Coding-theoretic constructions for (t, m, s)-nets and ordered orthogonal arrays

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1 Introduction

$(t, m, s)-$nets were defined by Niederreiter [17] in the context of quasi-Monte Carlo methods of numerical integration. Niederreiter pointed out close connections to certain combinatorial and algebraic structures. This was made precise in the work of Lawrence, Mullen and Schmid [11, 15, 24]. These

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authors introduce a large class of finite combinatorial structures, which we will call **ordered orthogonal arrays** OOA. These OOA contain orthogonal arrays as a subclass. $(t, m, s)_q$-nets (that is, (t, m, s)-nets in base q as in the original Definition 2.2 in [17]) are equivalent to another parametric subclass of OOA. Loosely speaking a $(t, m, s)_q$-net is **linear** if it is defined over the field $\mathbb{F}_q$ with q elements. The duality between linear codes and linear orthogonal arrays carries over to the more general setting of linear OOA (see [14] or [20]). Here OOA generalize orthogonal arrays (dual codes). The weight function generalizing Hamming weight was first described by Niederreiter in [16, 18]. It was systematically exploited by Rosenbloom-Tsfasman in [23]. We use the term NRT-space for the corresponding metric space. A description is in Section 2.

Our main results are generalizations of coding-theoretic construction techniques from Hamming space to NRT-space, most notably concatenation (equivalently: Kronecker products), the $(u, u+v)$-construction and the Gilbert-Varshamov bound.

Let $k = m-t$ denote the **strength** of a net. If a linear $(t, m, s)_q$-net exists, where $m < s$, then a linear code $[s, s-m, k+1]_q$ exists. From this point of view it is a basic problem (the problem of net-embeddability) to decide when a code $[s, s-m, k+1]_q$ can be completed to a linear $(m-k, m, s)_q$-net. More generally we ask when a linear OOA with certain parameters can be embedded in a larger OOA. We speak of a theorem of Gilbert-Varshamov type if the existence of the larger OOA can be guaranteed whenever the parameters satisfy a certain numerical condition. In the final section we apply our theoretical construction techniques as well as computer-generated net embeddings of error-correcting codes to improve upon net-parameters for nets of moderate strength and dimension defined over small fields.

2 Linear nets and linear ordered orthogonal arrays

A (t, m, s)-net is a subset of Euclidean $s-$space. We mentioned in the introduction that (t, m, s)-nets can equivalently be described by finite geometrical objects. More precisely (t, m, s)-nets are equivalent to a subclass of **ordered orthogonal arrays.** For our purposes this description is more natural. We use it as a definition. Moreover we concentrate on the linear case.

Definition 1. *Let $\Omega = \Omega^{(T,s)}$ be a set of Ts elements, partitioned into s* **blocks** *$B_i, i = 1, 2 \ldots, s$, where $B_i = \{\omega_1^{(i)}, \ldots, \omega_T^{(i)}\}$. Each block carries a total ordering:*

$$\omega_1^{(i)} < \omega_2^{(i)} < \cdots < \omega_T^{(i)}.$$

This gives Ω the structure of a partially ordered set, the union of s totally ordered sets of T points each. We consider Ω as a basis of a $Ts-$dimensional vector space $\mathbb{F}_q^{(T,s)}$. An **ideal** *in Ω is a set of elements closed under predecessors. An* **antiideal** *is a subset closed under followers. Observe that antiideals are precisely the complements of ideals.*

We visualize elements $x = (x_j^{(i)}) \in \mathbb{F}_q^{(T,s)}, i = 1, \ldots, s; j = 1, \ldots, T$ either as strings of length Ts, divided in s segments (the blocks) of length T each, or as matrices with T rows and s columns. Refer to these representations as **vector notation** and **matrix notation,** respectively. The interpretation of $x \in \mathbb{F}_q^{(T,s)}$ as a point in the $s-$dimensional unit cube is obtained by reading the $x_j^{(i)}$ for fixed i as the T first digits of the $q-$ary expansion of a real number between 0 and 1. As an example, the point

0	0	1	1
1	1	1	0
1	0	0	1

in $\mathbb{F}_2^{(3,4)}$ is mapped to the point $(\frac{3}{8}, \frac{1}{4}, \frac{3}{4}, \frac{5}{8}) \in [0, 1)^4$. This also motivates the hierarchical ordering inside the blocks.

We introduce some more terminology, which will be helpful in describing the basic parameters of NRT-space.

Definition 2. *We refer to coordinate positions of $\mathbb{F}_q^{(T,s)}$ as* **cells.** *They are in obvious bijection with the elements of Ω. The* **breadth** *$b = b(x)$ of a vector $x \in \mathbb{F}_q^{(T,s)}$ is the number of blocks $B_i, i = 1, 2, \ldots, s$ where x has a nonzero entry. The ideal $K = K(x)$ generated by x is the smallest ideal containing the support of x. The breadth of an ideal K is the number of blocks it intersects nontrivially. Let $n = |K|$ be the* **size** *of K. The* **type** *$\pi = \pi(K)$ is the partition n, where the multiplicity f_i of i as a part of π is the number of blocks, which intersect K in i points. The breadth $b(\pi)$ of a partition is the number of its nonzero parts. If $\pi = \pi(K(x))$, then $b(\pi) = b(x)$.*

Definition 3 (NRT-metric). *Let $x \in \mathbb{F}_q^{(T,s)}$. The* **weight** *of x is*

$$\rho(x) = \rho(x, 0) = \sum_{i=1}^{s} T - \max\{j \mid x_1^{(i)} = \ldots x_j^{(i)} = 0\}$$

The **distance** $\rho(x,y)$ *is defined as* $\rho(x,y)=\rho(x-y)$. *The* **minimum weight** *(=minimum distance) of a subspace* $\mathcal{C}\subseteq \mathbb{F}_q^{(T,s)}$ *is the minimum among the weights of its nonzero members*

We may visualize the weight $\rho(x)$ as follows: in each block let the leading zeroes evaporate. The number of remaining cells is $\rho(x)$. It is clear that ρ is a metric. Also, $Ts-\rho(x,y)$ is the size of the maximal ideal on which x and y agree.

