The adic tame site

Katharina Hübner

Einstein Institute of Mathematics
Hebrew University
Jerusalem

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1. Introduction

Etale cohomology of a scheme with torsion coefficients away from the residue characteristics yields a well behaved cohomology theory. For instance, there is a smooth base change theorem, a cohomological purity theorem, and the cohomology groups are $\mathbb{A}^1$-homotopy invariant. This breaks down, however, if we take the coefficients of the cohomology groups to be p-torsion, where p is a residue characteristic of the scheme in question. The problem can be seen already when looking at the cohomology group $H^1_{ét}(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z})$ for some algebraically closed field k. If the characteristic of k is not p, this cohomology group vanishes. But if the characteristic of k is p, $H^1_{ét}(\mathbb{A}^1_k, \mathbb{Z}/p\mathbb{Z})$ is infinite due to wild ramification at infinity.

In order to address these problems we introduce the tame site $(X/S)_t$ of a scheme X over some base scheme S which does not allow this wild ramification at the boundary. The rough idea is to consider only étale morphisms $Y \to X$ which are tamely ramified (in an appropriate sense) along the boundary $\bar{X} - X$ of a compactification $\bar{X}$ of X over S. The concept of tameness is a valuation-theoretic one. This makes it more natural to work in the language of adic spaces rather than in the language of schemes. For an étale morphism of adic spaces it is straightforward to define tameness: An étale morphism $\varphi : Y \to X$ is tame at a point $y \in Y$ with $\varphi(y) = x$ if the valuation on $k(y)$ corresponding to y is tamely ramified in the finite separable field extension $k(y)|k(x)$. Defining coverings to be the surjective tame morphisms, we obtain the tame site Z_t for every adic space Z. In addition, we define the strongly étale site $Z_{sét}$ by replacing "tame"with "unramified".

This construction also provides a tame site for a scheme X over a base scheme S by associating with $X \to S$ the adic space $\mathrm{Spa}(X, S)$ (see [Tem11]) and considering the tame site $\mathrm{Spa}(X, S)_t$. Note that $\mathrm{Spa}(X, S)$ is not an analytic adic space: If $X = \mathrm{Spec}\, A$ and $S = \mathrm{Spec}\, R$ are affine, we have $\mathrm{Spa}(X, S) = \mathrm{Spa}(A, A^+)$, where A^+ is the integral closure of the image of R in A and A is equipped with the *discrete* topology. The adic space $\mathrm{Spa}(X, S)$ should not be thought of an analytification of X/S but rather as a means of encoding the essential information on $X \to S$ in the language of adic spaces. We call adic spaces which are locally of this type discretely ringed.

Of course, tameness is not a new concept in algebraic geometry. Several approaches have been made to define the notion of a tame covering space of a scheme over a base scheme. These are summarized and compared in [KS10]. Having a notion of tameness for covering spaces we can define the corresponding tame fundamental group. In Section 9 we show that the fundamental group of the tame site coincides with the curve-tame fundamental group constructed in [Wie08], see also [KS10].

Also in other respects the tame site behaves the way it should: For an étale torsion sheaf with torsion away from the characteristic the tame cohomology groups coincide with the étale cohomology groups. If $X \to S$ is proper, the tame cohomology groups of $\mathrm{Spa}(X, S)$ coincide with the étale cohomology groups for all étale sheaves (see Section 8).

Having established these rather straightforward comparison results we move on to prove our first big theorem concerning the tame site, namely absolute cohomological purity in characteristic $p > 0$ (see Corollary 14.5): Let S be a quasi-compact, quasi-separated scheme of characteristic $p > 0$ and X a regular scheme which is separated and essentially of finite type over S. Assume that resolution of singularities holds over S. Then, if $U \hookrightarrow X$ is an inverse limit of open immersions, we have

$$H^i_t(\mathrm{Spa}(U, S), \mathbb{Z}/p\mathbb{Z}) \cong H^i_t(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z}).$$

This immediately implies that under the hypothesis of resolution of singularities the tame cohomology groups $H^i_t(\mathrm{Spa}(X, S), \mathbb{Z}p\mathbb{Z})$ are homotopy invariant for regular schemes X of finite type over S (see Corollary 14.6).

