Logarithmic differentials on discretely ringed adic spaces

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Contents

1. Introduction

Consider a smooth discretely ringed adic space $\mathscr{X}$ over a field k. Here, discretely ringed means that $\mathscr{X}$ is locally isomorphic to the spectrum of a Huber pair (A, A^+), where A and A^+ carry the discrete topology. The space $\mathscr{X}$ comes with two structure sheaves, $\mathscr{O}_{\mathscr{X}}$ and $\mathscr{O}^+_{\mathscr{X}}$. One might ask for a similar partner Ω^+ for the sheaf of differentials $\Omega_{\mathscr{X}} = \Omega^1_{\mathscr{X}/k}$. It should be a subsheaf of $\Omega := \Omega^1_{\mathscr{X}}$ defined by a condition $|\omega_x| \le 1$ for suitable $\mathscr{O}_{\mathscr{X},x}$-seminorms $|\cdot|$ on the stalks $\Omega_{\mathscr{X},x}$ for every point $x \in \mathscr{X}$. Such a sheaf Ω^+ will be useful for investigating cohomological purity for p-torsion sheaves in characteristic $p > 0$. As explained in the introduction to [Hüb20], the logarithmic deRham sheaves $\nu(r)$ play a crucial role in cohomological purity. They are defined by an exact sequence

$$0 \to \nu(r) \to \Omega^r_{d=0} \xrightarrow{C-1} \Omega^r \to 0,$$

in the étale topology. However, we expect purity to hold only for the tame topology (see [Hüb18]) and the above sequence is not exact in the tame topology. We hope to solve this problem by replacing Ω^r with $\Omega^{r,+}$. This will be subject to future investigations.

In this article we construct a sheaf Ω^+ as above using the Kähler seminorms (cf. [Tem16], § 4.1) on the stalks Ω_x defined by

$$|\omega|_\Omega := \inf_{\omega=\sum_i f_i dg_i} \max_i \{|f_i| \cdot |g_i|\},$$

where the infimum is taken over all representations of ω as a finite sum $\sum_i f_i dg_i$ (see Section 5.1). In Section 5.2 we prove that Ω^+ is indeed a sheaf on $\mathscr{X}$. In fact, it is even a sheaf on the tame site $\mathscr{X}_t$ of $\mathscr{X}$ but not on the étale site.

It turns out that Ω^+ has a description in terms of logarithmic differentials. After a preliminary section on the logarithmic cotangent complex (see Section 2), we study logarithmic differentials in Section 4. Let us specify the connection of logarithmic differentials with Ω^+. For a Huber pair (A, A^+) over k such that A is a localization of A^+, we equip A^+ with the total log structure $(A^+ \setminus \mathfrak{m}_A \to A^+)$ on A^+. The corresponding logarithmic differentials $\Omega^{\log}_{(A,A^+)}$ define a presheaf $\Omega^{\log}$ but not a sheaf. We prove that the sheafification of $\Omega^{\log}$ is Ω^+ in Section 5.2. An important input is that for a local Huber pair (A, A^+) over k the logarithmic differentials $\Omega^{\log}_{(A,A^+)}$ are torsion free over A^+, i.e., they imbed into Ω_A.

The last section is dedicated to a study of logarithmic differentials on adic spaces of the form $\operatorname{Spa}(Y, \bar{Y})$, where $\bar{Y}$ is a scheme over the field k and Y is an open subscheme such that the associated log structure on $\bar{Y}$ is log smooth. The main result (Theorem 6.12) constructs a natural isomorphism

$$\Omega^+(\operatorname{Spa}(Y, \bar{Y})) \cong \Omega^{\log}(Y, \bar{Y}),$$

where $\Omega^{\log}$ on the right hand side is the sheaf of logarithmic differentials on the log scheme associated with $(Y, \bar{Y})$. The crucial point is that on the adic space $\operatorname{Spa}(Y, \bar{Y})$ we do not need to sheafify $\Omega^{\log}$ in order to compute the global sections of Ω^+. This makes Ω^+ a lot more accessible and it is possible to use the theory of logarithmic differentials on log schemes to inverstigate Ω^+. We also want to stress that the above isomorphism is obtained without assuming resolution of singularities The proof relies on the theory of unramified sheaves (see Section 6.2), a notion adapted from [Mor12], and techniques similar to the ones applied in [HKK17] for studying cdh differentials.

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2. The logarithmic cotangent complex

In [Ols05] Olsson describes two approaches for a logarithmic cotangent complex. His own construction using log stacks has the advantage that it is trivial for log smooth morphisms. However, transitivity triangles only exist under certain conditions and the construction only works for fine log schemes, i.e. under strong finiteness conditions that are not satisfied in our situation. Gabber's version described in [Ols05], §8 is more functorial but it has the disadvantage that it is not trivial for all log smooth morphisms. We will use Gabber's log cotangent complex and compare it in special situations to Olsson's in order to make explicit computations. Slightly more generally we will define the log cotangent complex for simplicial prelog rings as described for instance in [Bha12], §5 or [SSV16], §4.

Let us start with reviewing some definitions. Recall that a prelog ring is a ring R and a (commutative) monoid M together with a homomorphism of monoids $M \to R$, where R is considered as a monoid with its multiplicative structure. A log ring is a prelog ring $\iota : M \to R$ inducing an isomorphism $\iota^{-1}(R^\times) \to R^\times$. The inclusion of the category of log rings into prelog rings has a left adjoint, logification (see [Ogu18], Chapter II, Proposition 1.1.5) We write $(M^a \to R)$ or $(M \to R)^a$ for the logification of $(M \to R)$.

Denote by Set, Mon, Ring, and $\mathrm{LogRing}^{\mathrm{pre}}$ the categories of sets, monoids, rings, and prelog rings. We write sSet, sMon, sRing, and $s\mathrm{LogRing}^{\mathrm{pre}}$ for the respective categories of simplicial objects. We endow sSet with the standard model structure, i.e. the weak equivalences are the maps inducing a weak homotopy equivalence on geometric realizations and the fibrations are the Kan fibrations. Defining the (trivial) fibrations to be the homomorphisms that are (trivial) fibrations on the underlying category of simplicial sets, we obtain a closed model structure on sRing and sMon (see [Bha12], §4). Now consider the forgetful functor

$$\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}} : s\mathrm{LogRing}^{\mathrm{pre}} \longrightarrow s\mathrm{Mon} \times s\mathrm{Ring}$$

mapping $(M \to A)$ to (M, A). By [SSV16], Proposition 3.3 there is a projective proper simplicial cellular model structure on $s\mathrm{LogRing}^{\mathrm{pre}}$ whose fibrations and weak equivalences are the maps that are mapped to fibrations and weak equivalences, respectively, under $\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}}$. With respect to this model structure $\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}}$ is a left and right Quillen functor ([Bha12], Propositions 5.3 and 5.5). Its left adjoint is the functor $\mathrm{Free}^{s\mathrm{Mon}\times s\mathrm{Ring}}_{s\mathrm{LogRing}^{\mathrm{pre}}}$ mapping (M, A) to $(M \to A[M])$.

For a homomorphism $(M \to A) \to (N \to B)$ of simplicial prelog rings we write $s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)}$ for the category of simplicial $(M \to A)$-algebras over $(N \to B)$. It inherits a model structure from $s\mathrm{LogRing}^{\mathrm{pre}}$. Consider the functor

$$\begin{aligned} \Omega : s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)} &\to \mathrm{Mod}_B \\ (L \to C) &\mapsto \Omega^1_{(L\to C)/(N\to B)} \otimes_C B \end{aligned}$$

where Ω^1 is defined by applying to each level the functor of log Kähler differentials (see [Ogu18], Chapter IV, Proposition 1.1.2; note that a log ring in loc. cit. is what we here call a prelog ring). Being a left Quillen functor ([SSV16], Lemma 4.6), it has a left derived functor

$$L\Omega : \mathrm{Ho}(s\mathrm{LogRing}^{\mathrm{pre}}_{(M\to A)//(N\to B)}) \to \mathrm{Ho}(\mathrm{Mod}_B)$$

on the respective homotopy categories. The image of $(N \to B)$ under $L\Omega$ is called the *cotangent complex* of $(N \to B)$ and denoted $\mathbb{L}_{(M\to A)/(N\to B)}$. For a homomorphism $(M \to A) \to (N \to B)$ of discrete log rings it can be computed as follows. For shortness write $F := \mathrm{Forget}^{\mathrm{LogRing}^{\mathrm{pre}}}_{\mathrm{Mon}\times\mathrm{Ring}}$ and $G := \mathrm{Free}^{\mathrm{Mon}\times\mathrm{Ring}}_{\mathrm{LogRing}^{\mathrm{pre}}}$ (the discrete versions of the above considered functors). We have a canonical free resolution

(1) {{canonical_resolution}} $$GFGF(N \to B) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; GF(N \to B) \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; (N \to B),$$

which we denote by $P_\bullet \to (N \to B)$. Then $\mathbb{L}_{(M\to A)/(N\to B)}$ is represented by $\Omega(P_\bullet)$. In particular, we recover Gabbers definition ([Ols05], Definition 8.5).

The cotangent complex has the following two important properties (see [SSV16], Proposition 4.12) {transitivi

Proposition 2.1. *(i)* Transitivity. *Let* $(M \to A) \to (N \to B) \to (K \to C)$ *be maps of simplicial prelog rings. Then there is a homotopy cofiber sequence in* $\mathrm{Ho}(\mathrm{Mod}_C)$

$$C \otimes_B^h \mathbb{L}_{(N\to B)/(M\to A)} \to \mathbb{L}_{(K\to C)/(M\to A)} \to \mathbb{L}_{(K\to C)/(N\to B)}.$$

(ii) Base change. *Let*

$$\begin{tikzcd} (N' \to B') & (N \to B) \arrow[l] \\ (M' \to A') \arrow[u] & (M \to A) \arrow[l] \arrow[u] \end{tikzcd}$$

be a homotopy pushout square in $s\mathrm{LogRing}^{\mathrm{pre}}$. *Then there is an isomorphism in* $\mathrm{Ho}(\mathrm{Mod}_{B'})$

$$B' \otimes_B^h \mathbb{L}_{(N\to B)/(M\to A)} \cong \mathbb{L}_{(N'\to B')/(M'\to A')}.$$

In order to apply these results in our setting of discrete prelog rings it would be useful to know when the homotopy pushouts appearing in (i) and (ii) coincide with the ordinary pushout. The homotopy pushout in (i) appearing in the cofiber sequence is taken in the homotopy category of Mod_C. Suppose that C and B are discrete. Then it is well known that

$$C \otimes_B \mathbb{L}_{(N\to B)/(M\to A)} \cong C \otimes_B^h \mathbb{L}_{(N\to B)/(M\to A)}$$

in case C is flat over B. In the base change setting for discrete prelog rings it turned out to be easier to prove the base change result from scratch instead of deducing it from the homotopy version Proposition 2.1 (ii) for simplicial prelog rings.

_cotangent}

Lemma 2.2. *Let*

$$\begin{tikzcd} (N' \to B') & (N \to B) \arrow[l] \\ (M' \to A') \arrow[u] & (M \to A) \arrow[l] \arrow[u] \end{tikzcd}$$

be a pushout square in $\mathrm{LogRing}^{\mathrm{pre}}$ *which is a homotopy pushout square in* $s\mathrm{LogRing}^{\mathrm{pre}}$. *Then*

$$\mathbb{L}_{(N'\to B')/(M'\to A')} \cong \mathbb{L}_{(N\to B)/(M\to A)} \otimes_A A'.$$

Proof. Let $(K \to P) \to (N \to B)$ be a simplicial resolution in the category of simplicial $(M \to A)$-algebras. Then the induced map

$$(K \to P) \otimes_{(M\to A)} (M' \to A') \to (N \to B) \otimes_{(M\to A)} (M' \to A') = (N' \to B')$$

represents the map from the homotopy pushout to the naive pushout, hence is a weak equivalence. It is therefore a simplicial resolution of $(N' \to B')$ in the category of simplicial $(M' \to A')$-algebras and we can use it to compute the cotangent complex of $(N' \to B')$ over $(M' \to A')$:

$$\begin{aligned} \mathbb{L}_{(N'\to B')/(M'\to A')} &= \Omega^1_{(K\to P)\otimes_{(M\to A)}(M'\to A')/(M'\to A')} \otimes_{P\otimes_A A'} (B \otimes_A A') \\ &= (\Omega^1_{(K\to P)/(M\to A)} \otimes_P B) \otimes_A A' \\ &= \mathbb{L}_{(N\to B)/(M\to A)} \otimes_A A'. \end{aligned}$$

$\square$

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Lemma 2.3. *Let*

$$\begin{tikzcd} (N' \to B') & (N \to B) \arrow[l] \\ (M' \to A') \arrow[u] & (M \to A) \arrow[l] \arrow[u] \end{tikzcd}$$

be a pushout square in $\mathrm{LogRing}^{\mathrm{pre}}$. *It is a homotopy pushout square if and only if the two pushout squares*

(2) {{pushout_monoids_rings}}
$$\begin{tikzcd} N' & N \arrow[l] \\ M' \arrow[u] & M \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} B' & B \arrow[l] \\ A' \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

are homotopy pushout squares.

