mathfrak{S} = (W_i)_{i\in I}$ of U. Since $\{V_1, V_2\}|_{W_{i_0\dots i_p}}$ is a simple Laurent covering of $W_{i_0\dots i_p}$ and $W_{i_0\dots i_p}$ is a rational subdomain of $\mathrm{Spa}(A, A^+)$, we can apply our assumption to $W_{i_0\dots i_p}$ instead of U and conclude that $\{V_1, V_2\}|_{W_{i_0\dots i_p}}$ is $\mathcal{F}$-acyclic. Consequently, $\{V_1, V_2\}$ has the *property* $\mathcal{P}$, i.e, without loss of generality we can assume that $\{V_1, V_2\}$ fulfills the *property* $\mathcal{P}$. This proofs c.)

Note that the *property* $\mathcal{P}$ implies $\mathcal{F}$-acyclicity for any covering that fulfills it, since we can always consider the trivial covering $\mathfrak{A}_0 = U$. Thus, by Lemma 2.5.18 the covering $\mathfrak{B}$ fulfills the *property* $\mathcal{P}$ and therefore is $\mathcal{F}$-acyclic.

Now we can apply Lemma 3.1.14, and obtain also $H^i(U, \mathcal{F}) = \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$ □

We successfully proved, that if we manage to proof acyclicity of the structure sheaf for simple Laurent coverings on a rational subdomain U of $\mathrm{Spa}(A, A^+)$, we could follow acyclicity for any rational covering on U. As we will see below the sheaf condition is all that we need to prove acyclicity for simple Laurent coverings on any rational subdomain. As stipulated in [17, §2.4] we could also see the next assertion as following: the only obstruction to the analogue of Tate's acyclicity Theorem is the failure of the structure presheaf to be a sheaf. To see this, consider the Čech complex associated to a simple Laurent covering $\{V_1, V_2\}$ of $\mathrm{Spa}(A, A^+)$, which is of the form

$$0 \longrightarrow \Gamma(\mathrm{Spa}(A, A^+), \mathcal{O}) \longrightarrow \Gamma(V_1, \mathcal{O}) \oplus \Gamma(V_2, \mathcal{O}) \longrightarrow \Gamma(V_1 \cap V_2, \mathcal{O}) \longrightarrow 0.$$

Since $\Gamma(\mathrm{Spa}(A, A^+), \mathcal{O}) \cong A$, this translates into the sequence

$$0 \longrightarrow A \longrightarrow B_1 \oplus B_1 \longrightarrow B_{12} \longrightarrow 0,$$

where $(A, A^+) \longrightarrow (B_1, B_1^+)$,$(A, A^+) \longrightarrow (B_2, B_2^+)$,$(A, A^+) \longrightarrow (B_{12}, B_{12}^+)$ correspond to the rational localizations of $V_1, V_2, V_1 \cap V_2$, respectively.

Lemma 3.2.2. *Let S_-, S_+ be the simple Laurent covering of* $\mathrm{Spa}(A, A^+)$ *define by some $f \in A$. Let $(A, A^+) \longrightarrow (B_1, B_1^+)$,$(A, A^+) \longrightarrow (B_2, B_2^+)$,$(A, A^+) \longrightarrow (B_{12}, B_{12}^+)$ be the rational localizations corresponding to $S_-, S_+, S_- \cap S_+$, respectively. Then the map $B_1 \oplus B_2 \longrightarrow B_{12}$ taking (b_1, b_2) to $(b_1 - b_2)$ is surjective.*

Proof. Lemma 2.5.10, tells us that any rational subspace U of $\mathrm{Spa}(A, A^+)$ defined by $f_1, \dots, f_n, g$ is represented by a rational localization $\varphi : (A, A^+) \longrightarrow (B, B^+)$, where B is the quotient of $A\{T_1, \dots, T_n\}$ for the closure of the ideal $(gT_1 - f_1, \dots, gT_n - f_n)$. In this way we obtain strict surjections

$$\varphi_1 : A\{T\} \longrightarrow B_1, \varphi_2 : A\{U\} \longrightarrow B_2, \varphi_{12} : A\{T, U\} \longrightarrow B_{12}$$

taking T to f and U to f^{-1}. This means

$$B_1 \cong A\{T\}/\overline{(T - f)}, B_2 \cong A\{U\}/\overline{(fU - 1)}, B_{12} \cong A\{T, U\}/\overline{(T - f, fU - 1)}$$

In particular, any $b \in B_{12}$ can be lifted to some $\sum_{i,j=0}^{\infty} a_{ij} T^i U^j \in A\{T, U\}$. Consider the sum of a_{ij} over all $i, j \geq 0$ with $i - j = n$. Remember that $\sum_{i,j=0}^{\infty} a_{ij} T^i U^j \in A\{T, U\}$ means that $\lim\limits_{i,j \to \infty} |a_{ij}| = 0$. Consequently, $\sum_{n=i-j}^{\infty} a_{ij}$ converges in A for all n, say to a_n'. Let b_1 be the image of $\sum_{n=0}^{\infty} a_n' T^n$ in B_1. Let b_2 be the image of $-\sum_{n=1}^{\infty} a_{-n}' U^n$ in B_2.

Then

$$
\begin{aligned}
b &= \sum_{i,j=0}^{\infty} a_{ij} f^i (f^{-1})^j \\
&= \sum_{i-j=0}^{\infty} a_{ij} + \sum_{i-j=1}^{\infty} a_{ij} f + \dots + \sum_{i-j=-1}^{\infty} a_{ij} f^{-1} + \sum_{i-j=-2}^{\infty} a_{ij} (f^{-1})^2 + \dots \\
&= \sum_{n=0}^{\infty} a'_n f^n + \sum_{n=1}^{\infty} a'_{-n} (f^{-1})^n \\
&= \varphi_{12} (\sum_{n=0}^{\infty} a'_n T^n + \sum_{n=1}^{\infty} a'_{-n} U^n) \\
&= \varphi_{12}(b_1 - b_2).
\end{aligned}
$$

This proofs the desired surjection. $\square$

Theorem 3.2.3. *Suppose that (A, A^+) is sheafy. Then for every rational covering $\mathfrak{A}$ of any rational subspace $U \cong \mathrm{Spa}(B, B^+)$ of $X = \mathrm{Spa}(A, A^+)$,*

$$H^i(U, \mathcal{O}_X) = \check{H}^i(U, \mathcal{O}_X; \mathfrak{A}) = \begin{cases} U & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. By Lemma 3.2.1, it suffices to check $\mathcal{O}_X$-acyclicity for simple Laurent coverings. We may as well consider only simple Laurent coverings of $\mathrm{Spa}(A, A^+)$ itself, since the condition that $\mathcal{O}_X$ is a sheaf implies that $\mathcal{O}_U \cong \mathcal{O}_X|_U$ is also a sheaf by restriction. In the notation of Lemma 3.2.2, the Čech complex $C^\bullet(\{V_1, V_2\}, \mathcal{O}_X)$, where $\{V_1, V_2\}$ is a simple Laurent covering of A, induces the sequence

$$0 \longrightarrow A \longrightarrow B_1 \oplus B_2 \longrightarrow B_{12} \longrightarrow 0,$$

which is exact at B_{12} due to Lemma 3.2.2. By the sheafy hypotesis, it is also exact at A and $B_1 \oplus B_2$. Thus, Lemma 3.2.1 yields the claim. $\square$

Definition 3.2.4. Let (A, A^+) be an adic Banach ring. Let $\mathcal{F}$ be a presheaf of topological rings on $\mathrm{Spa}(A.A^+)$. We say that $\mathcal{F}$ satisfies the *Tate sheaf property* if for every rational localization $(A, A^+) \longrightarrow (B, B^+)$ and every rational covering $\mathfrak{A}$ of $U \cong \mathrm{Spa}(B, B^+)$,

$$H^i(U, \mathcal{F}) = \check{H}^i(U, \mathcal{F}; \mathfrak{A}) = \begin{cases} \mathcal{F}(U) & i = 0 \\ 0 & i > 0. \end{cases}$$

In particular, this implies that $\mathcal{F}$ is a sheaf.