In order to prove the purity theorem we examine the Artin Schreier sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a^+ \longrightarrow \mathbb{G}_a^+ \to 0,$$

on $\mathrm{Spa}(X, S)_t$, where $\mathbb{G}_a^+$ is the sheaf defined by $\mathbb{G}_a^+(Z) = \mathscr{O}_Z^+(Z)$. It reduces us to the study of the cohomology of $\mathbb{G}_a^+$. In Section 10 we compare the cohomology groups $H^i_{\mathrm{top}}(\mathrm{Spa}(X, S), \mathscr{O}^+_{\mathrm{Spa}(X,S)})$ with $H^i_{\mathrm{top}}(S, \mathscr{O}_S)$. This is where we use resolution of singularities.

In Section 12 we show that for every strongly noetherian analytic or discretely ringed adic space Z we have a natural isomorphism

$$H^i_{\mathrm{top}}(Z, \mathbb{G}_a^+) \xrightarrow{\sim} H^i_{s\acute{e}t}(Z, \mathbb{G}_a^+)$$

for all $i \geq 0$. In preparation to this we examine in Section 11 Prüfer Huber pairs, i.e. Huber pairs (A, A^+) such that $A^+ \to A$ is a Prüfer extension. Prüfer Huber pairs are important in the study of the cohomology groups of $\mathbb{G}_a^+$ because $\mathbb{G}_a^+$ is acyclic on the adic spectra of Prüfer Huber pairs.

The final step is the comparison of the strongly étale with the tame cohomology of $\mathbb{G}_a^+$. More precisely, we show in Section 13 that for any noetherian, discretely ringed or analytic adic space Z we have natural isomorphisms

$$H^i_{s\acute{e}t}(Z, \mathbb{G}_a^+) \xrightarrow{\sim} H^i_t(Z, \mathbb{G}_a^+)$$

for all $i \geq 0$.

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2. Background on adic spaces

To fix notation let us briefly recall from [Hub93b] and [Hub94] some notions concerning adic spaces. A *Huber ring* (f-adic ring in Huber's terminology) is a topological ring A such that there exists an open subring A_0 carrying the I-adic topology for a finitely generated ideal $I \subseteq A_0$. The ring A_0 is called a *ring of definition* of A and the ideal I an *ideal of definition*. An example of a Huber ring is $\mathbb{Q}_p$ with ring of definition $\mathbb{Z}_p$ and ideal of definition $p\mathbb{Z}_p$.

An element a of a Huber ring A is *power-bounded* if the set $\{a^n \mid n \in \mathbb{N}\}$ is bounded, i.e. for any neighborhood $U \subset A$ of 0 there is a neighborhood V of 0 such that

$$V \cdot \{a^n \mid n \in \mathbb{N}\} \subseteq U.$$

An element a of A is called *topologically nilpotent* if the sequence a^n converges to 0. Every topologically nilpotent element is power-bounded. We denote the set of power bounded elements of A by A° and the set of topologically nilpotent elements by $A^{\circ\circ}$.

A *ring of integral elements* of A is an open, bounded, integrally closed subring A^+ of A. The rings of integral elements are precisely the subrings A^+ of A such that

$$A^{\circ\circ} \subseteq A^+ \subseteq A^\circ.$$

Moreover, every ring of integral elements is a ring of definition of A. A *Huber pair* (affinoid ring in Huber's terminology) is a pair (A, A^+) consisting of a Huber ring A and a ring of integral elements $A^+ \subseteq A$.

Given a Huber pair (A, A^+) we define its *adic spectrum*

$$X = \mathrm{Spa}(A, A^+) = \{\text{continuous valuations } v : A \to \Gamma \cup \{0\} \mid v(a) \leq 1 \ \forall\, a \in A^+\}$$

Notice that we write valuations multiplicatively. Furthermore, for an element $x \in X$ we write $f \mapsto |f(x)|$ for the valuation corresponding to X.