Proof. Let $(N'' \to B'')$ represent the homotopy pushout of $(M \to A) \to (M' \to A')$ and $(M \to A) \to (N \to B)$. We obtain a map $(N' \to B') \to (N'' \to B'')$. By the definition of the model structure on $s\mathrm{LogRing}^{\mathrm{pre}}$ it is a weak equivalence if and only if $N' \to N''$ and $B' \to B''$ are weak equivalences. The pushout in the category of prelog rings is compatible with the pushouts in the category of monoids and the category of rings:

$$B' \cong A' \otimes_A B \qquad \text{and} \qquad N' \cong M' \sqcup_M N,$$

i.e., the diagrams (2) are pushout squares. Moreover, as $\mathrm{Forget}^{s\mathrm{LogRing}^{\mathrm{pre}}}_{s\mathrm{Mon}\times s\mathrm{Ring}}$ is a left Quillen functor, it preserves homotopy colimits. Therefore, B'' and N'' represent the homotopy pushouts of

$$\begin{tikzcd} & N \\ M & M \arrow[l] \arrow[u] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & B \\ A' & A, \arrow[l] \arrow[u] \end{tikzcd}$$

respectively. We conclude that $(N' \to B') \to (N'' \to B'')$ is a weak equivalence if and only if both $(N' \to N'')$ and $(B' \to B'')$ are. $\square$

{condition_

Corollary 2.4. *Let*

(3) {{log_pushout_square}}
$$\begin{tikzcd} (N' \to B') & (N \to B) \arrow[l] \\ (M' \to A') \arrow[u] & (M \to A) \arrow[l] \arrow[u] \end{tikzcd}$$

be a pushout square in $\mathrm{LogRing}^{\mathrm{pre}}$. *Assume that either of the ring homomorphisms* $A \to B$ *or* $A \to A'$ *is flat and that either* $M \to N$ *or* $M \to M'$ *is an integral homomorphism of integral monoids. Then the square (3) is a homotopy pushout square.*

Proof. By Lemma 2.3 we have to show that the two diagrams in (2) are homotopy pushout squares. For the diagram of rings this is well known. For the diagram of monoids this is [Kat89], Proposition 4.1. $\square$

{localizati

Corollary 2.5. *Let*

$$(M \to A) \to (N \to B)$$

be a homomorphism of prelog rings and $S \subseteq A$ a multiplicative subset. Then

$$\mathbb{L}_{(N \to S^{-1}B)/(M \to S^{-1}A)} \cong S^{-1}(\mathbb{L}_{(N \to B)/(M \to A)}).$$

We finished our treatment of the compatibility of the logarithmic cotangent complex with base change. The rest of this section uses transitivity and base change to compute the log cotangent complex for certain well behaved prelog rings.

polynomial}

Proposition 2.6. *Let $M \to A$ be a prelog ring and N a finitely generated free monoid. Then*

$$H_i(\mathbb{L}_{(M \oplus N \to A[N])/(M \to A)})$$

vanishes for $i \geq 1$ and is isomorphic to $N^{\mathrm{gp}} \otimes A[N]$ for $i = 0$.

Proof. By [Ols05], Theorem 8.16 we know that taking the associated log ring does not change the cotangent complex:

$$\mathbb{L}_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})} \cong \mathbb{L}_{(N \to \mathbb{Z}[N])/(0 \to \mathbb{Z})}.$$

Since $(\{\pm 1\} \to \mathbb{Z})$ is (obviously) log flat over $\mathbb{Z}$ with trivial log structure, Gabber's cotangent complex $\mathbb{L}_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})}$ coincides with Olsson's (see [Ols05], Corollary 8.29), which we denote by $\mathbb{L}^{Ols}_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})}$. But

$$\mathbb{L}^{Ols}_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})} \cong \Omega^1_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})}$$

as $(\{\pm 1\} \to \mathbb{Z}) \to (N^a \to \mathbb{Z}[N])$ is log flat ([Ols05], 1.1 (iii)) and

$$\Omega^1_{(N^a \to \mathbb{Z}[N])/(\{\pm 1\} \to \mathbb{Z})} \cong \mathrm{Hom}_{\mathrm{Mon}}(N, \mathbb{Z}[N]).$$

Now consider the pushout square

$$\begin{tikzcd}
(M \oplus N \to A[N]) & (N \to \mathbb{Z}[N]) \arrow[l] \\
(M \to A) \arrow[u] & (0 \to \mathbb{Z}). \arrow[l] \arrow[u]
\end{tikzcd}$$

The ring homomorphism $\mathbb{Z} \to \mathbb{Z}[N]$ is flat and the monoid N is integral. Hence, by Corollary 2.4, the above square is a homotopy pushout square. Applying Lemma 2.2 yields an isomorphism

$$\mathbb{L}_{(M \oplus N \to A[N])/(M \to A)} \cong \mathbb{L}_{(N \to \mathbb{Z}[N])/(0 \to \mathbb{Z})} \otimes_{\mathbb{Z}} A.$$

From this and the above description of $\mathbb{L}_{(N \to \mathbb{Z}[N])/(0 \to \mathbb{Z})}$ we obtain the result. $\square$

ular_ideal}

Proposition 2.7. *In the situation of Proposition 2.6 let I be a regular ideal of $A[N]$. Then*

$$\mathbb{L}_{(M \oplus N \to A[N]/I)/(M \to A)} \cong (I/I^2 \xrightarrow{-d} \Omega^1_{(M \oplus N \to A[N])/(M \to A)} \otimes_{A[N]} A[N]/I),$$

where I/I^2 is placed in degree -1 and d is induced from the differential.

Proof. The proof is the same as for Olson's cotangent complex ([Ols05], Lemma 6.9): By Proposition 2.1 (i) we have a homotopy cofiber sequence

$$\mathbb{L}_{(M \oplus N \to A[N])/(M \to A)} \otimes^h_{A[N]} A[N]/I \to \mathbb{L}_{(M \oplus N \to A[N]/I)/(M \to A)} \to \mathbb{L}_{(M \oplus N \to A[N]/I)/(M \oplus N \to A[N])}.$$

Proposition 2.6 gives us

$$\mathbb{L}_{(M \oplus N \to A[N])/(M \to A)} \otimes^h_{A[N]} A[N]/I \cong \Omega^1_{(M \oplus N \to A[N])/(M \to A)} \otimes_{A[N]} A[N]/I.$$

Moreover,

$$\mathbb{L}_{(M\oplus N\to A[N]/I)/(M\oplus N\to A[N])} \cong \mathbb{L}_{(A[N]/I)/A[N]} \cong I/I^2[1].$$

We have the first isomorphism because the map on monoids is the identity ([Ols05], Lemma 8.17) and the second one is a classical result for the cotangent complex of rings ([Ill71], III, Proposition 3.2.4).

It remains to show that the resulting map $I/I^2 \to \Omega^1_{(M\oplus N\to A[N])/(M\to A)}$ is given by the negative of the differential. By functoriality we have a factorization

$$I/I^2 \to \Omega^1_{A[N]/A} \to \Omega^1_{(M\oplus N\to A[N])/(M\to A)}.$$

The first map is the negative of the differential by [Ill71], III Proposition 1.2.9 and the second map is the canonical one. □

{cotangent_

Corollary 2.8. *Let A be a ring and M a finitely generated free commutative submonoid of $A^\times$. Then $\mathbb{L}_{(M\to A)/(\{0\}\to A)}$ is concentrated in degree zero.*

Proof. We choose generators $m_1,\dots,m_r$ of M. This defines an isomorphism of $A[M]$ with $A[T_1,\dots,T_r]$. Let I be the ideal of $A[T_1,\dots,T_r]$ generated by $T_i - m_i$ for $i = 1,\dots,r$. This is clearly a regular ideal. By Proposition 2.7 we have

$$\mathbb{L}_{(M\to A)/(\{0\}\to A)} \cong (I/I^2 \xrightarrow{-d} \Omega^1_{(M\to A[M])/(\{0\}\to A)} \otimes_{A[M]} A).$$

We have a natural identification of I/I^2 with the free A-module with generators $(T_i - m_i)$. Moreover, by Proposition 2.6, $\Omega^1_{(M\to A[M])/(\{0\}\to A)} \otimes_{A[M]} A)$ is isomorphic to $M^{\mathrm{gp}} \otimes A$. The differential d maps $(T_i - m_i)$ to $dT_i = T_i(dT_i/T_i)$ (corresponding to $m_i \otimes m_i \in M^{\mathrm{gp}} \otimes A$). This map is injective. □

Finally, we will need that the cotangent complex is compatible with filtered colimits:

{colimits_c

Proposition 2.9. *Let $(M \to A) = \mathrm{colim}_{i\in I}(M_i \to A_i)$ and $(N \to B) = \mathrm{colim}_{i\in I}(N_i \to B_i)$ be filtered colimits in the category of prelog rings. Suppose we are given compatible homomorphisms $(M_i \to A_i) \to (N_i \to B_i)$. Then there is a natural isomorphism*

$$\mathbb{L}_{(N\to B)/(M\to A)} \cong \operatorname*{colim}_{i\in I} \mathbb{L}_{(N_i\to B_i)/(M_i\to A_i)}.$$

Proof. The functors F and G in the canonical resolution (1) commute with filtered colimits and so does the formation of log differentials. □

3. Unramified and tame extensions

For a valued field K we will adopt the following notation. The valuation of an element x in K is written $|x|_K$ or only $|x|$ when it does not cause confusion. We denote the valuation ring of K by K^+ and the value group of the valuation by Γ_K. We endow K^+ with the *total log structure* $(K^+ \setminus \{0\} \to K^+)$. For an extension $L|K$ of valued fields we define

$$\mathbb{L}^{\log}_{L/K} := \mathbb{L}_{(L^+\setminus\{0\}\to L^+)/(K^+\setminus\{0\}\to K^+)}.$$

Remember that a finite extension $L|K$ of valued fields is *unramified* if $L^{\mathrm{sh}} = K^{\mathrm{sh}}$ (strict henselization). It is *tamely ramified* (or tame for short) if $[L^{\mathrm{sh}} : K^{\mathrm{sh}}]$ is prime to the residue characteristic of K^+. In this case $[\Gamma_L : \Gamma_K] = [L^{\mathrm{sh}} : K^{\mathrm{sh}}]$. An algebraic extension $L|K$ of valued fields is tame if all its finite subextensions are tame.