Definition 4. *Let* $S_l^{(T,s)}$ *be the number of vectors of weight* l *in* $\mathbb{F}_q^{(T,s)}$ *and* $V_l^{(T,s)}=\sum_{i=0}^{l}S_i^{(T,s)}$ *the volume of a ball of radius* l *in* $\mathbb{F}_q^{(T,s)}$.

Proposition 1. *We have*

$$S_l^{(T,s)}=\sum_{\pi}\binom{s}{f_T,\dots,f_1,s-b}(q-1)^b q^{l-b},$$

where the sum is over all partitions π *of* l *of depth* $\leq T$, *and* $b=b(\pi), f_i=f_i(\pi)$.

Proof. $S_l^{(T,s)}$ counts the vectors of $\mathbb{F}_q^{(T,s)}$, whose support generates an ideal of size l. The type of such an ideal K is a partition π as above. The number of vectors generating a fixed K of breadth b clearly is $(q-1)^b q^{l-b}$. It remains to count the ideals K with a given type π. This number is

$$\binom{s}{f_T}\binom{s-f_T}{f_{T-1}}\cdots\binom{s-f_T-\cdots-f_2}{f_1}=\binom{s}{f_T,\dots,f_1,s-b}.$$

■

We now define the objects we are primarily interested in.

Definition 5. *A linear subspace (code)* $\mathcal{C}\subseteq\mathbb{F}_q^{(T,s)}$ *has* **strength** $k=k(\mathcal{C})$ *if* k *is maximal such that the projection from* $\mathcal{C}$ *to any ideal of size* k *is surjective. We also call such a subspace an* **ordered orthogonal array** *OOA, which is* $q-$*linear, has* **length** s, **depth** T, **dimension** $m=\dim(\mathcal{C})$ *and strength* k.

A linear $(m-k,m,s)_q$-net is equivalent to an $m-$dimensional code $\mathcal{C}\subseteq\mathbb{F}_q^{(k,s)}$ of strength k. Observe also that linear OOA of depth 1 are precisely linear orthogonal arrays, in other words an $m-$dimensional code in $\mathbb{F}_q^{(1,s)}$ of strength k is the dual (with respect to the ordinary dot product) of a code $[s,s-m,k+1]_q$.

Definition 6. *Define a symmetric bilinear form on* $\mathbb{F}_q^{(T,s)}$ *by*

$$\langle x, y\rangle = \sum_{i=1}^{s} x_1^{(i)} y_T^{(i)} + x_2^{(i)} y_{T-1}^{(i)} + \cdots + x_T^{(i)} y_1^{(i)}.$$

The dual $\mathcal{C}^{\perp}$ *is defined with respect to this scalar product.*

Observe that $\mathbb{F}_q^{(1,s)}$ is the usual Hamming space, with its metric, the dot product and the corresponding notion of duality. Generalizing the notion of Hamming space we may call $\mathbb{F}_q^{(T,s)}$ with the NRT-metric and the corresponding notion of strength the **NRT-space.** It is an important albeit elementary observation that the duality (in Hamming space) between strength and minimum distance can be extended to our setting (see [14] or [20]).

Theorem 1. *Let* $\mathcal{C} \subseteq \mathbb{F}_q^{(T,s)}$ *be a linear subspace (code). Then*

$$\rho(\mathcal{C}^{\perp}) = k(\mathcal{C}) + 1.$$

We are led to the natural problem of generalizing coding-theoretic bounds and constructions from Hamming space to NRT-space.

3 Trace codes

Theorem 2. *Let* $\mathcal{C} \subseteq \mathbb{F}_{q^r}^{(T,s)}$ *of dimension* m *and strength* k*. We can construct* $\tilde{\mathcal{C}} \subseteq \mathbb{F}_q^{(T,rs)}$ *of dimension* rm *and strength* k*.*

Proof. Let $\{b_1, \ldots, b_r\}$ be a basis of $F = \mathbb{F}_{q^r} \mid \mathbb{F}_q$. We describe an $\mathbb{F}_q-$isomorphism $\tilde{}: \mathcal{C} \longrightarrow \tilde{\mathcal{C}}$ as follows: Let $tr : F \longrightarrow \mathbb{F}_q$ be the trace and $x \in \mathcal{C}$. The entry of $\tilde{x}$ in coordinate (i, a), where $1 \leq i \leq s, 1 \leq a \leq r$ and depth j is $\tilde{x}_j^{(i,a)} = tr(x_j^{(i)} b_a)$. It is obvious that we have an $\mathbb{F}_q-$ isomorphism as the kernel is trivial. In particular $\dim(\tilde{\mathcal{C}}) = mr$. It is also obvious that $\tilde{\mathcal{C}}$ still has strength k. ■

The special case of nets was proved in [22].

4 Concatenation

The following construction may be seen as a concatenation construction or as a Kronecker product for linear codes in NRT-space. A different Kronecker product construction is in [21].

Theorem 3. *Let $\mathcal{C}_1 \subseteq \mathbb{F}_{q^r}^{(T_1,s_1)}$ of dimension m and $\mathcal{C}_2 \subseteq \mathbb{F}_q^{(T_2,s_2)}$ of dimension r. Let $\alpha : \mathbb{F}_{q^r} \longrightarrow \mathcal{C}_2$ be an $\mathbb{F}_q-$isomorphism. Define the concatenation $\mathcal{C}_2 \circ \mathcal{C}_1 = \alpha(\mathcal{C}_1) \subset \mathbb{F}_q^{(T_1T_2,s_1s_2)}$ as follows (in matrix notation): each $x \in \mathcal{C}_1$ yields $\alpha(x) \in \mathcal{C}_2 \circ \mathcal{C}_1$ by applying α to each entry of x. Then $\dim(\mathcal{C}_2 \circ \mathcal{C}_1) = mr$ and $k(\mathcal{C}_2 \circ \mathcal{C}_1) \geq \min\{k(\mathcal{C}_1), k(\mathcal{C}_2)\}$.*