For $f_1, \dots, f_n, g \in A$ such that the ideal of A generated by $f_1, \dots, f_n$ is open we define the *rational subset* $R\big(\frac{f_1,\dots,f_n}{g}\big)$ of X by

$$R\Big(\frac{f_1, \dots, f_n}{g}\Big) = \{x \in X \mid |f_i(x)| \leq |g(x)| \neq 0 \ \forall\, i = 1, \dots, n\}.$$

It is the adic spectrum of the Huber pair

$$(A(\frac{f_1, \dots, f_n}{g}), A(\frac{f_1, \dots, f_n}{g})^+),$$

where $A(\frac{f_1,\dots,f_n}{g})$ is the localization A_g of A endowed with the topology defined by the ring of definition $A^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ and the ideal of definition $IA^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ and $A(\frac{f_1,\dots,f_n}{g})^+$ is the integral closure of $A^+[\frac{f_1}{g}, \dots, \frac{f_n}{g}]$ in $A(\frac{f_1,\dots,f_n}{g})$. We endow X with the topology generated by the rational subsets as above.

On the topological space X we can define a presheaf $\mathscr{O}_X$ of complete topological rings (complete always comprises Hausdorff) such that for any rational subset $R\big(\frac{f_1,\dots,f_n}{g}\big)$ of X we have

$$\mathscr{O}_X(R\Big(\frac{f_1, \dots, f_n}{g}\Big)) = A\langle\frac{f_1, \dots, f_n}{g}\rangle,$$

the latter ring being the completion of $A(\frac{f_1,\dots,f_n}{g})$. In particular,

$$\mathscr{O}_X(X) = \hat{A}.$$

Furthermore, there is a subpresheaf $\mathscr{O}_X^+$ of $\mathscr{O}_X$ with

$$\mathscr{O}_X^+(R\Big(\frac{f_1, \dots, f_n}{g}\Big)) = A\langle\frac{f_1, \dots, f_n}{g}\rangle^+.$$

We say that a Huber pair (A, A^+) is *sheafy* if the corresponding presheaf $\mathscr{O}_X$ on $X = \mathrm{Spa}(A, A^+)$ is a sheaf. In this case we speak of the *structure sheaf* $\mathscr{O}_X$. If $\mathscr{O}_X$ is sheaf, $\mathscr{O}_X^+$ is a sheaf, as well. The Huber pair (A, A^+) is known to be sheafy in the following cases:

(1) $\hat{A}$ has a noetherian ring of definition over which $\hat{A}$ is finitely generated.
(2) A is a strongly noetherian Tate ring.
(3) The topology of $\hat{A}$ is discrete.

Throughout this article we will only consider Huber pairs satisfying one of the above conditions.

An adic space is a triple $(X, \mathscr{O}_X, (v_x)_{x\in X})$, where

- X is a topological space,
- $\mathscr{O}_X$ is a sheaf of complete topological rings whose stalks are local rings,
- for every $x \in X$, v_x is an isomorphism class of valuations on $\mathscr{O}_{X,x}$ whose support is the maximal ideal of $\mathscr{O}_{X,x}$,

which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ for a sheafy Huber pair (A, A^+).

Unfortunately, closed subsets of adic spaces do not carry the structure of an adic space in general. Therefore, following [Hub96], §1.10, we define *prepseudo-adic spaces* to be pairs $X = (\underline{X}, |X|)$, where $\underline{X}$ is an adic space and $|X|$ is a subset of (the underlying topological space of) $\underline{X}$. If Y is an adic space and Z is a subset of Y, we often use the same letter Z to denote the prepseudo-adic space (Y, Z). A prepseudo-adic space X is called *pseudo-adic space* if $|X|$ is convex and pro-constructible. In particular, any closed subset Z of an adic space Y defines a pseudo-adic space.

3. The strongly étale and the tame site

Recall from [Hub96], Definition 1.6.5 i) that a morphism of adic spaces $Y \to X$ is étale if it is locally of finite presentation and if, for any Huber ring (A, A^+), any ideal I of A with $I^2 = \{0\}$, and any morphism $\mathrm{Spa}(A, A^+) \to X$ the mapping

$$\mathrm{Hom}_X(\mathrm{Spa}(A, A^+), Y) \to \mathrm{Hom}_X(\mathrm{Spa}(A, A^+)/I, Y)$$

is bijective. Moreover, a finite extension of valued fields $L|K$ is called unramified if the strict henselizations in a fixed algebraic closure coincide: $L^{\mathrm{sh}} = K^{\mathrm{sh}}$. It is tame if the degree of $L^{\mathrm{sh}}|K^{\mathrm{sh}}$ is prime to the residue characteristic. In case $L|K$ is Galois, it is unramified (tame) if and only if the inertia group (the ramification group) is trivial (see [Ray70] and [EP05]).