{cotangent_

Lemma 3.1. *Let $L|K$ be unramified. Then $\mathbb{L}^{\log}_{L/K} \cong 0$. In particular, $\Omega^{\log}_{L/K} = 0$.*

Proof. Since $L|K$ is unramified, $\Gamma_L = \Gamma_K$, so the total log structure of L^+ is the logification of $(K^+ \setminus \{0\} \to L^+)$. We can thus compute the logarithmic cotangent complex as follows:

$$\mathbb{L}^{\log}_{L/K} \cong \mathbb{L}_{(K^+\setminus\{0\}\to L^+)/(K^+\setminus\{0\}\to K^+)} \cong \mathbb{L}_{L^+/K^+} \cong 0.$$

The left hand isomorphism is due to [Ols05], Theorem 8.16, the middle one to [Ols05], Lemma 8.17, and the right hand one to [GR03], Theorem 6.3.32 and the well known fact that the differentials vanish for unramified extensions. □

ngent_tame}

Proposition 3.2. *For any tame extension $L|K$ of valued fields the logarithmic cotangent complex is trivial: $\mathbb{L}^{\log}_{L/K} \cong 0$. In particular, $\Omega^{\log}_{L/K} \cong 0$.*

Proof. Using Lemma 3.1 and transitivity (Proposition 2.1 (i)) for the extensions in the diagram

$$\begin{tikzcd} & L^{\mathrm{sh}} \arrow[dl, dash] \arrow[dr, dash] & \\ K^{\mathrm{sh}} \arrow[dr, dash] & & L \arrow[dl, dash] \\ & K. & \end{tikzcd}$$

we reduce to the case where K is strictly henselian. Moreover, since the logarithmic cotangent complex is compatible with filterd colimits (Proposition 2.9),we can reduce to the case of a finite extension. We decompose the extension $L|K$ into a chain of subextensions of prime degree:

$$K = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_n = L$$

such that $[L_{i+1} : L_i]$ is a prime number. Transitivity (Proposition 2.1 (i)) allows us to treat each extension separately. We may thus assume $[L : K]$ is a prime number ℓ (prime to the residue characteristic as $L|K$ is tame).

We have $L = K[a^{1/\ell}]$ for some $a \in K$ with $|a| < 1$. The valuation ring L^+ is the filtered colimit of its subalgebras $R_b = K^+[ba^{1/\ell}]$ with $b \in K$ such that $|ba^{1/\ell}| < 1$ (see the proof of [GR03], Proposition.3.13 (i)). We equip R_b with the prelog structure

$$M_b := (K^+ \setminus \{0\} \oplus \mathbb{N})/\sim \to R_b,$$

where the equivalence relation is generated by $(b^\ell a, 0) \sim (1, \ell)$ (note that the first component is written multiplicatively and the second one additively) and $(x, r) \in M_b$ is mapped to $x(ba^{1/\ell})^r \in R_b$. We claim that the total log structure of L^+ is the logification of the colimit of the prelog rings $(M_b \to R_b)$. Since M_b and R_b are naturally contained in L^+ and we already know that $L^+ = \operatorname{colim}_b R_b$, this amounts to checking that every element $y \in L^+ \setminus \{0\}$ can be written in the form $y = ux(ba^{1/\ell})^r$ for $x \in K^+$, $r \in \mathbb{N}$, $b \in K$ such that $|ba^{1/\ell}|_L < 1$, and u a unit of L^+. We choose $b \in K$ such that $y \in R_b$. Then $|y|_L = |x(ba^{1/\ell})^r|$ for some x and r as above. Setting $u = yx^{-1}(ba^{1/\ell})^{-r}$, the claim follows.

Using that logification does not change the cotangent complex ([Ols05], Theorem 8.16) and Proposition 2.9 this reduces us to showing that $\mathbb{L}_{(M_b\to R_b)/(K^+\setminus\{0\}\to K^+)}$ is concentrated in degree 0.

We now consider the following pushout square of prelog rings:

$$\begin{tikzcd}
(M_b \to R_b) & (\mathbb{N} \to R_b) \arrow[l] \\
(K^+ \setminus \{0\} \to K^+) \arrow[u] & (\mathbb{N} \to K^+), \arrow[l] \arrow[u]
\end{tikzcd}$$

where the prelog structures on the right hand side are given by $r \mapsto (ba^{1/\ell})^r$ and $r \mapsto (b^\ell a)^r$, the right hand vertical monoid homomorphism is $r \mapsto \ell r$, the upper horizontal one $r \mapsto (1, r)$, and the lower one $r \mapsto (b^\ell a)^r$. The identity on K^+ is (obviously) flat and the right hand vertical map of monoids is integral. Hence, the diagram is also a homotopy pushout (see Corollary 2.4). We conclude that

$$\mathbb{L}_{(M_b \to R_b)/(K^+ \setminus \{0\} \to K^+)} \cong \mathbb{L}_{(\mathbb{N} \to R_b)/(\mathbb{N} \to K^+)}.$$

By [Ols05], Theorem 8.16

$$\mathbb{L}_{(\mathbb{N} \to K^+)/(\mathbb{N} \to R_b)} \cong \mathbb{L}_{(\mathbb{N} \to K^+)^a/(\mathbb{N} \to R_b)^a}.$$

The logification of $(\mathbb{N} \to K^+) \to (\mathbb{N} \to R_b)$ is a log étale, integral homomorphism of fine, integral log rings. Its cotangent complex is thus isomorphic to Olsen's cotangent complex ([Ols05], Corollary 8.29), which in turn is concentrated in degree zero by log smoothness ([Ols05], (1.1 (iii))). Moreover, it vanishes in degree zero by [Ogu18], Chapter IV, Propostion 3.1.3. □

4. Logarithmic differentials on adic spaces {section_lo

All Huber pairs in this section will be endowed with the discrete topology and all adic spaces will be discretely ringed, i.e., locally isomorphic to a Huber pair with the discrete topology. Recall from [Hüb18], Definition 6.1, that a Huber pair (A, A^+) is local if A is a local ring with maximal ideal $\mathfrak{m}_A$ and A^+ is the preimage in A of a valuation ring of $A/\mathfrak{m}_A$. Given a local Huber pair (A, A^+), we endow A^+ with the *total log structure* given by

$$(A^+ \cap A^\times \to A^+) = (A^+ \setminus \mathfrak{m}_A \to A^+).$$

This extends the definition of the total log structure of a valuation ring. For a morphism of Huber pairs $(A, A^+) \to (B, B^+)$, we define

$$\Omega^{n,\log}_{(B,B^+)/(A,A^+)} := \Omega^n_{(B^+ \cap B^\times \to B^+)/(A^+ \cap A^\times \to A^+)}.$$

If $n = 1$, we omit n and just write $\Omega^{\log}_{(B,B^+)/(A,A^+)}$. For this section we fix a field k and a valuation ring k^+ of k. We assume that one of the following properties is satisfied:

- the residue characteristic of k^+ is 0,
- k is algebraically closed,
- $k = k^+$ is perfect.

For a Huber pair (A, A^+) over (k, k^+) we use the short notation $\Omega^n_{A^+}$ for $\Omega^n_{A^+/k^+}$ and $\Omega^{n,\log}_{(A,A^+)}$ for $\Omega^{n,\log}_{(A,A^+)/(k,k^+)}$.

4.1. **Logarithmic differentials on local Huber pairs.** The following is a reformulation of results of [GR03], § 6.5.

{Omega_valu

Proposition 4.1. *Let (K, K^+) be any extension of valued fields of (k, k^+). Then $\Omega^{n,\log}_{(K,K^+)}$ and $\Omega^n_{K^+}$ are torsion free for all $n \geq 1$.*

Proof. The statement about Ω_{K^+} is [GR03], Theorem 6.5.15 and Corollary 6.5.21. In [GR03], § 6.5 Gabber and Ramero examine the natural homomorphism

$$\rho_{K^+/k^+} : \Omega^{\log}_{k^+/\mathbb{Z}} \otimes_{k^+} K^+ \to \Omega^{\log}_{K^+/\mathbb{Z}}.$$

The cokernel of ρ_{K^+/k^+} is isomorphic to $\Omega^{\log}_{(K,K^+)}$ ([Ogu18], Chapter IV, Proposition 2.3.1). Therefore the result for $n = 1$ follows from [GR03] Lemma 6.5.16, Theorem 6.5.20, and Corollary 6.5.21. The general case ($n > 1$) follows as well as over a valuation ring exterior products of torsion free modules are torsion free ([HKK17]). □

We want to extend this result to local Huber pairs over (k, k^+). To achieve this, we need some preparation.

orsionfree}

Lemma 4.2. *Let (A, A^+) be a local Huber pair and M^+ an A^+-module. Denote by $\mathfrak{m}$ the maximal ideal of A and set $K^+ = A^+/\mathfrak{m}$. Then M^+ is torsion free over A^+ if and only if $M^+_{\mathfrak{m}}$ is torsion free over A and $M^+/\mathfrak{m}M^+$ is torsion free over K^+.*

Proof. Suppose M^+ is torsion free over A^+. Torsion freeness of $M^+_{\mathfrak{m}}$ is clear as A is flat over A^+. In order to show that $M^+/\mathfrak{m}M^+$ is torsion free, take $a \in A^+ \setminus \mathfrak{m}$ and $m \in M^+$ such that $am \in \mathfrak{m}M^+$. Since $\mathfrak{m}$ is an ideal of A, the action of A^+ on $\mathfrak{m}M^+$ extends to A. But a is a unit in A, whence $m \in \mathfrak{m}M^+$.

Let us now assume that $M^+_{\mathfrak{m}}$ and $M^+/\mathfrak{m}M^+$ are both torsion free. Take $a \in A^+$ and $m \in M^+$ such that $am = 0$. From the torsion freeness of $M^+_{\mathfrak{m}}$ we obtain that either $a = 0$ or there is $s \in A^+ \setminus \mathfrak{m}$ such that $sm = 0$. In the latter case we use the torsion freeness of $M^+/\mathfrak{m}M^+$ to conclude that $m \in \mathfrak{m}M^+$. But $\mathfrak{m}M^+$ extends to an A-module and s is a unit in A, so $m = 0$. □

rsion_free}

Proposition 4.3. *Let (A, A^+) be a local (k, k^+)-algebra such that A is the localization of a smooth k-algebra. Then $\Omega^n_{A^+}$ and $\Omega^{n,\log}_{(A,A^+)}$ are flat A^+-modules. In particular, they torsion free.*

Proof. We give the proof for $\Omega^{n,\log}_{(A,A^+)}$. For $\Omega^n_{A^+}$ the argument is the same. We first show that $\Omega := \Omega^{n,\log}_A$ is torsion free. Let $\mathfrak{m}$ be the maximal ideal of A, $K = A/\mathfrak{m}$, and $K^+ = A^+/\mathfrak{m}$. By Lemma 4.2 we have to show that $\Omega/\mathfrak{m}\Omega$ is torsion free over K^+ and $\Omega_{\mathfrak{m}}$ is torsion free over A. Since $A^+ \to A$ is a localization, we have by [Ogu18], Chapter IV, Proposition 1.1.3

$$\Omega_{\mathfrak{m}} \cong \Omega^n_{A/k}$$

and this is torsion free (even free) as A is the localization of a smooth k-algebra. Now consider the short exact sequence (see [Ogu18], Chapter IV, Proposition 2.3.2 and Theorem 3.2.2)

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega/\mathfrak{m} \to \Omega^{\log}_{(K,K^+)} \to 0.$$

Since $\mathfrak{m}/\mathfrak{m}^2$ is a K-vector space, it is torsion free over K^+. Moreover, $\Omega^{\log}_{(K,K^+)}$ is torsion free by Proposition 4.1.

Knowing that Ω is torsion free and $\Omega_{\mathfrak{m}}$ is flat and we can now apply [Hüb18], Proposition 10.7 to conclude that Ω is flat. □

f_log_diff}

4.2. **The presheaf of logarithmic differentials.** The naive idea of defining logarithmic differentials on an adic space $\mathscr{X}$ is to set for an affinoid open $\operatorname{Spa}(A, A^+)$

$$\Omega^{\log}(\operatorname{Spa}(A, A^+)) = \Omega^{\log}_{(A,A^+)}$$

and to glue these for general open subspaces. This approach is too naive fo various reasons. Unfortunately the sheaf condition is not satisfied. Consider for instance the following

Example 4.4. Let $\mathscr{X}$ be the affinoid adic space $\operatorname{Spa}(k[T, T^{-1}], k)$ over an algebraically closed field k. On the one hand,

$$\Omega^{\log}_{(k[T,T^{-1}],k)} = \Omega_{(k\setminus\{0\}\to k)/(k\setminus\{0\}\to k)} = 0.$$

On the other hand, $\mathscr{X} = \operatorname{Spa}(\mathbb{G}_{m,k}, \mathbb{P}^1_k)$ is covered by the affinoid open subspaces $\operatorname{Spa}(k[T, T^{-1}], k[T])$ and $\operatorname{Spa}(k[T, T^{-1}], k[T^{-1}])$. The logarithmic differentials dT/T and $-dT^{-1}/dT^{-1}$ on $\operatorname{Spa}(k[T, T^{-1}], k[T])$ and $\operatorname{Spa}(k[T, T^{-1}], k[T^{-1}])$, respectively, coincide on the intersection but do not lift to a global section. Hence, the sheaf condition is not satisfied.