Theorem 3.2.5. *Let (A, A^+) be a sheafy adic Banach ring. Then the structure sheaf $\mathcal{O}_X$ on X =$\mathrm{Spa}(A, A^+)$ satisfies the Tate sheaf property.*

Proof. The Tate sheaf property is in this case Theorem 3.2.3. $\square$

4 Part: Kiehl glueing property

In this section we will use Lemma 2.5.18 to compare sheaves of locally free modules of finite rank with their global sections. In particular, we will be interested into glueing together finitely generated projective modules in order to attain a category equivalence, similar to the classical result of Kiehl; see Theorem 4.1.5*(ii)*.
We will proceed as in the previous section and prove the claim for simple Laurent coverings and then for any rational covering.

4.1 Coherent sheaves on affinoid spaces

We will present the equivalent of the classical results of Tate and Kiehl in the theory of coherent sheaves from our curring setting of adic spectra. In order to do so, we will have to restrict ourselves, in some cases, to the setting of affinoid spaces over an analytic field. Accordingly, some results here are analogous to results in rigid analytic geometry, which we mention to establish a comparison between these classical results and the ones in the setting of Banach algebras over an analytic field.

Definition 4.1.1. For (A, A^+) an adic Banach ring let M be an A-module. We define the *presheaf associated* to M on $X = \mathrm{Spa}(A, A^+)$, denoted by $\widetilde{M}$, as follows. For any rational subspace $U \cong \mathrm{Spa}(B, B^+)$ of X, we define the group $\widetilde{M}(U)$ as $M \otimes_A B$, i.e., $U \mapsto \widetilde{M}(U) := M \otimes_A B$.
For $V \subseteq \mathrm{Spa}(A, A^+)$ open, we define the group $\widetilde{M}(V)$ as $M \otimes_A \varprojlim_{\mathrm{Spa}(B_i, B_i^+) \subseteq V} B_i$, where the inverse limit is taken over all rational localizations $(A, A+) \longrightarrow (B_i, B_i^+)$ for which $\mathrm{Spa}(B_i, B_i^+) \subseteq V$,i.e., $V \mapsto \widetilde{M}(V) := M \otimes_A \varprojlim_{\mathrm{Spa}(B_i, B_i^+) \subseteq V} B_i$.

Note that although it would be desirable to work with sheaves on $\mathrm{Spa}(A, A^+)$ as for $\mathcal{O}_X$-modules in algebraic geometry, the presheaf $\widetilde{M}$ is in general not a sheaf, even if we restrict ourselves to finite coverings as in rigid analytic geometry; see [17, Example 2.8.7] for a counterexample. However, if A is an affinoid space over an analytic field or if we only consider finite projective modules over a sheafy Banach ring (A, A^+), we could assure acyclicity of the presheaf $\widetilde{M}$; see [6, Theorem 4.3/11] and [6, Corollary 5.2/4] for the first assertion and Lemma 4.2.10 for the second.

Definition 4.1.2. Let $X = (A, A^+)$ be an adic Banach ring. Let $\mathcal{F}$ be a sheaf of topological rings on $\mathrm{Spa}(A.A^+)$. We say that $\mathcal{F}$ satisfies the *Kiehl glueing property* if for every rational subdomain $U \cong \mathrm{Spa}(B, B^+)$ of $\mathrm{Spa}(A, A^+)$, the functor from the category of finite projective $\mathcal{F}(U)$-modules to the category of sheaves of $\mathcal{F}$-modules over U which are locally free of finite rank, defined by $\mathfrak{F} : M \mapsto \widetilde{M}$, is an equivalence of categories.

For the definition to make sense, we need to proof that the $\mathcal{F}$-module $\widetilde{M}$, associated to a finite projective B-module, is locally free of finite rank.

For this consider the elements $f_1, \ldots, f_n$ in B such that $f_1 u_1 + \cdots + f_n u_n = 1$ and $M \otimes_B B_{f_i}$ is a finitely generated free B_{f_i}-module for all $i = 1, \ldots, n$, which exist by

Theorem 2.1.4. It follows that $M \otimes_B \widehat{B_{f_i}}$ is a finitely generated free $\widehat{B_{f_i}}$-module for all $i = 1, \ldots, n$. Taking the standard rational covering $\mathfrak{A} = (U_i)_{1 \leq i \leq n}$ of U generated by the elements $f_1, \ldots, f_n$, where $U_i = X(\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}) \cong \mathrm{Spa}(B_i, B_i^+)$, we get that $\widehat{B_{f_i}} \cong B_i$, via $f_i^{-1} \mapsto (u_1 + u_2 T_2 + \cdots + u_n T_n)$, for all $i = 1 \ldots, n$. Thus, $\widetilde{M}|_{U_i} = (M \otimes_B B_i)\widetilde{}$ is a free $\mathcal{F}_U|_{U_i}$-module, making $\widetilde{M}$ into a locally free sheaf of finite rank; compare Lemma 2.5.10.

As remarked in [17, Theorem 2.2.8], the following theorem is an analogue of the Banach-Schauder open mapping theorem [19, Theorem 2.11] in the rigid analytic geometry. This theorem will help us proof that any simple Laurent covering fulfills the conditions of a glueing square (Definition 4.2.3).

Theorem 4.1.3. *Let R be a Banach ring containing a topologically nilpotent unit. Let $\varphi : V \longrightarrow W$ be a bounded surjective homomorphism of Banach modules over R. Then φ is open and strict.*

Proof. For R an analytic field, see [9, §I.3.3, Théorème 1]. For the general case, see [13]. ☐

The next Lemma is one of the key arguments for proving the classical theorems of Tate and Kiehl (Theorem 4.1.5), which we will only mention at the end of this section, in order to remark its similarity to our main Theorem 1.0.1.

Lemma 4.1.4. *Let (A, A^+) be an adic Banach ring in which A is an affinoid algebra over an analytic field K. Let $\{(A, A^+) \longrightarrow (B_i, B_i^+)\}_{i=1}^n$ be an affinoid covering. Then the ring homomorphism $A \longrightarrow B_1 \oplus \cdots \oplus B_n$ is faithfully flat.*

Proof. The homomorphism $A \longrightarrow B_1 \oplus \cdots \oplus B_n$ is flat by [1, Corollary 7.3.2/6]. It is faithful by Lemma 2.1.7 and the fact that every maximal ideal of A is closed (by Corollary 2.4.4). ☐

Theorem 4.1.5. *Let (A, A^+) be an adic Banach ring in which A is an affinoid algebra over an analytic field K. Let $\mathfrak{A}$ be an affinoid covering.*

(i) For any finite A-module M, let $\tilde{M}$ be the sheaf of $\mathcal{O}$-modules on $\mathrm{Spa}(A^+, A)$ induced by M. Then $\check{H}^i(\mathrm{Spa}(A, A^+), \tilde{M}; \mathfrak{A}) = M$ for $i = 0$ and 0 for $i > 0$. In particular, (A, A^+) is sheafy and $H^i(\mathrm{Spa}(A, A^+)) = M$ for $i = 0$ and 0 for $i > 0$.

