Bilinear DHOs with hyperplane induced subDHOs: an algorithm

Background for this text is the article [1]. We describe a simple algorithm which finds for a given bilinear DHO $\mathcal{T}$ of rank n the bilinear DHOs $\mathcal{S}$, such that $\mathcal{S}$ possesses a hyperplane which induces a subDHO isomorphic to $\mathcal{T}$.

Suppose $\dim U(\mathcal{T}) = n+m$. Then by [1, Thm. 4.7] a bilinear DHO $\mathcal{S}$ of the desired type has an ambient space of rank $n+1+M$ with $m \leq M \leq n$. We can assume that $U(\mathcal{S}) = X \oplus Y$ with $X = \langle e_0, e_1, \ldots, e_n \rangle$, $Y = \langle e_{n+1}, \ldots, e_M \rangle$, and that the hyperplane is $H = \langle e_1, \ldots, e_M \rangle$. Wlog. we can assume $U(\mathcal{T}) = X_0 \oplus Y_0$ with $X_0 = \langle e_1, \ldots, e_n \rangle$, $Y_0 = \langle e_{n+1}, \ldots, e_m \rangle$. We write (a, x) for an element of the form $ae_0 + \sum_{i=1}^{n} \in X$, $x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ and (y, z) for an element of the form $\sum_{i=n+1}^{m} y_i e_i + \sum_{j=m+1}^{M} z_j e_j \in Y$ with $y = (y_{n+1}, \ldots, y_m) \in \mathbb{F}_2^m$ and $z = (z_{m+1}, \ldots, z_M) \in \mathbb{F}_2^{M-m}$.

Let $\beta_0 : X_0 \to \mathrm{Hom}(X_0, Y_0)$ be a monomorphism defining $\mathcal{T}$ as $\mathcal{S}_{\beta_0}$ and let a monomorphism $\beta : X \to \mathrm{Hom}(X, Y)$ be a monomorphism which describes $\mathcal{S} = \mathcal{S}_\beta$ with respect to the given basis. We may assume that $\ker \beta(1,0) = \langle e_0 \rangle$. Then $\beta(a, e) \in \mathbb{F}_2^{(n+1)\times m \times (M-m)}$, and

$$\beta(1,0) = \begin{pmatrix} 0 & 0 \\ A_1 & A_2 \end{pmatrix}, \quad A_1 \in \mathbb{F}_2^{n\times m}, \quad A_2 \in \mathbb{F}_2^{n\times (M-m)} \tag{1}$$

and

$$\beta(0,e) = \begin{pmatrix} \delta(e) & \gamma(e) \\ \beta_0(e) & 0_{n\times(M-m)} \end{pmatrix}, \quad \delta(e) \in \mathbb{F}_2^m, \quad \gamma(e) \in \mathbb{F}_2^{M-m}. \tag{2}$$

As β_0 is given it remains to determine the matrix (A_1, A_2) and the linear mappings δ and γ. This sets up the following simple procedure:

Input: An additively closed DHO-set $\mathcal{D}_0 \in \mathbb{F}_2^{n\times m}$ (which is $\beta_0(\mathbb{F}_2^n)$) and a number M, $m \leq M \leq m+n$.

Qutput: The additively closed DHO-sets $\mathcal{D} \in \mathbb{F}_2^{(n+1)\times M}$ such that the associated DHOs have a hyperplane inducing the subDHO $\mathcal{T}$.

STEP 1. Let $\mathcal{D}_0^*$ be the $\mathbb{F}_2$-space of matrices $D^* = (D, 0_{n\times(M-m)}) \in \mathbb{F}_2^{n\times M}$, $D \in \mathcal{D}_0$. Determine the set of $\mathcal{A}^*$ of matrices $A = (A_1, A_2) \in \mathbb{F}_2^{n\times M}$ such that $A + D^*$ has rank $n+1$ for $D^* \in \mathcal{D}_0^*$.

STEP 2. Take $\mathcal{A}$ as a set of representatives for the cosets $A + \mathcal{D}_0^*$, $A \in \mathcal{A}^*$.

STEP 3. Let $\{D_1^*, \ldots, D_n^*\}$ be a basis of the $\mathbb{F}_2$-space $\mathcal{D}_0^*$ and $A \in \mathcal{A}$. Set $\overline{A} = \binom{0}{A} \in \mathbb{F}_2^{(n+1)\times M}$. Set $\mathcal{I}_j = \mathrm{Im}\,(D_j^* + A)$, $1 \leq j \leq n$. For each n-tuple $(v_1, \ldots, v_n) \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_n$ set $\overline{D}_j = \binom{v_j}{D_j^*}$, $1 \leq j \leq n$. If the $\mathbb{F}_2$-space

$$\mathcal{D} = \langle \overline{A}, \overline{D_1}, \ldots, \overline{D_n} \rangle$$

is a DHO-set **return** $\mathcal{D}$.

Remark. (a) If $M = m + n$ it is easy to see that the DHO $\mathcal{T}$ is isomorphic to a symmetric DHO and up to isomorphism there is a unique DHO $\mathcal{S}$, namely the DHO from the extension construction [1, Ex.4.9]. If $m = n - 1$, the linear mapping γ in (2) must be injective. This forces $M = m + n$. So in this case too our search only produces the DHO from the extension construction.

(b) The search was limited to the cases $n = 4$, $m = M = 4, 5, 6$ and $n = 4$, $m = 4$, $M = 5$.

(c) These computations show that that for $M \neq n + m$ a bilinear DHO $\mathcal{T}$ of rank n may not occur in a bilinear DHO $\mathcal{S}$ of rank $n + 1$ as a hyperplane induced subDHO. There also may be more than one pairwise non-isomorphic, bilinear DHOs of rank $n + 1$ containing $\mathcal{T}$ as a hyperplane induced subDHO.

References

[1] U. Dempwolff, Y. Edel: The Radical of Binary Dimensional Dual Hyperovals, to be submitted.