Apart from the fact that the above defined presheaf of logarithmic differentials is not a sheaf, its secions on $\operatorname{Spa}(k[T, T^{-1}], k)$ are not the ones we would expect. Intuitively there should be a global section lifting dT/T and $-dT^{-1}/dT^{-1}$.

To overcome the problem described in the example we only work with strict affinoids, which are defined as follows.

Definition 4.5. We say that a Huber pair (A, A^+) is strict if A is a localization of A^+. An affinoid adic space $\operatorname{Spa}(A, A^+)$ is strict if (A, A^+) is strict. For an adic space $\mathscr{X}$ we denote the category of strict affinoid open subspaces by $\mathscr{X}_{\mathrm{straff}}$.

{strict_nei

Lemma 4.6. *Let $\mathscr{X}$ be an adic space locally of the form $\operatorname{Spa}(A, A^+)$ with A/A^+ essentially of finite type. Then the strict affinoids of $\mathscr{X}$ form a basis of the topology.*

Proof. Without loss of generality we may assume that $\mathscr{X}$ is of the form $\operatorname{Spa}(A, A^+)$ with A/A^+ essentially of finite type. Given an affinoid open subspacee $\operatorname{Spa}(B, B^+)$ we have a diagram

$$\begin{tikzcd} B & A \arrow[l] \\ B^+ \arrow[u] & A^+ \arrow[l] \arrow[u] \end{tikzcd}$$

such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is an open immersion and B^+ is the normalization in B of an A^+-algebra of finite type. In particular, B is essentially of finite type over B^+. It thus has a compactification $Y \to \operatorname{Spec} B^+$. By [Hüb18], Lemma 7.5 we have an identification $\operatorname{Spa}(B, B^+) = \operatorname{Spa}(B, Y)$. Covering Y by affines $\operatorname{Spec} A_i^+$ and each $\operatorname{Spec} B \cap \operatorname{Spec} A_i^+$ by affines $\operatorname{Spec} A_{ij}$, we obtain a cover of $\operatorname{Spa}(B, Y)$ by the strict affinoids $\operatorname{Spa}(A_{ij}, A_i^+)$. $\square$

{tensor_pro

Lemma 4.7. *Let (A, A^+), (B, B^+), and (C, C^+) be strict affinoids. Then the tensor product*

$$(D, D^+) = (B, B^+) \otimes_{(A,A^+)} (C, C^+)$$

is strict.

Proof. Let $S = D^+ \cap D^\times$. We claim that $D = S^{-1}D^+$. Every element of S is invertible in D, whence the existence of a natural homomorphism $S^{-1}D^+ \to D$. Injectivity is clear as $D^+ \subset D$. Let $d \in D$. We want to write $d = d^+/s$ for $d^+ \in D^+$ and $s \in S$. Without loss of generality we may assume $d = b \otimes c$ for $b \in B$ and $c \in C$. But by assumption $b = b^+/s$ and $c = c^+/t$ for $b^+ \in B^+$, $s \in B^+ \cap B^\times$, and $b \in C^+ \cap C^\times$. Hence $d = (b^+ \otimes c^+)/(s \otimes t)$. □

Let $X \to S$ be a morphism of schemes which is essentially of finite type. We equip $\mathrm{Spa}(X,S)_{\mathrm{straff}}$ (see [Tem11], § 3.1 for the definition) with the topology whose coverings are surjective families. Note that by Lemma 4.7 the necessary fiber products for the structure of a site exist. We denote by $\mathrm{Spa}(X,S)_{\mathrm{top}}$ the site associated with the topological space $\mathrm{Spa}(X,S)$. By Lemma 4.6 the corresponding topoi of $\mathrm{Spa}(X,S)_{\mathrm{straff}}$ and $\mathrm{Spa}(X,S)_{\mathrm{top}}$ are equivalent. If $\mathcal{F}$ is a presheaf on $\mathrm{Spa}(X,S)_{\mathrm{straff}}$ we can view its sheafification as a sheaf $\mathcal{G}$ on all of $\mathrm{Spa}(X,S)$. Slightly abusing notation we will say that $\mathcal{G}$ is the sheafification of $\mathcal{F}$. We have thus justified the restriction to strict affinoids.

Our presheaf of interest is the presheaf of logarithmic differentials $\Omega^{\log}$. It is defined on $\mathrm{Spa}(X,S)_{\mathrm{straff}}$ as

$$\Omega^{\log}(\mathrm{Spa}(A,A^+)) := \Omega^{\log}_{(A,A^+)}.$$

Similarly we define $\Omega^{n,\log}$ by

$$\Omega^{n,\log}(\mathrm{Spa}(A,A^+)) := \Omega^{n\,\log}_{(A,A^+)}.$$

Even restricted to strict affinoids $\Omega^{\log}$ is not a sheaf as the following example shows.

Example 4.8. For positive integers d and r we consider the action of μ_d on $\mathbb{C}[X_0,\dots,X_r]$ induced by the diagonal embedding of μ_d in $\mathrm{Gl}_{r+1}(\mathbb{C})$. In other words, $\xi \in \mu_d$ acts by multiplying each coordinate with ξ. We consider the quotient spaces

$$X_{r,d} := (\mathrm{Spec}\,\mathbb{C}[X_0,\dots,X_r])/\mu_d = \mathrm{Spec}\,\mathbb{C}[X_0,\dots,X_r]^{\mu_d}.$$

They are normal and can also be described as the affine cone of the dth Veronese embedding of $\mathbb{P}^r_C$. Moreover, note that

$$A^+_{r,d} := \mathbb{C}[X_0,\dots,X_r]^{\mu_d}$$

is the $\mathbb{C}$-subalgebra of $\mathbb{C}[X_0,\dots,X_r]$ generated by all monomials of degree d. In [GR11], Proposition 4.1 it is shown that $\Omega_{X_{r,d}}$ has torsion if and only if $d \geq 3$.

Let $U_{r,d} = \mathrm{Spec}\,A_{r,d}$ be the open subscheme of $X_{r,d}$ defined by inverting X_0^d. Then $(A_{r,d},A^+_{r,d})$ is a strict Huber pair. By transitivity (Proposition 2.1 (i)) and the vanishing of $H_1(\mathbb{L}_{((X_0^d)^{\mathbb{N}}\to A^+_{r,d})/(\{0\}\to A^+_{r,d})})$ (Corollary 2.8) we know that

$$\Omega_{A^+_{r,d}/k} \to \Omega^{\log}_{(A_{r,d},A^+_{r,d})/(\mathbb{C},\mathbb{C})}$$

is injective. Hence, $\Omega^{\log}_{(A_{r,d},A^+_{r,d})/(\mathbb{C},\mathbb{C})}$ has torsion as well for $d \geq 3$.

Let $Y_{r,d} \to X_{r,d}$ be the blowup in the origin. Denote by D the Cartier divisor of $Y_{r,d}$ which is the pullback of the Cartier divisor of $X_{r,d}$ defined by X_0^d. Then $Y_{r,d}$ is smooth and D is a simple normal crossings divisor. In particular, $(U_{r,d},Y_{r,d})$ is log smooth, so $\Omega^{\log}_{(U_{r,d},Y_{r,d})/(\mathbb{C},\mathbb{C})}$ is torsion free.

We cover $Y_{r,d}$ by affine schemes $\mathrm{Spec}\,B_i^+$. As the complement of $U_{r,d}$ in $Y_{r,d}$ is the support of a principal Cartier divisor, the intersection of $\mathrm{Spec}\,B_i$ with $U_{r,d}$ is still affine. We denote the corresponding ring by B_i. The strict affinoids $\mathrm{Spa}(B_i,B_i^+)$ cover $\mathrm{Spa}(A_{r,d},A^+_{r,d})$.

Moreover, $\Omega^{\log}_{(B_i,B_i^+)/(\mathbb{C},\mathbb{C})}$ is a finitely generated free B_i^+-module as (B_i, B_i^+) is log smooth over $(\mathbb{C},\mathbb{C})$. In particular, $\Omega^{\log}_{(B_i,B_i^+)/(\mathbb{C},\mathbb{C})}$ is torsion free over $A^+_{r,d}$. But then

$$\Omega^{\log}_{(A_{r,d},A^+_{r,d})/(\mathbb{C},\mathbb{C})} \to \prod_i \Omega^{\log}_{(B_i,B_i^+)/(\mathbb{C},\mathbb{C})}$$

cannot be injective because $\Omega^{\log}_{(A_{r,d},A^+_{r,d})/(\mathbb{C},\mathbb{C})}$ has torsion. We conclude that the sheaf axiom is not satisfied.

The example already suggests that the problems lie in the singularities of $\operatorname{Spec} A^+$. Indeed we will see in Section 6 that the differentials are well behaved for log smooth Huber pairs.

We are now interested in the sheafification of $\Omega^{n,\log}$. For a strict Huber pair (A, A^+) over (k, k^+) and $n \geq 1$, we consider the natural map

$$\Omega^{n,\log}_{(A,A^+)} \to \Omega^{n,\log}_{(A,A^+)} \otimes_{A^+} A \xrightarrow{\sim} \Omega_A.$$

If (A, A^+) is local and A is the localization of a smooth k-algebra, it is injective. (Proposition 4.3). We conclude that the sheafification of $\Omega^{n,\log}$ is a subsheaf of Ω^n in case $\operatorname{Spa}(X, S)$ is smooth over (k, k^+). It will turn out that the sheafification can be described in terms of the Kähler seminorm which we study in the next subsection.

5. The Kähler seminorm {section_Ka

5.1. The Kähler seminorm for local Huber pairs. Fix (k, k^+) as before. Moreover, throughout this subsection (A, A^+) is a local Huber pair over (k, k^+) equipped with the discrete topology. We denote by $\mathfrak{m}$ the maximal ideal of A^+.

Definition 5.1. We define the Kähler seminorm on Ω_A by

$$|\omega|_\Omega := \inf_{\omega=\sum f_i dg_i} \max_i\{|f_i|_A |g_i|_A\},$$

where the infimum is over all representations of ω as a finite sum $\omega = \sum_i f_i dg_i$.

The Kähler seminorm has been studied in [Tem16], § 5 for real valued fields. As our setting is a little bit different, we give the proofs of the properties we need although they are similar as in loc. cit. By Proposition 4.3 we can consider $\Omega^{\log}_{(A,A^+)}$ as an A^+-submodule of Ω_A. {unit_ball}

Lemma 5.2. *We have*

$$\Omega^{\log}_{(A,A^+)} = \{\omega \in \Omega_A \mid |\omega|_\Omega \leq 1\}.$$

Proof. For $\omega = \sum_i f_i dg_i/g_i$ in $\Omega^{\log}_{(A,A^+)}$ we have

$$|\omega|_\Omega \leq \max_i\{|f_i|\} \leq 1.$$

Now take $\omega \in \Omega^1_A$ with $|\omega|_\Omega \leq 1$. By definition there is a representation $\omega = \sum_i f_i dg_i$ with

$$\max_i\{|f_i|_A |g_i|_A\} \leq 1,$$

i.e., $|f_i|_A |g_i|_A \leq 1$ for all i. So $f_i g_i \in A^+$.