(ii) The functor $M \mapsto \tilde{M}$ defines a tensor equivalence between finite A-modules and coherent sheaves of $\mathcal{O}$-modules on $\mathrm{Spa}(A, A^+)$.

The first part is non other than Tate's acyclicity theorem, which we already studied in the framework of adic spectra; see Theorem 3.2.5 and Corollary 4.2.10. The second part is an analogous of the Kiehl's glueing property for coherent sheaves and finitely generated modules; see Definition 4.1.2.

4.2 Glueing of finite projective modules

For $(A, A^+) \longrightarrow (B, B^+)$ a rational localization of adic Banach rings, the map $A \longrightarrow B$ is flat if A is an affinoid algebra over an analytic field; see Theorem [17, Lemma 2.5.7] or Lemma 4.1.4. But it need not be flat in general. For instance, flatness almost always fails for perfectoid algebras. Guided by this observation and by the analogy with the Beauville-Laszlo theorem ([17, Proposition 1.3.6]), we will limit our glueing ambitions to cases where the modules being glued are themselves flat.
Following this idea, we will focus our attention on the category of sheaves of locally free modules of finite rank over various sheaves of rings on adic spectra.

Definition 4.2.1. Let

$$\begin{array}{ccc} R & \longrightarrow & R_1 \\ \big\downarrow & & \big\downarrow \\ R_2 & \longrightarrow & R_{12} \end{array}$$

be a commuting diagram of ring homomorphisms such that the sequence

$$0 \longrightarrow R \longrightarrow R_1 \oplus R_2 \longrightarrow R_{12} \longrightarrow 0$$

of R-modules, in which the last nontrivial arrow takes (s_1, s_2) to $s_1 - s_2$, is exact. By a *glueing datum* over this diagram, we will mean a datum consisting of modules M_1,M_2,M_{12} over R_1,R_2,R_{12}, respectively, equipped with isomorphisms

$$\psi_1 : M_1 \otimes_{R_1} R_{12} \cong M_{12},\ \psi_2 : M_2 \otimes_{R_2} R_{12} \cong M_{12}.$$

We say such a glueing datum is *finite* or *finite projective* if the modules are finite or finite projective over their corresponding rings.
When considering a glueing datum, it is natural to consider the kernel M of the map

$$\psi_1 - \psi_2 : M_1 \oplus M_2 \longrightarrow M_{12} \qquad (m_1, m_2) \mapsto (\psi_1(m_1 \otimes 1) - \psi_2(m_2 \otimes 1)).$$

Note that since R_1 and R_2 are R-modules, then $M_1 \oplus M_2$ can be seen also as a R-module. It follows that M, as a submodule of $M_1 \oplus M_2$ is also a R-module.
There are natural maps $M \longrightarrow M_1$, $M \longrightarrow M_2$ of R-modules, which by tensoring with R_1 and R_2 respectively, correspond to the maps $M \otimes_R R_1 \longrightarrow M_1$, $M \otimes_R R_2 \longrightarrow M_2$.

Lemma 4.2.2. *Let $R_1 \to S, R_2 \to S$ be bounded homomorphisms of Banach rings (not necessarily containing topologically nilpotent units) such that the sum homomorphism $\psi : R_1 \oplus R_2 \to S$ of groups is strict and surjective. Then there exists a constant $c' > 0$ such that for every positive integer n, every matrix $U \in \mathrm{GL}_n(S)$ with $\|1 - U\| < c'$ can be written in the form $\psi(C_1)\psi(C_2)$ with $C_i \in \mathrm{GL}_n(R_i)$ $i = 1, 2$.*

Proof. By hypothesis, there exists a constant $d \geq 1$ such that every $x \in R_{12}$ lifts to a pair $(z, y) \in R_1 \oplus R_2$ with $|z|, |y| \leq d|x|$ and $|\psi((z, 0))|, |\psi((0, y))| \leq d|x|$. Indeed, since ψ is strict, there exist constants $c, l > 0$ such that,

$$\begin{aligned} |\psi((z,0))| &\leq c\max\{|z|, |0|\} = c|z| \\ |\psi((0,y))| &\leq c\max\{|y|, |0|\} = c|y| \quad \text{ and} \\ |z|, |y| &\leq \max\{|z|, |y|\} \leq l|x|, \end{aligned}$$

for some $(z, y) \in R_1 \oplus R_2$ with $\psi((z, y)) = x$.

If $c \leq 1$,

$$\begin{aligned} |\psi((z,0))|, |\psi((0,y))| &\leq c|z|, c|y| \\ &\leq |z|, |y|. \\ &\leq l|x|. \end{aligned}$$

Hence, put $d := \max\{l, 1\}$.

If $c \geq 1$,

$$\begin{aligned} |\psi((z,0))|, |\psi((0,y))| \leq c|z|, c|y| \leq cl|x| \quad &\text{and} \\ |z|, |y| \leq c|z|, c|y| \leq cl|x|. \end{aligned}$$

Hence, put $d := \max\{cl, 1\}$.

Choose $c' = d^{-2}$.

Given $U \in \mathrm{GL}_n(S)$ such that $|U - 1| < c'$, let $V = U - 1$. Lift every entry V_{ij} to a pair $(Z_{ij}, Y_{ij}) \in R_1 \oplus R_2$ with $|Z_{ij}|, |Y_{ij}| \leq d|V_{ij}|$ and $|\psi\left((Z_{ij}, 0)\right)|, |\psi\left((0, Y_{ij})\right)| \leq d|V_{ij}|$. Define the matrix

$$U_0 := \psi(1 - Z)U\psi(1 - Y),$$

which fulfills:

$$\begin{aligned} |U_0 - 1| &= |\psi(1 - Z)U\psi(1 - Y) - 1| \\ &= |\psi(1 - Z)(1 + (U - 1))\psi(1 - Y) - 1| \\ &= |(1 - \psi(Z))(1 + (U - 1))(1 - \psi(Y)) - 1| \\ &= |(1 + (U - 1) - \psi(Z) - \psi(Z)(U - 1))(1 - \psi(Y)) - 1| \\ &= |1 - \psi(Y) + (U - 1) - (U - 1)\psi(Y) - \psi(Z) + \psi(Z)\psi(Y) \\ &\qquad - \psi(Z)(U - 1) + \psi(Z)(U - 1)\psi(Y) - 1|. \end{aligned}$$

Since, $\psi(Y) + \psi(Z) = \psi((Z, Y)) = (U - 1)$, then

$$\begin{aligned} &|1 - \psi(Y) - (U - 1) - (U - 1)\psi(Y) - \psi(Z) + \psi(Z)\psi(Y) \\ &\qquad - \psi(Z)(U - 1) + \psi(Z)(U - 1)\psi(Y) - 1| \\ &= |1 - 1 - (\psi(Y) + \psi(Z)) + (U - 1) - (U - 1)\psi(Y) + \psi(Z)\psi(Y) \\ &\qquad - \psi(Z)(U - 1) + \psi(Z)(U - 1)\psi(Y)| \\ &= |1 - 1 - (U - 1) + (U - 1) - (U - 1)\psi(Y) + \psi(Z)\psi(Y) \\ &\qquad - \psi(Z)(U - 1) + \psi(Z)(U - 1)\psi(Y)| \\ &= |-\psi(Z)(U - 1) - (U - 1)\psi(Y) + \psi(Z)\psi(Y) + \psi(Z)(U - 1)\psi(Y)| \\ &\leq d|U - 1|^2, \end{aligned}$$

Here we used $\psi(Z) := \psi((Z,0))$ and $\psi(Y) := \psi((0,Y))$ to simplify the notation.
If $|U-1| \leq d^{-l} < 1$ for some integer $l \geq 2$, then $|U_0 - 1| \leq d^{-l-1}$. Consequently, we may iterate the construction and obtain a series of matrices $(U_n)_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &U_{n+1} := \psi(1-Z_n)U_n\psi(1-Y_n) \quad \text{and} \\ &|Z_n|, |Y_n| \leq d|U_n - 1|. \end{aligned}$$

It follows that

$$\begin{aligned} &\lim_{n\to\infty} U_{n+1} = 1 \\ &\lim_{n\to\infty} \psi(1-Z_n) = \psi(C_1') \\ &\lim_{n\to\infty} \psi(1-Y_n) = \psi(C_2'), \end{aligned}$$

since $(1-Z_n)_{n\in\mathbb{N}}, (1-Y_n)_{n\in\mathbb{N}}$ are Cauchy series in the complete spaces $M_n(R_1), M_n(R_2)$ respectively and ψ is in particular a bounded homomorphism.