Proof. As the elements of $\mathcal{C}_2 \circ \mathcal{C}_1$ are in bijection with those of $\mathcal{C}_1$, the statement concerning the dimension is obvious. Let $k = \min\{k(\mathcal{C}_1), k(\mathcal{C}_2)\}$. Consider an ideal K of size k in $\Omega^{(T_1T_2,s_1s_2)}$. The natural projection $\overline{K}$ to $\Omega^{(T_1,s_1)}$ is an ideal of size $\leq k$. We can therefore find $x \in \mathcal{C}_1$ such that $\alpha(x)$ has arbitrarily chosen entries from $\mathcal{C}_2$ in the positions of this ideal. For each $(i_1, j_1) \in \overline{K}$ the intersection of K with the corresponding $\Omega^{(T_2,s_2)}$ is itself an ideal, clearly of size $\leq k$. The claim follows. ∎

The special cases of Theorem 3 when either $\mathcal{C}_1$ or $\mathcal{C}_2$ is a net and the other is an OA ($T_2 = 1$ or $T_1 = 1$) is in [22].

5 The $(u, u+v)$-construction

Theorem 4. *For $i = 1, 2$ let $\mathcal{C}_i \subset \mathbb{F}_q^{(T,s_i)}$ be linear OOA of dimension m_i and strength k_i, where $s_1 \leq s_2$. We can construct $\mathcal{C} \subset \mathbb{F}_q^{(T,s_1+s_2)}$ of dimension $m_1 + m_2$ and strength $\min\{k_2, 2k_1 + 1\}$.*

Proof. This is a direct generalization of the famous $(u, u+v)$-construction in coding theory, which seems to go back to [26]. Consider the duals $\mathcal{C}_i^\perp$. These have dimension $Ts_i - m_i$ and distance $k_i + 1$. We apply the $(u, u+v)$-construction to $\mathcal{C}_i^\perp$. Our $\mathcal{C}$ will be obtained by dualizing (back). More precisely let $C_i = (C_i^{(1)}, C_i^{(2)}, \ldots, C_i^{(s_i)})$ be a generic element of $\mathcal{C}_i, i = 1, 2$. We define $\mathcal{C}^\perp$ as the image of the $(u, u+v)$-mapping

$$u : \mathcal{C}_1^\perp \oplus \mathcal{C}_2^\perp \longrightarrow \mathbb{F}_q^{(T,s_1+s_2)}$$

given by

$$u(C_1, C_2) = (C_1^{(1)}, C_1^{(1)} + C_2^{(1)}, \ldots, C_1^{(s_1)}, C_1^{(s_1)} + C_2^{(s_1)}, C_2^{(s_1+1)}, \ldots, C_2^{(s_2)}).$$

It is obvious that u is $\mathbb{F}_q$-linear and injective. In particular $\dim(\mathcal{C}^\perp) = (Ts_1 - m_1) + (Ts_2 - m_2) = T(s_1 + s_2) - (m_1 + m_2)$, hence $\dim(\mathcal{C}) = m_1 + m_2$. In order to find the strength of $\mathcal{C}$ we have to determine the distance of $\mathcal{C}^\perp$.

Let $C_2 = 0, C_1 \neq 0$. Then $\rho(C_1, 0) = 2\rho(C_1) \geq 2(k_1 + 1)$. Let $C_2 \neq 0$. For each $j = 1, 2, \dots, s_1$ the weight of the pair of columns $(C_1^{(j)}, C_1^{(j)} + C_2^{(j)})$ is at least the weight of the single column $C_2^{(j)}$. It follows $\rho(C_1, C_2) \geq k_2 + 1$ if $C_2 \neq 0$. ■

Let $k_2 = 2k_1 + 1$. In order to obtain a net as result, we must have $T = k_2$. This means that $\mathcal{C}_2$ is a $(t_2, m_2, s_2)_q$-net, $k_2 = m_2 - t_2$, whereas $\mathcal{C}_1$ has depth $T = k_2 > k_1$ and strength k_1. The effective depth of $\mathcal{C}_1$ is therefore k_1, and $\mathcal{C}_1$ is obtained from a net of strength k_1 by adding meaningless rows. We have seen the following:

Corollary 1. *Assume $k_2 \leq 2k_1 + 1$ and there exist linear $(t_1, m_1, s_1)_q$- and $(t_2, m_2, s_2)_q$-nets, where $k_i = m_i - t_i$ and $s_1 \leq s_2$. Then we can construct a linear $(m_1 + t_2, m_1 + m_2, s_1 + s_2)_q$-net.*

An application of Corollary 1 to nets $(16, 23, 127)_2$ and $(2, 5, 15)_2$ yields a $(21, 28, 142)_2$-net. As a ternary example we obtain an $(11, 22, 23)_3$-net from a $(4, 15, 12)_3$-net and a $(2, 7, 11)_3$-net. A different generalization of the $(u, u + v)$-construction is attempted in [20].

As an example start from $(6, 17, 10)_2$ and apply Corollary 1 with $(3, 8, 10)_2$ as second ingredient. The result is a $(14, 25, 20)_2$-net. More examples will show up in the last section. Just as in coding theory, it is possible to apply Corollary 1 in a recursive fashion.

The $(u, u + v)$-construction can be generalized from the linear case to not necessarily linear ordered orthogonal arrays. The following definition generalizes Definition 5.

Definition 7. *Let $\mathcal{A}$ be an alphabet of size $|\mathcal{A}| = q$. A multisubset $\mathcal{C} \subseteq \mathcal{A}^{(T,s)}$ of size q^m has* **strength** *$k = k(\mathcal{C})$ if k is maximal such that for every ideal K of size k and every k-tuple of entries in K precisely q^{m-k} elements of $\mathcal{C}$ have the prescribed projection to K. We call $\mathcal{C}$ an* **ordered orthogonal array** *OOA of* **length** *s,* **depth** *T,* **dimension** *m and strength k.*

Observe that in the nonlinear case the dimension m need not be integer.