Definition 3.1. A morphism of prepseudo-adic spaces $f : Y \to X$ is called *strongly étale* (resp. *tame*) at a point $y \in |Y|$ if f is étale at y and the valuation $|\cdot(y)|$ is unramified (resp. tame) over $|\cdot(f(y))|$. The morphism f is called strongly étale (resp. tame) if f is so at every point of Y.

Note that by the following lemma the ring theoretic and valuation theoretic notions of ramification are compatible.

Lemma 3.2. *Let (k, k^+) be a complete affinoid field. An étale morphism $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(k, k^+)$ is strongly étale if and only if $k^+ \to A^+$ is étale.*

Proof. By [Hub96], Cor. 1.7.3 iii) the ring homomorphism $k \to A$ is étale and A^+ is the integral closure of an open subring of A which is of finite type over k^+. (Note that since k is a field, every étale homomorphism $k \to B$ is finite étale. Hence, B is automatically complete). Therefore, we may assume that A is a field and $k \to A$ is a finite separable field extension. Let k^+_A be the integral closure of k^+ in A. It is a semi-local Prüfer domain (Recall that a Prüfer domain is an integral domain R such that its localization at each prime is a valuation ring). As A^+ is a subring of A containing k^+_A, A^+ is a semi-local Prüfer domain, as well. More precisely, it is a localization of k^+_A. This implies that $\mathrm{Spec}\, A^+ \to \mathrm{Spec}\, k^+_A$ is an open immersion, as A^+, being finitely generated over k^+, is finitely generated over k^+_A.

It suffices to check that $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(k, k^+)$ is strongly étale at the closed points of $\mathrm{Spa}(A, A^+)$. Similarly we can check the étaleness of $k^+ \to A^+$ at the maximal ideals of A^+. The closed points of $\mathrm{Spa}(A, A^+)$ correspond to the maximal ideals of A^+: If $\mathfrak{m}$ is a maximal ideal of A^+, the corresponding closed point of $\mathrm{Spa}(A, A^+)$ is given by the valuation ring $A^+_{\mathfrak{m}}$.

Let $K|k$ be a finite Galois extension dominating $A|k$ and write G for its Galois group. Let $\mathfrak{m}$ be a maximal ideal of A^+. Choose a valuation v' of K above the valuation v of A associated with $A^+_{\mathfrak{m}}$. It corresponds to a maximal ideal $\mathfrak{m}'$ of the integral closure of A^+ in K lying over $\mathfrak{m}$. Then, $A|k$ is unramified at v if and only if the inertia subgroup $I_{v'} \subseteq G$ associated with v' is contained in $\mathrm{Gal}(K|A)$. But $I_{v'}$ coincides with the inertia group $I_{\mathfrak{m}'}$ of $\mathfrak{m}'$ and by [Ray70], Théorème X.1 the morphism $\mathrm{Spec}\, k^+_A \to \mathrm{Spec}\, k^+$ is étale in a neighborhood of $\mathfrak{m}$ if and only if $I_{\mathfrak{m}'}$ is contained in $\mathrm{Gal}(K|A)$. As $\mathrm{Spec}\, A^+ \to \mathrm{Spec}\, k^+_A$ is an open immersion, this proves the result. $\square$

Let X be a prepseudo-adic space. We define the following sites over X called the *strongly étale* site $X_{s\acute{e}t}$ and the *tame site* X_t:

- The underlying categories of $X_{s\acute{e}t}$ and X_t are the categories of strongly étale and tame morphisms $f : Y \to X$, respectively.
- Coverings are families $\{f_i : Y_i \to Y\}_{i\in I}$ of strongly étale, respectively tame, morphisms such that

$$|Y| = \bigcup_{i\in I} f_i(|Y_i|).$$

In order to show that this definition makes sense, we have to convince ourselves that tameness and strong étaleness are stable under compositions and base change. But this follows by combining the corresponding stability results of étaleness ([Hub96], Proposition 1.6.7) and extensions of valued fields ([EP05], §5).

