History of Valuation Theory
Part I

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Abstract

The theory of valuations was started in 1912 by the Hungarian mathematician Josef Kürschák who formulated the valuation axioms as we are used today. The main motivation was to provide a solid foundation for the theory of p-adic fields as defined by Kurt Hensel. In the following decades we can observe a quick development of valuation theory, triggered mainly by the discovery that much of algebraic number theory could be better understood by using valuation theoretic notions and methods. An outstanding figure in this development was Helmut Hasse. Independent of the application to number theory, there were essential contributions to valuation theory given by Alexander Ostrowski, published 1934. About the same time Wolfgang Krull gave a more general, universal definition of valuation which turned out to be applicable also in many other mathematical disciplines such as algebraic geometry or functional analysis, thus opening a new era of valuation theory.

In the present article which is planned as the first part of more to come, we report on the development of valuation theory until the ideas of Krull about general valuations of arbitrary rank took roots. That is, we cover the pre-Krull era. As our sources we use not only the published articles but also the information contained in letters and other material from that time, mostly but not exclusively from the legacy of Hasse at the University library at Göttingen.

*This copy contains some minor corrections of the published version.

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1 Introduction

The origin of this paper was the manuscript for my lecture delivered at the Valuation Theory Conference in Saskatoon, August 1999.

For publication in the proceedings volume I had intended to enlarge that manuscript such as to contain a survey of the complete history of valuation theory until this day. The importance of valuation theory in various applications is not questioned any more, and it would have been an exciting story to report on the impact which valuation theoretic notions and results have had in those applications. However, it soon turned out that for such a project I would have needed more pages as were allowed here and, what is more important, I would also have needed much more time for preparation.

Well, here is at least the first part of my intended "History of Valuation Theory", the next part(s) to follow in due course. This first part covers the period of valuation theory which starts with Kürschák's defining paper [1912] and ends about 1940, when the ideas of Krull about general valuations of arbitrary rank took roots, opening a new era of valuation theory. The discussion of Krull's seminal paper [1932g] itself will be included in the second part, as well as the application of Krull's valuation theory to other fields of mathematics, including real algebra, functional analysis, algebraic geometry and model theory. In other words:

This first part covers pre-Krull valuation theory.

Let us make clear that "pre-Krull" is not meant to be understood in the sense of time, i.e., not "pre-1932". Rather, we mean those parts of valuation theory which do not refer to Krull's general concept of valuation of arbitrary rank and are based solely on Kürschák's notion of valuation, i.e., valuations of rank 1. [1]

For a more detailed description of the content of this paper we refer to the table of contents. The reader will notice that one important part of valuation theory is not covered, namely the so-called non-archimedean analysis. Again, the reason is lack of space and time. For non-archimedean analysis the reader might consult Ullrich's paper [1995]. For the rich contributions of Krasner to valuation theory we refer to Ribenboim's article "Il mondo Krasneriano" [1985b].

Valuation theory has become important through its applications in many fields of mathematics. Accordingly the history of valuation theory has to take into account its applications. In the pre-Krull period (in the sense as explained above), it was the application to number theory which triggered much of the development of valuation theory. In fact, valuation theory was created primarily with the aim of understanding number theoretical concepts, namely Hensel's p-adic numbers for a prime number p. Although the formal definition of valuation had been given by Kürschák [1912] it will appear that the ideas which governed valuation theory in its first (pre-Krull) phase all came from Hensel. Thus Hensel may be called the father of valuation theory. Well, perhaps better "grandfather" because he never cared about the formal theory of valuations but only for his p-adic number fields. In any case, he has to be remembered as the great figure standing behind all of valuation theory (in the pre-Krull period).

[1] This explains the fact that although here we do not discuss Krull's paper, we do discuss several papers which appeared after Krull's, i.e., after 1932. On the other hand, when in Part II we shall discuss Krull's paper [1932g], we shall have to consider a number of papers published before Krull's, belonging to strings of development leading to Krull's notion of general valuation, as for instance Hahn [1907a] and Baer [1927f] on ordered fields.

We have inserted a whole section devoted to the application of valuations to number theory. This exhibits the power of valuation theory and its role in the development of number theory. The reader will notice that in this section the name of Helmut Hasse will be dominant. Indeed, it was Hasse who successfully introduced and applied valuation theoretic ideas into number theory. He always propagated that the valuation theoretic point of view can be of help to better understand the arithmetic structure of number fields. [2] Van der Waerden, a witness of the times of the 1920s and 1930s, speaks of Hasse as "Hensel's best and great propagandist of p-adic methods" [1975a]. If today the knowledge of valuations is considered a prerequisite to anyone who wishes to work in algebraic number theory then, to a high degree, this is due to Hasse's influence.

The application of valuation theory to algebraic geometry will be dealt with later, in the second part of this project.

Besides of the original papers we have heavily used the information contained in letters and other material from that time. In those times mathematical information was usually exchanged by letters, mostly handwritten, before the actual publication of papers. Such letters often contain valuable information about the development of mathematical ideas. Also, they let us have a glimpse of the personalities behind those ideas. To a large degree we have used the legacy of Helmut Hasse contained in the University Library at Göttingen. Hasse often and freely exchanged mathematical ideas and informations with his correspondence partners; there are more than 6000 letters. Besides of the Göttingen library, we have also used material from the archive of Trinity College (Cambridge), and from the Ostrowski legacy which at present is in the hands of Professor Rita Jeltsch-Fricker in Kassel.

Remark. The bibliography contains the papers which we have cited in this article. We have sorted it by the year of publication, in order to give the reader some idea of the progress in time, concerning the development of valuation theory. But as we all know, the publication date of a result is usually somewhat later than the actual date when it was discovered, or when it was communicated to other mathematicians. Thus the ordering of our bibliography gives only a rough picture of the actual development. Sometimes the letters which were exchanged give more precise information; if so then we have mentioned it in the text.

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2 The beginning

2.1 Kürschák

Valuations have been around in mathematics since ancient times. When Euclid had established prime decomposition then this result permitted to code the natural numbers by the exponents with which the various primes p occur in these numbers; those exponents in fact represent the p-adic valuations used in number theory. Similarly in the theory of functions: the order of a holomorphic

[2] Perhaps it is not superfluous to point out that this has nothing to do with what is called "*Methodenreinheit*".

function at a given point P on a compact Riemann surface represents a valuation on the respective function field, and the function is uniquely determined, up to a constant factor, by its behavior at those valuations.

Valuations of this kind have been exploited heavily in number theory and complex function theory during the 19th century. However, *Valuation Theory* as a separate and systematic mathematical research, based on a set of axioms, started in the 20th century only, in the year 1912 when the Hungarian mathematician JOSEF KÜRSCHÁK (1864–1933) announced at the Cambridge International Congress of Mathematicians the first abstract structure theorems on valued fields [1912].

The paper itself, written in German, appeared one year later in Crelle's Journal [1913]. At the beginning of the paper we find the familiar four axioms for a valuation:

$$\|a\| > 0 \quad \text{if } a \neq 0, \ \text{ and } \|0\| = 0 \tag{2.1}$$

$$\|1+a\| \leq 1 + \|a\| \tag{2.2}$$

$$\|ab\| = \|a\| \cdot \|b\| \tag{2.3}$$

$$\exists a : \|a\| \neq 0, 1 \tag{2.4}$$

Here, a and b range over the elements of a given field K, and the values $\|a\|$ are supposed to be real numbers. Kürschak uses already the name "*Bewertung*" which is still used today, and which is translated into English as "valuation". He tells us that he had chosen this name in order to indicate that it is meant as a generalization of the notion of "*absoluter Wert*" (absolute value) which he understood as the ordinary absolute value defined in the real or the complex number field. [3]

The main purpose of Kürschák's paper was to present a proof of the following theorem:

> *Every valued field K admits a valued field extension $\mathbb{C}_K$ which is algebraically closed and complete.*

Here, "complete" refers to the given valuation and signifies that every Cauchy sequence is convergent. Kürschák uses the German terminology "*perfekt*" for "complete". Kürschák's terminology became widely used in the twenties and early thirties but was later abandoned in favor of "*komplett*" or "*vollständig*" in order to avoid misunderstandings with the English terminology. [4]

Kürschák says explicitly that he was inspired by Hensel's book on algebraic numbers [1908]. His aim is to give a solid foundation of Hensel's p-adic algebraic numbers, in a similar way as Cantor had given for the real and complex numbers. Thus we see that the main motivation to introduce valuation theory came from algebraic number theory while the model for the axioms and for the method of reasoning was taken from analysis. Kürschák's paper may be viewed as one of the first instances where "analytical algebra" or, as we prefer today, "topological algebra" was deliberately started.

[3] In modern (Bourbaki) terminology, all Kürschák valuations are called "absolute values", and the word "valuation" refers to the more general notion as defined later by Krull. In this article we use "valuation" in the sense of Kürschák.

[4] In English, the property "perfect" of a field signifies that the field does not have proper inseparable algebraic extensions; in German this property is called "*vollkommen*", as introduced by Steinitz [1910].

Today, we usually define the p-adic number field $\mathbb{Q}_p$ as the completion of the rational number field $\mathbb{Q}$ with respect to its p-adic valuation, and $\mathbb{C}_p$ as the completion of the algebraic closure of $\mathbb{Q}_p$, thereby using the fact that the valuation of the complete field $\mathbb{Q}_p$ extends uniquely to its algebraic closure. In so doing we follow precisely Kürschák's approach. Before Kürschák, Hensel had defined p-adic algebraic numbers through their power series expansions with respect to a prime element. This procedure was quite unusual since Hensel's power series do not converge in the usual sense, and hence do not represent "numbers" in the sense as understood at the time, i.e., they are not complex numbers. Accordingly there was some widespread uneasiness about Hensel's p-adic number fields and there were doubts whether they really existed. Kürschák's paper was written to clear up this point.

As to the choice of his axioms (2.1)–(2.4), Kürschák refers to Hensel's article [1907] where Hensel, in the case of p-adic algebraic numbers, had already defined some similar valuation function; the formal properties of that function are now used by Kürschák as his axioms. [5]

Kürschák's proof of his main theorem proceeds in three steps:

Step 1. Construction of the completion $\widehat{K}$ of a valued field K. [6]

Step 2. Extending the valuation from $\widehat{K}$ to its algebraic closure.

Step 3. Proving that the completion of an algebraically closed valued field is algebraically closed again.

In Step 1 he proceeds by means of what he calls Cantor's method, i.e., the elements of $\widehat{K}$ are defined to be classes of Cauchy sequences modulo null sequences. "Cantor's method" seems to have been well known at that time already since Kürschák does not give any reference. [7]

In Step 2, the existence of the algebraic closure of a field is taken from Steinitz' great Crelle paper [1910] which established the fundamentals of fields and the structure of their extensions. In fact, Kürschák's paper rests heavily on the paper of Steinitz and can be regarded as a natural continuation thereof. [8] In his introduction he refers to Steinitz and says that he (Kürschák) will now introduce into field theory a new notion, i.e., valuation. ("*In dieser Abhandlung soll in die Körpertheorie ein neuer Begriff eingeführt werden...*")

In Step 3, the method of proof is copied from the method of Weierstraß in [1891], where Weierstraß gave a new proof that the complex number field $\mathbb{C}$

[5] Hensel's definition was erroneous because of a missing minus sign in the exponent; Kürschák corrects this politely by saying that he has replaced p by p^{-1}.

[6] Concerning our notation: We do not follow necessarily the notation used by the authors of the papers discussed here. Instead, we try to use a unified notation for the convenience of the reader. In particular, $\widehat{K}$ is not the notation of Kürschák (he writes K' for the completion and calls it the "derived field" of K). Similarly, the notations $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{C}$ etc. which are generally used today, were not yet in use at the time of the papers discussed in our article.

[7] Compare Cantor's paper [1883], in particular pp. 567 ff.

[8] Such a continuation of the Steinitz paper seems to have been in the spirit of Steinitz himself. For, Steinitz says in the introduction of his paper [1910], that his article concerns the *foundations* of field theory only. He announces further investigations concerning the application of field theory to geometry, number theory and theory of functions. But those further articles have never appeared, for reasons which are not known to us. We may speculate that the reason is to be found in Kürschák's publication which was followed by those of Ostrowski.

is algebraically closed. That theorem was called, in those times, the "fundamental theorem of algebra". Accordingly, Kürschák now calls his theorem the "fundamental theorem of valuation theory". He points out that the arguments of Weierstraß are valid for an arbitrary algebraically closed valued field instead of the field of all algebraic numbers – but nevertheless he has to add an extra discussion of inseparability in the case of characteristic $p > 0$ which, of course, does not appear in Weierstraß' paper.

Kürschák does not always give detailed proofs; instead he says:

> *Da meine Untersuchungen beinahe ausnahmlos nur selbstverständliche Verallgemeinerungen bekannter Theorien sind, so scheint es mir zu genügen, wenn ich in den nächsten Kapiteln die einzelnen Definitionen und Sätze ausführlich darlege. Auf die Details der Beweise werde ich nur selten eingehen.*
>
> My investigations are, almost without exception, straightforward generalizations of known theories, and hence it seems to be sufficient that in the next chapters I present in detail the various definitions and theorems only. The details of the proofs will be given only occasionally.

One occasion for Kürschák to go into more detail of proof is in Step 2 when he discusses the possibility of extending the valuation from a complete field K to its algebraic closure $\widetilde{K}$. Let α be algebraic of degree n over K and $N\alpha$ its reduced norm, i.e., $\pm$ the constant coefficient of the irreducible monic polynomial of α over K. Then Kürschák gives the formula

$$\|\alpha\| = \|N\alpha\|^{1/n} \tag{2.5}$$

which defines an extension of the given valuation of K to its algebraic closure $\widetilde{K}$. This formula had been given by Hensel in the case of the p-adic numbers.

The main point in the proof is to derive the triangle inequality (2.2) from the definition (2.5). If the given valuation is non-archimedean then, as Kürschák observes, the method developed by Hensel in his book [1908] for p-adic numbers is applicable. Kürschák does not use the word "non-archimedean" which came into use later only [9]; but he says very clearly that Hensel's methods work:

> *... wenn die Bewertung von der besonderen Beschaffenheit ist, daß $\|a + b\|$ nicht größer ist als die größere der Zahlen $\|a\|$ und $\|b\|$.*
>
> ... if the valuation has the special property that $\|a+b\|$ is not greater than the greater of the numbers $\|a\|$ and $\|b\|$.

More precisely, Kürschák says (without proof) that the following lemma, which today is called "Hensel's Lemma", is valid in every complete non-archimedean valued field, and that Hensel's proof applies:

> LEMMA: *If $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is irreducible, and if $\|a_0\| \le 1$ then each coefficient $\|a_i\| \le 1$ $(0 \le i \le n-1)$.*

[9] The terminology "non-archimedean" and "archimedean" for valuations is introduced in Ostrowski's paper [1917]. Today the terminology "ultrametric" for "non-archimedean" is widely used.

Starting from the Lemma, Kürschák shows, the triangle inequality (2.2) is easily obtained as follows: In the above Lemma, take $f(x) \in K[x]$ to be the monic irreducible polynomial for α; then $a_0 = \pm N\alpha$. The irreducible polynomial for $1+\alpha$ is $f(x-1)$ with the constant coefficient

$$\pm N(1+\alpha) = f(-1) = (-1)^n + a_{n-1}(-1)^{n-1} + \cdots - a_1 + a_0$$

Using the Lemma and the fact that the given valuation on K is non-archimedean we see that

$$\|a_0\| = \|N\alpha\| \leq 1 \Longrightarrow \|N(1+\alpha)\| \leq 1\,.$$

This implies the triangle inequality (2.2) in the algebraic closure of the complete field K.

But Kürschák is looking for a unified proof which simultaneously covers all cases, archimedean and non-archimedean alike. He finds the appropriate method in the thesis of Hadamard [1892]. Hadamard had considered only the case of the complex field $\mathbb{C}$ as base field but Kürschák observes that his arguments are valid in an arbitrary complete valued field K. Given a monic polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

over the complete field K, Hadamard's method permits to determine the radius of convergence r_f of the power series

$$P_f(x^{-1}) = \frac{x^n}{f(x)} = 1 + c_1x^{-1} + c_2x^{-2} + \cdots$$

in the variable x^{-1}. If $f(x)$ is irreducible over K then Kürschák is able to conclude that

$$r_f = \|a_0\|^{1/n} = \|N\alpha\|^{1/n} \tag{2.6}$$

where α is a root of $f(x)$. Now, the triangle inequality (2.2) follows by comparing the radii of convergence of $P_f(x^{-1})$ and $P_g(x^{-1})$ with $g(x) = f(x-1)$.

This discussion in Kürschák's paper seems to be somewhat lengthy. Obviously, he could not yet know Ostrowski's theorem [1918], which says that for an archimedean valuation, the only complete fields are $\mathbb{R}$ (the reals) and $\mathbb{C}$ (the complex numbers). In these two cases the problem of extending the valuation is trivial, and so Kürschák's paper could indeed have been considerably shortened by concentrating on the non-archimedean case and using Hensel's Lemma – as is the usual procedure today. Nevertheless it seems interesting that a unified proof, applicable in the archimedean as well as the non-archimedean case, does exist, a fact which today seems to be forgotten. It would be of interest to simplify Kürschák's proof using what today is known from analysis in complete valued fields, and thus give a simple unified treatment for the archimedean and the non-archimedean case. Hensel's Lemma can then be deduced from this.

Concerning step 3, Kürschák wonders whether this step would really be necessary. Perhaps the algebraic closure of a complete field is complete again? He doubts that this is the case for $\mathbb{Q}_p$ but is not able to decide the question. But already in the same year, in the next volume of Crelle's Journal, the question will be settled in a paper by Ostrowski (see section 2.2).

When Kürschák published his valuation theory paper in 1912 he was 48. As far as we were able to find out, this was the last and only paper of Kürschák on valuation theory. [10] His list of publications comprises about 80 papers between the years 1887 and 1932, on a wide variety of subjects including analysis, calculus of variations and elementary geometry. He held a position at the Technical University in Budapest, and he became an influential academic teacher, recognizing and assisting mathematical talents. [11]

We have no knowledge of how Kürschák became interested in the subject of valuations. Had he been in contact with Kurt Hensel in Marburg? Or with some other colleague who knew Hensel, or with Steinitz? In any case, his paper appeared just at the right time, providing a solid base for the study of valued fields which started soon after. It seems that the axiomatization of the theory of Hensel's p-adic fields was overdue at that time, after the work of Hensel and Steinitz. Thus, if Kürschák had not written this paper then perhaps some other mathematician would have done it at about the same time and in a similar spirit. This of course does not diminish Kürschák's merits. His paper, like other publications of him, is very clearly written – certainly this contributed to the quick and wide distribution of his notions and results.

In this connection it may be not without interest that about the same time, A. Fraenkel had already presented another axiomatic foundation of Hensel's p-adic number fields [1912a]. At that time Fraenkel stayed in Marburg with Hensel. His paper had appeared in Crelle's Journal in the volume before Kürschák's. Neither Fraenkel nor Kürschák cites the other, and so both papers seem to have been planned and written independently from each other. Fraenkel's paper is completely forgotten today. Fraenkel in his memoirs [1967] explains this in rather vague terms by saying that his axiomatization had been *purely formal* whereas Kürschák's takes into account the *content* of Hensel's theory ("*inhaltliche Begründung*"). It is not clear what Fraenkel may have meant by this. In retrospective we see that Kürschák's paper, in contrast to Fraenkel's, opened up a new branch of mathematics, i.e., valuation theory, which turned out to be successfully applicable in many parts of mathematics. Let us quote a remark by Ostrowski from [1917]:

> *Überhaupt kann, wie uns scheint, ein vollständiger Einblick in die Natur dieser merkwürdigen Bildungen* [*der p-adischen Zahlen*] *nur vom allgemeinen Standpunkt der Bewertungstheorie gewonnen werden.*
>
> Anyway, in our opinion, a complete understanding of those curious constructions [the p-adic numbers] can only be obtained from the point of view of general valuation theory.

Ostrowski does not mention Fraenkel in this connection, but since both had been in Marburg at that time it is well conceivable that they had discussed the question of an adequate foundation of Hensel's theory of p-adic numbers, in the

[10] In [1923e] he published a method to determine irreducibility of a polynomial over $\mathbb{Q}_p$ by means of Newton's diagram method (which he calls the Puiseux diagram method). It is curious that Kürschák, the founder of valuation theory, restricted his investigation to polynomials with integer coefficients; he did not mention that the Newton diagram method can be applied without any extra effort to polynomials over an arbitrary non-archimedean complete valued field, as Ostrowski later [1934] observed.

[11] E.g., John von Neumann was one of his students. – I am indebted to Kálmán Györy for providing me with biographical information about Kürschák. See also the books [1992] and [1996a] on Mathematics in Hungary.

light of the two papers by Fraenkel and by Kürschák. And the above remark represents Ostrowski's opinion on that question.

2.2 Ostrowski

After Kürschák had started the theory of valued fields it was ALEXANDER OSTROWSKI (1893 – 1986) who took over and developed it further to a considerable degree.

Ostrowski was born in Kiev and had come to Marburg in 1911, at the age of 18, in order to study with Hensel. Abraham Fraenkel, who had been in Marburg at that time, recalls in his memoirs [1967] that Ostrowski showed unusual talent and originality ("*eine ungewöhnliche Begabung und Originalität*").

From Fraenkel we also learn that Ostrowski had been advised by his professor in Kiev to go to Marburg. At that time the dominant mathematician in Kiev was D.A. Grave. [12] It seems remarkable to us that Grave sent his extraordinarily gifted student to Marburg and not to Göttingen, although the latter was worldwide known for its inspiring mathematical atmosphere, and Landau in Göttingen had offered to accept him as a student. The reason for this advice may have been that he (Grave) was well acquainted with the theory of Hensel's p-adic numbers and considered it to be something which would become important in the future. In any case, we know that p-adics were taught in Kiev. [13]

In Marburg, the home of p-adics, Kürschák's paper was thoroughly studied and discussed. Soon the young Ostrowski found himself busily engaged in developing valuation theory along the tracks set by Kürschák.

2.2.1 Solving Kürschák's Question

In his first [14] paper [1913a], published in Crelle's Journal, Ostrowski solved the open question of Kürschák mentioned in the foregoing section, namely whether the algebraic closure of a complete valued field is complete again.

Ostrowski proves that a separable [15] algebraic extension of a complete valued field K is complete if and only if it is of finite degree over K. From this he concludes that the algebraic closure $\widetilde{K}$ of a complete field K is complete if and only if $\widetilde{K}$ is finite over K. This holds regardless of whether $\widetilde{K}$ is separable over

[12] A biography on Grave is contained in "The MacTutor History of Mathematics archive", at the internet address http://www-groups.dcs.st-and.ac.uk/ history/.

[13] It is known that N. Chebotarëv, who was a contemporary of Ostrowski and had studied in Kiev, was well acquainted with p-adic numbers. When Chebotarëv met Ostrowski 1925 in Göttingen, the latter posed him a problem coming from complex analysis, and Chebotarëv was able to solve it by means of p-adic numbers; see the article by Lenstra and Stevenhagen [1996].

[14] Actually, Ostrowski had already published another mathematical article, on finite fields, which had appeared (in Russian) in the S. B. Phys. Math. Ges. Kiew. See vol. 3 of Ostrowski's Collected Papers [1983].