In case $g_i \notin \mathfrak{m}$ we write $g_i = g_i'/g_i''$ with $g_i', g_i'' \in A^+ \setminus \mathfrak{m}$. Then

$$f_i dg_i = f_i g_i \frac{dg_i'}{g_i'} + f_i g_i \frac{dg_i''}{g_i''} \in \Omega^{\log}_{(A,A^+)}.$$

Suppose now that $g_i \in \mathfrak{m}$. If $f_i \in \mathfrak{m}$ as well, then in particular, f_i and g_i are both elements of A^+. Hence, $f_i dg_i \in \Omega_{A^+} \subset \Omega^{\log}_{(A,A^+)}$ (Ω_{A^+} is torsion free by Proposition 4.3). Finally, if $g_i \in \mathfrak{m}$ but $f_i \notin \mathfrak{m}$, we write

$$f_i dg_i = d(f_i g_i) - g_i df_i.$$

The first term is in $\Omega^1_{A^+} \subset \Omega^{\log}_{(A,A^+)}$ and the second term in $\Omega^{\log}_{(A,A^+)}$ by the same reasoning as above. We conclude that $\omega \in \Omega^{\log}_{(A,A^+)}$. □

on_Kaehler}

Lemma 5.3. *The Kähler seminorm is the maximal A-seminorm on Ω_A with $|\Omega^{\log}_A|_\Omega \leq 1$ and $|dx|_\Omega = 0$ for $x \in \mathfrak{m}$.*

Proof. We already know from Lemma 5.2 that the Kähler seminorm is less or equal to one on logarithmic differentials. It is also clear from the definition that $|dx|_\Omega = 0$ for $x \in \mathfrak{m}$. It remains to show the maximality. Let $|-|$ be a seminorm such that $|\Omega^{\log}_A| \leq 1$ and $|dx| = 0$ for $x \in \mathfrak{m}$. Let $\omega \in \Omega_A$ and pick a representation $\omega = \sum_i f_i dg_i$. For every i such that $g_i \notin \mathfrak{m}$ take $g'_i, g''_i \in A^+ \setminus \mathfrak{m}$ such that $g_i = g'_i/g''_i$. Then

$$f_i dg_i = f_i g_i (\frac{dg'_i}{g'_i} + \frac{dg''_i}{g''_i}).$$

For i with $g_i \in \mathfrak{m}$ we have $|f_i dg_i| = 0$. Hence, by the strong triangle inequality,

$$|\omega| \leq \max_{i, g_i \notin \mathfrak{m}} \{ |f_i g_i|_A |\frac{dg'_i}{g'_i}|, |f_i g_i|_A |\frac{dg''_i}{g''_i}| \}.$$

By our assumption $|\frac{dg'_i}{g'_i}| \leq 1$ and $|\frac{dg''_i}{g''_i}| \leq 1$, whence

$$|\omega| \leq \max_{i, g_i \notin \mathfrak{m}} \{ |f_i|_A |g_i|_A \} = \max_i \{ |f_i|_A |g_i|_A \}.$$

Since this holds for all representations $\omega = \sum_i f_i g_i$, we obtain $|\omega| \leq |\omega|_\Omega$. □

Definition 5.4. For a local Huber pair (A, A^+) and an A^+-module M we define the adic seminorm by

$$|x|_{ad} := \inf_{\substack{a^+ \in A^+ \\ x \in a^+ M}} |a^+|_A.$$

We can consider the adic seminorm on $\Omega^{\log}_A$. On the other hand, we have an inclusion $\Omega^{\log}_{(A,A^+)} \hookrightarrow \Omega_A$ (see Proposition 4.3). We thus obtain a seminorm on $\Omega^{\log}_{(A,A^+)}$ by restricting the Kähler seminorm to $\Omega^{\log}_{(A,A^+)}$.

ehler_adic}

Lemma 5.5. *For $\omega \in \Omega^{\log}_{(A,A^+)}$ we have $|x|_\Omega = |x|_{ad}$*

Proof. By Lemma 5.3 it suffices to show that $|\Omega^{\log}_{(A,A^+)}|_{ad} \leq 1$, $|dx|_{ad} = 0$ for $x \in \mathfrak{m}$, and $|\omega|_\Omega \leq |\omega|_{ad}$ for all $\omega \in \Omega^{\log}_{(A,A^+)}$. The first assertion is obvious as $|\Omega^{\log}_{(A,A^+)}|_{ad} \subseteq |A^+|_A$. For the second one take $x \in \mathfrak{m}$ and $a^+ \in A^+ \setminus \mathfrak{m}$. Then x is divisible by a^+ and

$$dx = d(a^+ \cdot \frac{x}{a^+}) - (\frac{x}{a^+}) da^+ = a^+ (d(\frac{x}{a^+}) - \frac{x}{(a^+)^2}) da^+,$$

i.e., $dx \in a^+ \Omega^{\log}_{(A,A^+)}$. By the definition of the adic seminorm, this means $|dx| \leq |a^+|$. As a^+ was arbitrary, this implies $|dx| = 0$.

Let us now prove the last assertion. Take $\omega \in \Omega^{\log}_{(A,A^+)}$ and $a^+ \in A^+$ with $\omega \in a^+\Omega^{\log}_{(A,A^+)}$. So there is a representation $\omega = \sum_i a^+ f_i dg_i/g_i$ with $f_i \in A^+$ and $g_i \in A^+ \setminus \mathfrak{m}$. Then

$$|\omega|_\Omega \leq \max_i \{|a^+|_A \cdot |f_i|_A\} \leq |a^+|_A.$$

Since this holds for all $a^+ \in A^+$ with $\omega \in a^+\Omega^{\log}_{(A,A^+)}$, we obtain

$$|\omega|_\Omega \leq |\omega|_{ad}.$$

□

{isometry}

Proposition 5.6. *Let $(B,B^+)/(A,A^+)$ be a tame extension of local Huber pairs. Then*

$$\Omega_A \otimes_A B \xrightarrow{\sim} \Omega_B$$

is an isometry (with respect to the Kähler seminorm).

Proof. Consider the following map of distinguished triangles

$$\begin{tikzcd} \mathbb{L}^{\log}_{(A,A^+)} \otimes^{\mathbb{L}}_{A^+} B^+ \arrow[r] \arrow[d] & \mathbb{L}^{\log}_{(B,B^+)} \arrow[r] \arrow[d] & \mathbb{L}^{\log}_{(B,B^+)/(A,A^+)} \arrow[r, "+1"] \arrow[d] & {} \\ \mathbb{L}_A \otimes^{\mathbb{L}}_A B \arrow[r] & \mathbb{L}_B \arrow[r] & \mathbb{L}_{B/A} \arrow[r, "+1"] & {} . \end{tikzcd}$$

Since B/A is étale, $\mathbb{L}_{B/A} \cong 0$ ([Ill71], Proposition 3.1.1). Moreover, $(B,B^+)/(A,A^+)$ is tame, whence $\mathbb{L}^{\log}_{(B,B^+)/(A,A^+)} \cong 0$ (see Proposition 3.2). Furthermore, B is flat over A and B^+ is flat over A^+ (see [Hüb18], Proposition 10.7). Hence the derived tensor products are naive tensor products. We thus obtain a diagram

$$\begin{tikzcd} \Omega^{\log}_{(A,A^+)} \otimes_{A^+} B^+ \arrow[r, "\sim", "\phi"'] \arrow[d] & \Omega^{\log}_{(B,B^+)} \arrow[d] \\ \Omega_A \otimes_A B \arrow[r, "\sim", "\psi"'] & \Omega_B. \end{tikzcd}$$

The vertical maps are localizations by Corollary 2.5. Hence, in order to show that ψ is an isometry, it suffices to show that ϕ is an isometry. But the restriction of the Kähler seminorm to logarithmic differentials coincides with the adic seminorm (Lemma 5.5) and the adic seminorm is unique for a given A^+-module. □

{section_Ka

5.2. **The Kähler seminorm on adic spaces.** For a discretely ringed adic space $\mathscr{X}$, a point $x \in \mathscr{X}$, and an open neighborhood $\mathscr{U} \subset \mathscr{X}$ of x we define the Kähler seminorm $|-|_x$ on $\Omega_{\mathscr{X}}(\mathscr{U})$ associated with x as follows. For $\omega \in \Omega_{\mathscr{X}}(\mathscr{U})$ let ω_x be the image of ω in $\Omega_{\mathscr{X},x} = \Omega_{\mathscr{O}_{\mathscr{X},x}}$. Then

$$|\omega|_x := |\omega_x|_\Omega,$$

where $|-|_\Omega$ is the Kähler seminorm on $\Omega_{\mathscr{O}_{\mathscr{X},x}}$ associated with the local Huber pair $(\mathscr{O}_{\mathscr{X},x}, \mathscr{O}^+_{\mathscr{X},x})$.

Recall from [Hüb18], that an étale morphism $f : \mathscr{X} \to \mathscr{Y}$ of adic spaces is strongly étale at a point $x \in \mathscr{X}$ if the residue field extension $k(x)|k(f(x))$ is unramified with respect to the valuation of $k(x)$ corresponding to x. Moreover, f is tame if $k(x)|k(f(x))$ is tamely ramified. The tame (strongly étale) morphisms to $\mathscr{X}$ together with surjective families form a site $\mathscr{X}_t$ ($\mathscr{X}_{s\acute{e}t}$), called the tame (strongly étale) site.

Definition 5.7. We define the subpresheaf Ω^+ of Ω on $\mathscr{X}_t$ by

$$\Omega^+(\mathscr{U}) := \{\omega \in \Omega(\mathscr{U}) \mid |\omega|_x \leq 1 \ \forall x \in \mathscr{U}\}.$$

Notice that this construction is indeed functorial: For $\mathscr{V} \to \mathscr{U}$ in $\mathscr{X}_t$, $\omega \in \Omega^+(\mathscr{U})$, and $x \in \mathscr{V}$ we have

$$|\omega|_{\mathscr{V}}|_x = |\omega|_{f(x)} \leq 1$$

by Proposition 5.6. By restriction, we obtain a presheaf on the topological space $\mathscr{X}$ and on the strongly étale site $\mathscr{X}_{sét}$, as well. We denote all of these Ω^+. Moreover, we set $\Omega^{n,+} := \bigwedge^n \Omega^+$.

Proposition 5.8. *The presheaf Ω^+ is a sheaf on $\mathscr{X}_t$.*

Proof. For a covering $(\varphi_i : \mathscr{U}_i \to \mathscr{U})$ in $\mathscr{X}_t$ consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^+(\mathscr{U}) \arrow[r] \arrow[d, hook] & \prod_i \Omega^+(\mathscr{U}_i) \arrow[r] \arrow[d, hook] & \prod_{ij} \Omega^+(\mathscr{U}_i \times_{\mathscr{U}} \mathscr{U}_j) \arrow[d, hook] \\
0 \arrow[r] & \Omega(\mathscr{U}) \arrow[r] & \prod_i \Omega(\mathscr{U}_i) \arrow[r] & \prod_{ij} \Omega(\mathscr{U}_i \times_{\mathscr{U}} \mathscr{U}_j).
\end{tikzcd}$$

The lower row is exact as Ω is a sheaf. We have to show that the upper row is exact. Since $\Omega^+ \to \Omega$ is injective, exactness on the left hand side is clear. Let $(\omega_i)_i \in \prod_i \Omega^+(\mathscr{U}_i)$ be such that

$$\omega_i|_{\mathscr{U}_i \times_{\mathscr{U}} \mathscr{U}_j} = \omega_j|_{\mathscr{U}_i \times_{\mathscr{U}} \mathscr{U}_j} \quad \forall i, j.$$

There is $\omega \in \Omega(\mathscr{U})$ such that $\omega|_{\mathscr{U}_i} = \omega_i$ for all i. In order to show that $\omega \in \Omega^+(\mathscr{U})$, take $x \in \mathscr{U}$. Since $(\varphi_i : \mathscr{U}_i \to \mathscr{U})$ is a covering, there is i and $x_i \in \mathscr{U}_i$ such that $\varphi_i(x_i) = x$. By Proposition 5.6

$$|\omega|_x = |\omega|_{\mathscr{U}_i}|_{x_i} = |\omega_i|_{x_i} \leq 1.$$

Hence $\omega \in \Omega^+(\mathscr{U})$. □

Remark 5.9. For a morphism $\varphi : \mathscr{V} \to \mathscr{U}$ in $\mathscr{X}_{ét}$ that is not tame, $\omega \in \Omega(\mathscr{U})$, and $x \in \mathscr{V}$, it is not true in general that $|\omega|_{\mathscr{V}}|_x = |\omega|_{\varphi(x)}$ (compare [Tem16], Theorem 5.6.4). We only have $|\omega|_{\mathscr{V}}|_x \leq |\omega|_{\varphi(x)}$. So Ω^+ is a presheaf on the étale site but not necessarily a sheaf.