Theorem 5. *Let $\mathcal{A}$ be an alphabet of size $|\mathcal{A}| = q$. For $i = 1, 2$ let $\mathcal{C}_i \subset \mathcal{A}^{(T,s_i)}$ of dimension m_i and strength k_i, where $s_1 \leq s_2$. We can construct $\mathcal{C} \subset \mathcal{A}^{(T,s_1+s_2)}$ of dimension $m_1 + m_2$ and strength $k = \min\{k_2, 2k_1 + 1\}$.*

Proof. We write the elements of $\mathcal{A}^{(T,s)}$ as Ts-tuples with s sections of length T (this is the vector notation mentioned in Section 2). For every pair u, v, where $u \in \mathcal{C}_2$ and $v \in \mathcal{C}_1$, we define a row in $\mathcal{A}^{(T,s_1+s_2)}$ by $r(u,v) = (u, u+v)$. Here we have chosen a structure of an abelian group on $\mathcal{A}$. The addition in $u+v$ is componentwise. The last $s_2 - s_1$ blocks of u have been removed before performing the addition. Let the array $\mathcal{C}$ consist of all these rows $r(u,v)$. We have to show that $\mathcal{C}$ has strength $\geq k$.

Denote the cells of $\mathcal{A}^{(T,s_1+s_2)}$ by (L,i,j), where $i \leq s_2, j \leq T$ (these form the **left part** L) and (R,i,j), where $i \leq s_1, j \leq T$ (the **right part** R). Let K be an ideal of size k. Let $C(K) = \{(i,j) | (R,i,j) \in K \text{ and } (L,i,j) \in K\}$ and $c = |C(K)|$. Let an arbitrary k-tuple be prescribed on the cells from K. The projection of u to the cells from $K \cap L$ are prescribed. Let x be a tuple on $(R, C(K))$ and U_x the set of elements $u \in \mathcal{C}_2$ having the prescribed projection on $K \cap L$ and projecting to x on $(R, C(K))$. Let further V_x be the set of elements $v \in \mathcal{C}_1$ such that $u+v$ has the prescribed projection on $(R, C(K))$. For every $v \in V_x$ let $U_{x,v}$ consist of those $u \in U_x$ such that $u+v$ has the prescribed projection on $(K \cap R) \setminus (R, C(K))$. The pairs (u,v) such that $r(u,v)$ has the required projection on K is then

$$\bigcup_x \bigcup_{v \in V_x} (U_{x,v}, \{v\}).$$

Observe that $c \leq k_1$ as $2c \leq k$. We are done. ■

It follows that Corollary 1 generalizes from the linear case to arbitrary nets.

6 The finite Gilbert-Varshamov bounds for OOA

Let a code $\mathcal{C} \subseteq \mathbb{F}_q^{(T-1,s)}$ of dimension m and strength k be given. It can be represented as follows: let $a(r), r = 1, \ldots m$ be a basis of $\mathcal{C}$. Write the $a(r)$ as rows of a matrix A. The section corresponding to block B_i is $a^{(i)} = (a_1^{(i)}, \ldots, a_{T-1}^{(i)})$, where $a_j^{(i)} \in \mathbb{F}_q^m$.

We want to find vectors $a_T^{(i)}$, which complement $\mathcal{C}$ to an m-dimensional code in $\mathbb{F}_q^{(T,s)}$ of strength k. It can be assumed that $a_T^{(i)}, i < s$ have been found already. Our counting condition must be strong enough to guarantee the existence of $a_T^{(s)}$.

Each ideal $K \subset \Omega^{(T,s-1)}$ of size $l \leq k-T$ yields a condition. The number of candidates for $a_T^{(s)}$ excluded by K is q^{T-1} times the number of vectors in $\mathbb{F}_q^{(T,s-1)}$ whose support generates K. We obtain the following:

Theorem 6. *Let $\mathcal{C} \subseteq \mathbb{F}_q^{(T-1,s)}$ of dimension m and strength $\geq k$ be given. Assume*

$$V_{k-T}^{(T,s-1)} < q^{m-T+1},$$

equivalently

$$\sum_{l \leq k-T} \sum_{\pi} (q-1)^b q^{l-b} \binom{s-1}{f_T, \ldots, f_1, s-1-b} < q^{m-T+1},$$

where the sum is over all partitions of l of depth $\leq T$, and b is the breadth of π. Then there is a code $\mathcal{D} \subseteq \mathbb{F}_q^{(T,s)}$ of dimension m and strength $\geq k$, which projects to $\mathcal{C}$.

We mention that Theorem 6 generalizes the strengthened Gilbert-Varshamov bound ([13], p. 34, Theorem 2 of [1]) from Hamming space to NRT-space. It is stronger than the generalization of the ordinary Gilbert-Varshamov bound obtained in [23]. Theorem 6 has the following obvious corollary:

Theorem 7. *Assume $V_{k-T}^{(T,s-1)} < q^{m-T+1}$ holds for $T = 1, 2, \ldots, k-1$. Then there is a linear $(m-k, m, s)_q$-net, equivalently a code $\mathcal{C} \subset \mathbb{F}_q^{(k,s)}$ of dimension m and strength $\geq k$.*

7 Net-embeddable error-correcting codes

Definition 8. *Let $\mathcal{C}$ be a linear code $[s, s-m, k+1]_q$ (equivalently: $\mathcal{C}^\perp \subseteq \mathbb{F}_q^{(1,s)}$ has dimension m and strength $\geq k$). We call $\mathcal{C}$* **net-embeddable** *if there is a linear $(m-k, m, s)_q$-net projecting to $\mathcal{C}^\perp$.*

Recall that we identify linear nets with the corresponding linear subspaces of $\mathbb{F}_q^{(k,s)}$. Net-embeddability is guaranteed if Theorem 6 can be applied recursively, for $T = 2, \ldots, k$. In this section we apply our method in the following form:

Theorem 8. *Assume a linear code $[s, s-m, k+1]_q$ exists and $V_{k-T}^{(T,s-1)} < q^{m-T+1}$ holds for $T = 2, \ldots, k-1$. Then there is an $(m-k, m, s)_q$-net.*

The following lemma simplifies the comparison between the corresponding conditions.