If $d=1$, then $|U-1| < d^{-2} = 1$. It follows that

$$|U_n - 1| \leq d^n |U-1|^{2^{n+1}} = |U-1|^{2^{n+1}}.$$

Consequently,

$$\lim_{n\to\infty} |U_n - 1| \leq \lim_{n\to\infty} |U-1|^{2^{n+1}} = 0$$

We conclude that for $d=1$ the series $(U_n)_{n\in\mathbb{N}}$ still converges to 1.
We conclude that $1 = \psi(C_1')U\psi(C_2')$, which yields $U = \psi(C_1)\psi(C_2)$ for some $C_i \in \mathrm{GL}_n(R_i)$ $i = 1, 2$.

□

Definition 4.2.3. Let

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R_1 \arrow[d] \\ R_2 \arrow[r] & R_{12} \end{tikzcd}$$

be a commutative diagram of Banach rings. (For the purposes of this definition, it is not necessary to assume the presence of topologically nilpotent units.) We call this diagram a *glueing square* if the following conditions hold.

a.) The sequence

$$0 \longrightarrow R \longrightarrow R_1 \oplus R_2 \longrightarrow R_{12} \longrightarrow 0$$

of R-modules, in which the last nontrivial arrow takes (s_1, s_2) to $s_1 - s_2$, is exact and the two maps in the middle are strict.

b.) The map $R \longrightarrow R_{12}$ has dense image.

c.) The map $\mathcal{M}(R_1 \oplus R_2) \longrightarrow \mathcal{M}(R)$ is surjective.

We define *glueing data* on a glueing square as in Definition 4.2.1.

Let us recall two important lemmata from the standard abstract descent formalism ([17, §1.3]), which we will use for the glueing of modules.

Lemma 4.2.4. *Consider a finite glueing datum as in Definition 4.2.1, for which $M \otimes_R R_1 \longrightarrow M_1$ is surjective. Then we have the following:*

(i) The map $\psi_1 - \psi_2 : M_1 \oplus M_2 \longrightarrow M_{12}$ is surjective.

(ii) The map $M \otimes_R R_2 \longrightarrow M_2$ is also surjective.

(iii) There exists a finitely generated R-submodule M_0 of M such that for $i = 1, 2$, $M_0 \otimes R_i \longrightarrow M_1$ is surjective.

Proof. (i) The surjection $M \otimes_R R_1 \longrightarrow M_1$ induces a surjection $M \otimes_R R_{12} \longrightarrow M_{12}$ by tensoring over R_1 with R_{12}. Hence, we obtain a surjection

$$(M \otimes_R R_1) \oplus (M \otimes_R R_2) \longrightarrow M_{12},$$

which factors trough $M_1 \oplus M_2$.

$$\begin{tikzcd}
(M \otimes_R R_1) \oplus (M \otimes_R R_2) \arrow[r, two heads] \arrow[d] & M_{12} \\
M_1 \oplus M_2 \arrow[ur, "\psi_1 - \psi_2"'] &
\end{tikzcd}$$

We conclude that $\psi_1 - \psi_2 : M_1 \oplus M_2 \longrightarrow M_{12}$ is surjective.

(ii) For each $v \in M_2$, $\psi(v)$ lifts to $(M \otimes_R R_1) \oplus (M \otimes_R R_2)$; we can thus find w_i in the image of $(M \otimes_R R_i) \longrightarrow M_{12}$ for $i = 1, 2$, such that $\psi_1(w_1) - \psi_2(w_2) = \psi_2(v)$. Consider $v' = (w_1, v + w_2) \in M_1 \oplus M_2$; note that $v' \in M$ by construction. Consequently, the image of $(M \otimes_R R_2) \longrightarrow M_{12}$ contains both w_2 and $v - w_2$, and hence also v. This yields (ii), from which (iii) is immediate since each M_i is a finite R_i-module for $i = 1, 2$. $\square$

Definition 4.2.5. For a map $\varphi : F \longrightarrow G$ of finite free modules over a ring R, we define $I_j\varphi$ as the image of the map

$$\wedge^j F \otimes \wedge^j G^* \longrightarrow R,$$

induced by $\wedge^j \varphi : \wedge^j F \longrightarrow \wedge^j G$. Here $\wedge^j$ denotes the j-th exterior power. If we choose bases for F and G, then φ may be represented by a matrix, and one sees that $I_j\varphi$ is generated by the minors (that is, determinants of submatrices) of size j of the matrix. We make the convention that the determinant of the 0×0 matrix is 1. In particular, $I_0\varphi = R$, and more generally $I_j\varphi = R$ for $j \leq 0$.
Let M be a finitely generated R-module, and let

$$F \xrightarrow{\varphi} G \to M \to 0,$$

be a finite presentation for M, with G of rank r. For each number $i < \infty$ we define the *i-th Fitting Ideal* of M as

$$\mathrm{Fitt}_i(M) = I_{r-i}\varphi \subset R.$$

The Fitting Ideals of a finitely presented R-module M, are invariants of the module with interesting properties:

1. The Fitting Ideals of M are finitely generated ideals of R satisfying $\mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \dots$ and $\mathrm{Fitt}_i(M) = R$ for i sufficiently large.

2. For any ring homomorphism $R \longrightarrow S$, we have $\mathrm{Fitt}_i(M \otimes_R S) = \mathrm{Fitt}_i(M)S$ [11, Corollary 20.5]. In particular, the Fitting Ideals commute under localization.

3. The R-module M is finite projective of constant rank n if and only if $\mathrm{Fitt}_i(M) = 0$ for $i = 0, \dots, n-1$ and $\mathrm{Fitt}_n(M) = R$ [11, Proposition 20.8].

Lemma 4.2.6. *Suppose that for every finite projective glueing datum as in Definition 4.2.1, the map $M \otimes_R R_1 \longrightarrow M_1$ is surjective. Then,*

(i) For any finite projective glueing datum, M is a finitely presented R-Module and $M \otimes_R R_1 \longrightarrow M_1, M \otimes_R R_2 \longrightarrow M_2$ are bijective.

(ii) Suppose in addition that the image of $\mathrm{Spec}(R_1 \oplus R_2) \longrightarrow \mathrm{Spec}(R)$ contains $\mathrm{Maxspec}(R)$. Then with notation as in (i), M is a finite projective R-module.