[15] He does not use the word "separable" which was coined only later by van der Waerden. Instead he uses "of the first kind" (*von 1. Art*) as introduced by Steinitz [1910]. Ostrowski then defines "of the second kind" (*von 2. Art*) to mean "purely inseparable" in today's terminology. It seems somewhat curious that Steinitz himself, although he introduced the notion "of the first kind", did not introduce the terminology "of the second kind" (*von 2. Art*). This seems to be due to Ostrowski. A purely inseparable extension field is called by Steinitz "root field" (*Wurzelkörper*).

K or not. [16] In particular it follows that the algebraic closure of $\mathbb{Q}_p$ is not complete, which answers the question of Kürschák.

Ostrowski could not know the Artin-Schreier theorem which was discovered much later in [1927a] only. According to Artin-Schreier, the algebraic closure $\widetilde{K}$ of a field K is almost never finite over K, the exceptions being the real closed fields – and of course the trivial cases $\widetilde{K} = K$.

Hence Ostrowski's result implies that the algebraic closure of a complete field K is almost never complete, with the only exceptions when K is real closed – and of course the trivial cases when K itself is algebraically closed.

Ostrowski was quite young when this paper appeared in 1913 (he was 20), and it is perhaps due to this fact that the paper appears somewhat long-winded. But Ostrowski continued to work on valuation theory. In his next papers he not only simplified the proofs but also produced a number of further fundamental results.

2.2.2 Revision: Non-archimedean Valuations

In his paper [1917], Ostrowski sets out to prove the results of his 1913 paper anew and, he says, with proofs much more elementary.

From the start he considers non-archimedean valuations only. This is permitted since in another paper [1918] he is able to classify the complete archimedean valued fields. (See section 2.2.3.) Here we find the following results:

(1) A new and simple proof of the fact that any finite extension of a complete field is complete again, this time including the case of inseparable extensions. This proof has become standard today, and with the same arguments one usually shows that any normed vector space of finite dimension over a complete field is complete again. By the way, in this proof the non-archimedean property of the valuation is not used, hence archimedean valuations are included.

(2) An addition to Kürschák's result by showing that for a complete ground field, the extension of the valuation to its algebraic closure is *unique* – a fact which, Ostrowski says, is implicitly contained also in Kürschák's construction but his own proof is now much easier.

(3) A Lemma which later became known as "Krasner's Lemma". This is a very useful version of Hensel's Lemma and says the following: Let K be complete with respect to a non-archimedean valuation, and let α be algebraic and separable over K. Denote by μ the minimal distance of α to its conjugates over K. Then:

> LEMMA: *If L is any valued algebraic extension of K such that the distance of α to L is $< \mu$ then $\alpha \in L$.*

It is true that Ostrowski states the contention for $\mu/2$ instead of μ but from his proof it is clear that indeed the lemma holds for μ because of the strong triangle inequality which Ostrowski does not use in this instance. [17]

---

[16] For, if $\widetilde{K}$ is infinite over the complete field K then he shows that the separable closure too is infinite over K. By the way, this implies that a separably closed complete field is algebraically closed – a result which later was rediscovered by F. K. Schmidt; see section 4.2.

[17] Ostrowski seems to claim that under the hypothesis of the Lemma, α is the *only one* among its K-conjugates whose distance to L is $< \mu$; he uses the words "*eine einzige*". Obviously this uniqueness assertion is false in general, there are trivial counterexamples. In his proof Ostrowski does not deal with this uniqueness, and when he applies the Lemma he does not

We see: Although the Lemma had been stated and used by Ostrowski in [1917] already, it has come to the attention of the mathematical community much later only, under the name of "Krasner's Lemma". This is another instance of a situation which we can observe in every corner of mathematics: The name of a mathematical result or notion does not always reflect the historical origin.

(4) A simple proof, using the above Lemma, that every infinite separable algebraic extension of a complete field is *not* complete. The same is proved for inseparable extensions if the inseparability exponent is infinite. From this Ostrowski concludes again his main result of his earlier paper [1913a]: that the algebraic closure $\widetilde{K}$ of a complete field K is complete if and only if $\widetilde{K}$ is finite over K.

2.2.3 Ostrowski's Theorems

Although the paper [1918] appeared after [1917] it was completed already before [1917]. It is dated April 1916. The paper contains two fundamental theorems.

In the first theorem, all possible valuations of the rational number field $\mathbb{Q}$ are determined. Ostrowski finds that, up to equivalence, these are precisely the known ones. Two valuations $\|\cdot\|_1$ and $\|\cdot\|_2$ of a field K are said to be equivalent if one is a power of the other, i.e. $\|a\|_2 = \|a\|_1^r$ for every $a \in K$, where the exponent r does not depend on a. It is clear that equivalent valuations generate the same metric topology of the field K and, hence, lead to isomorphic completions. Usually, equivalent valuations will not be distinguished. More precisely, an equivalence set of valuations will be called a *prime* of the field, usually denoted by a symbol like $\mathfrak{p}$, and the corresponding completion will be denoted by $\widehat{K}_{\mathfrak{p}}$ or simply $K_{\mathfrak{p}}$. [18]

Hence, what Ostrowski proves in [1918] can be expressed by saying that every prime $\mathfrak{p}$ of the rational number field $\mathbb{Q}$ either corresponds to a prime number p and thus belongs to the usual p-adic valuation of Hensel-Kürschák, or $\mathfrak{p} = \mathfrak{p}_\infty$ is an infinite prime and belongs to the ordinary absolute value.

Today this theorem belongs to the basics of a first course in valuation theory. The usual proof presented today is due to E. Artin [1932f] who uses essentially the same ideas as Ostrowski but is able to streamline the proof into 2 pages. Perhaps it may be mentioned that Hasse was so delighted about Artin's version of Ostrowski's proof that he included it into his book "*Zahlentheorie*" [1949] with the comment: "*Dieser schöne Beweis geht auf Artin zurück.*" (This beautiful proof is due to Artin.)

Actually, the above definition of a "prime" $\mathfrak{p}$ in an arbitrary field K is given by Ostrowski in the case of non-archimedean valuations only. In this case, he also switches to the additive notation of valuations by defininig $v(a) = -\log\|a\|$. [19]

use it. Hence we believe that this is not really a mathematical error, but it reflects a certain linguistic slip of the author. In a later paper [1934] he corrects it and says that "*eine einzige*" should be replaced by "*eine solche*", which gives the formulation as we have stated it. – In general, though, Ostrowski's papers are very clear and sharp; in [1988] they are praised as "true pearls in mathematical literature" (*wahre Perlen im mathematischen Schrifttum*).

[18] In the literature there are various other definitions of "primes" of a field, all of which are equivalent. Also, instead of the terminology "prime", other names are used too, e.g. "prime divisors", "prime spots", "places" or "points" of a field. For the purpose of this report we will take the definition of "prime" as given in the text. If the valuations are archimedean then the corresponding prime $\mathfrak{p}$ is called an "infinite" prime and denoted by a symbol like $\mathfrak{p}_\infty$.

[19] The minus sign is missing in his definition, and this is corrected in [1934].

Of course it is a question of taste whether one wishes to include archimedean valuations or not among what one calls "primes". As Hasse [1927], §2 has pointed out, the experiences in class field theory strongly indicate that archimedean valuations should be treated, as far as possible, on the same footing as non-archimedean ones, and this is generally accepted today.

In any case, it seems worthwhile to state that Ostrowski was the first one who formulated explicitly the notion of "prime" in any abstract field by means of valuation theory. Below we shall see that later in his long paper [1934] he will enlarge on this, studying the set of all those primes which he calls the "Riemann surface" of the field.

The second main result in this paper consists of what today is simply called "Ostrowski's theorem" in valuation theory. It is concerned with *archimedean* valuations and asserts that the only fields which are complete with respect to an archimedean valuation are the fields $\mathbb{R}$ of real numbers and $\mathbb{C}$ of complex numbers, up to isomorphisms as *topological fields*. The topological isomorphism implies that the given valuation becomes equivalent to the ordinary absolute value. As said already in section 2.1, this theorem permits a substantial simplification in Kürschák's discussion of extending the valuation from a complete field to an algebraic extension. For, in view of Ostrowski's theorem it is now possible to consider *non-archimedean* valuations only, and for these Hensel's Lemma can be used. Hence it will not be necessary any more to refer to Hadamard's results on the radius of convergence of certain power series. [20]

One of the first readers of Ostrowski's paper was Emmy Noether. In a postcard to Ostrowski [21] she writes:

> *Ihre Funktionalgleichungen habe ich angefangen zu lesen, und sie interessieren mich sehr. Kann man wohl den allgemeinsten Körper charakterisieren, der einem Teiler des Körpers aller reellen Zahlen isomorph ist?...*
>
> I have started to read your functional equations [22] and I am very interested in it. Is it perhaps possible to characterize the most general field which is isomorphic to a subfield [23] of the field of all real numbers?

Emmy Noether does not only express her interest in Ostrowski's work but immediately poses the correct question: *Which fields can be isomorphically embedded into* $\mathbb{R}$? Her question was answered later in [1927a] by Artin-Schreier's theory of formally real fields. We have cited this postcard in order to put into evidence that Emmy Noether has shown interest in the development of valuation theory

[20] Ostrowski points out that Hadamard's investigations can be regarded as a generalization of Bernoulli's method to approximate the roots of an algebraic equation. Ostrowski's proof, he says, uses another generalization of the same method – but only in the case of quadratic extensions which suffices for his theorem and therefore is much simpler.

[21] The postcard is not dated; it seems that it was written in early 1916, some months before Ostrowski submitted his manuscript [1918] to the Acta Mathematica. – I am indebted to Prof. Rita Jeltsch-Fricker to give me access to the legacy of Ostrowski where I found several postcards from Emmy Noether.

[22] By "functional equations" she refers to the title of Ostrowski's paper which reads: *Über einige Lösungen der Funktionalgleichung* $\varphi(x) \cdot \varphi(y) = \varphi(xy)$. (On some solutions of the functional equation ...)

[23] She writes "Teiler" which usually translates with "divisor". In the present context it obviously means "subfield" in today's terminology.

right from the beginning, although she herself was never active in this direction. Later in 1930/31 she actively participated, together with Richard Brauer and Helmut Hasse, in the proof of the Local-Global Principle for algebras; see section 3.4.2.

2.3 Rychlík and Hensel's Lemma

It was the Czech mathematician KAREL RYCHLÍK (1885–1968) who set out to present explicitly such simplification as mentioned by Ostrowski and Kürschák, for non-archimedean valuations only. His paper [1919] appeared in a Czech journal, soon after Ostrowski's paper [1918]. But since it was written in Czech language it seems that it was not properly noticed by the mathematical community (though it was refereed in the *Jahrbuch über die Fortschritte der Mathematik*, vol. 47). Later in 1923 Rychlík published essentially the same paper in German language, in Crelle's Journal [1923f]. [24] This paper certainly was read and appreciated.

Rychlík appears to have been strongly influenced by Hensel's theory of p-adic numbers, through Hensel's various papers and particularly through Hensel's books [1908], [1913b] which he knew well. As a result of his studies he had published, since 1914, some earlier papers already presenting Hensel's ideas of the foundation of algebraic number theory. Those papers were also in Czech language. Of particular interest to us is the paper [1916a] where Rychlík, in a postscript, presented Kürschák's construction for the p-adic fields $\mathbb{Q}_p$. [25] It is apparent that Rychlik was not a newcomer in valuation theory when he wrote the paper [1919], or its German version [1923f]. [26]

In these papers [1919], [1923f] under discussion, Rychlík cites Kürschák and Ostrowski and their results mentioned above. For a complete field with a non-archimedean valuation, he says, he is now going to present a simple proof for the prolongation of the valuation to an algebraic extension. [27] He will present in full detail the proof which Kürschák, by a sort of hand waving, had indicated but not explicitly presented.

Seen from today, Rychlík is taking the relevant results and proofs from Hensel's book [1908] and repeating them in the more general framework of an arbitrary complete, non-archimedean valued field. This seems rather trivial to us but we should keep in mind that at those times, valuation theory was quite

[24] Usually this paper is cited as having appeared in 1924. But volume 153 of Crelle's Journal consisted of two issues and the first issue, containing Rychlík's paper, appeared on Aug 27, 1923 already.

[25] More generally, Rychlík covered also the Hensel g-adic rings $\mathbb{Q}_g$ for an arbitrary integer $g > 1$. In this case the ordinary, Kürschák's, notion of valuation is not adequate; instead he had to use what later was called a "pseudo-valuation"; this is a function which satisfies all the conditions for a valuation except the multiplicative rule (2.3) which is to be replaced by $\|ab\| \leq \|a\|\|b\|$. Thus Rychlík preceded Mahler [1936]-[1936c], who introduced the formal notion of pseudo-valuation, by 20 years.

[26] I am indebted to Dr. Magdalena Hykšová, and also to Professor Radan Kučera, for translating part of the Czech papers of Rychlík. Dr. Hykšová also informed me about many interesting biographical details; she has published a paper [2001] on the life and work of Rychlík.

[27] I do not know who coined the word "prolongation" in this context; one could also say "extension" of a valuation to an algebraic extension field. By the way, Rychlík himself does not use any of these words; instead he says: "We obtain a valuation of every algebraic extension of the complete field K". (*Wir erhalten eine Bewertung jeder algebraischen Erweiterung des perfekten Körpers K.*)

new and people were not yet used to valuation theoretic arguments. Moreover, there was indeed some difference between Hensel's arguments for p-adic fields and those for general complete fields. Namely, the valuation of a p-adic field is *discrete* in the sense that its value group is a discrete subgroup of the positive reals; this implies that the maximal ideal of the valuation ring is generated by one element ("prime element"). The relevant approximation algorithm in Hensel's treatment proceeds sucessively with respect to the powers of a prime element. But in general the valuation will not be discrete and hence there does not exist a prime element. Therefore in Hensel's arguments, the powers of the prime element have to be replaced by the powers of some suitable element in the maximal ideal. For us, this seems quite natural and straightforward. As said above already, apparently this was not considered to be trivial in those times, when abstract valuation theory had just been started.

Sometimes in the literature there appears what is called the "Hensel-Rychlík Lemma". This terminology recalls that Hensel was the first to discover the validity and importance of the Lemma in the case of p-adic fields and their finite extensions, and that Rychlík did verify its validity for arbitrary complete non-archimedean valued fields. [28] Today, however, the name "Hensel's Lemma" has become standard, and the name "Hensel-Rychlik" is used sometimes only to distinguish a certain version of Hensel's Lemma from others. We see again, that such names do not reflect the historical background but are assigned somewhat arbitrarily by the mathematical community – as we have noticed already in the foregoing section with "Krasner's Lemma" which in fact is due to Ostrowski.

Now what does the statement say which is known as "Hensel-Rychlík Lemma"?

We work in a non-archimedean complete valued field K. Let $\mathcal{O}$ be its valuation ring. Consider a polynomial $f(x) \in \mathcal{O}[x]$. Suppose that $f(x)$ splits "approximately" into two factors, which means that there exist relatively prime polynomials $g_0(x), h_0(x) \in \mathcal{O}[x]$ of positive degrees and a small $\varepsilon > 0$ such that

$$\|f(x) - g_0(x)h_0(x)\| \le \varepsilon\,.$$

This condition is to be understood coefficient-wise, i.e., each coefficient of the polynomial on the left hand side is of value $\le \varepsilon$. It is also assumed that the degree of $f(x)$ equals the sum of the degrees of $g_0(x)$ and $h_0(x)$ and, morever, that the highest coefficient of $f(x)$ is the product of the highest coefficients of $g_0(x)$ and $h_0(x)$. [29] If ε is sufficiently small then it is claimed that $f(x)$ splits over $\mathcal{O}$. For, if $R = R(g_0, h_0) \in \mathcal{O}$ denotes the resultant of g_0 and h_0 we have:

> HENSEL-RYCHLÍK LEMMA: *If $\varepsilon < \|R\|^2$ then $f(x)$ splits over K. More precisely, there are polynomials $g(x), h(x) \in \mathcal{O}[x]$ of the same degrees as $g_0(x)$ and $h_0(x)$ respectively, such that*
>
> $$f(x) = g(x)h(x)\,,$$

[28] We shall see below in section 5.1 that Ostrowski had verified this even earlier, before 1917. But he did not publish this and therefore Rychlík could not have knowledge of Ostrowski's manuscript.

[29] This last condition is not mentioned in Rychlík's paper – probably because it is not mentioned in Hensel's book. Perhaps Rychlík had overlooked that Hensel, in his context, normalizes his polynomials such that the highest coefficients should be a power of the given prime element; then the condition is always satisfied. Rychlík's proof tacitly assumes that the condition is satisfied. Without the condition, the contention of the Lemma would have to be modified such that f admits a decomposition $f = c \cdot gh$ where c is a certain unit in the valuation ring, and g, h satisfy the approximation properties as stated. – Rychlík's error in the statement of the lemma was corrected by Ostrowski [1934].

and that $\|g(x) - g_0(x)\| \le \varepsilon \|R\|^{-1} < \|R\|$, *and similarly for* $h(x)$.

The latter relations say that the factors $g(x)$ and $h(x)$ admit $g_0(x), h_0(x)$ as approximations, arbitrarily close if ε is sufficiently small.

In particular it follows that $f(x)$, under the hypothesis of this Lemma, is reducible. From this it is easy and standard to deduce the Lemma of section 2.1, page 7 for irreducible polynomials, which Kürschák had used to extend the valuation to any algebraic extension of K.

In the special case when $g_0(x) = x - a_0$ is linear, the resultant of $x - a_0$ and of $h_0(x) = \dfrac{f(x) - f(a_0)}{x - a_0}$ is computed to be $f'(a_0)$ where $f'(x)$ denotes the derivative of $f(x)$. One obtains from the above Lemma:

> *If there is* $a_0 \in \mathcal{O}$ *such that* $\|f(a_0)\| \le \varepsilon < \|f'(a_0)\|^2$ *then there exists a root* $a \in \mathcal{O}$ *of* $f(x)$ *with* $\|a - a_0\| \le \varepsilon \|f'(a_0)\|^{-1} < \|f'(a_0)\|$.

Sometime this special case is called "Hensel-Rychlík Lemma" – but in fact it is mentioned explicitly in the Czech version [1919] of Rychlík's paper only, not in the German version [1923f]. In the still more special case when $\|f'(a_0)\| = 1$ we obtain what most of the time is now called "Hensel's Lemma":

> *If* $f(x)$ *has a simple root* $\overline{a}$ *in the residue field* $\overline{K} = \mathcal{O}/\mathcal{M}$ *modulo the maximal ideal* $\mathcal{M}$, *then* $\overline{a}$ *can be lifted to a simple root* $a \in \mathcal{O}$ *of* $f(x)$.

At this point we have to warn the reader that the name "Hensel's Lemma" is not used in a unique way. Different authors, or even the same author in different publications, use the name "Hensel's Lemma" for quite different statements. One gets the impression that every author working in valuation theory (or elsewhere) creates his own preferred version of Hensel's Lemma. This reflects the fact that the validity of Hensel's Lemma implies very strong and important structural properties which one would like to understand from various viewpoints.

Most versions of Hensel's Lemma are equivalent. A good overview of various Hensel's Lemmas (including Krasner's Lemma which is due to Ostrowski) is given in Ribenboim's instructive article [1985]. But the reader should be aware of the fact that in the literature there can be found numerous other variants of Hensel's Lemma, e.g., lifting of idempotents in algebras, or lifting of simple points on varieties, etc. It would be desirable to investigate the historical development of the ideas underlying the various versions of Hensel's Lemma, not only in valuation theory but in other branches of mathematics as well. [30]

The name of Kurt Hensel will be remembered in mathematics for a long time through the name of "Hensel's Lemma", whichever form it will obtain in future developments. And, we may add, this is entirely appropriate in view of the important mathematical notion of p-adic numbers with which Hensel enriched the mathematical universe.

[30] The so-called Newton approximation algorithm from analysis can be viewed as a special method to prove Hensel's Lemma. Also, we find that Hensel's Lemma had been stated and proved by Gauß, as observed by Günther Frei (in a forthcoming publication on the history of the theory of algebraic function fields).

With the paper [1923f] Karel Rychlík exits the scene of general valuation theory. [31] We have not found any later publication by him on valuations, although he was still active for quite a while on the foundation of divisor theory in algebraic number fields. He continued to play a role in the development and reorganisation of university teaching in his country. We read in [2000]:

> *Rychlík was the first to introduce methods and concepts of "modern" abstract algebra into this country – by means of the published treatises as well as his university lectures.*

Rychlík was a professor at Czech Technical University but he delivered lectures also at Charles University in Prague, as a private associate professor. See [2000] where one can find more detailed biographical information about Rychlik.

3 Valuations in number theory

The papers by Ostrowski and Rychlík discussed in section 2 can be viewed as appendices to Kürschák's opening paper, clearing up certain points which were left open by Kürschák including the role of archimedean valuations. After that there begins the period of expansion and application. This means that on the one hand, valuation theory was systematically expanded to study the structure of valued fields in more detail. On the other hand, it was more and more realized that valuation theory can be profitably used in various applications.

Naturally, these two strings of development – expansion and application – cannot be sharply separated. Sometimes the need for a certain application led to expand the general structure theory of valuations; on the other hand, after knowing more about the structure of valued fields it became apparent that this knowledge could be applied profitably.

The foremost and first applications belong to number theory. This has good historical reasons since the main motivation to create the notion of valuation came from Hensel's p-adic methods introduced in number theory.

Valuation theory has not only produced new methods which could be profitably used in number theoretical research, but it has also led to a change of viewpoint. For instance the transition from "local" to "global" became one of the central questions in number theory. In this section we shall discuss the first steps which started this development.

3.1 Hasse: The Local-Global Principle

3.1.1 The motivation

Consider the year 1920. In that year Helmut Hasse (1898–1979), a young student of 22, decided to leave his home university of Göttingen in order to go to Marburg and continue his studies with Hensel. The motivation for this decision was Hensel's book "*Zahlentheorie*" [1913b] where p-adic numbers are presented as a basis for an introductory course to Number Theory. Hasse had

[31] There had been an earlier paper [1923d] in Crelle's Journal where he gave an example of a continuous but non-differentiable function defined in the p-adic field of Hensel, but this does not concern us here.

found this book in a second hand Göttingen bookshop. [32] In the foreword to his Collected Papers [1975] he recalls:

> *Das Buch war mir vom ersten Augenblick an wegen seiner völlig neuartigen Methoden besonders reizvoll und eines gründlichen Studiums wert erschienen... Auf mich hatte es eine magische Anziehungskraft ausgeübt, und so ging ich nach dem "kleinen" Marburg.*
>
> From the first moment, this book was particularly appealing to me because of his completely new methods, and certainly it seemed to be worth of detailed study... I felt strongly attracted to it, and hence I went to the "small" Marburg.

Thus this elementary and, we may say, unpretentious booklet had instantly prompted Hasse to dive deeper into p-adic number theory – which proved to be decisive for his further work, and for the further development of algebraic number theory at large.

Hensel's book does not even mention Kürschák's abstract notion of valuation. [33] The p-adic number field $\mathbb{Q}_p$ is introduced in the Hensel way by means of p-adic power series. The valuation (additively written) of $\mathbb{Q}_p$ appears as "order" (*Ordnungszahl*) which is assigned to each p-adic number. Limits are introduced without explicit mention of the valuation, and completeness of $\mathbb{Q}_p$ is proved directly. Hensel's Lemma does not appear since the book does not discuss finite extensions of $\mathbb{Q}_p$.