Lemma 1. *Let* $V_q(r,n) = V_r^{(1,n)}$ *be the volume of a ball of radius* r *in Hamming space* $\mathbb{F}_q^{(1,n)}$. *If* $n \geq \frac{2q-1}{q-1}r + \frac{q}{q-1}$, *then*

$$V_q(r+1,n) \geq qV_q(r,n).$$

Proof. As $V_q(r+1,n) = V_q(r,n) + \binom{n}{r+1}(q-1)^{r+1}$ the claim is equivalent to $V_q(r,n) \leq (q-1)^r\binom{n}{r+1}$. We have $V_q(r,n) = V_q(r-1,n) + \binom{n}{r}(q-1)^r$. By induction we have $V_q(r-1,n) \leq (q-1)^{r-1}\binom{n}{r}$. It suffices to show

$$(q-1)^{r-1}\binom{n}{r} + (q-1)^r\binom{n}{r} \leq (q-1)^r\binom{n}{r+1},$$

equivalently $\binom{n}{r}q \leq \binom{n}{r+1}(q-1)$. We have

$$\binom{n}{r+1}/\binom{n}{r} = (n-r)/(r+1).$$

Our claim is therefore $q(r+1) \leq (q-1)(n-r)$, equivalently $n \geq \frac{2q-1}{q-1}r + \frac{q}{q-1}$.
■

It is easy to see that for strength $k < 3$ net-embeddability is always satisfied. In the case of strength 3 we are given a code $[s, s-m, 4]_q$. Geometrically this is an $s-$cap in projective space $PG(m-1,q)$. Depth 2 can be reached provided $V_1^{(2,s-1)} = 1 + (s-1)(q-1) < q^{m-1}$. The depth 3 condition is then automatically satisfied. We conclude that each code $[s, s-m, 4]_q$ is net-embeddable provided $s < 1 + (q^{m-1}-1)/(q-1)$. This has been proved in [24]. The first non-embeddable codes occur in this case when $q = 2$ (the extended binary Hamming code is non-embeddable) and in characteristic 2 when $m = 3$. The best binary strength 3 net parameters are $(m-3, m, 2^{m-1}-1)_2$.

For strength 4 the depth 2 condition is strongest. We conclude that a linear code $[s, s-m, 5]_q$ is net-embeddable provided $V_2^{(2,s-1)} < q^{m-1}$. As $V_2^{(2,s-1)} = V_q(2, s-1) + q(q-1)(s-1)$ we arrive at a statement first proved in [25]. The present paper grew out of an attempt to generalize this result.

7.1 Strength 5

Again the depth 2 condition is dominating. This implies that every linear code $[s, s-m, 6]_q$, which satisfies $V_q(3, s-1)+q(q-1)(s-1)V_q(1, s-2) < q^{m-1}$ is net-embeddable.

7.2 Strength 6

It follows from Lemma 1 that the condition for depth 3 is weaker than the depth 2 condition provided $s \geq 12$. The conditions for larger depths are weaker yet. This implies that each linear code $[s, s-m, 7]_q$, $s \geq 12$ which satisfies $V_q(4, s-1) + q(q-1)(s-1)V_q(2, s-2) + q^2(q-1)^2\binom{s-1}{2} < q^{m-1}$ is net-embeddable.

8 Net parameters

We present tables of net parameters $(m-k, m, s)_q$. For $q = 2, 3, 4, 5$ we list k, m, s. Observe that in case $s > m$ the underlying error-correcting code has parameters $[s, s-m, k+1]_q$. As a starting point we used the tables in [5] for $q = 2, 3, 5$. Label t refers to surviving entries from these tables. In some cases when there was a choice we replaced label t by one of the constructions below. Net parameters from [22, 19] are labelled a and b, respectively. The values for strength 3 in the non-binary case follow from cap constructions, see [9]. The label used for nets obtained from embeddings of caps is c. The caps leading to values $(6, 9, 1216)_3$ and $(8, 11, 6464)_3$ are constructed in [7]. Many values for strength $k = 4$ are derived from the families described in [2, 8]. More families of binary nets of moderate strengths based on cyclic codes will be constructed in a forthcoming publication. The corresponding table entries carry the subscript f. The nets with subscript e are computer-constructions obtained by the second author. When $s > m$ starting point is an error-correcting code. Subscript u indicates an application of the $(u, u+v)$-construction. When $s \leq m$ Theorem 7 (pure GV) is applied. The corresponding subscript is g. In case $s > m$ typically we start from a code parameter given in [3] and apply Theorem 8 to prove that it can be embedded in a net. The corresponding entries are marked h.

A class of interesting constacyclic quaternary codes with $d = 5$ were constructed in [10, 6]. We use parameters

$$[85, 77, 5]_4,\ [171, 162, 5]_4,\ [341, 331, 5]_4,\ [683, 672, 5]_4,\ [1365, 1353, 5]_4,$$

$$[2731, 2718, 5]_4,\ [5461, 5447, 5]_4,\ [10923, 10908, 5]_4.$$

In some cases, when no better construction seemed available, we used Theorem 7 (subscript g) also when $s > m$. Some good nets can be derived from Theorem 6 starting from $\mathcal{C} \subset \mathbb{F}_q^{(T-1,s)}$ of strength k for $T > 2$. All our examples have $T = 3$. These entries are labelled i. The depth 2 codes $\mathcal{C}$ are derived from linear OA of strength k and length $\geq 2s$ in the most obvious way, by identifying $2s$ coordinates of the space containing the OA with the coordinates of $\Omega^{(2,s)}$. The codes which we used as ingredients can either be obtained from the data base [3] or from primitive BCH-codes. As an example, a $(12, 16, 3125)_5$-net is based on a $[3125, 3109, 5]_5$-code, an extended primitive BCH-code.

While we focused attention on linear nets, the tables contain also parameters of nonlinear nets. The only surviving parameters based on nonlinear nets are Mark Lawrence's $(5, 21, 516)_2$ and $(5, 25, 2503)_2$ from [12]. They carry subscript L. Finally, we leave a blank for values (k, m) in the tables whenever either we cannot construct a net of length s exceeding the entry in cell $(k, m-1)$ or when we have reached a length of several thousand for a dimension $m' < m$ already.