Proposition 5.10. *Let $n \geq 1$ and $\mathscr{X}$ a (discretely ringed) adic space. As a sheaf on the topological space $\mathscr{X}$, $\Omega^{n,+}$ is the sheafification of $\Omega^{n,\log}$. In particular, $\Omega^{n,+}$ is a subsheaf of Ω^n.*

Proof. Let us first show the proposition for $n = 1$. The homomorphism $\Omega^{\log} \to \Omega$ factors through Ω^+ as for an open $\mathscr{U} \subseteq \mathscr{X}$, $\omega \in \Omega^{\log}(\mathscr{U})$ and $x \in \mathscr{U}$ we have

$$\omega_x \in \Omega^{\log}_{(\mathscr{O}_{\mathscr{X},x}, \mathscr{O}^+_{\mathscr{X},x})}$$

and

$$|\Omega^{\log}_{(\mathscr{O}_{\mathscr{X},x}, \mathscr{O}^+_{\mathscr{X},x})}|_\Omega \leq 1$$

by Lemma 5.2. It thus suffices to show that for all $x \in \mathscr{X}$ the induced homomorphism on stalks

$$\Omega^{\log}_x \to \Omega^+_x$$

is an isomorphism. This is precisely the assertion of Lemma 5.2.

For $n \geq 1$ it is clear by definition and from the result for $n = 1$ that the sheafification of $\Omega^{n,\log}$ is $\Omega^{n,+}$. It then follows from Proposition 4.3 that the natural homomorphism $\Omega^{n,+} \to \Omega^n$ is injective. $\square$

Note that the sheafification of $\Omega^{n,\log}$ on the topological space $\mathscr{X}$ also provides the sheafification on the strongly étale and on the tame site as $\Omega^{n,+}$ is a tame sheaf.

6. Differentials on smooth adic spaces

{section_di

6.1. **Setup.** Recall from [Hub96], Definition 1.6.5 that a morphism $\mathscr{X} \to \mathscr{Y}$ of adic spaces is *smooth* if it is locally of finite presentation and for every morphism $\mathrm{Spa}(A, A^+) \to \mathscr{Y}$ from an affinoid adic space and every ideal I of A with $I^2 = 0$, the homomorphism

$$\mathrm{Hom}_{\mathscr{Y}}(\mathrm{Spa}(A, A^+), \mathscr{X}) \to \mathrm{Hom}_{\mathscr{Y}}(\mathrm{Spa}(A, A^+)/I, \mathscr{X})$$

is surjective.

We fix a perfect field k and consider discretely ringed adic spaces over $\mathrm{Spa}(k, k)$. For short we will speak of adic spaces over k.

A pair of schemes $(X, \bar{X})$ is called *log smooth* if X is an open subscheme of $\bar{X}$ (we implicitly take the immersion $X \to \bar{X}$ as part of the datum) such that the associated log structure on $\bar{X}$ is log smooth over k. We say that $X \to \bar{X}$ is a *log smooth presentation* of an adic space $\mathscr{X}$ over k if $\mathscr{X} = \mathrm{Spa}(X, \bar{X})$ and $(X, \bar{X})$ is log smooth. In particular, if $\mathscr{X}$ has a log smooth presentation, it is smooth. The converse direction only holds under the assumption that resolutions of singularities exist over k.

For a morphism of schemes $X \to S$ such that $\mathrm{Spa}(X, S)$ is a smooth adic space over k, we consider the following site $(X, S)_{\log}$: The objects are finite disjoint unions of log smooth pairs $(Y, \bar{Y})$ fitting into a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ \bar{Y} \arrow[r] & S \end{tikzcd}$$

such that $Y \to X$ is an open immersion and $\bar{Y} \to S$ is the normalization in Y of a scheme of finite type over S. The morphisms are compatible morphisms of pairs over (X, S) (but we do not require the associated morphism of log schemes to be log smooth). If (X, S) itself is log smooth, it is a final object of $(X, S)_{\log}$. A morphism $(Y', \bar{Y}') \to (Y, \bar{Y})$ in $(X, S)_{\log}$ is called an *open immersion* if the associated morphism of log schemes is an open immersion, i.e., $\bar{Y}' \to \bar{Y}$ is an open immersion and $Y' = Y \times_{\bar{Y}} \bar{Y}'$. We define the coverings of $(X, S)_{\log}$ to be surjective families

$$((Y_i, \bar{Y}_i) \to (Y, \bar{Y}))_{i \in I}$$

of open immersions. In other words, the topology is the Zariski topology on $\bar{Y}$.

On $(X, S)_{\log}$ we consider the sheaf $\Omega^{n,\log}$ of logarithmic differentials (compare [Ogu18], Theorem 1.2.4). It is no coincidence that the symbol $\Omega^{n,\log}$ is the same as for the presheaf of logarithmic differentials on the site of strict affinoids studied in Section 4.2. In fact for a strict affinoid $\mathrm{Spa}(A, A^+)$ such that $(\mathrm{Spec}\, A, \mathrm{Spec}\, A^+)$ is log smooth, we have

$$\Omega^{n,\log}_{(A,A^+)} = \Omega^{n,\log}(\mathrm{Spec}\, A, \mathrm{Spec}\, A^+)$$

by construction. Because of this compatibility the use of $\Omega^{n,\log}$ in both situations will not cause confusion.

For an object $(Y,\bar{Y})$ of $(X,S)_{\log}$ the induced morphism

$$\mathrm{Spa}(Y,\bar{Y}) \to \mathrm{Spa}(X,S)$$

is an open immersion. We thus obtain a morphism of sites

$$\ell : \mathrm{Spa}(X,S)_{\mathrm{top}} \to (X,S)_{\log}.$$

For log smooth Huber pairs (A,A^+) Lemma 5.2 provides functorial homomorphisms

$$\Omega^{n,\log}(\mathrm{Spec}\,A, \mathrm{Spec}\,A^+) \longrightarrow \Omega^{n,+}_{(A,A^+)} = \ell_*\Omega^{n,+}(\mathrm{Spec}\,A, \mathrm{Spec}\,A^+).$$

Since the log smooth pairs of the form $(\mathrm{Spec}\,A, \mathrm{Spec}\,A^+)$ form a basis of the topology of $(X,S)_{\log}$ and both $\Omega^{n,\log}$ and $\ell_*\Omega^{n,+}$ are sheaves on $(X,S)_{\log}$, the above homomorphism extends to a homomorphism of sheaves

$$\varphi : \Omega^{n,\log} \to \ell_*\Omega^{n,+}.$$

Our goal is to prove that if $\mathrm{Spa}(X,S)$ is smooth, φ is an isomorphism. Since we do not want to use resolution of singularities, the argument is somewhat intricate. It is inpired from [HKK17]. However, we have adapted the constructions to our situation to produce a more streamlined argument.

ed_sheaves}

6.2. Unramified sheaves.

Definition 6.1. We say that a morphism of schemes $Y \to Z$ is an isomorphism in codimension one if there is an open subscheme $U \subseteq Z$ containing all points of codimension ≤ 1 such that the base change $Y \times_Z U \to U$ is an isomorphism. A morphism $(Y,\bar{Y}) \to (Z,\bar{Z})$ in $(X,S)_{\log}$ is an isomorphism in codimension one if $\bar{Y} \to \bar{Z}$ is an isomorphism in codimension one and $Y = Z \times_{\bar{Y}} \bar{Z}$. In this case we write $(Y,\bar{Y}) \sim_1 (Z,\bar{Z})$.

In a similar way as in [Mor12], Definition 2.1, we define unramified sheaves:

Definition 6.2. A sheaf $\mathscr{F}$ on $(X,S)_{\log}$ is called unramified if for any open immersion $(Y',\bar{Y}') \to (Y,\bar{Y})$ in $(X,S)_{\log}$ with dense image the restriction

$$\mathscr{F}(Y,\bar{Y}) \to \mathscr{F}(Y',\bar{Y}')$$

is injective and an isomorphism if $(Y',\bar{Y}') \sim_1 (Y,\bar{Y})$.

A presheaf $\mathscr{G}$ on $\mathrm{Spa}(X,S)$ is called unramified if $\ell_*\mathscr{G}$ is an unramified sheaf.

Lemma 6.3. *Let $\mathscr{F}$ be an unramified sheaf on $(X,S)_{\log}$. If $(Y',\bar{Y}') \to (Y,\bar{Y})$ in $(X,S)_{\log}$ induces an isomorphim $\mathrm{Spa}(Y',\bar{Y}') \to \mathrm{Spa}(Y,\bar{Y})$, then the restriction*

$$\mathscr{F}(Y,\bar{Y}) \to \mathscr{F}(Y',\bar{Y}')$$

is an isomorphism.

Proof. The morphism $\mathrm{Spa}(Y',\bar{Y}') \to \mathrm{Spa}(Y,\bar{Y})$ is an isomorphism if and only if $Y' \cong Y$ and $\bar{Y}' \to \bar{Y}$ is proper birational. Since $(Y,\bar{Y})$ is log smooth, $\bar{Y}$ is normal. Hence, the exceptional locus of $\bar{Y}' \to \bar{Y}$ in $\bar{Y}$ is of codimension ≥ 2. In other words, its complement $\bar{V} \subset \bar{Y}$ contains all points of codimension ≤ 1. By construction $Y \subseteq \bar{V}$ and the open immersion $\bar{V} \to \bar{Y}$ lifts to an open immersion $\bar{V} \to \bar{Y}'$. We thus obtain a diagram

$$\begin{tikzcd}
 & (Y', \bar{Y}') \arrow[dd] \\
(Y, \bar{V}) \arrow[ur, hook, "\circ"] \arrow[dr, hook, "\circ"'] & \\
 & (Y, \bar{Y}).
\end{tikzcd}$$

The diagonal arrows are open immersions with dense image and the image of the lower one contains all points of codimension ≤ 1. Applying $\mathscr{F}$ yields

$$\begin{tikzcd}
 & \mathscr{F}(Y', \bar{Y}') \arrow[dl, hook'] \\
\mathscr{F}(Y, \bar{V}) & \\
 & \mathscr{F}(Y, \bar{Y}). \arrow[ul, "\sim"'] \arrow[uu]
\end{tikzcd}$$

Since $\mathscr{F}$ is unramified, the lower diagonal arrow is an isomorphism and the upper one is injective. Hence, the vertical arrow is an isomorphism (and the upper diagonal one as well). $\square$

For an open subset $\mathscr{U}$ of $\mathrm{Spa}(X, S)$ we define the following full subcategory $\mathscr{U}_{\log}$ of $(X, S)_{\log}$. Its objects are the objects $(Z, \bar{Z})$ of $(X, S)_{\log}$ such that the morphism $\mathrm{Spa}(Z, \bar{Z}) \to \mathrm{Spa}(X, S)$ induced by the structure morphism factors through $\mathscr{U}$. Obviously, for $\mathscr{U}' \subseteq \mathscr{U}$ we have $\mathscr{U}'_{\log} \subseteq \mathscr{U}_{\log}$.

In case $\mathscr{U} = \mathrm{Spa}(Y, \bar{Y})$, all objects $(Z, \bar{Z})$ of $(X, S)_{\log}$ with a morphism $(Z, \bar{Z}) \to (Y, \bar{Y})$ are in $\mathrm{Spa}(Y, \bar{Y})_{\log}$. But $\mathrm{Spa}(Y, \bar{Y})_{\log}$ might be bigger. For instance, if $(Y, \bar{Y}) \to (Z, \bar{Z})$ is a morphism in $(X, S)_{\log}$ such that $\bar{Y} \to \bar{Z}$ is proper and not an isomorphism and $Z = Y$, then $(Z, \bar{Z}) \in \mathrm{Spa}(Y, \bar{Y})_{\log}$ but there is no morphism $(Z, \bar{Z}) \to (Y, \bar{Y})$. Only in the affine case we have the following lemma:

{final_obje

Lemma 6.4. *Let* (A, A^+) *be log smooth. Then* $(\mathrm{Spec}\, A, \mathrm{Spec}\, A^+)$ *is a final object of* $\mathrm{Spa}(A, A^+)_{\log}$.