It may not be without interest to know that, in Hensel's book, the style of writing was not Hensel's but for the most part Fraenkel's. See Fraenkel's report [1967] where he says:

> *Es fiel Hensel nicht leicht, seine originellen Gedanken in einer für den Druck geeigneten Form darzustellen, und dies war gerade meine Stärke. Er bat mich also, aufgrund der Vorlesung und regelmäßiger Gespräche über das Thema, das Buch niederzuschreiben, mit Ausnahme des letzten Kapitels.*
>
> Hensel had difficulties to put his original ideas into a form which was suitable for printing, but that was my strong point. Therefore he asked me to write down the text of the book, following the lectures and our regular conversations about the subject, except the last chapter. [34]

Hasse registered at Marburg University in May 1920 and started his seventh semester (fourth academic year). Already at the end of this month Hensel suggested to him a subject for his doctoral thesis: quadratic forms over $\mathbb{Q}$ and over $\mathbb{Q}_p$.

[32] It is possible to identify the date when Hasse had purchased the book, namely March 20, 1920. This date is written by Hasse's own hand onto the title page of his copy of the book, together with his name. (Hasse's copy is now contained in my library, thanks to Martin Kneser.)

[33] Compare the publication dates: Kürschák's paper had appeared in 1912, and Hensel's book in 1913.

[34] Fraenkel's help in editing the text is duly acknowledged in the foreword of Hensel's book. By the way, also the name of Ostrowski is mentioned; he had produced the index of Hensel's book.

3.1.2 The Local-Global Principle

Already in May 1921, one year after his moving to Marburg, Hasse had completed his thesis where he introduced and proved the famous *Local-Global Principle* for the representation of a rational number by a given quadratic form with rational coefficients.

The problem is as follows. Let $a \in \mathbb{Q}$, and $f(x_1, \ldots, x_n)$ be a quadratic form over $\mathbb{Q}$. To find necessary and sufficient conditions, in terms of p-adic numbers, such that a is representable by $f(x_1, \ldots, x_n)$, i.e., that there exist $x_1, \ldots, x_n \in \mathbb{Q}$ such that $a = f(x_1, \ldots, x_n)$. Hasse's Local-Global Principle says that a is representable by $f(x_1, \ldots, x_n)$ over $\mathbb{Q}$ if and only if it is representable over the p-adic completions $\mathbb{Q}_p$ for all primes p (including p_∞). In other words: *the statement*

$$\exists\, x_1, \ldots, \exists\, x_n : \quad a = f(x_1, \ldots x_n)$$

holds over $\mathbb{Q}$ if and only if it holds over $\mathbb{Q}_p$ for all p. The point is that in the complete fields $\mathbb{Q}_p$ there are explicit arithmetic criteria available for a to be representable by the quadratic form. If the number n of variables is > 4 then, for $p \neq p_\infty$ every $a \in \mathbb{Q}$ is representable by $f(x_1, \ldots, x_n)$ in $\mathbb{Q}_p$ [35], so that, by the Local-Global Principle, only the representability in $\mathbb{Q}_{p_\infty} = \mathbb{R}$ is relevant, and this can easily be checked according to whether the quadratic form q is definite or indefinite. If the number of variables is ≤ 4 then Hasse develops explicit criteria, involving the quadratic residue symbol, for the representability in $\mathbb{Q}_p$.

The case of binary quadratic forms ($n = 2$) had been discussed in chapter XII of Hensel's book [1913b]. [36] It seems that Hensel had not been satisfied with the restriction to $n = 2$, and therefore he had given to his student Hasse the task to generalize this at least for 3 or 4 variables. Finally Hasse was able to deal with an arbitrary number of variables.

Hasse reports in the preface to his Collected Works [1975] that the Local-Global Principle had been suggested to him by Hensel. He mentions a postcard which Hensel had sent to him, and which he (Hasse) preserves as a valuable keepsake. [37] There Hensel wrote:

> *Sehr geehrter Herr Hasse!... Ich habe immer die Idee, daß da eine bestimmte Frage zu Grunde liegt. Wenn ich von einer analytischen Funktion weiß, daß sie an allen Stellen rationalen Charakter hat, so ist sie rational. Wenn ich bei einer Zahl dasselbe weiß, daß sie für den Bereich jeder Primzahl p und für p_∞ p-adisch ist, so weiß ich noch nicht, ob sie eine rationale Zahl ist. Wie wäre das zu ergänzen?*

[35] This result has led to Artin's conjecture for forms of arbitrary degree over $\mathbb{Q}_p$, not only quadratic forms: Every such form of degree d with $n > d^2$ variables should admit a non-trivial zero in $\mathbb{Q}_p$. It seems that Artin himself never pronounced this conjecture in writing (see the preface to Artin's Collected Papers, written by S. Lang and J. Tate in 1965). Nevertheless it attracted great attention among number theorists. After a number of partial results, the conjecture was "almost" proved by Ax and Kochen in [1965], in the sense that for given degree, Artin's conjecture holds for all but finitely many prime numbers p. Their proof was remarkable since it was the first instance where valuation theory was combined with model theory. But soon thereafter Terjanian [1966] gave counter examples to the original conjecture of Artin.

[36] This is the chapter which had been edited by Hensel himself and not by Fraenkel.

[37] The postcard is dated Dec 2, 1920. It is now contained in the Hasse legacy at Göttingen.

> Dear Mr. Hasse!... I am always harboring the idea that there is a particular question at the bottom of these things. If I know of an analytic function that it is of rational type at each point, then it is a rational function. If I know the same of a number, that it is p-adic for each prime number p and for p_∞, then I do not yet know that it is rational. How would this have to be amended?

Hensel's question was not precisely formulated because he did not specify what he meant by "number". If "number" means "algebraic number" then his question can be answered affirmatively if the notion of a number a "to be p-adic" is interpreted such that there exists an isomorphism of the field $\mathbb{Q}(a)$ into $\mathbb{Q}_p$. This is a consequence of Frobenius' density results on the splitting of primes in an algebraic number field. [38] But that must have been known to Hensel. It seems that Hensel had vaguely something else in mind, where "number" and "to be p-adic" would have to be interpreted differently. [39] In any case, Hensel's question stimulated Hasse to find his Local-Global Principle. He writes in [1975]:

> *Es war die Frage am Schluß dieser Mitteilung* [auf der Henselschen Postkarte] *die mir die Augen geöffnet hat... Aus diesem Kern wuchs mir dann rasch... das Lokal-Global-Prinzip für alle Darstellungs- und Äquivalenzbeziehungen bei quadratischen Formen mit rationalen und dann allgemeiner auch mit algebraischen Koeffizienten. So verdanke ich die Entdeckung dieses Prinzips, wie so vieles andere, meinem verehrten Lehrer und späteren väterlichen Freunde Kurt Hensel.*
>
> It was the question at the end of this message [*on Hensel's postcard*] which opened my eyes... From this seed there grew quickly ... the Local-Global Principle for all representation- and equivalence relations for quadratic forms with rational and also with algebraic coefficients. Thus I owe the discovery of this principle, like so many other things, to my respected teacher and later my paternal friend, Kurt Hensel.

After his thesis, Hasse published in quick succession six other important papers (one jointly with Hensel) elaborating on the subject.

In the first of those papers [1923a] he developed a Local-Global Principle for the *equivalence* of quadratic forms over $\mathbb{Q}$. This deals with the following question: Two quadratic forms $f_1(x_1, \dots x_n)$ and $f_2(x_1, \dots x_n)$ with the same number n of variables, are called "equivalent" if one can be transformed into the other by means of a non-singular linear transformation of the variables. Here, the coefficients of the quadratic forms are supposed to be contained in a given base field, and the entries of the $n \times n$ transformation matrix should also be in that field. In the situation of Hasse's paper, the base field may be either $\mathbb{Q}$ or the completion $\mathbb{Q}_p$ with respect to any prime p of $\mathbb{Q}$.

Given two quadratic forms with coefficients in $\mathbb{Q}$, Hasse proved that they are equivalent over $\mathbb{Q}$ if and only if they are equivalent over $\mathbb{Q}_p$ for every prime

[38] Later, Frobenius' results were sharpened by Chebotarev's density theorem. We refer to § 24 of Hasse's class field report [1930].

[39] One is reminded of the similar words which Hensel had used 1905 in Meran when he presented his "proof" of the transcendency of the number e by means of p-adic methods. See the exposition by Peter Ullrich in [1998d]. Although Hensel's "proof" turned out to be erroneous, the broad idea of it was vindicated by Bézivin and Robba [1989].

p (including $p = \infty$). If we denote equivalence with the symbol $\sim$ then this means that *the statement*

$$f_1(x_1, \ldots x_n) \sim f_2(x_1, \ldots x_n)\,.$$

holds over $\mathbb{Q}$ *if and only if it holds over* $\mathbb{Q}_p$ *for all* p. Here again, the point is that in $\mathbb{Q}_p$ there are explicit arithmetic criteria available for two quadratic forms to be equivalent.

This subject had been studied earlier by Minkowski in [1890]; therefore Hasse's theorem is sometimes called the "Hasse-Minkowski theorem." However, it should be kept in mind that Minkowski did not work in the p-adic completions and accordingly he did not formulate a Local-Global Principle for those. Hasse in [1923a] cites Minkowski [1890] but he does not use Minkowski's result because, he says, Minkowski relies on his systematic theory of quadratic forms *over the ring of integers* $\mathbb{Z}$ which he (Hasse) does not need and which also seems inadequate for the present purpose since the problem is of rational and not integral nature. Instead, Hasse introduces new local invariants which are sufficient to characterize a quadratic form over $\mathbb{Q}_p$ up to equivalence. Besides of the number of variables n and the discriminant $d \neq 0$ (modulo squares) he introduces what today is called the "*Hasse invariant*" of a quadratic form. If the form f is transformed into diagonal form

$$f(x_1, \ldots, x_n) = \sum_{1 \le i \le n} a_i x_i^2 \qquad \text{with} \qquad a_i \in \mathbb{Q}_p^\times$$

(which is always possible) then the Hasse invariant is given by

$$c_p(f) = \prod_{i \le j} \left(\frac{a_i, a_j}{p} \right)$$

where $\left(\frac{a,b}{p}\right)$ denotes the quadratic Hilbert symbol for p which takes the value $+1$ or -1 according to whether a is a norm from $\mathbb{Q}_p(\sqrt{b})$ or not.

In today's terminology we usually prefer to talk about "quadratic vector spaces" rather than quadratic forms, and then the notion of equivalence of quadratic forms corresponds to the notion of isomorphism of quadratic spaces. [40] Thus the above Local-Global Principle can also be stated in the following form:

> *Two quadratic spaces* V *and* W *over* $\mathbb{Q}$ *are isomorphic if and only if all their localizations* $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ *and* $W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ *are isomorphic.* [41]

The Hasse invariants $c_p(f)$ appear now as invariants of the localized quadratic vector spaces V and therefore should be denoted by $c_p(V)$.

---

[40] The idea to consider quadratic spaces instead of quadratic forms is due to Witt [1937]. In his introduction Witt refers explicitly to Hasse's papers on quadratic forms.

[41] By the way, Hasse's first Local-Global Principle in [1923] can also be formulated in terms of quadratic spaces. For, the representability of a number a by a quadratic form $f(x_1, \ldots, x_n)$ is equivalent with the non-trivial representability of 0 by the quadratic form $f'(x_1, \ldots, x_{n+1}) = f(x_1, \ldots x_n) - a x_{n+1}^2$, which means that the quadratic space V' belonging to f' is *isotropic*. Hence the Local-Global Principle of [1923] can be expressed as follows: *A quadratic space over* $\mathbb{Q}$ *is isotropic if and only if all its localizations are isotropic.*

In his next paper [1924a] Hasse deals with a Local-Global Principle for symmetric matrices over $\mathbb{Q}$. Thereafter he takes the important and non-trivial step to generalize all these Local-Global Principles from the base field $\mathbb{Q}$ to an *arbitrary algebraic number field K of finite degree and its primes $\mathfrak{p}$* . This is done in [1924b], [1924c], after having dealt with preparatory theorems on the quadratic residue symbol in number fields in [1923b] and [1923c]. [42]

We want to emphasize that the above mentioned Hasse papers are of a different type from those of Kürschák, Ostrowski and Rychlík which we discussed in the foregoing sections. The latter are dealing with fundamentals, intending to provide a solid foundation for general valuation theory. But Hasse's first papers do not care about foundations. Studying those papers we find that Hasse does not even mention there the abstract notion of valuation. Of course, he investigates the completions $K_\mathfrak{p}$ of a number field K with respect to its primes $\mathfrak{p}$ defined by valuations (including the infinite primes $\mathfrak{p}_\infty$). But he takes the complete fields $K_\mathfrak{p}$ for granted, regardless of whether they are constructed via p-adic power series (as did Hensel) or by Cantor's method (as did Kürschák). In other words: Hasse *applied* valuation theory, whereas Kürschák, Ostrowski and Rychlík had started to *build* the proper foundation for it.

In view of these papers of Hasse, the theory of Hensel's $\mathfrak{p}$-adic fields (i.e., the completions of algebraic number fields at the various primes defined by valuations) became firmly and successfully established as a useful and powerful tool in number theory. Today this is not questioned; local fields and their structure play a dominant role. But before Hasse, this was not so obvious; there were still prominent mathematicians who considered valuation theory as not too interesting or at least superfluous for number theory. Hasse reports in [1975] that Richard Courant in 1920 had voiced the opinion that Hensel's book on p-adic numbers represented a fruitless side track only ("*ein unfruchtbarer Seitenweg*"). It may safely be assumed that this was not only Courant's private opinion but was shared by a number of other people in his mathematical neighborhood, in Göttingen and elsewhere. [43]

Let us briefly mention that Hasse's Local-Global Principle has turned out to be of importance far beyond the application to quadratic forms over number fields. For once, the Local-Global Principle was proved to hold also in various other situations in number theory, e.g., in the structure theory of central simple algebras (see section 3.4.2), and in the investigation of the embedding problem. Also in more general situations, for fields other than number fields, the Local-Global Principle has proved to be a powerful tool. Consider any multi-valued field, i.e., a field K equipped with a set V of valuations. Given any field theoretic statement A over K, one can ask whether the following is true:

> A *holds over K if and only if* A *holds over the completion $K_\mathfrak{p}$ for all primes $\mathfrak{p} \in V$.*

[42] We refer the reader to the self-contained treatment by O'Meara [1963]. There Hasse's theory of quadratic forms over number fields (and more) is presented *ab ovo* along Hasse's lines, preceded by an introduction to valuation theory à la Kürschák.

[43] In later years Courant seems to have had revised this opinion, obviously impressed by Hasse's work not only on quadratic forms but also on other problems in number theory where valuation theory was applied. We conclude this from a letter of F.K. Schmidt to Hasse dated Oct 3, 1934 in which F.K. Schmidt reports on a conversation of Courant with Abraham Flexner, the founder of the Institute of Advanced Study in Princeton.

If this is true then we say that A *admits the Local-Global Principle* over the multi-valued field K.

Such a Local-Global Principle has been established, meanwhile, in quite a number of cases also outside of number theory, e.g., in the so-called field arithmetic, and in the study of algebraic function fields over number fields. And even if, for a particular statement, the Local-Global Principle is not valid, the investigation of the obstruction to its validity often yields valuable information.

It is not possible in this article to follow up the development of the valuation theoretic Local-Global Principle in full detail. This will be the topic of a separate publication, and it is an exciting story. Here, let it suffice to observe that all this started in the years 1921-22, when Hasse proved the Local-Global Principles for quadratic forms, first over $\mathbb{Q}$ and then over an arbitrary algebraic number field of finite degree.

And we should not forget Hensel whose suggestion, although in rather vague terms, led Hasse to the formulation of his Local-Global Principles.

3.2 Multiplicative structure

3.2.1 The cradle of local class field theory

In order to obtain his Local-Global Principles for quadratic forms over a number field K, Hasse had to study and use Hilbert's quadratic reciprocity law in K. Following up an idea of Hensel in the case $K = \mathbb{Q}$ [1913b], he interpreted the quadratic Hilbert norm symbol $\left(\frac{a,b}{\mathfrak{p}}\right)$ in an arbitrary number field K in terms of the completion $K_\mathfrak{p}$ – where $\mathfrak{p}$ denotes a prime of K belonging to an archimedean or non-archimedean valuation, and $a, b \in K_\mathfrak{p}^\times$. As said above already, the local definition is to put $\left(\frac{a,b}{\mathfrak{p}}\right) = 1$ or -1 according to whether a is a norm from $K_\mathfrak{p}(\sqrt{b})$ or not.

In the older literature the terminology was "norm residue symbol" instead of "norm symbol", and $\left(\frac{a,b}{\mathfrak{p}}\right)$ was defined for $\mathfrak{p}$-integers $a, b \in K$ only. And the definition was that $\left(\frac{a,b}{\mathfrak{p}}\right) = 1$ if for any power $\mathfrak{p}^r$, with arbitrarily large exponent, a is congruent modulo $\mathfrak{p}^r$ to a norm from $K(\sqrt{b})$. It had been Hensel who observed that such congruence conditions for infinitely many powers of $\mathfrak{p}$ can be viewed, in the limit, as one single condition in the completion $K_\mathfrak{p}$.

Quite generally, observations of this kind have brought about an enormous conceptual simplification whose consequences, up to the present day, cannot be overestimated. The investigation of congruence properties for varying prime power modules $\mathfrak{p}^r$ can be replaced by the investigation of valuation theoretic properties of the completion $K_\mathfrak{p}$.

Hasse confirmed Hensel's idea that the locally defined symbol is quite suited to develop a criterion for the local representability of a number by a quadratic form. And he discovered that Hilbert's product formula

$$\prod_{\mathfrak{p}} \left(\frac{a,b}{\mathfrak{p}}\right) = 1 \qquad (a, b \in K^\times) \tag{3.1}$$

provides a tool to make the transition from local to global – in conjunction with the well known ordinary product formula for the valuations themselves:

$$\prod_{\mathfrak{p}} \|a\|_{\mathfrak{p}} = 1 \qquad\qquad (a \in K^{\times})\,. \tag{3.2}$$

Here, $\mathfrak{p}$ ranges over all primes of K and $\|\cdot\|_{\mathfrak{p}}$ is the corresponding valuation in suitable normalization.

One of the essential properties of the Hilbert symbol $\left(\frac{a,b}{\mathfrak{p}}\right)$ which had to be verified is the "*Vertauschungssatz*"

$$\left(\frac{a,b}{\mathfrak{p}}\right) = \left(\frac{b,a}{\mathfrak{p}}\right)^{-1}\,. \tag{3.3}$$

[44]

If $\mathfrak{p}$ does not divide 2 (including the case when $\mathfrak{p}$ is archimedean) then this is well known and straightforward. The only true difficulty arises in the case when $\mathfrak{p}|2$, i.e., when $\mathfrak{p}$ may be wildly ramified in $K_{\mathfrak{p}}(\sqrt{b})$. In the older, classical literature one tried to avoid this case by using Hilbert's product formula (3.1) in order to shift the problem to the tamely ramified primes. But then the explicit evaluation of $\left(\frac{a,b}{\mathfrak{p}}\right)$ was not possible in general, only for those a, b which satisfy certain restrictions. This was insufficient for Hasse's purpose. Finally in [1923c] he was able to show in purely local terms that

$$\left(\frac{a,b}{\mathfrak{p}}\right) = (-1)^{L(a,b)} \tag{3.4}$$

where $L(a,b)$ is a bilinear, non-degenerate form on the $\mathbb{Z}/2$-vector space $K_{\mathfrak{p}}^{\times}/K_{\mathfrak{p}}^{\times 2}$ of square classes in $K_{\mathfrak{p}}^{\times}$. And, what is essential, his construction showed that $L(a,b)$ is *symmetric* which yields the *Vertauschungssatz* (3.3).

This was of course well known in the case $K = \mathbb{Q}$, $p = 2$ where the square classes of $\mathbb{Q}_2$ are represented by the numbers $2^{x_0}(-1)^{x_1}5^{x_2}$ with $x = (x_0, x_1, x_2) \in (\mathbb{Z}/2)^3$. In this case $L(a,b) = x_0y_2 + x_1y_1 + x_2y_0$ if the vectors x, y represent a and b respectively. But in the case of an arbitrary number field K and an arbitrary prime $\mathfrak{p}$ dividing 2, this was not so easy for young Hasse. Today we would deduce (3.4) immediately from local class field theory. But we have to take into consideration that in 1921, local class field theory did not yet exist.

In fact, parallel to his work on the quadratic norm symbol, Hasse became interested, again on the suggestion of Hensel, in the similar problem for the m-th power norm symbol for an arbitrary positive integer m. His first paper [1923b] on this is written jointly with Hensel. After many years of work and quite a number of papers on this topic, Hasse's results led him finally to *discover* local class field theory in [1930a]. Here we cannot go into details and follow up the string of development which finally led to the establishment of local class field theory and its connection to global class field theory. But we would like to state that all this started in 1923 with Hasse's papers on the norm symbols in local fields. At that time Hasse was *Privatdozent* at the University of Kiel.

Thus Kiel in the year 1923 became the cradle of local class field theory.

[44]Since the *quadratic* norm symbol has the value 1 or -1, it is equal to its inverse. Hence we could have omitted the exponent -1 on the right hand side. Nevertheless we have included the exponent -1 since this becomes necessary for the norm symbols referring to an arbitrary positive integer; the m-th norm symbol assumes its values in the group of m-th roots of unity. The general norm symbol is antisymmetric; only in case of degree 2 this is the same as being symmetric.

3.2.2 Hensel: The basis theorem

The problem of whether an element $a \in K_{\mathfrak{p}}^{\times}$ is a norm from a given finite extension is of multiplicative nature. In order to deal with this problem it was necessary for Hasse to use the *fundamental basis theorem* for the multiplicative group $K_{\mathfrak{p}}^{\times}$ which *Kurt Hensel* had given already in [1916]. The general problem which Hensel dealt with can be described as follows.

Consider first the well known situation in the field $\mathbb{R}$ of real numbers. In this case, the multiplicative group $\mathbb{R}^{\times}$ admits -1 and e as basis elements in the sense that every $a \in \mathbb{R}^{\times}$ can be written uniquely in the form

$$a = (-1)^k \cdot e^{\alpha} \quad \text{where} \begin{cases} k \in \mathbb{Z}/2 \\ \alpha = \log(|a|) \in \mathbb{R} \end{cases} \tag{3.5}$$

Here, e^{α} is defined, as usual, by the exponential series

$$\exp(\alpha) = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots \tag{3.6}$$

and $e = \exp(1)$. Formula (3.5) gives the structure theorem

$$\mathbb{R}^{\times} \approx \mathbb{Z}/2 \times \mathbb{R} \tag{3.7}$$

where on the right hand side $\mathbb{R}$ means the additive group of real numbers. The important feature of this isomorphism is that the multiplication in $\mathbb{R}^{\times}$ is converted to addition in the groups $\mathbb{Z}/2$ and $\mathbb{R}$; this is essentially given by the exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}^{\times}$ and its inverse, the logarithm.

Now, Hensel asked whether a similar result holds in the case of a local number field $K_{\mathfrak{p}}$ with respect to a non-archimedean prime $\mathfrak{p}$. Is the multiplicative group $K_{\mathfrak{p}}^{\times}$ naturally isomorphic to some additive group which is canonically determined by the field? He had treated this question for the rational p-adic fields $\mathbb{Q}_p$ already in his number theory book [1913b]. One year later in [1914], Hensel started to deal with the case of an arbitrary local field $K_{\mathfrak{p}}$ which is a finite extension of $\mathbb{Q}_p$.