Notation in tables	
indices	explanation
t	tables from [5]
a	Niederreiter-Xing [22]
b	Niederreiter [19]
c	embedding of caps, see [9]
e	computer embeddings
f	Families from [2, 8]
g	Theorem 7
h	code embedding Theorem 8
i	Theorem 6
u	$(u, u+v)$-construction
L	M. Lawrence's nonlinear nets

q=2

$k\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	7_t	15_t	31_t	63_t	127_t	255_t	511_t	1023_t								
3	3_t	7_t	15_t	31_t	63_t	127_t	255_t	511_t	1023_t	2047_t	4095_t	8191_t				
4		3_t	5_t	8_t	11_t	17_f	23_e	32_e	47_e	65_f	81_h	128_e	151_h	257_f		510_f
5			3_t	5_t	7_t	10_e	14_e	20_e	26_e	36_e	45_e	69_e	77_e	129_f	140_e	257_f
6				3_t	5_t	6_t	9_t	11_t	15_e	21_e	23_e	26_e	36_e	42_e	48_e	64_e
7					3_t	5_t	6_t	7_t	11_t	13_e	16_e	20_e	23_e	28_e	34_e	41_e
8						3_t	5_t	6_t	7_t	9_t	11_e	14_t	16_e	19_e	22_e	26_e
9							3_t	5_t	6_t	7_t	8_t	10_e	12_e	14_e	17_t	20_e
10								3_t	5_t	6_t	7_t	8_t	9_t	11_e	13_e	15_e
11									3_t	5_t	6_t	7_t	8_t	9_t	10_t	12_e
12										3_t	5_t	6_t	7_t	8_t	9_t	10_t
13											3_t	5_t	6_t	7_t	8_t	9_t
14												3_t	5_t	6_t	7_t	8_t
15													3_t	5_t	6_t	7_t

$k\backslash m$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
4	513_f	1025_f		2046_f	2049_f	4097_f		8190_f	8193_f							
5		513_f	516_L	1025_f		2049_f	2053_L	4097_f		8193_f						
6	72_e	79_e	127_e			130_u	137_h	164_h	196_h	511_f			1023_f			2047_f
7	47_e	58_e	64_e		127_f			133_i	137_i	142_u	511_f			514_u	518_u	526_u
8	30_e	35_e	39_e				42_g	47_g	54_h	64_i	69_h	78_h	89_h	128_i	132_i	133_i
9	23_e	26_e	29_e				34_u	37_u	40_u	46_u	60_i	68_i	71_i	84_i	100_i	113_i
10	17_e	20_t	23_e			24_t	25_t	28_b	31_i	33_i	35_g	40_u	43_g	47_g	52_g	57_g
11	14_e	16_e	18_e				20_u	22_u	24_g	28_b	30_u	33_i	36_u	37_g	40_g	45_i
12	11_e	13_e	15_e				16_g	18_b	19_g	21_g	23_g	28_b		31_g	33_g	36_g
13	10_t	11_e	12_e		13_t	14_t		16_u	18_u	19_u	20_u	22_u	23_g	28_b		30_b
14	9_t	10_t		11_t	12_t	13_t	14_t		15_t		17_t	18_b	20_g	21_g	23_g	28_b
15	8_t	9_t	10_t		11_t	12_t	13_t	14_t		15_t		17_t	18_u	20_u	20_g	22_g

$k\backslash m$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
6			2050_u	2054_u	2062_u	8191_f										
7	2047_f			2050_u	2054_u	2062_u	8191_f									
8	145_h	163_h	184_h	208_h	234_h	263_h	273_h	274_h	276_h	277_h	294_g	324_g	357_g	394_g	435_g	480_g
9	128_i	134_i		135_i	137_h	152_h	169_h	187_h	208_h	230_h	255_h	282_h	285_h	286_h	287_h	289_g
10	64_i	69_g	76_h	84_h	95_i	128_i	132_i	136_i		146_h	161_h	176_h	193_h	211_h	231_h	253_h
11	50_i	62_i	68_i	71_i	75_i	91_i	100_i	110_i	121_i			122_h	133_h	144_h	156_h	170_h
12	39_g	42_g	45_g	49_g	53_g	58_g	62_g	67_g	73_g	78_g	85_g	91_g	99_g	106_g	115_h	124_h
13	33_g	36_i	38_g	41_g	44_g	47_g	51_g	54_g	64_i	68_i	71_i	72_g	77_g	83_g	89_g	96_g
14		30_b	32_g	35_g	38_i	40_b	43_g	46_g	49_g	52_g	56_g	60_g	64_g	68_g	72_g	77_g
15	23_g	28_b		30_b	32_g	34_b	37_g	40_b	43_i	45_g	48_g	51_g	54_g	57_g	61_g	65_g

q=3

$k\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	13_t	40_t	121_t	364_t	1093_t											
3	4_c	10_c	20_c	56_c	112_c	248_c	532_c	1216_c	2744_c	6464_c						
4		4_t	8_t	14_e	26_f	41_f	80_f	121_f	242_f	365_f	728_f	1093_f	2186_f	3281_f	6560_f	9841_f
5			4_t	7_t	11_e	18_e	28_e	38_e	77_e	95_e	103_e	104_h	151_h	219_h	244_h	245_h
6				4_t	7_t	8_t	13_e	19_e	25_e	33_e	42_e		49_h	65_h	87_h	110_h
7					4_t	7_t	8_t	11_e	15_e	20_e	26_e	34_e		41_i	43_i	51_h
8						4_t	7_t	8_t	10_t	14_e	17_e	22_e			25_g	32_b
9							4_t	7_t	8_t	10_t	12_t	15_e		16_t	18_u	21_g
10								4_t	7_t	8_t	10_t	12_t	13_t	14_t	16_t	
11									4_t	7_t	8_t	10_t	12_t	13_t	14_t	16_t
12										4_t	7_t	8_t	10_t	12_t	13_t	14_t
13											4_t	7_t	8_t	10_t	12_t	13_t
14												4_t	7_t	8_t	10_t	12_t
15													4_t	7_t	8_t	10_t