Proof. Let $(Y, \bar{Y})$ be an object of $\mathrm{Spa}(A, A^+)_{\log}$ and set $\mathscr{Y} = \mathrm{Spa}(Y, \bar{Y})$. Then $\mathscr{O}_{\mathscr{Y}}(\mathscr{Y}) = \mathscr{O}_Y(Y)$ and $\mathscr{O}^+_{\mathscr{Y}}(\mathscr{Y}) = \mathscr{O}_{\bar{Y}}(\bar{Y})$. By [Hub94], Proposition 2.1 there is a natural isomorphism

$$\mathrm{Hom}((A, A^+), (\mathscr{O}_{\mathscr{Y}}(\mathscr{Y}), \mathscr{O}^+_{\mathscr{Y}}(\mathscr{Y}))) \cong \mathrm{Hom}(\mathscr{Y}, \mathrm{Spa}(A, A^+)).$$

We thus obtain ring homomorphisms $A \to \mathscr{O}_Y(Y)$ and $A^+ \to \mathscr{O}_{\bar{Y}}(\bar{Y})$. By functoriality they fit into a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_Y(Y) & A \arrow[l] \\
\mathscr{O}_{\bar{Y}}(\bar{Y}) \arrow[u] & A^+ \arrow[l] \arrow[u]
\end{tikzcd}$$

The characterization of morphisms to affine schemes by homomorphisms of global sections of the structure sheaves yields a commutative diagram of schemes

$$\begin{array}{ccc} Y & \longrightarrow & \operatorname{Spec} A \\ \big\downarrow & & \big\downarrow \\ \bar{Y} & \longrightarrow & \operatorname{Spec} A^{+}. \end{array}$$

This defines a morphism $(Y, \bar{Y}) \to (\operatorname{Spec} A, \operatorname{Spec} A^{+})$ in $\operatorname{Spa}(A, A^{+})_{\log}$. $\square$

torization}

Lemma 6.5. *Let $\operatorname{Spa}(Y, \bar{Y}) \subset \operatorname{Spa}(X, S)$ be open coming from a diagram of schemes*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \bar{Y} & \longrightarrow & \bar{X}. \end{array}$$

Moreover, let $(Z, \bar{Z}) \in \operatorname{Spa}(Y, \bar{Y})_{\log}$. Then there is an open subscheme $\bar{U} \subseteq \bar{Z}$ isomorphic in codimension one, containing Z, and such that $(Z, \bar{U}) \to (X, S)$ factors through $(Y, \bar{Y})$.

Proof. Replacing, if necessary, $\bar{Y}$ with a compactification of Y over $\bar{Y}$, we may assume that $Y \to \bar{Y}$ is an open immersion with dense image. The morphism $\operatorname{Spa}(Z, \bar{Z}) \to \operatorname{Spa}(Y, \bar{Y})$ provides an open immersion $\varphi : Z \to Y$ and a birational map $\bar{\varphi} : \bar{Z} \to \bar{Y}$. Since $\bar{Z}$ is normal, $\bar{\varphi}$ is defined over an open subscheme $\bar{U} \subseteq \bar{Z}$ containing all points of codimension ≤ 1. Moreover, we may assume that $\bar{U}$ contains Z. By construction $(Z, \bar{U}) \to (X, S)$ factors through $(Y, \bar{Y})$. $\square$

We want to remind the reader of the concept of Riemann-Zariski morphisms (see [HS20]). A point x of an adic space $\mathscr{X}$ is called Riemann-Zariski, if it has no non-trivial horizontal specialization. A Riemann-Zariski morphism is a morphism of adic spaces mapping Riemann-Zariski points to Riemann-Zariski points. Let us now consider morphisms of adic spaces $\operatorname{Spa}(Y, T) \to \operatorname{Spa}(X, S)$ arising from diagrams of schemes

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & S. \end{array}$$

The above diagram is said to have universally closed diagonal if the induced morphism $Y \to X \times_S T$ is universally closed. In this case the morphism $\operatorname{Spa}(Y, T) \to \operatorname{Spa}(X, S)$ is Riemann Zariski and the converse holds if Y is quasi-compact and all residue field extensions of $Y \to X$ are algebraic (see [HS20], Lemma 12.7). In case S is integral, Y is quasi-compact, and $X \to S$ and $Y \to X$ (and hence also $Y \to T$) are open immersions with dense image, being Riemann Zariski is equivalent to $Y \cong X \times_S T$.

dimension1}

Lemma 6.6. *Let $(Y, \bar{Y})$ be in $(X, S)_{\log}$. Let $(\operatorname{Spa}(Y_i, \bar{Y}_i) \to \operatorname{Spa}(Y, \bar{Y}))_{i \in I}$ be a finite Riemann-Zariski covering and $(Z, \bar{Z}) \in \operatorname{Spa}(Y, \bar{Y})_{\log}$. Then there is an open immersion of the form $(Z, \bar{U}) \to (Z, \bar{Z})$ which is an isomorphism in codimension one such that*

- *$(Z, \bar{U}) \to (X, S)$ factors through $(Y, \bar{Y})$ and*
- *setting $Z_i = Z \times_Y Y_i$ and $\bar{U}_i = \bar{U} \times_{\bar{Y}} \bar{Y}_i$, the family $((Z_i, \bar{U}_i) \to (Z, \bar{U}))_{i \in I}$ is a covering in $(X, S)_{\log}$.*

Proof. Using Lemma 6.5 we find an open subscheme $\bar{V} \subseteq \bar{Z}$ containing Z and isomorphic in codimension one such that $(Z, \bar{V}) \to (X, S)$ factors through $(Y, \bar{Y})$. Set $Z_i = Z \times_Y Y_i = Z \cap Y_i$ and $\bar{V}_i = \bar{V}_i \times_{\bar{Y}} \bar{Y}_i$. Since $\bar{V}$ is normal, for each i the morphism $\bar{V}_i \to \bar{V}$ is an open immersion when restricted to a suitable open subscheme of $\bar{V}$ isomorphic in codimension one and containing Z. Denote by $\bar{U}$ the intersection of all of these subschemes for all i. Setting $\bar{U}_i = \bar{U} \times_{\bar{V}} \bar{V}_i$ we obtain a diagram

$$\begin{tikzcd}
(Z_i, \bar{U}_i) \arrow[r, hook, "\circ"] \arrow[d] & (Z_i, \bar{V}_i) \arrow[r] \arrow[d] & (Y_i, \bar{Y}_i) \arrow[d] \\
(Z, \bar{U}) \arrow[r, hook, "\circ", "\sim_1"'] & (Z, \bar{V}) \arrow[r] \arrow[d, hook, "\circ", "\sim_1"'] & (Y, \bar{Y}) \\
 & (Z, \bar{Z}). &
\end{tikzcd}$$

All required properties of $\bar{U}$ are clear except maybe that $(\bar{U}_i \to \bar{U})_{i \in I}$ is a surjective family. But by construction the family $(\mathrm{Spa}(Z_i, \bar{U}_i) \to \mathrm{Spa}(Z, \bar{U}))_{i \in I}$ is the pullback of the covering $(\mathrm{Spa}(Y_i, \bar{Y}_i) \to \mathrm{Spa}(Y, \bar{Y}))_{i \in I}$ by $\mathrm{Spa}(Z, \bar{U}) \to \mathrm{Spa}(Y, \bar{Y})$. In particular, it is surjective. This implies that $(\bar{U}_i \to \bar{U})_{i \in I}$ has to be surjective. □

Definition 6.7. For an unramified sheaf $\mathscr{F}$ on $(X, S)_{\log}$ we define a presheaf $\mathscr{F}_{\lim}$ on $\mathrm{Spa}(X, S)$ as follows:

$$\mathscr{F}_{\lim}(\mathscr{U}) = \lim_{(Y, \bar{Y}) \in \mathscr{U}_{\log}} \mathscr{F}(Y, \bar{Y}).$$

We want to emphazise that in the above definition we are taking a limit and not a colimit. The presheaf $\mathscr{F}_{\lim}$ is not related to the pullback $\ell^* \mathscr{F}$. Notice moreover, that the definition of $\mathscr{F}_{\lim}$ is indeed functorial: For open subsets $\mathscr{U}' \subseteq \mathscr{U}$ in $\mathrm{Spa}(X, S)$ we need a restriction $\mathscr{F}_{\lim}(\mathscr{U}) \to \mathscr{F}_{\lim}(\mathscr{U}')$ in

$$\mathrm{Hom}(\mathscr{F}_{\lim}(\mathscr{U}), \mathscr{F}_{\lim}(\mathscr{U}')) = \lim_{(Y', \bar{Y}')} \operatorname*{colim}_{(Y, \bar{Y})} \mathrm{Hom}(\mathscr{F}(Y, \bar{Y}), \mathscr{F}(Y', \bar{Y}')).$$

In other words, we have to find for each $(Y', \bar{Y}')$ in $\mathscr{U}'_{\log}$ a $(Y, \bar{Y})$ in $\mathscr{U}_{\log}$ and define a homomorphism

$$\mathscr{F}(Y, \bar{Y}) \to \mathscr{F}(Y', \bar{Y}').$$

Moreover, these homomorphisms need to be compatible. But for given $(Y', \bar{Y}')$ we can just take $(Y, \bar{Y}) = (Y', \bar{Y}')$ and the identity homomorphism on $\mathscr{F}(Y', \bar{Y}')$. This is clearly functorial.

{restrictio

Lemma 6.8. *Let $\mathcal{F}$ be an unramified sheaf on $(X, S)_{\log}$. Then for all open subspaces $\mathscr{U}' \subseteq \mathscr{U} \subseteq \mathrm{Spa}(X, S)$ the restriction*

$$\mathscr{F}_{\lim}(\mathscr{U}) \to \mathscr{F}_{\lim}(\mathscr{U}')$$

is injective.

Proof. Suppose $s = (s_{(Y, \bar{Y})})_{(Y, \bar{Y})} \in \mathscr{F}_{\lim}(\mathscr{U})$ maps to zero in $\mathscr{F}_{\lim}(\mathscr{U}')$. This means that $s_{(Y, \bar{Y})} = 0$ for all $(Y, \bar{Y}) \in \mathscr{U}'_{\log}$. Take any $(Y, \bar{Y})$ in $\mathscr{U}_{\log}$. We have to show that $s_{(Y, \bar{Y})} = 0$. Without loss of generality we may assume that $(Y, \bar{Y})$ is connected. There is a dense open $\bar{Y}' \subseteq \bar{Y}$ such that, setting $Y' = \bar{Y}' \cap Y$, the morphism $\mathrm{Spa}(Y', \bar{Y}') \to \mathrm{Spa}(X, S)$ factors through $\mathscr{U}'$. Then $\mathscr{F}(Y, \bar{Y}) \to \mathscr{F}(Y', \bar{Y}')$ is injective and $s_{(Y, \bar{Y})}|_{(Y', \bar{Y}')} = 0$, whence $s_{(Y, \bar{Y})} = 0$. □

{F_lim_shea

Proposition 6.9. *With the above notation $\mathscr{F}_{\lim}$ is a sheaf on* $\mathrm{Spa}(X, S)$.

Proof. It suffices to show the sheaf condition for coverings of the form $(\varphi_i : \mathrm{Spa}(Y_i, \bar{Y}_i) \to \mathrm{Spa}(Y, \bar{Y}))_{i\in I}$ with finite index set I coming from diagrams

$$\begin{array}{ccc} Y_i & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ \bar{Y}_i & \xrightarrow{\bar{\varphi}} & \bar{Y}. \end{array}$$

Moreover, by [HS20], Lemma 12.10, every such covering has a refinement which is Riemann-Zariski. Therefore, we may assume that our covering is Riemann Zariski. We need to show that the sequence

$$\mathscr{F}_{\lim}(\mathrm{Spa}(Y, \bar{Y})) \to \prod_i \mathscr{F}_{\lim}(\mathrm{Spa}(Y_i, \bar{Y}_i)) \rightrightarrows \prod_{ij} \mathscr{F}_{\lim}(\mathrm{Spa}(Y_i, \bar{Y}_i) \cap \mathrm{Spa}(Y_j, \bar{Y}_j))$$

is exact. Exactness on the left is assured by Lemma 6.8. Suppose we are given $s_i = (s_{i,(Z_i,\bar{Z}_i)})_{(Z_i,\bar{Z}_i)}$ in $\mathscr{F}_{\lim}(\mathrm{Spa}(Y_i, \bar{Y}_i))$ such that the restrictions of s_i and s_j to $\mathrm{Spa}(Y_i, \bar{Y}_i) \cap \mathrm{Spa}(Y_j, \bar{Y}_j)$ coincide. By definition this means that

$$s_{i,(Z,\bar{Z})} = s_{j,(Z,\bar{Z})}$$

for all $(Z, \bar{Z}) \in (\mathrm{Spa}(Y_i, \bar{Y}_i) \cap \mathrm{Spa}(Y_j, \bar{Y}_j))_{\log}$.