The main difficulty in the non-archimedean case is that the exponential series (3.6) does not converge everywhere, it is convergent for $v_p(\alpha) > \frac{1}{p-1}$ only. [45] This condition defines an ideal of the valuation ring of $K_{\mathfrak{p}}$, say $\mathcal{A}$. On $\mathcal{A}$ the exponential function satisfies the functional equation

$$\exp(\alpha + \beta) = \exp(\alpha)\exp(\beta)$$

and defines an isomorphism

$$\exp : \mathcal{A} \approx 1 + \mathcal{A}$$

whose inverse is given by the p-adic logarithm which is defined through the power series expansion

$$\log(\beta) = (\beta - 1) - \frac{(\beta-1)^2}{2} + \frac{(\beta-1)^3}{3} \mp \cdots \tag{3.8}$$

[45] In this context v_p denotes the additively written valuation of $K_{\mathfrak{p}}$, normalized in such a way that $v_p(p) = 1$. In this normalization a prime element $\pi \in K_{\mathfrak{p}}$ has the value $v_p(\pi) = \frac{1}{e}$ where e is the ramification degree of $K_{\mathfrak{p}}$ over $\mathbb{Q}_p$.

The logarithmic series converges not only for $\beta \in 1+\mathcal{A}$ but also for $\beta \in 1+\mathcal{M}$, where $\mathcal{M}$ denotes the maximal ideal of the valuation ring. $U_1 := 1+\mathcal{M}$ is the multiplicative group of all those elements $\beta \in K_\mathfrak{p}$ which satisfy $\beta \equiv 1 \mod \mathfrak{p}$, called the 1-units of $K_\mathfrak{p}$. On U_1 we have

$$\log(\alpha\beta) = \log(\alpha) + \log(\beta)\,.$$

However, on U_1 the logarithm function is in general *not injective*. For, there may be p-power roots of unity in $K_\mathfrak{p}$. It is injective on $1+\mathcal{A}$ where, as said above already, log is the inverse function of exp.

Hensel in his paper [1914] considers first the case when the ramification degree $e < p-1$. In this case $\mathcal{A} = \mathcal{M}$ and hence $1+\mathcal{A} = U_1$. Hence every element in U_1 is of the form $\exp(\alpha)$ with $\alpha \in \mathcal{M}$. But the structure of $K_\mathfrak{p}^\times$ modulo U_1 is well known and easy to establish: a basis is given by the elements π and ω where π is a prime element of $K_\mathfrak{p}$ and ω a primitive w-th root of unity where w is the number of roots of unity in $K_\mathfrak{p}$. [46] Hensel concludes that every $a \in K_\mathfrak{p}^\times$ has a unique representation of the form [47]

$$a = \pi^n \omega^k \exp(\alpha) \qquad \text{where} \begin{cases} n \in \mathbb{Z} \\ k \in \mathbb{Z}/w \\ \alpha \in \mathcal{M} \end{cases}$$

The valuation ring of $K_\mathfrak{p}$ is a free $\mathbb{Z}_p$-module of rank

$$r = [K_\mathfrak{p} : \mathbb{Q}_p]$$

and hence every of its non-zero ideals is so too. In particular it follows $\mathcal{M} \approx \mathbb{Z}_p^r$. Using the continuous isomorphism $\exp : \mathcal{M} \to U_1$ we see that $U_1 \approx \mathbb{Z}_p^r$ too (as $\mathbb{Z}_p$-modules). Hence there exist r basis elements $\eta_1, \ldots, \eta_r \in U_1$ such that every $a \in K_\mathfrak{p}$ admits a unique representation of the form

$$a = \pi^n \omega^k \eta_1^{\alpha_1} \cdots \eta_r^{\alpha_r} \qquad \text{where} \qquad \begin{cases} n \in \mathbb{Z} \\ k \in \mathbb{Z}/w \\ \alpha_i \in \mathbb{Z}_p \end{cases} \tag{3.9}$$

This gives

$$K_\mathfrak{p}^\times \approx \mathbb{Z} \times \mathbb{Z}/w \times \mathbb{Z}_p^r\,. \tag{3.10}$$

These formulas are the $\mathfrak{p}$-adic analogues to (3.5) and (3.7) in the real case. In [1914] Hensel could prove them in the case $e < p-1$ only. Hensel deals with the general case in the second part of [1914] and in [1915]. In modern terms, his argument can be summarized as follows:

Consider the p-adic logarithm function $\log : U_1 \to \mathcal{M}$. The kernel consists of the p-power roots of unity of $K_\mathfrak{p}$, say, μ_{p^s} for some integer s. The image $\mathcal{I}$ is a $\mathbb{Z}_p$-module containing $\mathcal{A}$ and, hence, free of rank r. Therefore there exists a section $\mathcal{I} \to U_1$ of the logarithm function, and this defines an extension of the exponential function exp to $\mathcal{I}$ as a domain of definition. This extension is not canonical; anyhow it follows that $U_1 = \mu_{p^s} \times \exp(\mathcal{I}) \approx \mathbb{Z}/p^s \times \mathbb{Z}_p^r$. The total number of roots of unity in $K_\mathfrak{p}$ is $w = (q-1)p^s$.

[46] If $e < p-1$, the field $K_\mathfrak{p}$ does not contain proper p-power roots of unity and therefore $w = q-1$, the number of non-zero elements in the residue field of $K_\mathfrak{p}$.

[47] Hensel writes e^α instead of $\exp(\alpha)$ but this may be misleading since $\exp(1)$ is not defined and hence e^α is not the exponentiation by α of the number $\exp(1)$.

In this way Hensel had proved that the formulas (3.9) and (3.10) are valid in general, for an arbitrary local non-archimedean number field $K_{\mathfrak{p}}$.

But this proof does not give an effective method to produce the basis elements η_i of (3.9). One has to choose a $\mathbb{Z}_p$-basis $\xi_1, \ldots, \xi_r$ of $\mathcal{I}$ and put $\eta_i = \exp(\xi_i)$. This is not constructive since the module $\mathcal{I}$, the image of the logarithm function, is in general not known explicitly. Moreover, the analytically defined exponential function produces the η_i as possibly transcendental numbers whose arithmetic properties are not easily established. Therefore Hensel looked for a better construction of the basis elements.

He found this in still another paper [1916]. There he presented an effective construction of the basis elements η_i which is adapted to the natural filtration $U_1 \supset U_2 \supset U_3 \supset \cdots$, where U_ν consists of those $\eta \in K_{\mathfrak{p}}$ for which $\eta \equiv 1 \mod \mathfrak{p}^\nu$. This construction was used later in the work of Hasse and others to study the local norms in the context of class field theory.

With this explicit construction of $\eta_1, \ldots, \eta_r$*, the basis theorem* (3.9) *is to be regarded as one of Hensel's important contributions to number theory.*

The ensuing structure theorem (3.10) deserves its place in general valuation theory, independent of its application to number theory. Note that the local number fields $K_{\mathfrak{p}}$ can be abstractly characterized, independent from their origin as completions of number fields of finite degree. Namely, these fields are the *locally compact valued fields of characteristic* 0. [48]

Sometimes the opinion is voiced that Kurt Hensel did not prove deep theorems on local fields, and that his main contribution to Number Theory was his *idea* that p-adic numbers exist and should be studied; the main credit for *putting his ideas to work* should go to Hasse and other students of Hensel. Well, Hensel's proof of the basis theorem is evidence that this view is not correct. But certainly it was Hasse who used the basis theorem in its full power in the discussion of the local Hilbert symbol, in particular in the wildly ramified case, in connection with explicit reciprocity and class field theory.

REMARK: Perhaps we should point out that the basis theorem, with a finite basis, does *not* hold in the analogous case of power series fields with finite coefficient fields. This seems to be one of the reasons why in the latter case, the theory of those fields has not yet been proven to be decidable, whereas in the case of local number fields this question is settled by the work of Ax-Kochen and Ershov. [49]

3.3 Remarks on p-adic analysis

In the foregoing section we had occasion to mention functions which were analytic in the p-adic sense: the exponential function and the logarithm. There are other p-adically analytic functions which were successfully employed in number theory. There have been a number of attempts to develop a systematic theory of analytic functions in the context of non-archimedean valuation theory, analogous to the theory of complex valued analytic functions. This development, including Tate's theory of rigid analysis, is an indispensable part of the history of valuation theory.

However, for the reasons stated in the introduction it was not possible to include it into this manuscript. Let it suffice to say that the first paper in this

[48] See e.g., Pontryagin's book [1957], or Warner [1989b].

[49] See e.g. our Lecture Notes [1984].

direction was the thesis of Schöbe [1930d], a doctorand of Hasse in Halle. For more information we refer the reader to the treatment by Ullrich [1995] and the literature cited there. In addition, the reader might consult [1985b] concerning the contributions of Marc Krasner.

Also, we would like to point out that by means of p-adic analysis several proofs of transcendency have been given. The first was by Kurt Mahler in [1932i] when he proved that the value of the p-adic exponential function $\exp(\alpha)$ is transcendental for any algebraic α in the domain of convergence of the p-adic exponential series (3.6). Later in [1935a] he proved the p-adic analogue of Gelfand's result: the quotient of p-adic logarithms $\frac{\log(\alpha)}{\log(\beta)}$, if irrational, is transcendental for algebraic p-adic numbers α, β which are in the domain of convergence of the logarithmic series (3.8). There followed a long series of p-adic transcendental theorems whose survey indeed would require a separate article.

Hensel's idea to prove the transcendency of the real number e by p-adic analytic methods [1905a], which contained an error, has been vindicated to some extent by Bézivin and Robba [1989]. For a historic review we refer to Ullrich's paper [1998d] and the literature cited there.

3.4 Valuations on skew fields

During the late 1920's and early 1930's there was growing awareness that the theory of non-commutative algebras could be used to obtain essential information about the arithmetic structure of commutative number fields. This view was forcibly and repeatedly brought forward by Emmy Noether; see e.g., her report at the Zürich International Congress of Mathematicians [1932] and the literature cited there.

The story started with the appearance of Dickson's book [1923g] "*Algebras and their Arithmetics*" in which the author did the first steps towards an arithmetic theory of maximal orders in an algebra over number fields. [50] This book gained increasing interest among the German algebraists [51], in particular since Speiser had arranged a German translation [1927c].

Emil Artin presented a complete and exhaustive theory of the arithmetic of maximal orders of semisimple algebras over number fields in his papers [1928a], [1928b], [1928c]. [52] He cited Dickson as a source of inspiration but his theory went far beyond, describing such a maximal order and its groupoid of ideals (in the sense of Brandt), in a similar way as Noether had done in the commutative case for what are now called Dedekind rings. [53]

Shortly afterwards Hasse [1931] gave a new treatment, and this time *on a valuation theoretic basis*. In the same way as a Dedekind ring can be treated

[50] The book also contained the first text book treatment of the Wedderburn structure theorems for semisimple algebras.

[51] By this I mean e.g., Emil Artin, Helmut Hasse, Emmy Noether, Andreas Speiser, Bartel van der Waerden and the people around them. Perhaps it is not superfluous to state that the word "German" in this connection is not meant to have implications in the direction of political doctrines. (Artin was Austrian, Speiser was Swiss, van der Waerden was Dutch.)

[52] Artin had divided his article into three separate papers, appearing successively in the same volume 5 of *Hamburger Abhandlungen*.

[53] However Artin did not work on an axiomatic basis like Noether for Dedekind rings (she called them "*Fünf-Axiome-Ringe*" (five axioms rings)). In later years one of the Ph.D. students of Artin, Karl Henke, presented an axiomatic description [1935] modeled after Noether's axiomatic treatment of Dedekind rings.

by means of its localizations with respect to the valuations belonging to its prime ideals, Hasse showed that in the non-commutative case, the arithmetic of a maximal order can be similarly described by its localizations with respect to the primes of the center. Moreover, by including the infinite primes belonging to the archimedean valuations, Hasse was able to proceed much further towards non-commutative foundation of commutative number theory – following the *desideratum* of Emmy Noether.

Thus once more Hasse showed that valuation theory provides for useful and adequate methods to deal with questions of higher algebraic number theory. This was well acknowledged by his colleagues. [54] As an example let us cite Emmy Noether, in a postcard to Hasse of June 25, 1930:

> *Ihre hyperkomplexe* $\mathfrak{p}$*-adik hat mir sehr viel Freude gemacht...*
>
> Your hypercomplex $\mathfrak{p}$-adics has given me much pleasure...

And Artin in a letter of Nov 27, 1930:

> *...Ich danke Ihnen auch für die Übersendung der Korrekturen Ihrer Arbeit über hyperkomplexe Arithmetik. Dadurch ist wirklich alles sehr einfach geworden...*
>
> ...Also, I would like to thank you for sending the proof sheets of your paper on hypercomplex arithmetics. With this, everything really has become very simple...

Let us describe Hasse's theorems, at least for local fields, in some more detail.

3.4.1 Hasse's theorems

Let K be an algebraic number field and A a central simple algebra over K. For any prime $\mathfrak{p}$ of K let $K_\mathfrak{p}$ be the corresponding completion and $A_\mathfrak{p} = A \otimes_K K_\mathfrak{p}$ the localization of A. This is a central simple algebra over $K_\mathfrak{p}$ and so, by Wedderburn's theorem, $A_\mathfrak{p}$ is a full matrix algebra over a central division algebra $D_\mathfrak{p}$ over $K_\mathfrak{p}$. Hence, as a first step Hasse considers valuations of division algebras $D_\mathfrak{p}$ over local fields. Suppose that $\mathfrak{p}$ is non-archimedean. The following theorems are proved in [1931]:

> **1.** *The canonical valuation of* $K_\mathfrak{p}$ *admits a unique extension to a valuation of* $D_\mathfrak{p}$*.*

Here, a valuation of the skew field $D_\mathfrak{p}$ is defined by Kürschák's axioms (2.1)–(2.4) in the same way as for commutative fields. The formula (2.5) for the extensions is also valid in the non-commutative case; but usually one replaces the norm by the so-called reduced norm for $D_\mathfrak{p}|K_\mathfrak{p}$. The proof of **1.** is quite the same as in the case of a commutative extension field, using Hensel's Lemma.

The ramification degree e and the residue degree f of $D_\mathfrak{p}|K_\mathfrak{p}$ are defined as in the commutative case. Then:

> **2.** *The ramification degree* e *and the residue degree* f *of* $D_\mathfrak{p}|K_\mathfrak{p}$ *are both equal to the index* m *of* $D_\mathfrak{p}$*, so that* $[D_\mathfrak{p} : K_\mathfrak{p}] = ef = m^2$*.*

[54] As mentioned in the introduction already, van der Waerden [1975a] speaks of Hasse as "Hensel's best and great propagandist of p-adic methods". Hasse, he said, came often to Göttingen and so van der Waerden was inspired by Hasse, as well as by Ostrowski, while dealing with valuations during the write up of the first volume of his book "*Moderne Algebra*".

The surprising thing is that, on the one hand, statement **1.** is valid for division algebras in the same way as for commutative field extensions, whereas statement **2.** shows a completely different behavior of valuations of division algebras when compared to commutative field extensions. Namely, ramification degree and residue field degree are both determined by the degree m^2 of $D_{\mathfrak{p}}$. This has important consequences.

Let π be a prime element and ω a primitive $q-1$-th root of $K_{\mathfrak{p}}$ (where q is the number of elements in the residue field). Let $\Phi_m(X)$ denote an irreducible polynomial of degree m which divides $X^{q^m-1}-1$ (such divisor does exist since there is an unramified extension of $K_{\mathfrak{p}}$ of degree m). Then:

> **3.** $D_{\mathfrak{p}} = K_{\mathfrak{p}}(u, \alpha)$ *is generated by two elements* u *and* α *with the defining relations:*
>
> $$\Phi_m(\alpha) = 0\,, \qquad u^m = \pi\,, \qquad u^{-1}\alpha u = \alpha^{q^r}$$
>
> *where* r *is some integer prime to* m*, uniquely determined by* $D_{\mathfrak{p}}$ *modulo* m*.*

In particular it is seen that $D_{\mathfrak{p}}$ is a *cyclic* algebra. Moreover, $D_{\mathfrak{p}}$ is uniquely determined by the index m and the number r modulo m. The quotient $j_{\mathfrak{p}}(D_{\mathfrak{p}}) := \frac{r}{m}$ modulo 1 is called the *Hasse invariant* of the skew field $D_{\mathfrak{p}}$.

This theorem then leads to the determination of the Brauer group $Br(K_{\mathfrak{p}})$ of all central division algebras over $K_{\mathfrak{p}}$ or, equivalently, of the similarity classes of central simple algebras over $K_{\mathfrak{p}}$. Using the above theorems, Hasse could prove that

$$Br(K_{\mathfrak{p}}) \approx \mathbb{Q}/\mathbb{Z} \tag{3.11}$$

and this isomorphism is obtained by assigning to each division algebra $D_{\mathfrak{p}}|K_{\mathfrak{p}}$ its Hasse invariant.

If $K_{\mathfrak{p}}$ is a local field for an infinite (archimedean) prime then $K_{\mathfrak{p}} = \mathbb{R}$ or $K_{\mathfrak{p}} = \mathbb{C}$. In the first case there is only one non-trivial skew field with center $\mathbb{R}$, namely the quaternion field $\mathbb{H}$. To this is assigned the Hasse invariant $j_{\mathfrak{p}}(\mathbb{H}) = \frac{1}{2}$, and so $Br(\mathbb{R}) \approx \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is of order 2. For the complex field we have $Br(K_{\mathfrak{p}}) = 0$.

Remark. Hasse's paper [1931] was the first one where valuations of division algebras have been systematically constructed and investigated, on the basis of Kürschák's axioms. This started a long series of investigations of valuations of arbitrary division algebras over a valued field, not necessarily over a local number field. We refer to the excellent report by A. W. Wadsworth [2002].

3.4.2 Consequences

Hasse's theorems have important implications for local class field theory. Observe that just one year earlier the main theorems on local class field theory had been presented by Hasse [1930a] and F. K. Schmidt [1930b] as a consequence of global class field theory. [55] But from the very first discovery of local class field theory, Hasse had looked for a foundation of local class field theory on purely local terms, independent from global class field theory. Emmy Noether shared this opinion; let us cite from the postcard (mentioned above already) of Noether to Hasse of June 25, 1930, when Hasse had sent her the manuscript of [1931]:

[55] Hasse's [1930a] had been received by the editors on March 16, 1929 while [1931] was received on June 18, 1930.

... Aus der Klassenkörpertheorie im Kleinen folgt: Ist Z zyklisch n-ten Grades über einem $\mathfrak{p}$-adischen Grundkörper K, so gibt es in K mindestens ein Element $a \neq 0$, derart daß erst a^n Norm eines Z-Elementes wird. Können Sie das direkt beweisen? Dann könnte man aus Ihren Schiefkörperergebnissen umgekehrt Klassenkörpertheorie im Kleinen begründen...

From local class field theory follows: If Z is cyclic of degree n over a $\mathfrak{p}$-adic base field K then there exists at least one element $a \neq 0$ in K such that only the a^n is a norm of a Z-element. Can you prove this directly? If so then one could deduce from your skew field results backwards the local class field theory...

In fact, it is easy and straightforward to answer Noether's question positively, using Hasse's result **3.** on skew fields. On the other hand, it seems that Emmy Noether had jumped too early to the conclusion that this alone already provides a foundation for the whole body of local class field theory. The inclusion of arbitrary abelian field extensions instead of cyclic ones required some more work. Later, Noether admitted this in her Zürich address [1932]. After referring to Hasse's canonical definition of his norm symbol (which he could manage because of the theorems above) she mentions

... eine hyperkomplexe Begründung der Klassenkörpertheorie im Kleinen, auf derselben Grundlage beruhend, die neuerdings Chevalley gegeben hat, wobei aber noch neue algebraische Sätze über Faktorensysteme zu entwickeln waren.

... a hypercomplex foundation of local class field theory, based on the same principles, which has recently been given by Chevalley, where however, in addition, new algebraic theorems on factor systems had to be developed.

The "new algebraic theorems on factor sets" can be found in Chevalley's Crelle paper [1933e]. From today's viewpoint, these "theorems" belong to the standard prerequisites of Galois cohomology but in those times cohomology theory had not yet been established on an abstract level.

Based on his local results, Hasse was able in a subsequent paper [1932e] to determine the Brauer group $Br(K)$ also of a global number field K, by means of valuation theoretic notions. Let $\mathfrak{p}$ range over all primes of K and $\mathfrak{J}$ denote the direct sum of all local groups of Hasse invariants, i.e.,

$$\mathfrak{J} = \sum_{\mathfrak{p} \text{ finite}} \mathbb{Q}/\mathbb{Z} \oplus \sum_{\mathfrak{p} \text{ real}} \frac{1}{2}\mathbb{Z}/\mathbb{Z}\,.$$

Then we have a natural map $j : Br(K) \to \mathfrak{J}$, assigning to each central simple algebra A over K the vector of the Hasse invariants of its local components. More precisely, let $A_\mathfrak{p} = A \otimes_K K_\mathfrak{p}$, and let $D_\mathfrak{p}$ be the division algebra which is similar to $A_\mathfrak{p}$, so that $A_\mathfrak{p}$ is a full matrix algebra over $D_\mathfrak{p}$. Then $j_\mathfrak{p}(A)$ is defined to be the Hasse invariant $j_\mathfrak{p}(D_\mathfrak{p})$. And $j(A) \in \mathfrak{J}$ is the vector consisting of the components $j_\mathfrak{p}(A)$.

In addition, there is the natural map $s : \mathfrak{J} \to \mathbb{Q}/\mathbb{Z}$ by adding the components of every vector of $\mathfrak{J}$.

Then the main theorems of [1932e] *can be expressed by saying that the following sequence is exact:*

$$0 \to Br(K) \xrightarrow{j} \mathfrak{J} \xrightarrow{s} \mathbb{Q}/\mathbb{Z} \to 0 \tag{3.12}$$

The exactness at the term $Br(K)$ means that the map j is injective. This fact is the famous **Local-Global Principle for central simple algebras**. It had been first proved in a joint paper of Hasse with Richard Brauer and Emmy Noether [1932c], and it was based on class field theory. [56] The exactness at the term $\mathfrak{J}$ says that the local invariants of a central simple algebra A over K satisfy the relation

$$\sum_{\mathfrak{p}} j_{\mathfrak{p}}(A) \equiv 0 \mod 1 \tag{3.13}$$

and that this is the *only* relation between the local Hasse invariants.

Many years later, Artin and Tate presented in their Seminar Notes [1952a] an axiomatic foundation of class field theory. Their axioms were given in the language of cohomology which by then was well developed. There are two main axioms. Their Axiom I is essentially the cohomological version of the exactness in (3.12) at the term $Br(K)$. And their Axiom II is essentially coming from the exactness at $\mathfrak{J}$. Thus the work of Hasse, Brauer and Noether which led to the exact sequence (3.12), had become the very base for the development of class field theory in an axiomatic framework. And this had been achieved by following the ideas of Kürschák and Hensel about valuations and their localizations.

3.5 Artin-Whaples: Axioms for Global Number Theory

As we have seen, the introduction of valuation theory into number theory has brought about drastic changes of view point and enormous advances. There arose the question whether this could be explained somehow. What are the special valuation theoretic properties which characterize number fields in contrast to other fields?