Proof. Choose M_0 as in Lemma 4.2.4(iii). Choose a surjection $F \longrightarrow M_0$ of R-modules with F finite free, and put $F_1 = F \otimes_R R_1$, $F_2 = F \otimes_R R_2$, $F_{12} = F \otimes_R R_{12}$, $N = \ker(F \longrightarrow M)$, $N_1 = \ker(F_1 \longrightarrow M_1)$, $N_2 = \ker(F_2 \longrightarrow M_2)$, $N_{12} = \ker(F_{12} \longrightarrow M_{12})$. From Lemma 4.2.4 we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & N_1 \oplus N_2 \arrow[r] \arrow[d] & N_{12} \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F_1 \oplus F_2 \arrow[r] \arrow[d] & F_{12} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, dashed] & M_1 \oplus M_2 \arrow[r] \arrow[d] & M_{12} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns, excluding the dashed arrows. Since M_i is projective for $i = 1, 2$, the exact sequence

$$0 \longrightarrow N_i \longrightarrow F_i \longrightarrow M_i \longrightarrow 0$$

splits, so

$$0 \longrightarrow N_i \otimes_{R_i} R_{12} \longrightarrow F_{12} \longrightarrow M_{12} \longrightarrow 0$$

is again exact for $i = 1, 2$. Thus N_i is finite projective over R_i and admits an isomorphism $N_i \otimes_{R_i} R_{12} \cong N_{12}$ for $i = 1, 2$. By Lemma 4.2.4 again, the dashed horizontal arrow in the diagram above is surjective. By diagram chasing, the dashed vertical arrow is also surjective; that is, we may add the dashed arrows to the diagram above while preserving exactness of the rows and columns. In particular, M is a finitely generated R-module; we may repeat the argument with M replaced by N to deduce that M is finitely presented.

For $i = 1, 2$ we obtain a commutative diagram

$$\begin{tikzcd} & N \otimes_R R_i \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d] & M \otimes_R R_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N_i \arrow[r] & F_i \arrow[r] & M_i \arrow[r] & 0 \end{tikzcd}$$

with exact arrows: the first row is derived from

$$0 \longrightarrow N \longrightarrow F_i \longrightarrow M \longrightarrow 0$$

by tensoring over R with R_i and the second row is derived from

$$0 \longrightarrow N_1 \oplus N_2 \longrightarrow F_1 \oplus F_2 \longrightarrow M_1 \oplus M_2 \longrightarrow 0.$$

By Lemma 4.2.4 the left vertical arrow is surjective. Then by the five lemma, the right vertical arrow must be injective. We thus conclude that the map $M \otimes_R R_i \longrightarrow M_i$, which was previously shown (Lemma 4.2.4) to be surjective, is bijective for $i = 1, 2$.

For n a nonnegative integer and $i \in \{1, 2, 12\}$, let $U_{n,i}$ be the closed-open subset of $\mathrm{Spec}(R_i)$ on which M_i has rank n, where the rank at a prime $\mathfrak{p}$ is given by

$$\begin{aligned} &\mathrm{Spec}(R_i) \longrightarrow \mathrm{Card} \\ &\mathfrak{p} \mapsto \dim_{\mathrm{Quot}(R_i/p)}\{M \otimes_{R_i} \mathrm{Quot}(R_i/p)\}. \end{aligned}$$

This is the nonzero locus of some idempotent $e_{n,i} \in R_i$. Since $M_{12} \cong M_1 \otimes_{R_1} R_{12} \cong M_2 \otimes_{R_2} R_{12}$, $U_{n,12}$ can be characterized as the pullback of either $U_{n,1}$ or $U_{n,2}$; this means that the images of $e_{n,1}$ and $e_{n,2}$ in R_{12} are both equal to $e_{n,12}$. It follows that $e_n = e_{n,1} \oplus e_{n,2}$ is an idempotent in R mapping to $e_{n,i}$ in R_i; its nonzero locus is an open subset U_n of $\mathrm{Spec}(R)$ whose pullback to $\mathrm{Spec}(R_i)$ is $U_{n,i}$. This means, that to prove that M is projective , we may reduce to case where M_1 and M_2 are finite projective of some constant rank n.

Since M is finitely presented, we may define its Fitting Ideals $\mathrm{Fitt}_i(M)$ as in Definition 4.2.5. Since $M_1 \oplus M_2$ is finite projective over $R_1 \oplus R_2$ of constant rank n, $\mathrm{Fitt}_i(M)(R_1 \oplus R_2) = \mathrm{Fitt}_i(M_1 \oplus M_2) = 0$ for $i = 0, \ldots, n-1$ and $\mathrm{Fitt}_i(M)(R_1 \oplus R_2) = \mathrm{Fitt}_i(M_1 \oplus M_2) = R_1 \oplus R_2$ for $i = n$. Since the map $R \longrightarrow R_1 \oplus R_2$ is injective, this implies that $\mathrm{Fitt}_i(M) = 0$ for $i = 0, \ldots, n-1$.

Now assume that the image of $\mathrm{Spec}(R_1 \oplus R_2) \longrightarrow \mathrm{Spec}(R)$ contains $\mathrm{Maxspec}(R)$. Then for each $\mathfrak{p} \in \mathrm{Maxspec}(R)$, M must have rank n at $\mathfrak{p}$ by comparison with some point in $\mathrm{Spec}(R_1 \oplus R_2)$, so $\mathrm{Fitt}_n(M)_{\mathfrak{p}} = \mathrm{Fitt}_n(M_{\mathfrak{p}}) = R_{\mathfrak{p}}$. It follows that $\mathrm{Fitt}_n(M) = R$. Hence M is a finite projective R-module. □

We proceed by proving that any simple Laurent covering fulfills the conditions of a glueing square.

Lemma 4.2.7. *Let R_1,R_2,R_{12} correspond to a simple Laurent covering S_-,S_+,$S_- \cap S_+$ of a rational subdomain $U \cong \mathrm{Spa}(B, B^+)$ of $\mathrm{Spa}(A, A^+)$, given by $f \in B$. If (A, A^+) is sheafy, then B,R_1,R_2,R_{12} define a glueing square.*

Proof. We check the conditions on Definition 4.2.3.
a.) As in the proof of Theorem 3.2.3, we see that the sequence

$$0 \to B \xrightarrow{h} R_1 \oplus R_2 \xrightarrow{g} R_{12} \to 0 \qquad (*)$$

is exact.
Since we are in the category of Banach rings, g is bounded by definition and surjective by exactness. Thus, by Theorem 4.1.3, g is strict and open. Applying Lemma 2.5.10, we get

$$R_1 \cong B\{T\}/\overline{(T-f)} \quad \text{and} \quad R_2 \cong B\{U\}/\overline{(fU-1)},$$

equipped each with the quotient norm and strict surjections

$$\theta_1 : B\{T\} \longrightarrow R_1, \theta_2 : B\{U\} \longrightarrow R_2, \theta_{12} : B\{T, U\} \longrightarrow R_{12}$$

Since the Tate algebra $B\{W\}$ is equipped with the Gauss norm,

$$|\sum_I a_I W^I| = \sup_I \{|a_I|\}, \quad \text{with} \quad a_I \in B,$$

We can consider the morphism $\varphi : B \longrightarrow B\{T\} \oplus B\{U\}$, given by $b \mapsto (b, b)$. For any $(b, b) \in B\{T\} \oplus B\{U\}$, the norm is given by $|(b, b)|_{B\{T\}\oplus B\{U\}} = \sup\{|b|, |b|\} = |b|$, which is equivalent to the norm of $b \in B$ and therefore φ is strict.
We conclude that