In [Tem11] Temkin associates with a morphism of schemes $X \to S$ an adic space $\mathrm{Spa}(X, S)$. The points of $\mathrm{Spa}(X, S)$ are triples (x, R, ϕ), where x is a point of X, R is a valuation ring of $k(x)$ and $\phi : \mathrm{Spec}\, R \to S$ is a morphism compatible with $\mathrm{Spec}\, k(x) \to S$. In case S is separated, ϕ is uniquely determined (if it exists) by (x, R). The topology of $\mathrm{Spa}(X, S)$ is generated by the subsets $\mathrm{Spa}(X', S')$ of $\mathrm{Spa}(X, S)$ coming from commutative diagrams

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

with X' and S' affine, $X' \to X$ an open immersion and $S' \to S$ of finite type. This construction is compatible with Huber's definition of the adic spectrum given in [Hub93b]: If $X = \mathrm{Spec}\, A$ and $S = \mathrm{Spec}\, A^+$ are affine and the homomorphism $A^+ \to A$ is injective with integrally closed image, $\mathrm{Spa}(X, S)$ coincides with Huber's $\mathrm{Spa}(A, A^+)$ (where A is equipped with the discrete topology).

Pulling back the structure sheaf of X via the support morphism

$$\mathrm{supp} : Z := \mathrm{Spa}(X, S) \to X, \quad (x, R, \phi) \mapsto x$$

we obtain a sheaf of rings $\mathcal{O}_Z$ on $Z = \mathrm{Spa}(X, S)$ making Z a locally ringed space with

$$\mathcal{O}_{Z,(x,R,\phi)} = \mathcal{O}_{X,x}.$$

For each point $z = (x, R, \phi)$ denote by v_z the equivalence class of valuations on $k(x)$ corresponding to R. We obtain an adic space $(Z, \mathcal{O}_Z, (v_z \mid z \in Z))$ such that for each rational subset U the topology on $\mathcal{O}_Z(U)$ is the discrete one. We call this type of adic spaces *discretely ringed* adic spaces. Checking functoriality yields:

Lemma 3.3. *The above assignment defines a functor*

$$\mathrm{Spa} : \{\textit{morphisms of schemes}\} \longrightarrow \{\textit{discretely ringed adic spaces}\}$$
$$(X \to S) \mapsto (Z = \mathrm{Spa}(X, S), \mathcal{O}_Z, (v_z \mid z \in Z)).$$

mapping morphisms of affine schemes to affinoid adic spaces.

Where no confusion can arise we write $\mathrm{Spa}(X, S)$ for the adic space

$$(Z = \mathrm{Spa}(X, S), \mathcal{O}_Z, (v_z \mid z \in Z)).$$

For a morphism of schemes $X \to S$ the *tame site* of $X \to S$ is defined to be the tame site of $\mathrm{Spa}(X, S)$.

4. Openness of the tame locus

Our aim is to show that the strongly étale and the tame locus of an étale morphism of adic spaces is open. The argument is similar to the one for Riemann Zariski spaces given in [Tem17]. First we prove that strongly étale morphisms are locally of a standardized form just as étale morphisms of schemes are locally standard étale. The proof of this statement follows the arguments given in [SP, Tag 00UE].

Proposition 4.1. *Let $\varphi : Y \to X$ be an étale morphism of schemes, $y \in Y$ and w a valuation of $k(y)$. Set $x = \varphi(y)$ and $v = w|_{k(x)}$. Suppose that w is unramified in the finite separable field extension $k(y)|k(x)$. Then there exists an affine open neighborhood* $\operatorname{Spec} A$ *of x and $f, g \in A[T]$ with $f = T^n + f_{n-1}T^{n-1} + \ldots + f_0$ monic and f' a unit in*

$$B = (A[T]/(f))_g$$

such that $\operatorname{Spec} B$ *is isomorphic over A to an open neighborhood of y and $v(f_i) \leq 1$ for all $i = 0, \ldots, n-1$ and $w(g) = 1$ (viewing g as an element of B and w as a valuation of B).*

Proof. We may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Denote by $\mathfrak{p} \subseteq A$ and $\mathfrak{q} \subseteq B$ the prime ideals corresponding to x and y.