If K is a field and $\mathfrak{p}$ a prime of K then we denote by $\|\cdot\|_{\mathfrak{p}}$ a real valued valuation belonging to $\mathfrak{p}$. Suppose that K has the following properties:

> **I.** *For any $a \in K^{\times}$, there are only finitely many primes $\mathfrak{p}$ of K such that $\|a\|_{\mathfrak{p}} \neq 1$.*
>
> **II.** *Every prime of K is either archimedean or discrete with finite residue field.* [57]

Then, if K is of characteristic 0, it follows that K is an algebraic number field of finite degree. If K is of characteristic $p > 0$ then K is a finitely generated algebraic function field of one variable over a finite field – if it is assumed that K admits at least one prime to avoid trivialities.

Today these fields are called "*global fields*". Thus global fields can be characterized by the valuation theoretic properties **I.** and and **II.** The proof is quite easy.

[56] Another proof was given later by Zorn [1933d], based on analytic number theory as developed in the thesis of Käte Hey [1929]; both had been Ph.D. students of Artin in Hamburg.

[57] Both cases can be subsumed under the unified condition that the completion $K_{\mathfrak{p}}$ is *locally compact* with respect to the topology induced by the valuation.

The above characterization of global fields was included in Hasse's textbook "*Zahlentheorie*" [1949]. That book had been completed in 1938 already but due to external circumstances it could appear in 1949 only. Thus its contents represent more or less the state of the art in 1938, at least in its first edition. In the second edition (which appeared in 1963) Hasse added a proof of the theorem of Artin-Whaples which yields a stronger and much more striking result.

EMIL ARTIN had announced this theorem in an address delivered on April 23, 1943 to the Chicago meeting of the American Mathematical Society. The published version, jointly with his student GEORGE WHAPLES appeared in [1945]. Their main result can be regarded in some sense to be the final justification of valuation theory in its application to number theory.

Artin considers a field K equipped with a non-empty set of primes S, which may or may not be the set of all primes of K. His axioms are:

> **I.**$'$ *Axiom* **I.** *holds for primes which are contained in S, i.e., for any $a \in K^\times$ there are only finitely many $\mathfrak{p} \in S$ such that $\|a\|_\mathfrak{p} \neq 1$.*
>
> **II.**$'$ *Axiom* **II.** *holds for at least one prime $\mathfrak{p} \in S$, i.e., $\mathfrak{p}$ is archimedean or discrete with finite residue field.*
>
> **III.**$'$ *In addition, the valuations $\| \cdot \|_\mathfrak{p}$ for $\mathfrak{p} \in S$ satisfy the product formula*

$$\prod_{\mathfrak{p} \in S} \|a\|_\mathfrak{p} = 1 \qquad (a \in K^\times)\,. \tag{3.14}$$

If these axioms are satisfied then again, K is either an algebraic number field of finite degree, or a finitely generated algebraic function field of one variable over a finite field of constants.

Moreover, the set S indeed consists of *all* primes of K, and the selected valuations $\|\cdot\|_\mathfrak{p}$ are *uniquely determined* up to a substitution of the form $\|\cdot\|_\mathfrak{p} \mapsto \|\cdot\|_\mathfrak{p}^r$ where the exponent r is independent of $\mathfrak{p}$. [58]

In view of this theorem it is possible, in principle, to derive all theorems of number theory from the Artin-Whaples valuation theoretic axioms. In fact, the authors do this for two of the fundamental theorems of number theory, namely Dirichlet's unit theorem and the finiteness of class number. The reader is struck not only by the economy but also by the beauty of those proofs.

We observe that not only algebraic number fields appear in Artin-Whaples' theorem but also algebraic function fields with finite field of constants. This reflects a long standing observation, going back to Dedekind and Gauss, on the analogy between algebraic number theory and the theory of algebraic functions with finite base field. [59]

The authors also discuss a somewhat more general setting, in which K contains a base field k for S, which is to say that all primes $\mathfrak{p} \in S$ are trivial on k. In this case it is required that at least one $\mathfrak{p} \in S$ is discrete, and its residue

[58] Later [1946] the authors showed that Axiom **I.**$'$ could be omitted if the product formula **III.**$'$ is understood to imply that for every $a \in K^\times$ the product (3.14) was absolutely convergent with the limit 1. But the result remains the same.

[59] See our manuscript [1998b], and also a forthcoming manuscript of Günther Frei concerning Gauss' contributions to this question. See also Ullrich's manuscript [1999a].

field is of finite degree over the base field k. [60] But k is not necessarily finite. In this more general case, the product formula (3.14) implies that K is a field of algebraic functions of one variable over k, and that S consists of *all* primes of K over k.

4 Building the foundations

In the foregoing section 3 we have reported about the impact of valuation theory to number theory during the 1920's and 1930's. Because of those striking results there was growing interest to develop general valuation theory beyond the first steps which we discussed in section 2. The motivation for this direction of research was to create an arsenal of new notions and methods which could be profitably applied to situations other than number fields, e.g., function fields of one or more variables, fields of power series and of Dirichlet series, functional analysis, but also to number fields of infinite degree – and more.

4.1 Dedekind-Hilbert theory

Let $L|K$ be a finite Galois field extension whose base field K carries a prime $\mathfrak{p}$ corresponding to a non-archimedean valuation. Let us fix one prime $\mathfrak{P}$ of L which is an extension of $\mathfrak{p}$. The Dedekind-Hilbert theory establishes a connection of the structure of the Galois group G and the structure of $\mathfrak{P}$ as an extension of $\mathfrak{p}$. This manifests itself by a decreasing sequence of subgroups

$$G \supset Z \supset T \supset V = V_1 \supset \cdots V_i \supset V_{i+1} \supset \cdots \supset 1$$

whose members are defined by valuation theoretic conditions with respect to $\mathfrak{P}$. Here, Z, T and V are the decomposition group (*Zerlegungsgruppe*), the inertia group (*Trägheitsgruppe*) and the ramification group (*Verzweigungsgruppe*) of $\mathfrak{P}$. These three groups are the "basic local groups" of $\mathfrak{P}$ whereas the V_i are the "higher ramification groups". [61] Usually one puts $V_0 = T$ and $V_{-1} = Z$. The groups T, V and the V_i are normal in Z.

The ramification groups reflect the decomposition type of the prime $\mathfrak{p}$ of the base field in the extension $L|K$.

Hilbert [1894] had developed the theory of these ramification groups in case K and k are number fields of finite degree. [62] Since then one often speaks of "Hilbert theory". Deuring [1931a] however says "Dedekind-Hilbert theory". Indeed, Dedekind had already arrived much earlier at the same theory, but restricted to the basic local groups Z, T, V, without the higher ramfication groups V_i. However he had published this only after Hilbert in [1894a]. [63] Here, let us follow Deuring and use "Dedekind-Hilbert theory". This seems particularly adequate in this context since in general valuation theory, the higher ramification groups (which are due to Hilbert) do not play a dominant role, and often cannot be defined in a satisfying way; thus our discussion will be mostly concerned with the basic local groups (which are due to Dedekind).

[60] This is equivalent to saying that the completion $K_\mathfrak{p}$ is *locally linearly compact* over k.

[61] Hilbert [1894] called them the "*mehrfach überstrichene Verzweigungsgruppen.*"

[62] More precisely, Hilbert assumed that $k = \mathbb{Q}$. But it is understood that from this, the case of an arbitrary number field of finite degree is immediate.

[63] See also the literature mentioned by Ore in Dedekind's *Gesammelte mathematische Werke*, vol.2 p.48.

Hilbert had included this theory in his "*Zahlbericht*" [1897]. Hence we may safely assume that the Dedekind-Hilbert ramification theory was well known among the number theorists during that time and did not need any explanation or motivation.

4.1.1 Krull, Deuring

In the years 1930-1931 there appeared two papers almost simultaneously, which generalized Dedekind-Hilbert theory to an *arbitrary* finite Galois field extension $L|K$ and to an *arbitrary* non-archimedean prime $\mathfrak{p}$ of K, including the case when $\mathfrak{p}$ is non-discrete: Wolfgang Krull [1930c] and Max Deuring [1931a].

Each of the two authors mentions the other paper in his foreword and states that his paper was written independently. The mere fact that two authors worked on the same subject at the same time, may indicate a widespread feeling of the necessity to have Dedekind-Hilbert theory available in the framework of general valuation theory. Deuring was a student of Emmy Noether [64] and Krull too was close to the circle around her. [65] We know that Emmy Noether always freely discussed her ideas with whoever was listening to her; hence it is conceivable that both, Deuring and Krull, were stimulated by her to deal with this problem.

Both authors came to similar conclusions. It turned out that the theorems in the general case are essentially the same as in the classical case considered by Dedekind and Hilbert, at least for the basic local groups Z, T, V – with some natural modifications though. In today's terminology the following was proved.

> **1.** *All primes $\mathfrak{P}'$ of L which extend the given prime $\mathfrak{p}$ of K are conjugate to $\mathfrak{P}$ under the Galois group G. More precisely, those $\mathfrak{P}'$ are in $1-1$ correspondence with the cosets of G modulo the decomposition group Z. Consequently, the index $(G : Z)$ equals the number r of different primes of L extending $\mathfrak{p}$.*

This is precisely as in the classical case of number fields.

In the following let $\overline{K}$ and $\overline{L}$ denote the residue fields of $\mathfrak{p}$ and of $\mathfrak{P}$ respectively and $f = [\overline{L} : \overline{K}]$ the residue degree. Let f_{sep} denote its separable part, i.e., $f_{\mathrm{sep}} = [\overline{L}_{\mathrm{sep}} : \overline{K}]$. Similarly $f_{\mathrm{ins}} = [\overline{L} : \overline{L}_{\mathrm{sep}}]$ denotes the inseparable part of f.

> **2.** *$\overline{L}_{\mathrm{sep}}$ is a Galois field extension of $\overline{K}$. Each $\sigma \in Z$ induces an automorphism of $\overline{L}|\overline{K}$, and this defines an isomorphism of the factor group Z/T with the full Galois group of $\overline{L}|\overline{K}$. Consequently $(Z : T) = f_{\mathrm{sep}}$.*

In the classical case Z/T is cyclic. This reflects the fact that in the classical case $\overline{K}$ is finite and hence admits only cyclic extensions.

In the following let Γ and Δ denote the value groups of the valuations belonging to $\mathfrak{p}$ and $\mathfrak{P}$, so that $\Gamma \subset \Delta$. The factor group Δ/Γ is a torsion group; its order $e = (\Delta : \Gamma)$ is the ramification degree of $\mathfrak{P}$ over $\mathfrak{p}$. Let e' denote the part

[64] Deuring's paper [1931a] was not yet his doctoral thesis. It was written before Deuring got his Ph.D.

[65] Krull held a position as associate professor in Erlangen at the time when he wrote this paper.

of e which is relatively prime to the residue characteristic p, thus $e' = (\Delta' : \Gamma)$ where Δ' denotes the group of those elements in Δ whose order modulo Γ is prime to p. Similarly $e_p = (\Delta : \Delta')$ is the p-part of e. [66]

> **3.** *For $\sigma \in T$ and $a \in L^\times$ let $\chi_\sigma(a)$ be the residue class of $a^{\sigma-1}$. This induces a character $\chi_\sigma \in Hom(\Delta'/\Gamma, \overline{L}^\times)$, and $\sigma \mapsto \chi_\sigma$ defines an isomorphism of T/V with the full character group of Δ'/Γ. Consequently T/V is abelian and $(T : V) = e'$.*

In the classical case T/V is cyclic. This reflects the fact that in the classical case $\mathfrak{P}$ is discrete which implies $\Delta \approx \mathbb{Z}$, hence every proper factor group of Δ is cyclic. Both Krull and Deuring found it remarkable that in the general case T/V is not necessarily cyclic, and they produced examples.

> **4.** *The ramification group V is a p-group and hence solvable. Its order $|V|$ is a multiple of $e_p \cdot f_{\rm ins}$.* [67]

In the classical case, $|V| = e_p$.

Both Krull and Deuring state that $|V|$ may be a *proper* multiple of $e_p \cdot f_{\rm ins}$. But they did not introduce the quotient

$$\delta = \frac{|V|}{e_p \cdot f_{\rm ins}} = \frac{|Z|}{e \cdot f}$$

(which is a p-power) as an invariant which was worthwhile to study. It was later called by Ostrowski the *defect* of $\mathfrak{P}$ over $\mathfrak{p}$. [68] Thus we have finally

$$[L : K] = \delta \cdot e \cdot f \cdot r \qquad (4.1)$$

where r is (as above) the number of different primes $\mathfrak{P}'$ which extend $\mathfrak{p}$. [69]

Deuring gave also a definition of higher ramification groups but it seems that these, except in the discrete case where they were introduced by Hilbert, did not play a significant role in future developments. (But see below in section 4.1.2 about Herbrand's work.)

Remark. Deuring's paper contains a new proof of the fact that a prime $\mathfrak{p}$ of K can be extended to any algebraic extension L of K. Of course, this had been proved in the very first paper on valuation theory by Kürschák [1913], with the formula (2.5) giving explicitly the value of an algebraic element over K. Deuring's proof does not use the valuation function but rather the valuation ring $\mathcal{O}$ belonging to the prime $\mathfrak{p}$. Deuring first observed that $\mathcal{O}$ is a *maximal* subring of K. [70] Conversely, every maximal subring of K belongs to a prime of K. Now, given an algebraic extension $L|K$, Deuring considers a subring $\mathcal{O}_L \subset L$

[66] If the residue field is of characteristic 0 then $\Delta' = \Delta$, $e' = e$, $e_p = 1$, $V = 1$.

[67] In view of this it seems to be adequate, and is often done nowadays, to view $f_{\rm ins}$ as part of the ramification degree and, accordingly, call $f_{\rm sep}$ the residue degree.

[68] See section 5.

[69] Sometimes in the literature, the defect is denoted by d instead of δ (although the letter d is often reserved for the discriminant). Also, the number r may be denoted by g (although the letter g is often reserved for the genus). With this notation, if n denotes the field degree, formula (4.1) acquires the cute form $n = d \cdot e \cdot f \cdot g$.

[70] This depends on the fact that the value group is of rank 1. For arbitrary Krull valuations this is not true.

which is maximal with the property that $\mathcal{O}_L \cap K = \mathcal{O}$; [71] then he showed that $\mathcal{O}_L$ is a maximal subring of L, hence belongs to a prime $\mathfrak{P}$ of L extending $\mathfrak{p}$. In this proof we see clearly the influence of Emmy Noether who preferred the ring-theoretic viewpoint. The extreme simplicity and elegance of this proof stands against the fact that it is not constructive – contrary to Kürschák's proof. It should be mentioned, however, that the method of Deuring's proof is the same which today is used in the existence proof for arbitrary Krull valuations which are centered at a given prime ideal of a given ring.

Deuring does this existence proof for arbitrary algebraic extensions $L|K$ including those of infinite degree. Accordingly, statement **1.** also covers infinite Galois extensions. It seems to us that in the other statements too he would have liked to include infinite extensions but somehow hesitated to do so. Note that Krull's paper [1928] on the profinite topology of infinite Galois groups had appeared not long ago, and so Deuring may have thought that some readers of his article would not yet be acquainted with compact topological groups as Galois groups. But it is surprising that Krull too considered finite extension only; after all, he had shown the way how to handle infinite Galois extensions in the first place. Moreover, Krull did not discuss higher ramification groups which is the only instance where the generalization of Dedekind-Hilbert theory to infinite Galois extensions is not straightforward.

4.1.2 Herbrand

Soon later Jacques Herbrand filled this gap in [1932b], [1933f]. [72] He fully realized that in number theory one would have to consider infinite extensions too. His method is quite natural, namely he considers infinite extensions as limits of finite ones.

He restricts his discussion to infinite extensions of number fields. In this case the value groups of the non-archimedean valuations, although not necessarily discrete, have rational rank 1. There was one difficulty, however, which is connected with the enumeration of the higher ramification groups V_i. Consider the case where the value group Γ is discrete. If $L|K$ is of finite degree then Δ is discrete too. Let π denote a prime element for $\mathfrak{P}$ and normalize the corresponding additive valuation such that $v_{\mathfrak{P}}(\pi) = 1$. Then the i-th ramification group V_i consists by definition of those $\sigma \in V$ for which $v_{\mathfrak{P}}(\pi^{\sigma-1} - 1) \geq i$. Now let $M|K$ be a Galois subextension of $L|K$ with Galois group H which is a factor group of G. It is true that the image of V_i in H is some higher ramification group for $M|K$ but in general it is not the i-th one.

Herbrand had given in [1931b] a method how to change the enumeration of the ramification groups such as to become coherent with the projection of G onto its factor groups H. Today we would describe it by the Hasse-Herbrand function

$$\varphi(t) = \int_0^t \frac{dx}{(V_0 : V_x)}$$

where V_x is defined for real $x \geq -1$ to be V_i when i is the smallest integer $\geq x$. (Here, $x \in \mathbb{R},\ x \geq -1$.) This function $\varphi(t)$ is piecewise linear, monotonous and

[71] The existence of such $\mathcal{O}_L$ would today be proved by Zorn's Lemma; at the time of Deuring's paper he used the equivalent theorem that every set can be well ordered.

[72] These papers were published posthumously, with a preface of Emmy Noether. In August 1931 Jacques Herbrand had died in a fatal accident in the mountains.

continuous; let $\psi(u)$ denote its inverse function. Then put

$$V_t = V^{\varphi(t)}\,, \qquad \text{hence} \qquad V^u = V_{\psi(u)}\,.$$

This is called the "upper enumeration" of the ramification groups; note that $u \in \mathbb{R}$ is not necessarily an integer. With this notation, Herbrand's theorem now says the following. Consider the situation as described above, i.e., H is a factor group of G, the Galois group of a Galois subextension $M|K$ of $L|K$. Then:

> *The image of $V^u \subset G$ in the factor group H is the u-th ramification group (in upper enumeration) of $M|K$.*

Thus Herbrand's upper enumeration of the ramification groups is coherent with respect to projections $G \to H$. Hence, for an infinite Galois extension $L|K$ Herbrand can define the V^u as the *projective* limit of the corresponding ramification groups (upper enumeration) of the finite factor groups of G.

This method works for any discrete valued field K with perfect residue field, and any Galois extension $L|K$, finite or infinite. [73]

Remark: As we know today, the full power of the Dedekind-Hilbert theory unfolds itself only if infinite Galois extensions are taken into consideration. For instance, we may take for L the separable algebraic closure of K. In that case $G = G_K$ is the absolute Galois group of the field K. For an algebraic number field K of finite degree it has been shown by Neukirch [1969] that the whole arithmetic structure of K, which manifests itself by the primes $\mathfrak{p}$ of K and their interconnections, is already coded in the absolute Galois group G_K and the structure of its basic local groups belonging to the various primes. Similar result in [1969a] for the Galois group of the maximal solvable extension. In this connection we would also like to mention the work of Florian Pop who proved similar statements for arbitrary finitely generated infinite fields, not necessarily algebraic. For a survey see [1997].

The origin of all this newer development can be seen in the papers by Krull, Deuring and Herbrand of 1931–1933.

4.2 F. K. Schmidt: Uniqueness theorem for complete fields

We have mentioned above Neukirch's result about the determination of an algebraic number field K of finite degree by its absolute Galois group G_K. There are two main ingredients (among others) in Neukirch's proof: One is Dedekind-Hilbert theory of ramification for infinite algebraic extensions as discussed above. The other is the uniqueness theorem of F. K. Schmidt in [1933] which says:

> **Uniqueness theorem.** *Let K be a complete valued field and suppose that K is not algebraically closed. Then the valuation of K is unique in the sense that it is the only valuation (up to equivalence) in which K is complete. For any other valuation of K (not equivalent to the given one) its completion is algebraically closed.*

The paper carries the title "*Mehrfach perfekte Körper*" (multi-complete fields) which means "fields which are complete with respect to more than one valuation". This title is somewhat misleading because F. K. Schmidt's theorem says

[73] I. Zhukov [1998e] has given a modified method which in certain cases also works if the residue field is not perfect.

that a complete field is *never* multi-complete except in the trivial case when K is algebraically closed. [74] However the title can be understood from the original aim of F.K. Schmidt's work.

Let us cite from letters of F. K. Schmidt to Hasse. In early 1930 F. K. Schmidt (who held a position of *Privatdozent* in Erlangen) had visited Hasse (who at that time was full professor in Halle). After his return to Erlangen F. K. Schmidt wrote on Feb 14, 1930 to Hasse:

> *Was ich Ihnen über die Fragen, die wir in Halle erörterten, schreiben wollte, betrifft zweierlei: Einmal handelt es sich um den Satz: Ist der Körper K hinsichtlich der diskreten Bewertung v perfekt, so ist v*
>
> 1. *die einzige diskrete Bewertung von K ,*
> 2. *die einzige Bewertung von K hinsichtlich der K perfekt ist...*
>
> Regarding the questions which we discussed in Halle, I wanted to write you two things: For one, there is the theorem: If a field K is complete with respect to a discrete valuation v then v is
>
> 1. the only discrete valuation of K ,
> 2. the only valuation of K with respect to which K is complete...

Here, "only" is meant to be "only up to equivalence".

From this letter it appears that F.K. Schmidt started this work on the suggestion of Hasse. In any case, this work was done parallel to his joint work with Hasse, determining the structure of complete discrete valued fields [75], and in this connection it was of course of high interest to know whether a field could be complete with respect to two different valuations.

At the end of the letter, after sketching his proof of the above statements 1. and 2., F.K. Schmidt says:

> *Ich möchte mir nun noch überlegen, ob ein Körper hinsichtlich zweier verschiedener Bewertungen, die dann natürlich nicht diskret sind, perfekt sein kann, und falls das möglich ist, die Beschaffenheit eines solchen Körpers kennzeichnen.*
>
> Next I wish to investigate whether a field could be complete with respect to two different valuations, which of course cannot be discrete. And if this is possible then I want to characterize the structure of such field.

We conclude that F.K. Schmidt apparently did not have a conjecture what to expect for non-discret valuations and, indeed, he was looking for a characterization of those fields which are complete with respect to two different valuations, expecting possibly non-trivial instances of this situation – as the title of the paper suggests.

Already one month later, on March 29, 1930, F.K. Schmidt writes that he has solved the problem; he states his theorem as given above and sends a corresponding manuscript to Hasse. [76]

[74] An algebraically closed field whose cardinality is sufficiently large is always multi-complete.

[75] See section 4.3. This topic was the second of the two things which F. K. Schmidt wanted to discuss in his letter to Hasse.

[76] It took more than 2 years until F. K. Schmidt submitted the final version for publication. From the correspondence it appears that he was frequently changing and rewriting his manuscript until finally it was in a form which satisfied his sense of elegance.

The main part of F. K. Schmidt's proof is based on Hensel's Lemma, hence is valid for arbitrary Henselian fields. The completeness property is used only in the proof of the following lemma:

If a complete field K is separably closed then it is algebraically closed. [77]

Indeed, for $a \in K^{\times}$ consider the separable polynomials $x^p - cx - a$ where p is the characteristic and the parameter $c \in K^{\times}$. If c converges to 0 then the roots of $x^p - cx - a$ converge to the p-th root of a and therefore $a^{1/p} \in K$.

All the other arguments in F. K. Schmidt's proof are valid for arbitrary Henselian fields. Thus his proof in [1933] yields the following theorem:

The above uniqueness theorem remains valid if "complete" is replaced by "Henselian", and "algebraically closed" by "separably closed". [78]

However, F. K. Schmidt does not use the notion of Henselian field; this notion appeared only in Ostrowski's paper [1934] under the name of "relatively complete" field (see section 5).