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "\varphi"] & R_1 \oplus R_2 \\ B\{T\} \oplus B\{U\} \arrow[ur, "\theta"] & \end{tikzcd}$$

is a commutative diagram of Banach rings, with $f = \varphi \circ \theta$, where θ is define by $\theta = (\theta_1, \theta_2)$. As remarked in Definition 2.2.3, h has to be strict, since it is a composition of strict morphisms and θ is surjective. This proofs *a.)*

b.) Consider the natural homomorphism of Banach rings $\phi : R_2 \longrightarrow R_{12}$. Let $b_{12} \in R_{12}$ be the image of $\sum_{i,j=0}^{\infty} a_{ij} T^i U^j \in B\{T, U\}$. Call b_2 the image of $\sum_{i,j=0}^{\infty} a_{ij} f^i U^j \in B\{U\}$ in R_2. It follows that $\phi(b_2) = b_{12}$. Hence ϕ is surjective and continious. Thus, $\phi(R_1)$ is dense in R_{12}. This proofs *b.)*

c.) Applying Lemma 2.5.10 as above, we get again

$$R_1 \cong B\{T\}/\overline{(T-f)} \quad \text{and} \quad R_2 \cong B\{U\}/\overline{(fU-1)}.$$

For any $\alpha \in \mathcal{M}(B)$, consider the following multiplicative seminorm on $B\{T\} \oplus B\{U\}$,

$$\beta\Big(\big(\sum_I a_I T^I, \sum_J b_J U^J\big)\Big) := \sup_{I,J}\{\alpha(a_I), \alpha(b_J)\}, \quad \text{with} \quad a_I, b_J \in B.$$

We can define $\tilde{\beta}$ as the quotient seminorm of β on $R_1 \oplus R_2$. This again is a multiplicative seminorm and we see that

$$\tilde{\beta}\Big((b,b)\Big) = \sup_{I,J}\{\alpha(b), \alpha(b)\} = \alpha(b),$$

for all $b \in B$.
Since $\alpha(b) \leq |b|$ for all $b \in B$, it follows that

$$\begin{aligned}\beta\Big(\Big(\sum_I a_I T^I, \sum_J b_J U^J\Big)\Big) &= \sup_{I,J}\{\alpha(a_I), \alpha(b_J)\} \\ &\leq \sup_{I,J}\{|a_I|, |b_J|\} \\ &= \Big|\Big(\sum_I a_I T^I, \sum_J b_J U^J\Big)\Big|_{B\{T\}\oplus B\{U\}}\end{aligned}$$

for all $\Big(\sum_I a_I T^I, \sum_J b_J U^J\Big) \in B\{T\} \oplus B\{U\}$. Hence, $\tilde{\beta} \in \mathcal{M}(R_1 \oplus R_2)$ and the map $\mathcal{M}(R_1 \oplus R_2) \longrightarrow \mathcal{M}(B)$ maps $\tilde{\beta}$ to α. We conclude that the map $\mathcal{M}(R_1 \oplus R_2) \longrightarrow \mathcal{M}(B)$ is surjective, which proves *c.)*

$\square$

In order to complete the proof of the Kiehl glueing property for simple Laurent coverings, a combination from Lemma 4.2.4 and Lemma 4.2.6 in the setting of glueing squares is needed, since then we would be able to glue finitely generated projective modules corresponding to a simple Laurent covering into a finitely generated projective module; specially important is Lemma 4.2.6*(ii)*.

Lemma 4.2.8. *Consider a glueing square as in Definition 4.2.3., an let M_1, M_2, M_{12} be a finite glueing datum. Let M be the kernel of the map $M_1 \oplus M_2 \longrightarrow M_{12}$ taking (m_1, m_2) to $\psi_1(m_1 \otimes 1) - \psi_2(m_2 \otimes 1)$.*

(i) For $i = 1, 2$, the natural map $M \otimes_R R_i \longrightarrow M_i$ is surjective.

(ii) The map $M_1 \oplus M_2 \longrightarrow M_{12}$ is surjective.

Proof. Choose generating sets $v_1, \dots, v_n$ and $w_1, \dots, w_n$ of M_1 and M_2, respectively, of the same cardinality. We may then choose $n \times n$ matrices A, B over R_{12} such that $\psi_2(w_j \otimes 1) = \sum A_{ij}\psi_1(v_i \otimes 1)$ and $\psi_1(v_j \otimes 1) = \sum B_{ij}\psi_2(w_i \otimes 1)$. Notice that $A, B \in \mathrm{GL}_n(R_{12})$, since $AB = \mathbb{I}$.
Following Definition 4.2.3 consider the maps $\phi : R_2 \to R_{12}$ and $\psi : R_1 \oplus R_2 \to R_{12}$. By hypothesis $\phi : R_2 \to R_{12}$ has dense image. We may thus choose $B' \in M_n(R_2)$ such that $B'' := \phi(B')$ fulfills:

$$\begin{aligned}&B'' \in \mathrm{GL}_n(R_{12}) \text{ and} \\ &\|B'' - B\| < \frac{c}{||A||},\end{aligned}$$

where c is the constant of Lemma 4.2.2. We may then write $\mathbb{I} + A(B'' - B) = \psi(C_1)\psi(C_2^{-1})$ with $C_i \in \mathrm{GL}_n(R_i)$ $i = 1, 2$, since $\|A(B'' - B)\| < c$ and Lemma 4.2.2.

We now define the elements $x_j \in M_1 \oplus M_2$ by the formula:

$$x_j = (x_{1,j}, x_{2,j}) = \Big(\sum_i (C_1)_{ij} v_i, \sum_i (B'C_2)_{ij} w_i \Big) \qquad j = 1, \dots, n.$$

Then

$$\begin{aligned} \psi_1(x_{1,j} \otimes 1) - \psi_2(x_{2,j} \otimes 1) &= \sum_i (\psi(C_1) - AB''\psi(C_2))_{ij} \psi_1(v_i \otimes 1) \\ &= \sum_i ((\mathbb{I} - AB)\psi(C_2))_{ij} \psi_1(v_i \otimes 1) \\ &= 0. \end{aligned}$$

so $x_j \in M$. Since $C_1 \in \mathrm{GL}_n(R_1)$, the $x_{1,j}$ generate M_1 over R_1, so the map $M \otimes_R R_1 \to M_1$ is surjective. We may now apply Lemma 4.2.4 and deduce (i) and (ii). $\square$

Theorem 4.2.9. *Consider a glueing square as in Definition 4.2.3, and let M_1, M_2, M_{12} be a finite projective glueing datum. Let M be the kernel of the map $M_1 \oplus M_2 \longrightarrow M_{12}$ taking (m_1, m_2) to $\psi_1(m_1 \otimes 1) - \psi_2(m_2 \otimes 1)$. Then M is a finite projective R-module and for $i = 1, 2$ the natural maps $M \otimes_R R_i \longrightarrow M_i$ are isomorphisms.*

Proof. By Lemma 4.2.8, the hypotheses of Lemma 4.2.6 are satisfied. It thus suffices to check that the additional hypothesis of Lemma 4.2.6*(ii)* is satisfied, i.e., that the image of $\mathrm{Spec}(R_1 \oplus R_2) \longrightarrow \mathrm{Spec}(R)$ contains $\mathrm{Maxspec}(R)$. Given $\mathfrak{p} \in \mathrm{Maxspec}(R)$, choose $\alpha \in \mathcal{M}(R)$ with $\mathfrak{p}_\alpha = \mathfrak{p}$ (see Definition 2.4.7). By assumption, α lifts to some $\beta \in \mathcal{M}(R_1 \oplus R_2)$; then $\mathfrak{p}_\beta$ is a prime ideal of $\mathrm{Spec}(R_1 \oplus R_2)$ lifting $\mathfrak{p}$. $\square$

Now, let us prove that if M is a finite projective module over a sheafy adic Banach ring (A, A^+), then the presheaf $\widetilde{M}$ defined in Definition 4.1.1 is in fact a sheaf on $X := \mathrm{Spa}(A, A^+)$. This would mean, that the functor

$$\begin{aligned} \mathfrak{F} : \mathfrak{Mod}_{fp}(A, A^+) &\longrightarrow \Big\{ \text{Sheaves of } \mathcal{O}_X\text{-modules} \Big\} \\ M &\longmapsto \mathfrak{F}(M) := \widetilde{M}, \end{aligned}$$

from the category of finite projective (A, A^+)-modules to the category of sheaves of $\mathcal{O}_X$-modules, is well defined.