There exists an étale ring homomorphism $A_0 \to B_0$ with A_0 of finite type over $\mathbb{Z}$ and a ring homomorphism $A_0 \to A$ such that $B = A \otimes_{A_0} B_0$. Denote the image of y in $\operatorname{Spec} B_0$ by y_0 and the restriction of w to $k(y)$ by w_0. Then it suffices to prove the lemma for $\operatorname{Spec} B_0 \to \operatorname{Spec} A_0$ and (y_0, w_0) instead of φ and (y, w). Hence, we may assume that A is noetherian.

By Zariski's main theorem there is a finite ring homomorphism $A \to B'$, an A-algebra map $\beta : B' \to B$, and an element $b' \in B'$ with $\beta(b') \notin \mathfrak{q}$ such that $B'_{b'} \to B_{\beta(b')}$ is an isomorphism. Thus we may assume that $A \to B$ is finite and étale at $\mathfrak{q}$.

By Lemma 3.2 the valuation ring $\mathscr{O}_w \subseteq k(y)$ associated with w is a local ring of an étale $\mathscr{O}_v$-algebra, where $\mathscr{O}_v \subseteq k(x)$ is the valuation associated with v. Hence, there are polynomials $\bar{f}, \bar{g} \in \mathcal{O}_v[T]$ with $\bar{f}$ monic and and

$$\bar{f}' \in \left(\mathcal{O}_v[T]/(\bar{f})\right)_{\bar{g}}^{\times} \tag{1}$$

such that $\mathcal{O}_w$ is isomorphic over $\mathcal{O}_v$ to a local ring of $\left(\mathcal{O}_v[T]/(\bar{f})\right)_{\bar{g}}$. Then $v(\bar{f}(T)) \leq 1$, $v(\bar{g}(T)) \leq 1$, $w(\bar{g}) = 1$, and the image $\beta \in \mathcal{O}_w$ of T generates the field extension $k(\mathfrak{q})|k(\mathfrak{p})$.

Write

$$B \otimes_A k(\mathfrak{p}) = \prod_{i=1}^{n} B_i \tag{2}$$

with local, Artinian rings B_i such that $\mathfrak{q}$ is the maximal ideal of B_1, i.e. $B_1 = B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = k(\mathfrak{q})$. Denote by $\mathfrak{q}_2, \ldots, \mathfrak{q}_n$ the prime ideals of B corresponding to the maximal ideals of $B_2, \ldots, B_n$, respectively. Consider the element

$$\bar{b} = (\beta, 0, \ldots, 0) \in \prod_{i=1}^{n} B_i = B \otimes_A k(\mathfrak{p}).$$

There is $\lambda \in A$ whose residue class $\bar{\lambda} \in k(\mathfrak{p})$ is non-zero such that $\bar{\lambda}\bar{b}$ lies in the image of B. After replacing A by A_λ, we may assume that $\lambda \in A^\times$. We can thus lift $\bar{b}$ to an element $b \in B$.

Let I be the kernel of the A-algebra homomorphism $A[T] \to B$ mapping T to b. Set $B' = A[T]/I$ and denote by $\mathfrak{q}'$ the preimage of $\mathfrak{q}$ in B'. Then in the same way as in [stacks project, Tag 00UE] we obtain $B'_{\mathfrak{q}'} \cong B_{\mathfrak{q}}$. Therefore, we may replace B by B' and henceforth assume that

$$B = A[T]/I.$$

The image $\bar{I}$ of I in $k(\mathfrak{p})[T]$ is a principal ideal generated by a monic polynomial $\bar{h}$. According to the decomposition (2) we obtain a decomposition of $\bar{h}$ into monic irreducible factors:

$$\bar{h} = \bar{h}_1 \cdot \bar{h}_2^{e_2} \cdot \ldots \cdot \bar{h}_n^{e_n}.$$

In particular, $\bar{h}_1 = \bar{f}$, which is a separable polynomial.