$k\backslash m$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
5	660_h	730_h	731_h	1985_h	2188_h	2189_h	5959_h	6562_h	6563_h	6564_h	6565_h	6566_h	6567_h	7262_g	9557_g	
6	120_h	201_h	244_h	245_h	246_h	610_h	729_h	730_h	731_h	1836_h	2187_h	2188_h	2189_h	5514_h	6561_h	6562_h
7	64_h	81_h	121_i	128_h	160_h	200_h	364_i	365_i	390_h	487_h	1093_i	1094_i	1094_i	1179_h	3280_i	3281_i
8	37_h	45_h	54_h	66_h	80_h	96_h	117_h	141_h	170_h	205_h	246_h	247_h	364_i	432_h	520_h	625_h
9	25_g	32_b	35_g	41_g	49_h	58_h	68_h	80_h	95_h	112_h	131_h	155_h	182_h	214_h	245_h	246_h
10	19_t	22_i	25_g	32_b	34_g	40_b	46_g	56_b	61_h	71_h	82_h	95_h	121_i	127_h	146_h	169_h
11		19_t	20_g	23_u	26_g	32_b	34_g	40_b	44_g	56_b	57_g	65_g	74_h	85_h	96_h	110_h
12	16_t		19_t		21_g	24_i	27_i	32_b	34_g	40_b	43_g	56_b		61_g	69_g	78_g
13	14_t	16_t		19_t		20_t	22_t	24_t	28_i	32_b	34_g	40_b	43_g	56_b		59_g
14	13_t	14_t	16_t		19_t		20_t	22_t	24_t	25_g	28_t	32_b	35_g	40_b	42_g	56_b
15	12_t	13_t	14_t	16_t		19_t		20_t	22_t	24_t		28_t	29_g	32_b	35_g	40_b

$k\backslash m$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
6	6563_h	6564_h	6565_h	6566_h	6862_g	8548_g										
7	3281_i	3284_u	3545_h	4417_h	5503_h	6568_h	6569_h	6570_h	6571_h	6572_h	6573_h	6812_g	8180_g	9824_g		
8	733_h	1093_i	1094_i	1306_h	1569_h	1885_h	3280_i	3281_i			3284_u	3288_u	6567_h	6568_h	6569_h	6570_h
9	247_h	405_h	475_h	556_h	652_h	732_h	733_h			796_g	912_g	1046_g	1200_g	1376_g	1578_g	1809_g
10	194_h	224_h	364_i	365_i	365_i	392_h	451_h	518_h				573_g	647_g	731_g	825_g	932_g
11	125_h	142_h	161_h	183_h	207_h	235_h	248_h	249_h	364_i				400_g	446_g	497_g	555_g
12	88_h	98_h	111_h	118_h	127_h	156_h	175_h	197_h	220_h	247_h	248_h	249_h	273_g	301_g	333_g	367_g
13	66_g	73_g	82_g	91_g	101_h	112_h	125_h	139_h	154_h	171_h	190_h	211_h	234_h	250_h	251_h	263_g
14		57_g	63_g	70_g	77_g	85_g	94_g	104_g	115_h	125_h	139_h	153_h	169_h	186_h	204_h	225_h
15	43_g	56_b			62_g	68_g	74_g	82_g	89_g	98_g	107_g	117_g	128_h	139_h	149_h	168_h

q=4

$k\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	21_t	85_t	341_t	1365_t	5461_t											
3	5_c	17_c	41_c	126_c	288_c	756_c	2110_c	4938_c								
4		5_a	10_e	19_e	32_e	85_e	171_e	341_e	683_e	965_h	1366_h	3861_h	5462_h			
5			5_a	9_a	16_e	26_e	36_e	64_e	81_e	96_h	154_h	245_h	258_h	619_h	983_h	1026_h
6				5_a	9_a	12_e	18_e	26_e	34_e		45_h	65_h	81_h	89_h	187_h	257_h
7					5_a	9_a	10_a	15_e	20_e		22_g	30_u	40_h	53_h	71_h	94_h
8						5_a	9_a	10_a	13_a	17_e		20_a	23_g	29_g	37_h	47_h
9							5_a	9_a	10_a	13_a	15_a	17_a	20_a	21_a	24_g	29_g
10								5_a	9_a	10_a	13_a	15_a	17_a	20_a	21_a	
11									5_a	9_a	10_a	13_a	15_a	17_a	20_a	21_a
12										5_a	9_a	10_a	13_a	15_a	17_a	20_a
13											5_a	9_a	10_a	13_a	15_a	17_a
14												5_a	9_a	10_a	13_a	15_a
15													5_a	9_a	10_a	13_a

$k\backslash m$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
5	2479_h	3937_h	4098_h	4099_h	4103_u	4119_u	4278_u	6046_g	8549_g							
6	258_h	259_h	753_h	1025_h	1026_h	1027_h	3020_h	4097_h	4098_h	4099_h	4103_u	4114_u	4696_g	6195_g	8174_g	
7	111_h	119_h	219_h	257_h	258_h	259_h	325_g	886_h	1025_h	1026_h	1027_h	1031_u	3554_h	4097_h	4098_h	4099_h
8	60_h	73_h	85_h	95_h	112_h	199_h	252_h	257_h	258_h	292_g	355_g	807_h	1018_h	1025_h	1026_h	1027_h
9	36_g	45_h	55_h	68_h	84_h	94_h	128_i	155_h	190_h	232_h	260_h	261_h	512_i	518_h	632_h	772_h
10	26_a	30_g	36_g	44_g	52_g	63_h	76_h	89_h	101_h	111_h	155_h	186_h	222_h	259_h	260_h	264_g
11		26_a	27_a	31_g	37_g	43_g	51_g	60_g	71_h	83_h	98_h	115_h	125_h	132_g	185_h	217_h
12	21_a		26_a	27_a	28_g	32_g	37_g	43_g	50_g	58_g	68_g	78_h	87_g	98_g	110_g	124_g
13	20_a	21_a		26_a	27_a		29_a	33_a	38_g	44_g	50_g	57_g	66_g	75_g	86_h	96_h
14	17_a	20_a	21_a		26_a	27_a		29_a	33_a	35_g	39_g	44_g	50_g	57_g	65_g	73_g
15	15_a	17_a	20_a	21_a		26_a	27_a		29_a	33_a		36_g	40_g	45_g	51_g	57_g