We have mentioned the uniqueness for Henselian fields because it was stated and proved by Kaplansky and Schilling in [1942]. They reduced it to the uniqueness theorem for complete fields, citing F. K. Schmidt's paper [1933] – apparently without noticing that F. K. Schmidt's proof itself yields the result for Henselian fields too. [79]

4.2.1 The Approximation Theorem

Today, the proof of F. K. Schmidt's uniqueness theorem is usually presented with the help of the following

Approximation Theorem. *Let K be a field and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be different primes of K, with valuations $\| \cdot \|_1, \ldots, \| \cdot \|_r$ respectively. Given any elements $a_1, \ldots, a_r \in K$ there exists $x \in K$ which approximates each a_i arbitrarily close with respect to $\mathfrak{p}_i$. This means that*

$$\|x - a_i\|_i < \varepsilon_i \qquad (1 \le i \le r)$$

for arbitrary prescribed numbers $\varepsilon_i \in \mathbb{R}$.

In order to deduce the uniqueness theorem from this, consider a field K which is Henselian with respect to two different primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. One has to prove that K is separably closed, i.e., that every separable polynomial splits completely over K. Consider two monic separable polynomials $f_1(X), f_2(X)$ of the same degree, say n. From the approximation theorem we conclude that there exists a monic polynomial $g(X)$ of degree n which approximates $f_1(X)$ with respect to $\mathfrak{p}_1$ and $f_2(X)$ with respect to $\mathfrak{p}_2$, coefficientwise and arbitrarily close. Now from Hensel's Lemma (for $\mathfrak{p}_1$) it follows that after sufficiently close $\mathfrak{p}_1$-approximation,

[77] As mentioned in section 2.2.1 this result had been already obtained by Ostrowski in his first paper [1913a]. F. K. Schmidt uses the same argument as did Ostrowski.

[78] According to F. K. Schmidt's proof, this theorem includes the case of a field with an archimedean valuation, if in such case the property "Henselian" is interpreted as being isomorphic to a real closed or algebraically closed subfield of $\mathbb{C}$.

[79] Kaplansky and Schilling consider non-archimedean valuations only but F. K. Schmidt's theorem includes archimedean valuations too, as pointed out above already.

$g(X)$ has the same splitting behavior over K as does $f_1(X)$. Similarly for $\mathfrak{p}_2$, and so $g(X)$ has also the same splitting behavior as $f_2(X)$. We conclude that every two monic separable polynomials $f_1(X), f_2(X) \in K[X]$ of the same degree have the same splitting behavior over K. But certainly there do exist, for any degree n, separable monic polynomials $f(X)$ which split completely. One has to choose n different elements $a_i \in K$ and put $f(X) = \prod_{1 \leq i \leq n}(X - a_i)$.

However, F. K. Schmidt did not use the general approximation theorem, perhaps because he was not aware of its validity. Instead, he gave an ad hoc construction for the approximating polynomial $g(X)$ to be able to conclude as we have done above.

Remark. According to our knowledge, the first instance where the approximation theorem had been formulated and proved, including archimedean primes, was the Artin-Whaples paper [1945]. Hasse reproduced the proof in the second edition of his textbook "Zahlentheorie". In the first edition, which had been completed in 1938 already, he proves the approximation theorem for algebraic number fields and algebraic function fields only. More precisely: He first verifies the theorem for $\mathbb{Q}$ and for the rational function field $k(X)$ over a field, and then proves: If the approximation theorem holds for K and its primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ then it holds for any finite algebraic extension field L of K with respect to the finitely many primes of L which are extending some $\mathfrak{p}_i$.

From F. K. Schmidt's arguments in [1933] a proof of the approximation theorem can be extracted for two primes, i.e., $r = 2$. This is essentially the start of Artin-Whaples' induction proof for arbitrary r.

If only non-archimedean primes are concerned, the theorem was formulated and proved by Ostrowski in [1934] already. If one follows Ostrowski's argument carefully then one can see that his proof remains valid if archimedean primes are included. But he did not formulate his result in this generality.

Even before Ostrowski's paper was published, Krull in [1930c] had given a proof. Ostrowski cites Krull but observes that Krull formulated only two special cases. This is true but it is clear that from those special cases the general theorem of Ostrowski follows immediately.

Deuring in [1931a] did not formulate the approximation theorem in general, but his arguments implicitly contain a proof in the situation which he considers: namely for a finite Galois extension $L|K$ and the finitely many primes $\mathfrak{P}$ of L which extend a given non-archimedean prime $\mathfrak{p}$ of K.

According to Ostrowski's words, his manuscript for [1934] had been essentially completed in 1916 already, much earlier than the papers by Krull, Deuring, F. K. Schmidt and Artin-Whaples.

The essential arguments in all those proofs are similar. So, who should be credited to have been the first one for proving the approximation theorem?

One should also recall that in the case of the rational number field, the approximation theorem is essentially equivalent to what is called the "Chinese remainder theorem" which has been known to Sun-Tzǐ in the 3rd century.

Perhaps it is best to keep the name "Artin-Whaples' approximation theorem" which Hasse has used.

4.3 Structure of complete discrete fields.

4.3.1 The approach of Hasse and F. K. Schmidt

Let us start with a free translation of the first sentences of the paper [1933c] by Hasse and F. K. Schmidt.

In modern algebra one can observe two lines of thought. The first one is the tendency towards axiomatization and generalization, in order to understand the mathematical phenomena as being part of a general theory which depends on a few simple, far reaching hypotheses only. The second one is the desire to characterize the structures which are formed by those abstract axioms and notions, thus returning from the general to the possible special cases. The first is often called the abstract *or* formal *point of view while the second can be described as the* concrete *viewpoint in the framework of modern algebra.*

The present paper belongs to the second of those ideas. The authors start from the general notion of complete valued field in the sense of Kürschák, which had evolved from Hensel's investigations of p-adic numbers. Now they are going to characterize all possible complete valued fields, at least in the case of discrete valuations.

From these words it appears that the authors regarded their work as being fundamental, and as part of the general development of what at that time was called "modern algebra". The introduction is very elaborate, spanning over 14 pages. The whole paper has 60 pages. [80] Its aim is to present an explicit description of all complete discrete valued fields K.

But it was not the last word on the subject. Three years later there appeared the papers by Teichmüller and Witt [1936d], [1937a], [1937b] which contained great simplifications. Other simplifications were given by Mac Lane [1939]. Today the paper of Hasse and F. K. Schmidt is almost forgotten, being superseded by those of Witt, Teichmüller and Mac Lane. In particular the so-called Witt vector calculus has obtained universal significance. The Hasse-F. K. Schmidt paper had served as a starter for this development.

There are two cases to be considered: First, the equal characteristic case, where the characteristic of the complete field K equals the characteristic of its residue field $\overline{K}$, and secondly the unequal characteristic case, where $\mathrm{char}(K) = 0$ and $\mathrm{char}(\overline{K}) = p > 0$. Hasse-F. K. Schmidt and also Teichmüller try to deal with both cases simultaneously, as far as possible. But here, for better understanding, let us discuss these two cases separately. We begin with equal characteristic case.

The cooperation of Hasse and F. K. Schmidt for the paper [1933c] started in early February 1930 when, as reported in section 4.2 already, F. K. Schmidt visited Hasse in Halle. On that occasion Hasse told him that he was working on the classification of complete discrete valued fields. After his return to Erlangen F. K. Schmidt asked Hasse, in a letter of Feb 14, 1930, to send him more details. Hasse did so and on Feb 23, F. K. Schmidt thanked him for his "beautiful proof" (*schönen Beweis*).

Apparently Hasse's proof concerned only the equal characteristic case. For F. K. Schmidt, in his reply of Feb 23, refers to this case only. In addition he

[80] This length is partly caused by the fact that the authors include a detailed explanation of the fundamental notions of general valuation theory, starting from Kürschák. By the way, the editing of the text of the paper was done by F. K. Schmidt. This is unusual since Hasse usually was very careful in the wording of his papers and, accordingly, mostly undertook himself the writing and editing of joint papers.

observes:

> *Bei der Lektüre fiel mir sogleich auf, daß der wesentliche Teil Ihres Satzes von der Voraussetzung, die Bewertung sei diskret, ganz unabhängig ist.*
>
> On the first reading I observed that the essential part of your theorem is independent of the hypothesis that the valuation is discrete.

And he proceeded to formulate the statement which he (F. K. Schmidt) thought he was able to prove:

> *If the complete field K and its residue field $\overline{K}$ have the same characteristic then K contains a subfield K which represents the residue field $\overline{K}$ – irrespective of whether the valuation is discrete or not.*

In today's terminology:

> *Let $\mathcal{O}$ denote the valuation ring of K. Then the residue map $\mathcal{O} \to \overline{K}$ admits a section $\overline{K} \xrightarrow{\approx} \mathsf{K} \subset \mathcal{O}$.*

F. K. Schmidt adds that in order to prove this statement for arbitrary, not necessarily discrete valuations, one has to modify Hasse's arguments in the case of inseparabilities but otherwise Hasse's proof can be used word for word. We do not know Hasse's proof but the above seems to indicate that Hasse considered only the case of residue fields of characteristic zero where there are no inseparabilities. Or, maybe in characteristic p he considered finite residue fields only which of course would be the most interesting case for number theory.

Also we do not know what F. K. Schmidt had in mind when he mentioned "modifications in case of inseparabilities". It seems to us that he mentioned this somewhat in haste but later, in the course of time, he realized that dealing with inseparabilities was more intricate than he originally had thought. For, in a later letter dated March 29, 1930, he mentions briefly that there has appeared a new difficulty which however he hopes to overcome.

Let us review the situation: The residue field $\overline{K}$ is said to be *separably generated* if there exists a transcendence basis $\overline{T} = (\bar{t}_i)$ such that $\overline{K}$ is separable algebraic over $\mathbb{F}_p(\overline{T})$. If this is the case then a section $\overline{K} \to \mathcal{O}$ can be constructed as follows: For each element $\bar{t}_i$ of the transcendence basis choose an arbitrary foreimage $t_i \in \mathcal{O}$ and let $T = (t_i)$; then the assignment $\overline{T} \mapsto T$ defines an isomorphism $\mathbb{F}_p(\overline{T}) \xrightarrow{\approx} \mathbb{F}_p(T)$, hence a section $\mathbb{F}_p(\overline{T}) \to \mathcal{O}$. Because of Hensel's Lemma this has a *unique* prolongation to a section $\overline{K} \to \mathcal{O}$. For, every $\overline{a} \in \overline{K}$ is a zero of a monic *separable* polynomial $\overline{f}(X) \in \mathbb{F}_p(\overline{T})[X]$; let $f(X)$ denote the image of that polynomial in $\mathbb{F}_p(T)[X]$. By Hensel's Lemma there exists a *unique* foreimage $a \in \mathcal{O}$ of $\overline{a}$ such that $f(a) = 0$. The assignment $\overline{a} \mapsto a$ (for $\overline{a} \in \overline{K}$) then yields a section $\overline{K} \to \mathcal{O}$.

We see: *If the residue field $\overline{K}$ is separably generated then indeed, there exists a section $\overline{K} \to \mathcal{O}$ of the residue map $\mathcal{O} \to \overline{K}$, and this does not depend on the structure of the value group.*

Concerning the extra hypothesis about $\overline{K}$ being separably generated, we read in the Hasse-F. K. Schmidt paper [1933c]:

> *... Das ist im charakteristikgleichen Fall zuerst von dem Älteren von uns [Hasse] erkannt worden, daran anschließend allgemein von dem*

Jüngeren [Schmidt], wobei sich überdies die Möglichkeit ergab, die diskrete Bewertung von K durch eine beliebige Exponentenbewertung zu ersetzen, hinsichtlich der nur K perfekt sein muß.

...In the case of equal characteristic, this has been discovered first by the elder of us [Hasse], subsequently in general by the younger [Schmidt], whereby it turned out to be possible to replace the discrete valuation of K by an arbitrary non-archimedean valuation, with the only condition that K is complete.

This seems to indicate that indeed, Hasse in his letter to F. K. Schmidt had discussed the equal characteristic case only, as we have suspected above already. Moreover we see that F. K. Schmidt now has taken back his former assertion in his letter of Feb 23 to Hasse, and that he has realized that this method works only for *separably generated* residue fields . Under this extra hypothesis it works not only for discrete valuations, but generally.

But perhaps every field is separably generated? Certainly this is true in characteristic 0; in this case the method of Hasse-F. K. Schmidt is quite satisfactory. In the following discussion let us assume that the characteristic is $p > 0$.

If the field is finitely generated (over its prime field) then F. K. Schmidt himself had proved in [1931c] that there exists a separating basis of transcendency. [81] But for arbitrary fields this seemed at first to be doubtful. In the next letters after Feb 23, 1930 this problem is not mentioned. Only in a later letter, of Sep 14, 1930, F. K. Schmidt returns to this question. In the meantime he had met Hasse in Königsberg at the meeting of the DMV. [82] They had done a walk together at the beach of the Baltic sea and discussed their joint paper in progress. Now, after F. K. Schmidt's return to Erlangen, he writes:

Nun muß ich Ihnen leider doch hinsichtlich der Franzschen Vermutung in der Theorie der unvollkommenen Körper etwas sehr Schmerzliches schreiben... Ich kann nämlich nun durch ein Beispiel zeigen, daß die Franzsche Vermutung tatsächlich bereits bei endlichem Transzendenzgrad nicht zutrifft. Genauer: ich kann einen Körper K von Primzahlcharakteristik p angeben mit folgenden Eigenschaften:
1. K ist von endlichem Transzendenzgrad n über seinem vollkommenem Kern k . 2. Ist $t_1, \dots t_n$ irgendeine Transzendenzbasis von $K|k$, so ist stets $K|k(x_1, \dots, x_n)$ von zweiter Art, wie man auch die Transzendenzbasis wählen mag.

Unfortunately I have to write you something painful with respect to the conjecture of Franz ... For, I am now able to show by an example that the conjecture of Franz does not hold even for finite degree of transcendency. More precisely: I can construct a field K of prime characteristic p with the following properties:
1. K is of finite transcendence degree over its perfect kernel k. 2. If $t_1, \dots t_n$ is any transcendence basis of $K|k$ then $K|k(x_1, \dots, x_n)$ is

[81] Actually, he proved it for transcendence degree 1 only but his proof can be modified for arbitrary finite transcendence degree. See, e.g., van der Waerden's textbook [1930e].

[82] DMV = *Deutsche Mathematiker Vereinigung* (German Mathematical Society). The annual DMV-meeting in Königsberg was scheduled in the first week of September, 1930. At that meeting, F. K. Schmidt gave a talk on his results about his uniqueness theorem for complete fields; see section 4.2. Hasse's talk was about the arithmetic in skew fields; see section 3.4.1.

always of the second kind, regardless of how one chooses the transcendence basis. [83]

From this and other parts of the letter it appears that:

- Franz [84] had written a manuscript in which he conjectured that any field of finite transcendence degree is separably generated,
- Hasse had told this to F. K. Schmidt in Königsberg,
- F. K. Schmidt had generalized this on the spot to arbitrary transcendence degree,
- after having returned to Erlangen F. K. Schmidt found an error in Franz' as well as in his own argument,
- and he got a counter example to Franz' conjecture.

But for the residue field of a *discrete* complete valued field, F. K. Schmidt apparently found a way to overcome that difficulty. In the published paper [1933c] a method is presented to construct a section $\overline{K} \xrightarrow{\approx} \mathsf{K} \subset \mathcal{O}$, regardless of whether $\overline{K}$ is separably generated or not. Consequently, if π is a prime element of K then it follows that $K = \mathsf{K}((\pi))$, the power series ring over K. This leads to the following theorem of the Hasse-F. K. Schmidt paper [1933c]. [85]

> **Structure Theorem (equal characteristic case)**. *Every complete discrete valued field K with* $\mathrm{char}(K) = \mathrm{char}(\overline{K})$ *is isomorphic to the power series field $\overline{K}((X))$ over its residue field $\overline{K}$.*

F. K. Schmidt's construction is somewhat elaborate and not easy to check. Indeed, it seems that nobody (including his co-author Hasse) did check the details at the time, and his proof was generally accepted. The first who really worked on the proof and checked the details seems to have been Mac Lane. In [1939] he pointed out an error in the proof. (This error concerned both the equal and the unequal characteristic case). Mac Lane politely speaks of an "unproven lemma", and he gives a new proof of the Structure Theorem (in both cases), not depending on that lemma. We shall discuss Mac Lane's proof in section

[83] F. K. Schmidt still uses the old terminology "first kind" and "second kind". In the published paper [1933c] already the new terminology "separable" and "inseparable" is used.

[84] Wolfgang Franz was a student of Hasse. He obtained his Ph.D. 1930 at the University of Halle. The subject of his thesis [1931d] was about Hilbertian fields, i.e., fields in which Hilbert's irreducibility theorem holds. One of Franz' results was that a separably generated function field of one or several variables over a base field, is Hilbertian. This explains Franz' interest in separably generated fields. The thesis of Franz initiated a new direction of research, namely the investigation of the Hilbert property of fields in connection with valuation theory. (For a survey see e.g., Jarden's book [1986] on Field Arithmetic.) – Franz edited the "*Marburger Vorlesungen*" of Hasse on class field theory [1933g]. Later he left number theory and algebra, and started to work in topology.

[85] If the residue field $\overline{K}$ is finite then the following theorem, as well as the corresponding structure theorem in the unequal characteristic case, is contained in the Groningen thesis of van Dantzig [1931e] already. This is acknowledged in a footnote of the Hasse-F. K. Schmidt paper [1933c]. There the authors say that the essential difficulties which appear in their paper do not yet show up if the residue field is finite. Indeed, from today's viewpoint the structure theorems of Hasse-F. K. Schmidt are almost immediate in case of finite residue field. But it should be mentioned that the thesis of van Dantzig contained much more than this, it was the first attempt of a systematic development of topological algebra.

4.3.4 below. Here, let us mention only that the error occurred in the handling of inseparabilities of the residue field; this concerns precisely the difficulty which F. K. Schmidt had mentioned in his letters to Hasse.

The "unproven lemma" turned out to be false but a certain weaker result was already sufficient to carry through the construction of F. K. Schmidt. Mac Lane planned to write a joint paper with F. K. Schmidt, proving that weaker result and thus correcting the error in [1933c]. Their paper was to appear in the *Mathematische Zeitschrift.* [86] But due to the outbreak of world war II their contact was cut off, and so Mac Lane single-handedly wrote a brief report in [1941], naming F. K. Schmidt as co-author and stating the correct lemma which when used in [1933c] saves F. K. Schmidt's construction. Today this note as well as F. K. Schmidt's construction is forgotten since, as said above, it is superseded by the results of Witt and Teichmüller, and later those of Mac Lane.

The difficulties which F. K. Schmidt faced while struggling with the details of his proof can be measured by the time needed for completion of the manuscript. The authors had started their work on this topic in February 1930, and in November that year F. K. Schmidt sent to Hasse the first version. But some time later he decided to write a new version. There followed a series of exchanges back and forth of this manuscript. In October 1931 F. K. Schmidt sent the first part of another version to Hasse who had wished to include the paper into the Crelle volume dedicated to Kurt Hensel. But it was too late for inclusion since the Hensel dedication volume was scheduled to appear in December 1931 already. [87] The published version carries the date of receipt as April 24, 1932. But even after that date there were substantial changes done. In a letter dated July 11, 1932 F. K. Schmidt announced that now the whole paper has to be rewritten. And even on April 4, 1933 F. K. Schmidt inquired whether it is still possible to make some changes in the manuscript. Finally, in an undated letter (written in May or June 1933) F. K. Schmidt gave instructions for the last and final changes while Hasse was reading the galley proofs of the manuscript.

4.3.2 The Teichmüller character

We are now going to describe Teichmüller's beautiful construction in [1936d] which superseded the Hasse-F. K. Schmidt construction.

Consider first the case when the residue field $\overline{K}$ is perfect. Its characteristic is a prime number p. Given an element α in the residue field $\overline{K}$, Teichmüller chooses, for $n = 0, 1, 2, \ldots$, an arbitrary representative a_n of $\alpha^{p^{-n}}$. Then each $a_n^{p^n}$ is a representative of the given α. Teichmüller [1936d] observed that

$$\chi(\alpha) := \lim_{n\to\infty} a_n^{p^n} \tag{4.2}$$

exists and is independent of the choice of the a_n. (Here he used the fact that

[86] They had discussed this plan when F. K. Schmidt visited the United States in 1939 as a representative of the Springer Verlag, in connection with the American plans to establish *Mathematical Reviews*. See [1979a]. – It is to be assumed that Mac Lane and F. K. Schmidt had met earlier between 1931 and 1933, when Mac Lane studied in Göttingen.

[87] It seems that F. K. Schmidt was not unhappy with this outcome because he wished to make still other changes. Hasse, on the other hand, had in the meantime (November 9, 1931) succeeded, together with R. Brauer and E. Noether, to prove the Local-Global Principle for algebras, and so he put that new manuscript into the Hensel dedication volume instead.

the valuation is discrete.) It is seen that

$$\chi(\alpha \cdot \beta) = \chi(\alpha) \cdot \chi(\beta) . \tag{4.3}$$

The multiplicative map $\chi : \overline{K} \to \mathcal{O}$ is called the *Teichmüller character*; its image $\mathfrak{R} = \chi(\overline{K})$ is the *Teichmüller representative set* for the residue map. It is the only representative set for which $\mathfrak{R}^p = \mathfrak{R}$.

This construction works in both cases, the equal characteristic case and the unequal characteristic case. In the equal characteristic case we have besides of (4.3) the corresponding formula for the addition:

$$\chi(\alpha + \beta) = \chi(\alpha) + \chi(\beta) . \tag{4.4}$$

Therefore $\mathfrak{R}$ is a field and $\chi : \overline{K} \xrightarrow{\approx} \mathfrak{R}$ is a section.

The beauty of this construction is not only its simplicity but also that it is canonical.

If $\overline{K}$ is not perfect then Teichmüller uses the theory of p-bases of $\overline{K}$ which he had introduced in a recent paper in another context [1936e]. Choose a p-basis $\overline{M}$ of $\overline{K}$ and a foreimage $M \subset \mathcal{O}$; then consider the union of the fields $K(M^{p^{-n}})$, $(n = 0, 1, 2, \ldots)$. Its completion L is a discrete valued complete field whose residue field $\overline{L}$ is the perfect hull of $\overline{K}$. This then allows to reduce the problem to the case of perfect residue fields. More precisely: The Teichmüller character $\chi : \overline{L} \to L$ maps $\overline{M}$ into M and $\overline{K}$ into K.

In [1937b] Teichmüller gives a more detailed (and in some points corrected) version of this construction. Note that the construction of this section $\overline{K} \to K$ is not canonical but depends on the choice of the foreimage M of a p-basis of $\overline{K}$. Nevertheless it is quite simple and straightforward when compared with the Hasse-F. K. Schmidt construction [1933c], even when the latter is taken in its corrected form [1941].

4.3.3 Witt vectors

Next we consider the unequal characteristic case, which means that $\mathrm{char}(K) = 0$ and $\mathrm{char}(\overline{K}) = p > 0$. In this situation K is called *unramified* if p is a prime element of K. The main result of Hasse-F. K. Schmidt for unramified fields is as follows:

> **Structure Theorem (unequal characteristic case)** *An unramified, complete discrete field K of characteristic 0 and residue characteristic $p > 0$ is uniquely determined by its residue field $\overline{K}$ (up to isomorphism). For any given field in characteristic p there exists an unramified, complete discrete field whose residue field is isomorphic to the given one.*

Again, the proof given by Hasse-F. K. Schmidt in [1933c] is quite elaborate and not easy to check. The error which Mac Lane had pointed out in [1939] and corrected in [1941], concerns the unequal characteristic case too.