Corollary 4.2.10. *Suppose that (A, A^+) is sheafy. Let M be a finite projective A-module. Then for every rational covering $\mathfrak{A}$ of any rational subspace $U \cong \mathrm{Spa}(B, B^+)$ of $X = \mathrm{Spa}(A, A^+)$,*

$$H^i(U, \widetilde{M}) = \check{H}^i(U, \widetilde{M}; \mathfrak{A}) = \begin{cases} M \otimes_A B & i = 0 \\ 0 & i > 0. \end{cases}$$

Proof. Let $\mathfrak{A} = (U_i)_{i \in I}$ be a rational covering of U. Assume first, that M is a free A-module, i.e., $M \cong A^{(J)}$, for some index J, where $A^{(J)} := \bigoplus_{j \in J} A$. In this case the Čech complex associated to the covering $\mathfrak{A}$ is given by

$$0 \longrightarrow \widetilde{M}(U) \xrightarrow{\epsilon} \prod_i \widetilde{M}(U_i) \xrightarrow{d^0} \prod_{i_0 < i_1} \widetilde{M}(U_{i_0 i_1}) \xrightarrow{d^1} \dots$$

which is the same as

$$0 \longrightarrow A^{(J)} \xrightarrow{\epsilon^{(J)}} \Big(\prod_i \mathcal{O}_X(U_i)\Big)^{(J)} \xrightarrow{d^{0(J)}} \Big(\prod_{i_0<i_1} \mathcal{O}_X(U_{i_0 i_1})\Big)^{(J)} \xrightarrow{d^{1(J)}} \dots$$

and therefore exact by Theorem 3.2.3.
If M is not free, consider a simple Laurent covering

$$V_1 \cong \mathrm{Spa}(B_1, B_1^+) \quad V_2 \cong \mathrm{Spa}(B_2, B_2^+)$$

of U. By Lemma 4.2.7, B, B_1, B_2, B_{12}, where $V_1 \cap V_2 \cong \mathrm{Spa}(B_{12}, B_{12}^+)$ define a glueing square. Applying Lemma 4.2.9 for the finite projective glueing datum define by $M \otimes_A B_1, M \otimes_A B_2, M \otimes_A B_{12}$, we get that the short sequence of finite projective A-modules

$$0 \longrightarrow M \otimes_A B \longrightarrow (M \otimes_A B_1) \oplus (M \otimes_A B_2) \longrightarrow M \otimes_A B_{12} \longrightarrow 0$$

is exact, due to the fact that M is a flat A-module.
Since this sequence corresponds to the Čech complex associated to the covering $\{V_1, V_2\}$ of U

$$0 \longrightarrow \widetilde{M}(U) \xrightarrow{\epsilon} \prod_{1,2} \widetilde{M}(V_i) \xrightarrow{d^0} \widetilde{M}(V_{12}) \xrightarrow{d^1} 0$$

we conclude that

$$H^i(U, \widetilde{M}) = \check{H}^i(U, \widetilde{M}; \{V_1, V_2\}) = \begin{cases} M \otimes_A B & i = 0 \\ 0 & i > 0. \end{cases}$$

The result follows after invoking Lemma 3.2.1. □

Analog to the proof of the Tate sheaf property we will define a property $\mathcal{P}$ that fulfills the conditions of Lemma 2.5.18 in order to derive the Kiehl glueing property.

Lemma 4.2.11. *Let $X = (A, A^+)$ be a sheafy Banach ring and let $\mathcal{F}$ be a locally free $\mathcal{O}_X|_U$-Module of finite rank over a rational subspace $U \cong \mathrm{Spa}(B, B^+)$ of $\mathrm{Spa}(A, A^+)$. Then the property $\mathcal{P}$:*

> Given a covering $\mathfrak{A} = (U_i)_{i \in I}$ of U with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$, where $U_i \cong \mathrm{Spa}(B_i, B_i^+)$ and M_i is a finite projective B_i-module, then $\mathcal{F} \cong \widetilde{M}$ for a finite projective B-module,

fulfills the conditions of Lemma 2.5.18.

Proof. a.) Consider a covering $\mathfrak{A} = (U_i)_{i \in I}$ of U with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$, where $U_i \cong \mathrm{Spa}(B_i, B_i^+)$ and M_i is a finite projective B_i-module. Given a refinement $\mathfrak{B} = (V_{ij})_{j \in J}$ with the property $\mathcal{P}$, then $\widetilde{M_i}|_{V_{ij}} \cong (M_i \otimes_{B_i} C_j)^\sim$, where $V_{ij} \subset U_i$ and $V_{ij} \cong \mathrm{Spa}(C_j, C_j^+)$. We claim that $M_i \otimes_{B_i} C_j$ is a finite projective C_j-module.
Since M_i is a finite projective B_i-module, there exist a finite free B_i-module Z, such that $Z = M_i \oplus N_i$. Thus tensoring with C_j over B_i, we get

$$Z \otimes_{B_i} C_j = (M_i \otimes_{B_i} C_j) \oplus (N_i \otimes_{B_i} C_j).$$

So if $Z \otimes_{B_i} C_j$ is a finite free C_j-module, we would have proven the claim by Theorem 2.1.4.
Since Z is a finite free B_i-module, we can write Z as $Z \cong \bigoplus_{k=0}^n B_i$, then

$$Z \otimes_{B_i} C_j \cong \Big(\bigoplus_{k=0}^n B_i\Big) \otimes_{B_i} C_j \cong \bigoplus_{k=0}^n (B_i \otimes_{B_i} C_j) \cong \bigoplus_{k=0}^n C_j.$$

We conclude that $M_i \otimes_{B_i} C_j$ is a finite projective C_j-module. Since $\mathfrak{B}$ has the *property* $\mathcal{P}$, it follows that $\mathcal{F} \cong \widetilde{M}$, with M a finite projective B-module. This proofs a.)

b.) Let $\mathfrak{A} = (U_i)_{i\in I}$ be a covering of U with the *property* $\mathcal{P}$ and let $\mathfrak{B}_\mathfrak{i} = (V_{ij})_{j\in J}$ be a covering of each U_i, also with the *property* $\mathcal{P}$. Consider $\mathfrak{B} = \bigcup_{i\in I} \mathfrak{B}_\mathfrak{i}$ as a covering of U, such that for each $V_{ij} \cong \mathrm{Spa}(C_j, C_j^+)$, $\mathcal{F}|_{V_{ij}} \cong \widetilde{M_{ij}}$ with M_{ij} a finite projective C_{ij}-module.
All the $V_{ij} \in \mathfrak{B}_\mathfrak{i}$ have then the premises of the *property* $\mathcal{P}$. Hence $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$, with M_i a finite projective B_i-module, for each $U_i \cong \mathrm{Spa}(B_i, B_i^+) \in \mathfrak{A}$. This again makes $\mathfrak{A}$ fulfill the premises of the property $\mathcal{P}$ and we conclude that $\mathcal{F} \cong \widetilde{M}$, with M a finite projective B-module. This proofs b.)