Possibly replacing A by A_λ for $\lambda \in A$ as before we can lift $\bar{h}$ to a monic polynomial $f \in I$. Similarly, by (1), we can lift some power of $\bar{g} \in k(\mathfrak{p})[T]$ to a polynomial $g \in A[T]$ of the form $g = a_1 f + a_2 f'$ for some $a_1, a_2 \in A[T]$. We obtain a surjection

$$\varphi : A[T]/(f) \to B = A[T]/I$$

mapping g to an element b of $B\backslash\mathfrak{q}$ with $w(b) = 1$.

Since $A \to B$ is étale at $\mathfrak{q}$, there is $b' \in B\backslash\mathfrak{q}$ such that $A \to B_{bb'}$ is étale. We can find $a' \in A$ such that $v(a') = w(b')$ as $w|v$ is unramified. Upon replacing A by $A_{a'}$ we may assume that $a' \in A^\times$. Then $w(bb'/a) = 1$ Choose a preimage g' under φ of bb'/a'. Then φ induces an étale surjection

$$\varphi_{g'} : (A[T]/(f))_{g'} \longrightarrow B_{\varphi(g')} = B_{bb'/a'},$$

which is thus a localization. Modifying g' further in the same way as above we achieve that $\varphi_{g'}$ is an isomorphism. □

Corollary 4.2. *Let $\varphi : Y \to X$ be an étale morphism of adic spaces and $y \in Y$ a point where φ is strongly étale. Then there exist an affinoid open neighborhood $\mathrm{Spa}(A, A^+)$ of $x := \varphi(y)$, an affinoid open neighborhood V of y, and $f, g \in A[T]$ with $f = T^n + f_{n-1}T^{n-1} + \ldots + f_0$ monic and f' a unit in*

$$B = (A[T]/(f))_g$$

such that $|f_i(x)| \le 1$, $|g(y)| = 1$ and V is X-isomorphic to $\mathrm{Spa}(B, B^+)$ where B^+ is the integral closure of an open subring of B which is algebraically of finite type over A^+.

Proof. We may assume that $X = \mathrm{Spa}(R, R^+)$ and $Y = \mathrm{Spa}(S, S^+)$ are affinoid. By [Hub96], Corollary 1.7.3 iii) étale morphisms are locally of algebraically finite type. More precisely, for every étale morphism $Z \to \mathrm{Spa}(R, R^+)$ of affinoid adic spaces there is an étale ring map $R \to C$ of finite type and a ring of integral elements $C^+ \subseteq C$ which is the integral closure of a subring of C of finite type over C^+ such that $Z \cong \mathrm{Spa}(S, S^+)$ over (R, R^+). Hence, we may assume that $(R, R^+) \to (S, S^+)$ is of algebraically finite type and $R \to S$ is étale (in the algebraic sense). Denote by x the image point of y in X. By Proposition 4.1 there exist an affine open neighborhood $\mathrm{Spec}\, A$ of $\mathrm{supp}\, x \in \mathrm{Spec}\, R$ and $f, g \in A[T]$ with $f = T^n + f_{n-1}T^{n-1} + \ldots f_0$ monic and f' a unit in

$$B = (A[T]/(f))_g$$

such that $\mathrm{Spec}\, B$ is isomorphic over A to an open neighborhood of $\mathrm{supp}\, y$, $|f_i(x)| \le 1$ and $|g(y)| = 1$.

Set $U = \mathrm{Spa}(R, R^+) \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, A$. This is an open subspace of $X = \mathrm{Spa}(R, R^+)$. By construction of the fiber product (see [Hub94], Proposition 3.8), U is glued together from affinoid adic spaces of the form $\mathrm{Spa}(A, A_i^+)$ for $i \in \mathbb{N}$ and where A_i^+ is the integral closure in A of a finite type R^+-subalgebra of A. Choose $i \in \mathbb{N}$ such that

$x \in \operatorname{Spa}(A, A_i^+)$ and set $A^+ := A_i^+$. Similarly, we find an open affinoid neighborhood of y in $V = \operatorname{Spa}(A, A^+) \times_{\operatorname{Spec} A} \operatorname{Spec} B$ of the form $\operatorname{Spa}(B, B^+)$ such that B^+ is the integral closure in B of a finite type A^+-subalgebra of B. This finishes the proof. □