We have to find $s \in \mathscr{F}_{\lim}(\mathrm{Spa}(Y, \bar{Y}))$ with $s|_{\mathrm{Spa}(Y_i,\bar{Y}_i)} = s_i$ for all i. Let $(Z, \bar{Z})$ be in $\mathrm{Spa}(Y, \bar{Y})_{\log}$. In the following we explain how to define $s_{(Z,\bar{Z})}$. Lemma 6.6 provides us with an open subscheme $\bar{U} \subseteq \bar{Z}$ isomorphic in codimension one and containing Z such that $(Z, \bar{U}) \to \mathrm{Spa}(X, S)$ factors through $(Y, \bar{Y})$ and $((Z_i, \bar{U}_i) \to (Z, \bar{U}))_{i\in I}$ is a covering in $(X, S)_{\log}$ (where $Z_i = Z \times_Y Y_i$ and $\bar{U}_i = \bar{U} \times_{\bar{Y}} \bar{Y}_i$). Since $\mathscr{F}$ is a sheaf on $(X, S)_{\log}$, the sequence

$$0 \to \mathscr{F}(Z, \bar{U}) \to \prod_i \mathscr{F}(Z_i, \bar{U}_i) \to \prod_{i,j} \mathscr{F}(Z_i \cap Z_j, \bar{U}_i \cap \bar{U}_j)$$

is exact. The sections $s_{i,(Z_i,\bar{U}_i)} \in \mathscr{F}(Z_i, \bar{U}_i)$ coincide on the intersections $(Z_i \cap Z_j, \bar{U}_i \cap \bar{U}_j)$. They thus lift to a unique section $s_{(Z,\bar{U})}$ of $\mathscr{F}(Z, \bar{U})$. We define $s_{(Z,\bar{Z})}$ to be the preimage of $s_{(Z,\bar{U})}$ under the isomorphism $\mathscr{F}(Z, \bar{Z}) \to \mathscr{F}(Z, \bar{U})$. It follows from the fact that $\mathscr{F}$ is unramified that the $s_{(Z,\bar{Z})}$ are compatible and define an element of $\mathscr{F}_{\lim}(\mathrm{Spa}(Y, \bar{Y}))$. We leave the details to the reader.

Let us show that $s|_{\mathrm{Spa}(Y_i,\bar{Y}_i)} = s_i$. This is equivalent to showing that for all $(Z, \bar{Z}) \in \mathrm{Spa}(Y_i, \bar{Y}_i)_{\log}$ we have $s_{(Z,\bar{Z})} = s_{i,(Z,\bar{Z})}$. By unramifiedness we can check this equality after restricting to $(Z, \bar{U})$ for an open subscheme $\bar{U} \subseteq \bar{Z}$ isomorphic in codimension one and containing Z. By Lemma 6.6 we may thus assume that $(Z, \bar{Z}) \to (X, S)$ factors through $(Y_i, \bar{Y}_i)$ and $((Z_j, \bar{Z}_j) \to (Z, \bar{Z}))_{j\in I}$ (for $Z_j = Z_i \times_{Y_i} Y_j$ and $\bar{Z}_j = \bar{Z}_i \times_{\bar{Y}_i} \bar{Y}_j$) is a covering in $\mathrm{Spa}(X, S)_{\log}$. By construction, $s_{(Z,\bar{Z})}$ is uniquely defined by the condition $s_{(Z,\bar{Z})}|_{(Z_j,\bar{Z}_j)} = s_{j,(Z_i,\bar{Z}_j)}$ for all $j \in I$. In particular, $s_{(Z,\bar{Z})}|_{(Z_i,\bar{Z}_i)} = s_{i,(Z_i,\bar{Z}_i)}$. But $(Z_i, \bar{Z}_i) = (Z, \bar{Z})$, so $s_{(Z,\bar{Z})} = s_{i,(Z,\bar{Z})}$. ∎

Lemma 6.10. $\Omega^{n,\log}$ *is unramified.*

Proof. Theorem 38 in [Mat70] says that a noetherian normal domain is the intersection of the localizations at its height one prime ideals. It follows from this that the sheaf $\mathscr{O}$

on $(X,S)_{\log}$ defined by

$$(Y,\bar{Y}) \mapsto \mathscr{O}_{\bar{Y}}(\bar{Y})$$

is unramified. Since the objects of $(X,S)_{\log}$ are log smooth, $\Omega^{n,\log}$ is a locally free $\mathscr{O}$-module. Hence, it is unramified as well. $\square$

6.3. **The comparison theorem.** We have a natural map $\Omega^{n,\log} \to \Omega^{n,\log}_{\lim}$ of presheaves on the site of strict affinoids $\operatorname{Spa}(X,S)_{\text{straff}}$. Sine $\Omega^{n,\log}_{\lim}$ is a sheaf by Proposition 6.9, this map factors through the sheafification $\Omega^{n,+}$ of $\Omega^{n,\log}$:

$$\Omega^{n,\log} \to \Omega^{n,+} \to \Omega^{n,\log}_{\lim}.$$

{comparison

Proposition 6.11. *Let (A,A^+) be log smooth. Then the natural homomorphism*

$$\Omega^{n,\log}_{(A,A^+)} \to \Omega^{n,+}(\operatorname{Spa}(A,A^+))$$

is an isomorphism.

Proof. Consider the chain of homomorphisms

$$\Omega^{n,\log}_{(A,A^+)} \xrightarrow{\varphi_1} \Omega^{n,+}(\operatorname{Spa}(A,A^+)) \xrightarrow{\varphi_2} \Omega^{n,\log}_{\lim}(\operatorname{Spa}(A,A^+)) \xrightarrow{\varphi_3} \Omega^n_A.$$

By Lemma 6.4 we know that $(\operatorname{Spec} A, \operatorname{Spec} A^+)$ is a final object of $\operatorname{Spa}(A,A^+)_{\log}$. Hence, we can identify $\Omega^{n,\log}_{\lim}(\operatorname{Spa}(A,A^+))$ with $\Omega^{n,\log}_{(A,A^+)}$ and then $\varphi_2 \circ \varphi_1$ is the identity. Moreover, $\varphi_3 \circ \varphi_2$ is the natural inclusion. We obtain

$$\begin{tikzcd}
\Omega^{n,\log}_{(A,A^+)} \arrow[r, "\varphi_1"] \arrow[rr, bend left, "\mathrm{id}"] & \Omega^{n,+}(\operatorname{Spa}(A,A^+)) \arrow[r, "\varphi_2"] \arrow[rr, bend right, "\text{inclusion}"'] & \Omega^{n,\log}_{\lim}(\operatorname{Spa}(A,A^+)) \arrow[r, "\varphi_3"] & \Omega^n_A.
\end{tikzcd}$$

A diagram chase shows that φ_1 and φ_2 are isomorphisms. $\square$

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Theorem 6.12. *Let $(Y,\bar{Y})$ be in $(X,S)_{\log}$. Then*

$$\Omega^{n,+}(\operatorname{Spa}(Y,\bar{Y})) \cong \Omega^{n,\log}(Y,\bar{Y}),$$

where $\Omega^{n,\log}$ denotes the sheaf of logarithmic differentials on $(X,S)_{\log}$.

Proof. Consider the subcategory $\mathscr{C}$ of $(X,S)_{\log}$ of objects of the form $(\operatorname{Spa} A, \operatorname{Spa} A^+)$. It is a site with the induced topology and the topoi associated with $\mathscr{C}$ and $(X,S)_{\log}$ are equivalent. We consider the following morphism of sites

$$\pi^{\text{straff}} : \operatorname{Spa}(X,S)_{\text{straff}} \longrightarrow \mathscr{C}$$
$$\operatorname{Spa}(A,A^+) \hookleftarrow (\operatorname{Spec} A, \operatorname{Spec} A^+),$$

It fits into the following commutative diagram of morphisms of sites

$$\begin{tikzcd}
\operatorname{Spa}(X,S)_{\text{top}} \arrow[r, "\pi"] \arrow[d, "\iota^{\text{straff}}"] & (X,S)_{\log} \arrow[d, "\iota^{\mathscr{C}}"] \\
\operatorname{Spa}(X,S)_{\text{straff}} \arrow[r, "\pi^{\text{straff}}"] & \mathscr{C}.
\end{tikzcd}$$

It follows by construction that $\pi_*^{\mathrm{straff}}\iota_*^{\mathrm{straff}}\mathscr{F} = \iota_*^{\mathscr{C}}\pi_*\mathscr{F}$ for any presheaf $\mathscr{F}$ on $\mathrm{Spa}(X,S)_{\mathrm{top}}$. Applying π_*^{straff} to the homomorphism $\Omega^{n,\log} \to \iota_*^{\mathrm{straff}}\Omega^{n,+}$ of presheaves on $\mathrm{Spa}(X,S)_{\mathrm{straff}}$, we obtain a homomorphism

$$\pi_*^{\mathrm{straff}}\Omega^{n,\log} \to \iota_*^{\mathscr{C}}\pi_*\Omega^{n,+}.$$

Unraveling the definitions, we see that $\pi_*^{\mathrm{straff}}\Omega^{n,\log}$ equals $\iota_*^{\mathscr{C}}\Omega^{n,\log}$ (where now $\Omega^{n,\log}$ denotes the sheaf of logarithmic differentials on $(X,S)_{\log}$). By Proposition 6.11 the above homomorphism is an isomorphism. Since the topoi associated to $\mathscr{C}$ and $(X,S)_{\log}$ are equivalent, we obtain an isomorphism

$$\Omega^{n,\log} \to \pi_*\Omega^{n,+}$$

of sheaves on $(X,S)_{\log}$. Evaluating at an object $(Y,\bar{Y})$ in $(X,S)_{\log}$ yields the result. $\square$

References

[Bha12] Bhargav Bhatt. p-adic derived de rham cohomology. *arXiv:1204.6560 [math.AG]*, 2012.

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[GR11] Daniel Greb and Sönke Rollenske. Torsion and cotorsion in the sheaf of Kähler differentials on some mild singularities. *Mathematical Research Letters*, 18(6):1259–1269, 2011.

[HKK17] Annette Huber, Stefan Kebekus, and Shane Kelly. Differential forms in positive characteristic avoiding resolution of singularities. *Bulletin de la Société Mathématique de France*, 145(2):305–343, 2017.

[HS20] Katharina Hübner and Alexander Schmidt. The tame site of a scheme. *arXiv:2004.05468 [math.AG]*, 2020.

[Hub94] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Mathematische Zeitschrift*, 217:513–551, 1994.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics. Vieweg, 1996.

[Hüb18] Katharina Hübner. The adic tame site. *arXiv:1801.04776 [math.AG]*, 2018.

[Hüb20] Katharina Hübner. Tame and strongly étale cohomology of curves. *arXiv:1911.05595 [math.AG]*, 2020.

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I*. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971.

[Kat89] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. pages 191–224, 1989.

[Mat70] Hideyuki Matsumura. *Commutative algebra*. W. A. Benjamin, Inc., New York, 1970.

[Mor12] Fabien Morel. *$\mathbb{A}^1$-algebraic topology over a field*, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[Ogu18] Arthur Ogus. *Lectures on Logarithmic Algebraic Geometry*. Cambridge University Press, 2018.

[Ols05] Martin C. Olsson. The logarithmic cotangent complex. *Mathematisch Annalen*, 333:859–931, 2005.

[SSV16] Steffen Sagave, Timo Schürg, and Gabriele Vezzosi. Derived logarithmic geometry I. *Journal de l'Institut de Mathématiques de Jussieu*, 15(2):367–405, 2016.

[Tem11] Michael Temkin. Relative Riemann-Zariski spaces. *Israel Journal of Mathematics*, pages 1–42, 2011.

[Tem16] Michael Temkin. Metrization of differential pluriforms on Berkovich analytic spaces. In *Nonarchimedean and tropical geometry*, Simons Symp., pages 195–285. Springer, [Cham], 2016.