c.) Let $\{V_1 \cong \mathrm{Spa}(C_1, C_1^+), V_2 \cong \mathrm{Spa}(C_2, C_2^+)\}$ be a simple Laurent covering of U, such that $\mathcal{F}|_{V_1} = \widetilde{M_1}$ and $\mathcal{F}|_{V_2} = \widetilde{M_2}$, with M_1 a finite projective C_1-module and M_2 a finite projective C_2-module. Consider $V_1 \cap V_2 \cong \mathrm{Spa}(C_{12}, C_{12}^+)$. As proven above, we get that the module M_{12}, defined by

$$\psi_i : M_i \otimes_{C_i} C_{12} \cong M_{12} \quad i = 1, 2,$$

is a finite projective C_{12}-module. Thus, by Lemma 4.2.7, B, M_1, M_2, M_{12} define a finite projective glueing data. Now we can apply Theorem 4.2.8 and conclude that the kernel of the map $M_1 \oplus M_2 \longrightarrow M_{12}$ taking (m_1, m_2) to $\psi_1(m_1 \otimes 1) - \psi_2(m_2 \otimes 1)$ is a finite projective B-module with isomorphisms $M \otimes_B C_i \longrightarrow M_i$, $i = 1, 2$. Consider the $\mathcal{O}_X|_U$-module $\widetilde{M}$ associated to the B-module M. For any rational subspace $V \cong \mathrm{Spa}(D, D^+)$ of U with $V \cap V_i \cong \mathrm{Spa}(D_i, D_i^+)$ for $i = 1, 2$, we have

$$\begin{aligned}\widetilde{M}(V \cap V_i) &= M \otimes_B D_i \\ &\cong (M \otimes_B C_i) \otimes_{C_i} D_i \\ &\cong M_i \otimes_{C_i} D_i \\ &= \widetilde{M_i}(V \cap V_i) \\ &\cong \mathcal{F}(V \cap V_i).\end{aligned}$$

Since $\mathcal{F}$ is an $\mathcal{O}_X|_U$-module we conclude that $\mathcal{F} \cong \widetilde{M}$. This proofs c.) □

Theorem 4.2.12. *Let (A, A^+) be a sheafy adic Banach ring. Then the structure sheaf on* $\mathrm{Spa}(A, A^+)$ *satisfies the Kiehl glueing property.*

Proof. Let $U \cong \mathrm{Spa}(B, B^+)$ be a rational subdomain of $\mathrm{Spa}(A, A^+)$. Now consider the functor

$$\mathfrak{F} : \Big\{\text{Finite projective } \mathcal{O}(U)\text{-modules.}\Big\} \to \left\{\begin{array}{l}\text{Sheaves of } \mathcal{O}_U\text{-modules over } U \\ \text{which are locally free of finite} \\ \text{rank.}\end{array}\right\}$$

given by $\mathfrak{F}(M) := \widetilde{M}$.

Let $\mathcal{F}$ be a locally free of finite rank $\mathcal{O}_U$-module over U. Choose a rational covering $\mathfrak{A} = (U_i)_{i\in I}$ of U, then $\mathfrak{A}$ has the *property* $\mathcal{P}$ by Lemma 4.2.11 and Lemma 2.5.18; thus $\mathcal{F} \cong \widetilde{M}$, where M is a finite projective B-module. By Corollary 4.2.10, $\mathcal{F}(U) \cong \widetilde{M}(U) = M$.
Consequently, define the functor

$$\mathfrak{G} : \left\{\begin{matrix}\text{Sheaves of } \mathcal{O}_U\text{-modules over } U \\ \text{which are locally free of finite} \\ \text{rank.}\end{matrix}\right\} \to \left\{\text{Finite projective } \mathcal{O}(U)\text{-modules.}\right\}$$

by $\mathfrak{G}(\mathcal{F}) := \mathcal{F}(U)$.

It is clear that both functors are quasi-inverse to each other by the discussion held above and so $\mathfrak{F}$ defines an equivalence of categories. $\square$

As a corollary of Theorem 3.2.3 and Theorem 4.2.12 we derive our main result: Theorem 1.0.1.

Corollary 4.2.13. *Let (A, A^+) be a sheafy adic Banach ring. Then the structure sheaf on* $\mathrm{Spa}(A, A^+)$ *satisfies the Tate sheaf property and the Kiehl glueing property.*

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References

[1] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean Analysis*. Grundlehren der Mathematik. Wiss. 261. Berlin: Springer-Verlag, 1984.

[2] The Stacks Project Authors. *Stack Project*. 2014. URL: `http://stacks.math.columbia.edu`.

[3] V. Berkovich. "Spectral Theory and Analytic Geometry over Non-Archimedean Fields". In: *Amer. Math. Soc.* (1990). Surveys and Monographs 33.

[4] V. Berkovich. "Étale cohomology for non-Archimedean analytic spaces". In: *Pub. Math.* (1993). IHÉS **78**, pp. 5–161.

[5] S. Bosch. *Algebraic Geometry and Commutative Algebra*. ISSN 0172-5939 Universitext. London: Springer-Verlag, 2013.

[6] S. Bosch. *Lectures on Formal and Rigid Geometry*. ISSN 978-3-319-04417-0 eBook. Switzerland: Springer, 2014.

[7] N. Bourbaki. *Topologie Générale, Chapitres 1 á 4*. Reprint of the 1985 original. Berlin: Springer, 2006.

[8] N. Bourbaki. *Algèbre Commutative, Chapitres 1 á 4*. Reprint of the 1981 original. Berlin: Springer, 2007.

[9] N. Bourbaki. *Spaces Vectoriels Topologiques*. Reprint of the 1981 original. Springer-Berlin, 2007.

[10] G. Christol and Z. Mebkhout. "Sur le théorème de l'indice des équations différentielles". In: *Invent. Math. 143* (2001). IV, pp. 629–672.

[11] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*. Graduate Texts in Math. 150. Springer-New York, 1995.

[12] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Math. 52. New York: Springer-Verlag, 1977.

[13] T. Henkel. "An Open Mapping Theorem for rings with a zero sequence of units". In: *arXiv:1407.5647v1* (2014).

[14] R. Huber. “Continuous valuations”. In: *Math. Z.* ***212*** (1993), pp. 455–477.

[15] R. Huber. “A generalization of formal schemes and rigid analytic varieties”. In: *Math. Z.* ***217*** (1994), pp. 513–551.

[16] R. Huber. *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces.* Aspects of Mathematics, E30. Braunschweig: Friedr. Vieweg & Sohn, 1996.

[17] K. Kedlaya, and R. Liu. “Relative p-adic Hodge theory: Foundations”. In: *arXiv:1301.0792v4* (2014).

[18] K.S. Kedlaya. “Nonarchimedean geometry of Witt vectors”. In: *Nagoya Math J. 209* (2013), pp. 111–165.

[19] W. Rudin. *Functional Analysis.* ISBN 0-07-054236-8. McGraw-Hill, 1973.

[20] P. Schneider. *Nonarchimedean Functional Analysis.* ISBN 3-540-42533-0. Berlin, Heidelberg, New York: Springer-Verlag, 2000.

[21] G. Tamme. *Introduction to Étale Cohomology.* ISBN 978-3-540-57116-2. Berlin, Heidelberg: Springer-Verlag, 1994.