We now discuss the papers [1936e] and [1937] by Teichmüller and by Witt which greatly simplified the situation.

Similar as in the equal characteristic case, suppose first that $\overline{K}$ is perfect, so that the Teichmüller character $\chi : \overline{K} \to \mathcal{O}$ is defined as in (4.2), with the multiplicative property (4.3). (Recall that $\mathcal{O}$ denotes the valuation ring of K.)

Again let $\mathfrak{R} = \chi(\overline{K})$. Since we are discussing the unramified case, p is a prime element of K and therefore every element $a \in \mathcal{O}$ admits a unique expansion into a series

$$a = a'_0 + a'_1 p + a'_2 p^2 + \cdots \qquad \text{with} \qquad a'_n \in \mathfrak{R}\,.$$

Since $\mathfrak{R} = \mathfrak{R}^p$ there are $a_n \in \mathfrak{R}$ with $a'_n = a_n^{p^n}$. We obtain an expansion of the form

$$a = a_0 + a_1^p p + a_2^{p^2} p^2 + \cdots \qquad \text{with} \qquad a_n \in \mathfrak{R}\,. \tag{4.5}$$

Let a_n correspond to $\alpha_n \in \overline{K}$ via the Teichmüller character, i.e., $a_n = \chi(\alpha_n)$. We see that a is uniquely determined by the vector

$$W(a) = (\alpha_0, \alpha_1, \alpha_2, \ldots) \tag{4.6}$$

with components α_n in the residue field $\overline{K}$.

There arises the question how the components of $W(a \pm b)$ are computed from the components of $W(a)$ and $W(b)$. And similar for $W(a \cdot b)$. Teichmüller [1936e] and also H. L. Schmid [1937c] had discovered that this computation proceeds via certain polynomials with coefficients in $\mathbb{Z}$, not depending on the residue field $\overline{K}$ but only on p, its characteristic.

At this point the discovery of Witt becomes crucial. He could show that those polynomials can be obtained by a universal algorithm which is easy to describe and easy to work with. Thus Witt gave an explicit, canonical construction of a complete discrete valuation ring $W(\overline{K})$ of characteristic 0, based on the given field $\overline{K}$ of characteristic p, and he showed that $\mathcal{O} \approx W(\overline{K})$. [88]

In my opinion, in order to give a brief description of the algorithms of Witt vectors, the best way to do so is to cite his words which he himself used in his seminal paper [1937a]. We do this in a free English translation:

For a vector

$$x = (x_0, x_1, x_2, \ldots)$$

with countably many components x_n *we introduce* ghost components [89]

$$x^{(n)} = x_0^{p^n} + p\, x_1^{p^{n-1}} + \cdots + p^n x_n\,.$$

Inversely, x_n *can be expressed as a polynomial in* $x^{(0)}, x^{(1)}, \ldots, x^{(n)}$ *with rational coefficients, and therefore the vector* x *is already determined by its ghost components. We define Addition, Subtraction and Multiplication of two vectors via the ghost components:*

$$(x \mathbin{\dot{\pm}} y)^{(n)} = x^{(n)} \mathbin{\dot{\pm}} y^{(n)} \qquad\qquad (n = 0, 1, 2, \ldots).$$

Satz 1. $(x \mathbin{\dot{\pm}} y)_n$ *is a polynomial in* $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_n$ *with integer coefficients.*

In view of his "Satz 1" Witt observes that one can substitute for the indeterminates $x_0, x_1, \ldots, y_0, y_1, \ldots$ elements of an arbitrary ring A and obtain a new ring $W(A)$, the ring of Witt vectors over A. In particular for $A = \overline{K}$ we obtain $W(\overline{K})$. Witt proves this to be a complete discrete valuation ring of characteristic 0, with p as prime element. Witt shows further that the map $a \mapsto W(a)$ defined by (4.5), (4.6) is an isomorphism of the valuation ring $\mathcal{O}$ of K with $W(\overline{K})$. Thus:

[88] Perhaps it is not superfluous to point out that the ring $W(\overline{K})$ of Witt vectors has nothing to do with what is called the "Witt ring" of the field $\overline{K}$ which is defined in the theory of quadratic forms.

[89] Witt says "*Nebenkomponenten*".

Witt's Theorem: *Every unramified complete discrete valuation ring $\mathcal{O}$ of characteristic 0 with perfect residue field $\overline{K}$ of characteristic $p > 0$ is isomorphic to the Witt vector ring $W(\overline{K})$.*

Hence the quotient field K of $\mathcal{O}$ is isomorphic to the quotient field $QW(\overline{K})$ of $W(\overline{K})$. We see that $QW(\overline{K})$ plays the same role in the unequal characteristic case, as the power series field $\overline{K}((X))$ does in the equal characteristic case.

If the residue field $\overline{K}$ is not perfect then $W(\overline{K})$ is not a valuation ring. But Teichmüller in [1937b] was able to prove the structure theorem also in this case, with the help of a construction using a p-basis of $\overline{K}$, similarly as in the equal characteristic case.

Finally, if K is ramified (in the unequal characteristic case) then it is shown that $QW(\overline{K})$ is canonically contained in K, and that K is a finite algebraic, purely ramified extension of $QW(\overline{K})$, and that it can be generated by a root of an Eisenstein equation. This is routinely derived from what we have seen above.

REMARK 1. The calculus of Witt vectors has implications throughout mathematics, not only in the construction of complete discrete valuation rings. In fact, it was first discovered by Witt in connection to another problem, namely the generalization of the Artin-Schreier theory to cyclic field extensions of characteristic p and degree p^n. This is explained in Witt's paper [1937a]. Later, Witt generalized his vector calculus such as to become truly universal, i.e., not referring to one particular prime number p. Instead of considering the powers $p^0, p^1, p^2, \ldots$ of one prime number p, he considers those of all prime numbers and their products, which is to say the natural numbers $1, 2, 3, \ldots$. The ghost components of a vector

$$x = (x_1, x_2, x_3, \ldots)$$

are now defined to be

$$x^{(n)} = \sum_{d|n} d \cdot x_d^{n/d} .$$

This is a truly universal calculus, and the Witt vectors in the sense of his original paper [1937a] represent, in a way, only a certain part (not a subring!) of the universal Witt vectors. See, e.g., Witt's article which is published posthumously in his Collected Works [1998f] and the essay of Harder therein. Certainly, the discovery of Witt vectors is to be rated as one of the highlights of mathematics in the 20th century – independent of the application to valuation theory which we have discussed here.

REMARK 2. Both the papers of Witt [1937] and of Teichmüller [1937b] were published in the same issue of Crelle's Journal, namely *Heft 3, Band 176*. [90] This whole issue consists of 8 papers by Witt, Teichmüller, H. L. Schmid and Hasse. The latter, as the editor of Crelle's Journal, had opened this issue for the *Göttinger Arbeitsgemeinschaft* (workshop) in which, as we can see, remarkable results had been achieved. [91]

[90] At that time, a volume of Crelle's Journal consisted of 4 issues (*4 Hefte*).

[91] The names of the members of the *Arbeitsgemeinschaft* in the years 1935/36 are mentioned in [1936d]. The two most outstanding members (besides of Hasse) were Teichmüller and Witt. In a letter of Hasse to Albert dated Feb 2, 1935, Hasse mentioned "*Dr. Witt, our best man here*". About Teichmüller he wrote to Davenport on Feb 3, 1936, that he (Hasse) was working with Teichmüller on cyclic fields of degree p. And he added: "*His paper on the Wachs-Raum is as queer as the whole chap.*"

4.3.4 Mac Lane

Saunders Mac Lane (1909–) studied from 1931 to 1933 in Göttingen where he came in contact with "modern" mathematics and algebra. His Ph.D. thesis was written under the supervision of Bernays, on a topic from mathematical logic. [92] Here we are concerned with his contributions to valuation theory. Kaplansky tells us in [1979a] that Mac Lane's interest in valuation theory can be traced quite directly to the influence of Oystein Ore at Yale, where Mac Lane had studied in 1929/1930 and to where he returned in 1933/1934 after his Göttingen interlude.

Already in section 4.3.1 we have mentioned the paper [1939] of Mac Lane where it was pointed out that the Hasse-F. K. Schmidt paper [1933c] contained an error, and where a new, correct proof of the Structure Theorem of Hasse-F. K. Schmidt was presented. But why was it necessary in the year 1939 to give a new proof? After all, a beautiful new proof had already been given in 1937 by Teichmüller and Witt, as reported in the foregoing sections 4.3.2 and 4.3.3.

Mac Lane mentions that Witt [1937a] uses a "*sophisticated vector analysis construction*"; he wants to avoid Witt vectors and prove the Structure Theorem by more elementary means. He does not attempt, however, to avoid the use of the Teichmüller character; it is only the Witt vectors which he wishes to circumvent. Hence Mac Lane discusses the unequal characteristic case only, since the equal characteristic case had been solved by Teichmüller without Witt vectors, as reported in section 4.3.2.

Mac Lane's new idea is to divide the proof into two steps: first the existence proof, i.e., the construction of a discrete valued unramified complete field of characteristic 0 with a given residue field of characteristic p; second, the uniqueness proof, i.e., the construction of an isomorphism between two such fields with the same residue field. The separation of these two steps shows, Mac Lane says, "*that the previous constructions* [he means those by Witt] *have been needlessly involved and can be replaced by an elementary stepwise construction*".

The existence part is easily dealt with: the given field of characteristic p which is to become the residue field, is an extension of the prime field $\mathbb{F}_p$ of characteristic p which in turn is the residue field of $\mathbb{Q}_p$. Therefore one has to verify the following general

> **Relative Existence Theorem.** *Let k be a non-archimedean complete valued field with residue field $\overline{k}$, and let $\overline{K}$ be an extension field of $\overline{k}$. Then there exists a complete unramified extension K of k whose residue field is the given $\overline{K}$.*

By Zorn's Lemma [93] it suffices to discuss the case when $\overline{K} = \overline{k}(\alpha)$ is a simple extension. If α is algebraic over $\overline{k}$ let $\overline{f}(X)$ be the monic irreducible polynomial for α over $\overline{k}$, and let $f(X) \in k[X]$ be a monic foreimage of $\overline{f}(X)$. Then $f(X)$ is irreducible over k. If a is a root of $f(X)$ then the field $K = k(a)$ solves the problem; the valuation of k extends uniquely to K and K is complete by the very first results of Kürschák and Ostrowski. Moreover, $K|k$ is unramified since $[K : k] = [\overline{K} : \overline{k}]$. This holds regardless of whether $\overline{f}(X)$ is separable or not. On

[92]Biographical information about Mac Lane, who became one of the leading mathematicians in the United States, can be found in his "Selected Papers" [1979b].

[93]Mac Lane does not yet use Zorn's Lemma of [1935b]; he refers to "known" methods using well orderings.

the other hand, if α is transcendental over $\overline{k}$ then consider the rational function field $k(x)$ in an indeterminate x over k, with its functional valuation. (For this see page 58.) The completion K of $k(x)$ then solves the problem.

We see that this existence part holds not only for discrete valuations; the arguments are valid quite generally and they do not give any problem with respect to inseparabilities of the residue field. And they were well known even at the time of Mac Lane. Thus the main point of Mac Lane's paper is the second part about uniqueness. Here he shows the following:

> **Relative Uniqueness Theorem.** *Let k be a complete valued field and assume that the valuation is discrete. Let K and K' be two complete unramified extensions of k. Suppose that there exists a $\overline{k}$-isomorphism of the residue fields $\overline{K} \to \overline{K}'$. Then this can be lifted to a k-isomorphism $K \to K'$ – provided $\overline{K}$ is separable over $\overline{k}$.*

Here, the essential new notion was that of "separable" field extension without the assumption that $\overline{K}$ is algebraic over $\overline{k}$. Mac Lane had discovered that for an arbitrary field extension $\overline{K}|\overline{k}$ of characteristic p, the adequate notion of "separable" is not the one used by Hasse and F. K. Schmidt (which was "*separably generated*") but it should be defined as "*$\overline{k}$-linear disjoint to $\overline{k}^{1/p}$*". Equivalently, this means that p-independent elements of $\overline{k}$ (in the sense of Teichmüller [1936e]) remain p-independent in $\overline{K}$. Mac Lane in [1939a] was the first one to study this notion systematically and pointing out its usefulness in dealing with field extensions of characteristic p. If there exists a separating transcendence basis then $\overline{K}|\overline{k}$ is separable in the above sense, but the converse is not generally true. Every field in characteristic p is separable over its prime field $\mathbb{F}_p$ – and this, as Mac Lane points out, is the fact which allows to deduce the "absolute" uniqueness theorem which says that an unramified complete field is uniquely determined by its residue field.

If the residue field $\overline{K}$ is perfect then the proof of the relative uniqueness theorem is almost immediate when the Teichmüller character $\chi : \overline{K} \to K$ is used. For, choose a basis of transcendency $\overline{T}$ of $\overline{K}|\overline{k}$, and put $T = \chi(\overline{T})$. Similarly, after identifying $\overline{K} = \overline{K}'$ by means of the given isomorphism, put $T' = \chi'(\overline{T})$ where $\chi' : \overline{K} \to K'$ is the Teichmüller character for K'. Then the map $T \mapsto T'$ yields an isomorphism of $k(T)$ onto $k(T')$ as valued fields. Since $\overline{K}$ is perfect, every $\overline{t} \in \overline{T}$ has a unique p-th power in $\overline{K}$, hence $\chi(\overline{T}^{1/p}) = T^{1/p}$ is well defined in K; similarly $\chi'(\overline{T}^{1/p}) = T'^{1/p}$ in K'. In this way the above isomorphism extends uniquely to an isomorphism $k(T^{1/p}) \to k(T'^{1/p})$. Similarly for $k(T^{1/p^2}), k(T^{1/p^3}), \ldots$, then to the union of these fields, and then to its completion. Further extensions involve only separably algebraic extensions of the residue field and hence can be dealt with by Hensel's Lemma.

If the residue field $\overline{K}$ is not perfect then Mac Lane uses the Teichmüller construction for imperfect residue fields as given in [1936d].

The above discussion shows that the main merit of Mac Lane's paper is not so much a new proof of the Witt-Teichmüller theorem about complete discrete valued fields in the unequal characteristic case. [94] The specific and highly interesting description of those fields by means of the Witt vector calculus is completely lost in Mac Lane's setup. In our opinion, the main point of Mac

[94] Mac Lane calls this "the p-adic case".

Lane's paper is the discovery of the new notion of "separable field extension" which indeed became an important tool in field theory in characteristic p. It is not only applicable in the proof of Mac Lane's Uniqueness Theorem but it has proved to be useful in many other situations, e.g., in algebraic geometry of characteristic p. In this respect Mac Lane's paper [1939] and the follow up [1939a] are indeed to be called a "classic", as does Kaplansky in [1979a].

Moreover, Mac Lane's method of describing the structure of discrete complete valuation rings has since served as a model for further similar descriptions in topological algebra where Witt vectors cannot be used. See, e.g., Cohen's paper [1946a] on the structure of complete regular rings, as well as Samuel [1953], Geddes [1954], and also [1959].

Let us note a minor but curious observation. In his paper [1939] Mac Lane mentions that the Hasse-F. K. Schmidt paper [1933c] contains an error and he sets out to give a new, correct proof. The proof is correct but a certain more general statement is not. Consider the situation of the relative uniqueness theorem. If the residue field $\overline{K}$ admits a separating basis of transcendency then, Mac Lane claims, the conclusion of that theorem holds even without the hypothesis that the valuation of k is discrete. Certainly this is not generally true in view of the existence of immediate extensions of complete fields with non-discrete valuations. In a subsequent paper [1940] he corrects this error, mentioning that his student I. Kaplansky had called his attention to the counter examples. We note that Mac Lane's error is of the same type as F. K. Schmidt's in his letter to Hasse of Feb 23, 1930! (See section 4.3.1.)

5 Ostrowski's second contribution

We will now discuss Ostroswki's great paper "*Untersuchungen zur arithmetischen Theorie der Körper*" [1934]. This paper has 136 pages; it almost looks like a monograph on valuation theory. It seems to be written as a follow up to Ostrowski's first papers [1913a],[1917], [1918]. Indeed, Ostrowski tells us in the introduction that a substantial part of its contents was already finished in the years 1915-1917. This was even earlier than Rychlík's [1919]. And we shall see that the Ostrowski paper contains very important and seminal ideas.

I do not know why Ostrowski waited more than 14 years with the publication of these results. The author says it was because of the adversity of the times and the great length of the manuscript ("*die Ungunst der Verhältnisse und wegen des großen Umfangs von 110 Manuskriptseiten*"). Perhaps, we may guess, his interest had partly shifted to other problems in mathematics, for within those 14 years he published about 70 other papers of which only a few can be said to have been influenced by valuation theory, and only one had a closer connection to valuation theory proper, namely [1933a] on Dirichlet series.

Ostrowski's former valuation theory papers were written and published during the years of the first world war 1914–1918. We are told by Jeltsch-Fricker [95] in [1988] that those years were very special for Ostrowski who, as said above, studied in Marburg (Germany) at that time. Since he was a Russian citizen and Russia was at war with Germany, he was confined to internment. On the intervention of Kurt Hensel however, he was granted certain privileges, among them the use of the university library at Marburg. In later years Ostrowski

[95] She had been assistant to Ostrowski in Basel.

said that this outcome had been reasonably satisfactory for him and he did not consider the four years of the war as wasted, for:

> *The isolation enabled him to concentrate fully on his investigations. He read through mathematical journals, in his own words, from cover to cover, occupied himself with the study of foreign languages, music and valuation theory (almost completely on his own).* [96]

We see that valuation theory is explicitly mentioned. On the other hand, it is also mentioned that he had learned a lot more by reading mathematical journals from cover to cover [97], and hence he may have discovered that his interests were not confined to valuation theory. In fact, after the war was ended and he was free again to move around in Germany, he left Marburg and went to Göttingen. [98] There he was absorbed with quite different activities. He obtained his Ph.D. in the year 1920 with E. Landau and D. Hilbert. The title of his thesis was "*Über Dirichletsche Reihen und algebraische Differentialgleichungen.*" [99]

We suspect that by 1932 Ostrowski realized that valuation theory had developed considerably in the meantime, and that a number of published results (e.g., by Deuring and by Krull) were contained as special cases in his old, unpublished manuscript. But perhaps it was his work on Dirichlet series [1933a] which had prompted him to dive into valuation theory again; in that paper he had used some facts about Newton diagrams for polynomials over Henselian fields. Moreover, there had appeared Krull's fundamental paper [1932g] which contained much of the basic ideas of Ostrowski concerning what he calls the abstract "Riemann surface" of a field. In any case he decided to finally publish his old manuscript, enriched with certain new ideas concerning the abstract Riemann surface which he dates to April–July 1932. He submitted the manuscript for publication to the "Mathematische Annalen" in 1932, and it was published in 1934.

"Ostrowski's Theorem" in [1917] (see section 2.2.3) had stated that every archimedean valued field is isomorphic, as a topological field, to a subfield of the complex number field $\mathbb{C}$. (See section 2.2.3.) In view of this result, his new paper [1934] is devoted exclusively to *non-archimedean* valuations. Instead of valuations he also speaks of *prime divisors* of a field K – a notion which, as we have seen above, he had already introduced in his earlier paper [1918]. He uses valuations in the additive form; thus a prime divisor $\mathfrak{p}$ of the field K is given by

[96] Cited from Jeltsch-Fricker [1988].

[97] It is reported that Ostrowski had a phenomenal memory capacity. In later years, when he was studying in Göttingen, he was used by his fellow students as a living encyclopedia. If one wanted to know details about when and where a particular mathematical problem had been discussed and who had done it, one could ask Ostrowski and would obtain the correct answer, including the volume number and perhaps also the page number of the respective articles.

[98] There is a postcard, dated June 12, 1918, from Emmy Noether in Göttingen to Ostrowski in Marburg, expressing her delight that he will soon move to Göttingen. The question arises whether Ostrowski had shown his valuation theoretic manuscript of [1934], which he said to have been essentially completed in 1917 already, to Emmy Noether. But there is no evidence of this, and we believe he rather did *not*. Otherwise she would probably have told her student Deuring about it, 12 years later when Deuring wrote his paper [1931a] (see section 4.1.1). After all, Deuring's results were contained in Ostrowski's (see section 5.2). But Deuring does not cite Ostrowski's manuscript, and so we believe that he did not know about it.

[99] This was connected with one of Hilbert's problems of 1900, and hence stirred much interest.

a map $v : K \to \mathbb{R} \cup +\infty$ such that

$$\begin{aligned} v(a) &< +\infty \quad \text{if } a \neq 0, \text{ and } v(0) = +\infty && (5.1) \\ v(a+b) &\geq \min(v(a), v(b)) && (5.2) \\ v(ab) &= v(a) + v(b) && (5.3) \\ \exists a : v(a) &\neq 0, +\infty && (5.4) \end{aligned}$$

Ostrowski calls $v(a)$ the "order" (*Ordnungszahl*) of a at the prime $\mathfrak{p}$. Through this notation and terminology he refers to the analogies from the theory of complex functions and Riemann surfaces – a viewpoint which penetrates the whole of Ostrowski's theory in this paper. It seems that Ostrowski was guided to a large extent by his experiences with the handling of Dirichlet series in his former papers. In fact, the first example of valued fields which Ostrowski presents here are fields of Dirichlet series.

According to the definitions, the valuation v is uniquely determined by its prime divisor $\mathfrak{p}$ up to equivalence, which means up to a real positive factor.

Ostrowski's paper consists of three parts. Each part is rich in interesting results, and we discuss them in three separate sections.

5.1 Part I. Henselian fields.

After the basic definitions and facts of valuation theory, Ostrowski introduces what today is called a "Henselian field", i.e., a valued field K in which Hensel's Lemma holds. In this connection it is irrelevant which of the various equivalent forms of Hensel's Lemma is used in the definition. Ostrowski uses the following:

> **Fundamental Lemma**: *Consider a polynomial*
>
> $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$
>
> *whose coeffcients belong to the valuation ring of the prime $\mathfrak{p}$ of K. Assume that*
>
> $$f(x) \equiv a_n x^n + \cdots + a_{m+1} x^{m+1} + x^m \mod \mathfrak{p} \qquad \text{with} \qquad n > m > 0$$
>
> *where the congruence is to be understood coefficient-wise modulo the maximal ideal of the valuation ring. Then $f(x)$ is reducible in K.*

If this Fundamental Lemma holds for a valued field K then Ostrowski calls the field "relatively complete" (*relativ perfekt*). Today the terminology is "Henselian". For the convenience of the reader we prefer to use today's terminology; thus we will always say "Henselian" when Ostrowski says "relatively complete". [100] Apart from the terminology, Ostrowski treats these fields in the same way as we would do today. He clearly sees that the important *algebraic* property of complete valued fields is not their completeness but the validity of Hensel's Lemma, and he creates his theory accordingly.

[100] We do not know when and by whom the name "Henselian field" had been coined; probably it was much later. Schilling in his book [1950] still uses the terminology of Ostrowski. Azumaya [1951] introduces the notion of "Henselian ring" in the context of local rings, but he does not tell whether this terminology had been used before for valuation rings or valued fields.