$k\backslash m$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
7	4103_u	4114_u	5153_g	6491_g	8178_g											
8	1157_g	3240_h	4084_h	4097_h	4098_h	4099_h	4100_h	4618_g	5629_g	6861_g	8363_g					
9	942_h	1029_h	2048_i	2048_i	2085_h	2543_h	3101_h	3781_h	4102_h	4103_h	4104_h	4105_h	4326_g	5143_g	6116_g	7272_g
10	308_g	535_h	637_h	758_h	903_h	1028_h	1029_h	1030_h	2048_i	2155_h				4101_h	4102_h	4103_h
11	254_h	259_h	260_g	298_g	341_g	391_g	649_h	758_h	885_h	1027_h	1028_h			1176_g	1350_g	1550_g
12	140_g	187_h	215_h	248_h	258_h	259_h	292_g	331_g	374_g	424_g	666_h	766_h			792_g	898_g
13	107_h	119_h	146_h	166_h	189_h	216_h	246_h	261_h	262_h	290_g	325_g	512_i	531_h			574_g
14	83_g	93_g	104_h	115_h	127_g	141_g	171_h	193_h	218_h	245_h	260_h	261_g	290_g	322_g	357_g	397_g
15	64_g	72_g	80_g	90_g	101_g	113_g	124_g	136_g	149_g	164_g	197_h	221_h	246_h	259_h	265_g	291_g

q=5

$k\backslash m$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	31_t	156_t	781_t	3906_t												
3	6_c	26_c	66_c	186_c	675_c	1715_c	4700_c									
4		6_t	12_e	27_e	44_e	78_e	137_e	138_h	167_g	285_g	625_h	831_g	1421_g	3125_h	4152_g	7099_g
5			6_t	10_t	21_e	33_e	46_e	68_e	96_h	124_h	130_h	156_g	233_g	624_h	625_h	775_g
6				6_t	10_t	14_e	27_e	33_e		44_h	67_h	102_h	130_h	131_h	344_h	515_h
7					6_t	10_t	12_t	25_e			31_u	50_u	57_h	79_h	110_h	131_h
8						6_t	10_t	12_t	16_t	18_t	20_t	22_g	29_g	39_h	52_h	69_h
9							6_t	10_t	12_t	16_t	18_t	20_t	21_t	24_g	30_g	39_g
10								6_t	10_t	12_t	16_t	18_t	20_t	21_t	22_a	26_g
11									6_t	10_t	12_t	16_t	18_t	20_t	21_t	22_a
12										6_t	10_t	12_t	16_t	18_t	20_t	21_t
13											6_t	10_t	12_t	16_t	18_t	20_t
14												6_t	10_t	12_t	16_t	18_t
15													6_t	10_t	12_t	16_t

$k\backslash m$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
5	1158_g	3124_h	3125_h	3868_g	5784_g	8648_g										
6	627_h	628_h	1727_h	2584_h	3127_h	3128_h	3129_h	3133_u	3877_g	5348_g	7378_g					
7	132_h	163_g	404_h	559_h	627_h	628_h	629_h	2033_h	2805_h	3127_h	3128_h	3129_h	3133_u	4002_g	5233_g	6842_g
8	90_h	119_h	132_h	136_g	171_g	214_g	464_h	608_h	627_h	628_h	668_g	840_g	2335_h	3055_h	3127_h	3128_h
9	50_h	63_h	79_h	102_h	129_h	132_h	148_g	180_g	219_g	267_g	524_h	626_h	627_h	628_h	723_g	883_g
10	32_g	39_g	49_g	60_h	75_h	92_h	113_h	131_h	134_g	159_g	190_g	226_g	269_g	476_h	583_h	626_h
11	23_a	27_g	33_g	40_g	49_g	59_g	71_h	86_h	103_h	125_h	132_h	146_g	171_g	200_g	312_i	373_h
12	22_a	23_a	26_a	29_g	35_g	41_g	49_g	58_g	69_g	82_h	97_h	115_h	132_h	138_g	159_g	183_g
13	21_t	22_a	23_a	26_a	27_a	32_a	37_g	43_g	50_g	58_g	68_g	80_g	93_h	108_h	126_h	133_h
14	20_t	21_t	22_a	23_a	26_a	27_a	32_a	33_g	38_g	44_g	51_g	59_g	68_g	78_g	90_g	104_h
15	18_t	20_t	21_t	22_a	23_a	26_a	27_a	32_a		36_a	40_g	46_g	52_g	60_g	68_g	78_g

$k\backslash m$	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
8	3129_h	3326_g	4186_g	5267_g	6627_g	8340_g										
9	1079_g	2636_h	3126_h	3127_h	3128_h	3129_h	3600_g	4401_g	5381_g	6580_g	8045_g	9837_g				
10	627_h	652_g	779_g	931_g	1112_g	2400_h	2936_h	3126_h	3127_h	3128_h	3244_g	3879_g	4637_g	5545_g	6629_g	7927_g
11	448_h	536_h	629_h	630_h	631_h	712_g	1562_i	1579_h	1889_h	2260_h	2704_h	3130_h	3131_h	3132_h	3133_h	3543_g
12	211_g	243_g	366_h	430_h	507_h	596_h	629_h	630_h	668_g	772_g	893_g	1576_h	1852_h	2177_h	2558_h	3006_h
13	150_g	171_g	195_g	222_g	253_g	288_g	420_h	488_h	565_h	628_h	629_h	638_g	728_g	832_g	951_g	1086_g
14	120_h	132_h	145_g	163_g	183_g	207_g	233_g	263_g	297_g	335_g	476_h	545_h	624_h	628_h	629_h	698_g
15	89_g	101_g	115_h	131_h	141_g	157_g	175_g	196_g	219_g	245_g	274_g	306_g	342_g	469_h	531_h	602_h

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