Ostrowski shows that every valued field K admits a Henselization K^h, unique up to isomorphism, and he proves the standard and well known properties of it. He does this in several steps.

Step 1. It is proved that the completion $\widehat{K}$ is Henselian, which is to say that the above mentioned Fundamental Lemma holds in $\widehat{K}$. Ostrowski remarks that this is a special case of the more general reducibility theorem in Rychlík's paper [1923f], but since he does not go for the full Hensel-Rychlík Lemma the proof of the Fundamental Lemma is particularly simple.

Step 2. It is proved that for any Henselian field, its valuation can be extended uniquely to the algebraic closure. Thus Ostrowski repeats a third time what Kürschák [1912] and Rychlík [1923f] had already proved. But the emphasis here is that the result depends on the Henselian property only, and completeness is not required. Of course, this could have been seen immediately from the proof given by Kürschák, or that by Rychlík – provided the notion of "Henselian field" would have been established. But neither Kürschák nor Rychlík had conceived that notion, and it was Ostrowski who introduced it.

At the same time, the important fact is proved that any algebraic extension $L|K$, finite or infinite, of a Henselian field K is Henselian again. (Recall that for extensions of infinite degree, this not true if one replaces "Henselian" by "complete"; this had been shown by Ostrowski in his first paper [1913a] in reply to Kürschák's question.) Moreover, given any valued field L then the intersection of two valued Henselian subfields of L is Henselian again.

Step 3. Now, for an arbitrary valued field K, Ostrowski constructs its Henselization K^h as the separable-algebraic closure of K within the completion $\widehat{K}$. It is proved that K^h, if defined in this way, is Henselian and, moreover, that every Henselian field extension of K contains a K-isomorphic image of K^h.

Step 4. At the same time, Ostrowski connects this notion with Galois theory, as follows. He presents a description of the prolongations of the valuation v of K to an arbitrary algebraic extension L of K, finite or not, in the following manner. Those prolongations are all given by the K-isomorphisms [101] of L into the algebraic closure of the completion $\widehat{K}$, observing that the image is uniquely valued by Steps 1 and 2. If $L|K$ is a Galois extension [102] then it follows, after fixing one prolongation w of v to L, that all other prolongations are obtained from w by the automorphisms of the Galois group. In fact, the other prolongations correspond to the cosets of the Galois group G modulo the decomposition subgroup Z, consisting of those automorphisms which leave w fixed. And the fixed field of Z in L is precisely the intersection $K^h \cap L$ – more precisely, it is isomorphic to it as valued field. Thus, $K^h \cap L$ is the *decomposition field* of v in L (unique up to conjugates over K).

Taking $L = K^s$ to be the separable algebraic closure, it follows that K^h is the decomposition field of v in K^s. This explains the terminology of Ostrowski who does not speak of "Henselization" but calls K^h the "*reduced universal decomposition field*". The word "reduced" refers to the fact that K^h is separable over K, and "universal" means that for any Galois extension $L|K$ the intersection $K^h \cap L$ is the decomposition field.

[101] Ostrowski uses the word "permutation" instead of "isomorphism", thereby following the old terminology of Dedekind.

[102] Ostrowski also considers what he calls a "normal extension", which is a Galois extension followed by some purely inseparable extension.

All this is quite the standard procedure today. [103] We have mentioned it in such detail because it seems remarkable that these ideas can be found in Ostrowski's paper [1934] which, according to the author, had been written in its essential parts in 1915 already. The main development of valuation theory before Ostrowski's [1934] was influenced by the applications to number fields, in which case an important role was played by the completion because it permitted to use analytic arguments in the pursuit of arithmetical problems – for instance, the use of exponential and logarithmic function, of the exponentiation with p-adic integer exponents etc. Thus from the arithmetic viewpoint, there was more interest, in those years, in the *analytic* properties of valued fields which implies working with *completions*, rather than with *Henselizations*. The Henselization is the proper notion if one concentrates on the *algebraic* properties of valuations.

With this background, the essential ideas of Ostrowski's paper were perhaps not adequately appreciated by the mathematical public of that time, at least not by those coming from number theory. In fact, in the *Zentralblatt* review about Ostrowski's paper, F. K. Schmidt did not even mention the notion of Henselian (or relatively complete) field and says about Part I: "*Seine Ergebnisse sind im wesentlichen bekannt.*". This indicates that his attention was fixed on the completion which he was used to, and he did not appreciate the importance of the new notion of Henselization. [104]

In view of this we find Ostrowski's algebraic theory of Henselizations remarkable indeed.

In later times the notion of Henselization turned out to be even more important in the theory of general Krull valuations. In that case the completion is not necessarily Henselian and hence not useful for discussing the prolongation of valuations. Thus Ostrowski, although his paper is concerned with valuations of rank 1 only, prepares the way for the discussion of higher rank valuations too by creating the notion of Henselization and pointing out its importance.

This Part I contains the Approximation Theorem for finitely many non-archimedean primes of a field K. We have discussed this theorem in section 4.2.1. Ostrowski speaks of "independence" (*Unabhängigkeit*) of finitely many primes in a field.

5.1.1 Newton diagrams

In some of his arguments, Ostrowski uses the method of *Newton diagram* of a polynomial over a valued field K. He says:

If the field K is Henselian then the sides of the Newton diagram correspond to the irreducible factors of the polynomial.

This last statement can be used as the definition of Hensel fields.

The use of the graphic Newton diagram method is not really necessary but it is lucid and the arguments become brief. Ostrowski takes the main statements about Newton diagrams for granted; he does not even give the definition of the Newton diagram of a polynomial and cites his earlier paper [1933a]. Although there he discusses complete fields only, it is clear from the context that the theory of Newton diagrams of [1933a] applies in the same way to arbitrary Henselian fields in his present paper. It seems that this is the main aim of Ostrowski when

[103] Except perhaps that in Step 1 one would today avoid the completion altogether by showing directly that the "reduced universal decomposition field" (see Step 4) is Henselian.

[104] The review appeared in vol. 5 of the *Zentralblatt*.

he discusses Newton diagrams in his paper. He wishes to put into evidence that these can be useful over arbitrary Henselian fields. [105]

5.2 Part II: Ramification and defect

And he sets out to show this in the second part of [1934]. There he discusses an algebraic extension L of a valued field K. One of his main results, striking because of its generality, is the following relation, valid for an arbitrary algebraic extension field L of finite degree n. There are only finitely many extensions of the given valuation of K to L; let r denote their number. For the i-th extension let e_i denote the corresponding ramification index and f_i the residue degree. With this notation, Ostrowski proves the important degree relation

$$n = \sum_{1 \le i \le r} \delta_i e_i f_i \tag{5.5}$$

where δ_i denotes the "defect" of the i-th extension. In the case of classical algebraic number theory, where the non-archimedean valuations are defined by prime ideals of Dedekind rings and algebraic extensions are separable, the relation (5.5) was well known at the time (with trivial defects $\delta_i = 1$) through the work of Dedekind, Hilbert, Weber and, later, Emmy Noether. In the general case it was known that $n \ge \sum_i e_i f_i$ (as far as this question had been studied). The achievement of Ostrowski was that he could define the defect δ_i as a local invariant (i.e., depending only on the local extension $L_i^h|K^h$ of the respective Henselizations) and show that it is always a power of the characteristic exponent p of the residue field.

For Galois extensions, all the prerequisites of defining the defect and deriving the formula (5.5) were contained already in the papers by Deuring [1931a] and Krull [1930c], as we had pointed out already in section (4.1.1). But neither of them took the step to introduce the defect as an invariant which is worthwhile to investigate. Perhaps this can be explained by the fact that in the classical cases in number theory the defect is always trivial and, hence, the appearance of a non-trivial defect was considered to be some pathological situation which was of no particular interest.

But not for Ostrowski. He goes to great length to investigate the structure of the defect; in particular to give criteria for $\delta_i = 1$, in which case the extension $L_i^h|K^h$ is *defectless*. Today, Ostrowski's notion of defect and its generalizations have become important in questions of algebraic geometry; see, e.g., Kuhlmann's thesis [1989a], and also his paper [2000a].

Ostrowski derives in detail a generalization of a good part of the Dedekind-Hilbert ramification theory: he defines and studies decomposition field, inertia field, ramification field etc. and their corresponding automorphism groups. But Galois theory does not play a dominant role in this paper; mostly the author prefers to consider fields instead of their Galois groups. [106] The reason for this is that he wishes to include, in a natural way, the inseparable field extensions which cannot be handled by Galois theory. Nevertheless he cites the recent

[105] Newton diagrams, in the framework of valuation theory, had also been studied by Rella [1927d].

[106] He even avoids the use of Sylow's theorem from group theory and, instead, uses a trick ("*Kunstgriff*") on algebraic equations, attributed to Foncéneux.

papers by Krull [1930c], Deuring [1931a], and Herbrand [1932b] where similar questions are discussed in a different way, relying on Galois theory and ideal theory. (See section 4.1.) As to ideal theory, Ostrowski says explicitly:

> *Wir haben vom Idealbegriff keinen Gebrauch gemacht, da es vielleicht einer der Hauptvorzüge der hier dargestellten Theorie ist, daß durch sie der Idealbegriff eliminierbar wird.*
>
> We have not made use of the notion of ideal since it is perhaps the main advantage of the theory as presented here, that the notion of ideal can be avoided.

Indeed, valuation theory had been created by Kürschák, following Hensel's ideas, in order to have a directly applicable tool to measure field elements with respect to their size or their divisibility properties, and in this respect to be free from ring or ideal theory.

5.3 Part III: The general valuation problem

5.3.1 Pseudo-Cauchy sequences

In this part, Ostrowski first considers the problem of extending a valuation of a field K to a purely transcendental field extension $K(x)$. To this end he develops his theory of *pseudo-Cauchy* sequences [107] $a_1, a_2, a_3, \dots$ with $a_i \in K$; the defining property is that finally (i.e., for all sufficiently large n) we have either

$$\|a_{n+1} - a_n\| < \|a_n - a_{n-1}\| \qquad (5.6)$$

or $a_{n+1} = a_n$. (Here, Ostrowski switches to the multiplicative form of valuations in the sense of Kürschák.)

If a_n is a pseudo-Cauchy sequence in K, or in some valued algebraic extension of K, and $0 \neq f(x) \in K(x)$ then $f(a_n)$ is also a pseudo-Cauchy sequence. Moreover, $\lim_{n\to\infty} \|f(a_n)\|$ exists and is called the limit value (*Grenzbewertung*) of $f(x)$ with respect to the pseudo-Cauchy sequence a_n. If this limit value does not vanish for all non-zero polynomials then Ostrowski puts

$$\|f(x)\| = \lim_{n\to\infty} \|f(a_n)\|$$

and obtains a valuation of $K(x)$ extending the given valuation in K. In this situation x is called a *pseudo limit* of the sequence a_n. Moreover, Ostrowski shows:

> *Every extension of the valuation of K to $K(x)$ is obtained in this way by a suitable pseudo-Cauchy sequence in the valued algebraic closure of K, so that x becomes a pseudo limit of the sequence.*

This is a remarkable result indeed. The question of how the valuation of K extends to a purely transcendental extension is quite natural and certainly had been asked by a number of people. Traditionally there had been the solution

$$\|f(x)\| = \|c_0 + c_1 x + \cdots + c_n x^n\| = \max(\|c_0\|, \dots, \|c_n\|)$$

[107] Ostrowski uses the terminology "pseudo convergent" instead of "pseudo-Cauchy". This reflects the terminology in analysis of that time. What today is called "Cauchy sequence" in analysis, was at that time often called "convergent sequence" (meaning convergent in some larger space). Alternatively, the terminology "fundamental sequence" was used, following G. Cantor [1883], p.567.

which is called the "functional valuation" of the rational number field, with respect to the given valuation of the base field K. More generally, there was known the solution

$$\|f(x)\| = \max(\|c_0\|, \|c_1\|\mu, \ldots, \ldots, \|c_n\|\mu^n)$$

if μ is any given positive real number; this is called by Ostrowski the extension "by an invariable element". These solutions are now contained as special cases in Ostrowski's construction, namely, if the pseudo-Cauchy sequence a_n is "of second kind" which means that finally $\|a_{n+1}\| < \|a_n\|$; we then have $\mu = \lim_{n\to\infty} \|a_n\|$.

Ostrowski investigates in some detail how the various properties of the pseudo-Cauchy sequence influences the properties of the extended valuation. Of particular interest is the case when both the residue field and the value group are preserved, i.e., when $K(x)$ becomes an *immediate extension* of K. If K is algebraically closed then this is the case if x is a "proper" pseudo limit of the sequence a_n. This is defined by the condition that the distance δ of x to K is not assumed, i.e., to every $b \in K$ there exists $c \in K$ such that $\|x - c\| < \|x - b\|$. Equivalently, $\|a_n - c\| < \|a_n - b\|$ for sufficiently large n.

The notion of pseudo-Cauchy sequence and its use for extending valuations to the transcendental extension $K(x)$ was a completely new idea which perhaps originated in Ostrowski's work on fields of Dirichlet series [1933a]. Anyhow, this aspect of valuation theory went far beyond what was known and studied in classical algebraic number theory where discrete valuations dominated. The introduction and study of pseudo-Cauchy sequences and their pseudo limits is to be viewed as a milestone in general valuation theory. It opened the path to the more detailed study of Krull valuations of which the well known Harvard thesis of Irving Kaplansky [1942] was to be the first instance.

5.3.2 Remarks on valuations of rational function fields

The problem of determining all possible extensions of a valuation from K to the rational function field $K(x)$ had been studied already by F. K. Schmidt. In a letter to Hasse dated March 7, 1930 he investigated all *unramified* extensions of the valuation to $K(x)$. In doing so he answered a question put to him by Hasse. He came to the conclusion that, if the residue field $\overline{K}$ is algebraically closed then every unramified extension of the valuation is a functional extension with respect to a suitable generator of $K(x)$, and conversely. Of particular interest is a postscript to his letter where he drops the hypothesis that K is algebraically closed but assumed that the valuation is discrete. He wrote:

> *Nach Abschluß dieses Briefes bemerke ich, daß ich die Prozesse, die zu jeder möglichen diskreten Fortsetzung führen, vollständig übersehen kann. Außer den in vorstehenden Überlegungen explizit oder implizit enthaltenen kommt noch ein wesentlich neuer Typus hinzu, bei dem der Restklassenkörper der Fortsetzung unendliche algebraische Erweiterung von $\overline{K}$ ist. Falls Sie diese vollständige Übersicht interessiert, so schreibe ich Ihnen gerne darüber.*
>
> After writing this letter I discover that I am able to completely describe all processes which lead to all possible discrete prolongations. Besides of those which are contained explicitly oder implicitly in the

above considerations, there appears a new type, in which the residue field is an infinite algebraic extension of $\overline{K}$. If you are interested in this complete description then I will be glad to write you about this.

Unfortunately this topic appears neither in later letters of F. K. Schmidt nor in his publications. So we will never know presicsely what F. K. Schmidt had discovered. But from those remarks it does not seem impossible that he had already the full solution which later was given by Mac Lane in his paper [1936f].

That paper of Mac Lane appeared two years after Ostrowski's [1934]. Mac Lane cites Ostrowski but, he says, his description is different. However, although formally it is indeed different, it is not difficult to subsume Mac Lane's results under those of Ostrowski. Let us briefly report about [1936f]:

Let us write the valuations additively. The given valuation on K is denoted by v. Suppose v is extended to a valuation w of $K(x)$. Then w can be approximated as follows:

Let $\mu = w(x)$. First consider the valuation v_1 on $K[x]$ defined for polynomials $f(x) = \sum_{0\leq i\leq n} a_i x^i$ by

$$v_1(f(x)) = \min_{0\leq i\leq n} (v(a_i) + i\mu) \ .$$

This is the valuation with x as an "invariable element" in the terminology of Ostrowski, as mentioned earlier already. Due to the non-archimedean triangle inequality we have

$$v_1(f(x)) \leq w(f(x)) \tag{5.7}$$

for every polynomial $f(x) \in K[x]$. This v_1 is called the first approximation to w. We have either $v_1 = w$, or there is a polynomial $\varphi(x)$ such that

$$v_1(\varphi(x)) < w(\varphi(x)).$$

Suppose that $\varphi(x)$ is monic of smallest degree with this property; in particular $\varphi(x)$ is irreducible. Put $\mu_1 = w(\varphi(x))$. Every polynomial $f(x) \in K[x]$ admits a unique expansion of the form

$$f(x) = \sum_{0\leq i\leq n} a_i(x)\varphi(x)^i$$

with $a_i(x) \in K[x]$ and $\deg a_i(x) < \deg \varphi(x)$. Then put

$$v_2(f(x)) = \min_{0\leq i\leq n} (v_1(a_i(x)) + i\mu_1) \ .$$

This is a valuation of $K[x]$, called the second approximation of w. Now we have

$$v_1(f(x)) \leq v_2(f(x)) \leq w(f(x)) \, . \tag{5.8}$$

If $v_2 \neq w$ then we can repeat this process, thus arriving at a sequence $v_1 \leq v_2 \leq v_3 \leq \cdots \leq w$ of valuations of $K[x]$. Then we have the following alternatives. Either $v_k = w$ for some k; in this case w is called a k-fold augmented valuation. [108] Or we have that $v_\infty = \lim_{k\to\infty} v_k$ is a valuation of $K[x]$. Moreover, *if the original valuation v of K is discrete then Mac Lane shows that $v_\infty = w$.*

[108] Mac Lane [1936f] speaks of "values" instead of "valuations".

This leads to a constructive description of all possible valuations of $K(x)$ if the original valuation is discrete; Mac Lane carefully describes the properties of those polynomials $\varphi(x)$ which appear in the successive approximations; he calls them "key polynomials". If the valuation w is not of augmented type then the value group of w is *commensurable* with the value group of v, and the residue field of $K(x)$ with respect to w is the union of the increasing sequence of the residue fields of the approximants v_k, each of which is a finite extension of K.

We see that F. K. Schmidt's observations, as mentioned in his 1930 letter to Hasse, are quite in accordance with these results of Mac Lane.

The motivation of Mac Lane for this investigation was to obtain a systematic explanation of various irreducibility criteria for polynomials, like the Eisenstein criterium; such irreducibility criteria had been given e.g., by Kürschák [1923e], Ore [1927e], Rella [1927d]. For details we refer to Mac Lane's paper [1938] which, as Kaplansky [1979a] points out, stands as definitive to this day.

We also remark that the said limit construction of valuations w of $K(x)$ is of the same type as Zariski has constructed in his investigations on algebraic surfaces, in particular by obtaining valuations of a two-dimensional function field by succesive blow-ups. But this belongs to the subject of the second part of our project, so we will not discuss it here.

5.3.3 The general valuation problem

Let us return to Ostrowski's paper [1934], Part III. There, Ostrowski puts the "general valuation problem" as follows: Given a valued field K and an arbitrary extension field L of K, one should give a method to construct all possible extensions of the valuation from K to L. The solution is, he says, as follows: Let $T = (t_i)_{i \in I}$ be a basis of transcendency of $L|K$. Then $L|K(T)$ is algebraic. The extensions of a valuation to an algebraic field extension are decribed in Part II. Hence it suffices to deal with the purely transcendental extension $K(T)$. Now, he says, the index set I may be assumed to be well ordered, and so $K(T)$ is obtained from K by successively adjoining one transcendental element t_i to the field constructed in the foregoing steps. Hence it suffices to deal with just one transcendental element, and this case is covered by Ostrowski's theory of pseudo-Cauchy sequences.

F. K. Schmidt in his *Zentralblatt*-review says this solution is purely genetic (*rein genetisch*). And he compares it with the description which Hasse and himself had given of the complete discrete valued fields. This is of quite different nature (see section 4.3.1). He says that these results are not contained in Ostrowski's paper, which aims primarily at non-discrete valuations.

Of course F. K. Schmidt is correct. It is clear that both view points, the structural one of Hasse-F. K. Schmidt and the "genetic" one by Ostrowski are of interest, and that none supersedes the other. The question is why F. K. Schmidt found it necessary to stress this self-evident point in his review?

There is a letter of F. K. Schmidt to Hasse dated Sep 9, 1933. F. K. Schmidt had been in Altdorf at the Swiss Mathematical Congress and had met Ostrowski there. The latter had informed him that just now he is reading the galley proofs of a long paper on valued fields in which he (Ostrowski) gives a precise description how to construct those fields. (Obviously these were the galley proofs for Ostrowski's paper [1934].) F. K. Schmidt reports to Hasse that upon questioning it seems that Ostrowski "had nothing essential of our things" (*"... daß er von* *unseren entscheidenden Sachen nichts hat*"). And he says that most probably there is no overlap between the paper of Ostrowski and their joint paper (which was expected to appear in the next days). Nevertheless F. K. Schmidt seems to have been somewhat worried because he did not find out what precisely was contained in Ostrowski's paper. For Ostrowski in their conversation had claimed not to remember details. With this background we may perhaps understand why F. K. Schmidt had included the above mentioned sentence in his review of Ostrowski's paper. It sounds to me like a sigh of relief, because his long paper [1933c] had not been superseded by that of Ostrowski. Of course he did not yet know that [1933c] contained an error, nor that this paper was to be superseded within two years by Teichmüller and Witt.

5.3.4 The Riemann surface of a field

Ostrowski starts the last §12 of his paper with the definition of what today is called *place* of a field. (His terminology is "*Restisomorphie*"). [109] He reports that to every such place there belongs a "general valuation" as defined by Krull. He defines a "simple" place to be one whose value group has archimedean ordering and shows that to such a "simple" place there belongs a valuation in his sense, i.e., the value group is a subgroup of the real numbers. Thus the simple places correspond to the "primes" as defined earlier.

If $K|k$ is an algebraic function field of several variables over an algebraically closed field then the set of all places $K \to k$ is called by Ostrowski the "absolute Riemann surface" of K over k. [110] Ostrowski shows that every place of this absolute Riemann surface can be obtained as a composition of simple places, i.e., of primes. If the number of these primes equals the degree of transcendency of $K|k$ then the point is called "Puiseux point". At such a point every element in K admits a Puiseux expansion – provided $K|k$ is of characteristic 0.

We see that this last §12 is of different flavor than the foregoing sections. The reader gets the impression that §12 had been inserted only after Ostrowski had seen the famous paper of Krull [1932g] introducing general valuation theory. We observe that Krull's paper [1932g] had appeared in 1932, and in the same year Ostrowski submitted his article (which appeared in 1934). Ostrowski strongly recommends to "read the rich article of Krull" ("*. . . die Lektüre der gehaltreichen Abhandlung von Herrn Krull aufs angelegentlichste empfohlen. . .*").

In reading Ostrowski's paper (not only this §12) one cannot help to wonder that the nature and the form of his results are such that they point directly to their generalization within the framework of the theory of general Krull valuations.

We may imagine that after having seen Krull's paper Ostrowski looked again at his old, never published manuscript of 1917 and realized the coherence of his ideas with the new ones introduced by Krull. And this then induced him to have it finally published, with relatively small changes, in the hope that it may prove helpful in further development. And certainly it did.

[109] Today we would say "*Resthomomorphie* ". But Ostrowski still uses the older terminology where the notion of "isomorphy" is used when we would say "homomorphy". In this terminology one distinguishes between "holoedric isomorphy" (meaning "isomorphy" in today's terminology) and "meroedric isomorphy" (which means "homomorphy").

[110] Today it is also called "Zariski space" or "Zariski-Riemann manifold".

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