Corollary 4.3. *Let $\varphi : Y \to X$ be an étale morphism of adic spaces. The subset of Y where φ is strongly étale, is open.*

Proof. Let $y \in Y$ be a point where φ is strongly étale and set $x = \varphi(y)$. By Corollary 4.2 we may assume that $X = \operatorname{Spa}(A, A^+)$ and $Y = \operatorname{Spa}(B, B^+)$ as in the statement of the corollary. Then φ is strongly étale at any point $y' \in Y$ with $|f_i(\varphi(y'))| \le 1$ and $|g(y')| = 1$. Indeed, set $x' = \varphi(y')$ and denote by $\bar{f}$ and $\bar{g}$ the residue classes of f and g in $k(x')[T]$. We obtain an étale ring extension $k(x')^+ \to \big(k(x')^+[T]/(\bar{f})\big)_{\bar{g}}$. Since $|g(y')| = 1$, $k(y')^+$ is a localization of $\big(k(x')^+[T]/(\bar{f})\big)_{\bar{g}}$. The subset $\{y' \in Y \mid |f_i(y')| \le 1 \ \forall i, |g(y')| = 1\}$ of Y is open and thus we are done. □

Corollary 4.4. *Let $\varphi : Y \to X$ be an étale morphism of adic spaces. The subset of Y where φ is tame, is open.*

Proof. We may assume that $X = \operatorname{Spa}(A, A^+)$ and $Y = \operatorname{Spa}(B, B^+)$ are affinoid. Let $y \in Y$ be a point where φ is tame and set $x := \varphi(y)$. By Abhyankar's lemma ([SGA1], Exp. XIII, Proposition 5.2), there are non-zero elements $\bar{a}_1, \ldots, \bar{a}_n \in k(x)$ and an integer m prime to the residue characteristic of $k(x)^+$ such that any lift to $k(x)[\mu_m, \sqrt[m]{\bar{a}_1}, \ldots, \sqrt[m]{\bar{a}_n}]$ of the valuation corresponding to x is unramified in

$$k(x)[\mu_m, \sqrt[m]{\bar{a}_1}, \ldots, \sqrt[m]{\bar{a}_n}] \otimes_{k(x)} k(y) \;\big|\; k(x)[\mu_m \sqrt[m]{\bar{a}_1}, \ldots, \sqrt[m]{\bar{a}_n}].$$

We may choose the $\bar{a}_i$ as images of some $a_i \in A$. Replacing $\operatorname{Spa}(A, A^+)$ by a rational open neighborhood of x we may further assume that $a_i \in A^\times$ and that m is invertible on $\operatorname{Spec} A^+$. The ring homomorphism

$$A \to A' := A[T_0, T_1, \ldots, T_n]\big/(T_0^m - 1, T_1^m - a_1, \ldots, T_n^m - a_n)$$

is finite étale. Set $X' := \operatorname{Spa}(A', A'^+)$ where A'^+ is the integral closure of A^+ in A'. Then $X' \to X$ is tame. Moreover,

$$Y' := Y \times_X X' \to X'$$

is strongly étale at any lift of x to X'. Fix such a lift $x' \in X'$. We find a point $y' \in Y'$ lying over x' as well as y ([Hub96], Corollary 1.2.3 iii) d)). Denote by φ' the morphism $Y' \to X'$ and by ψ the morphism $X' \to X$. By Corollary 4.3 there is an open neighborhood $V' \subseteq Y'$ of y' such that $V' \to X'$ is strongly étale. Then $V' \to X$ is tame. Since étale morphisms are open ([Hub96], Proposition 1.7.8), the image V of V' in Y is an open neighborhood of y and moreover, $V \to X$ is tame. □

5. Limits of adic spaces

In [Hub96], § 2.4 Huber defines the notion of a projective limit of adic spaces: Let $\mathcal{A}$ be the category of quasi-compact, quasi-separated pseudo-adic spaces with adic morphisms. We consider a functor p from a cofiltered category I to $\mathcal{A}$ and write X_i for $p(i)$. Let $c : I \to \mathcal{A}$ be the constant functor to some object X of $\mathcal{A}$ and

$$\varphi : c \to p, \quad i \mapsto (\varphi_i : X \to X_i)$$

a morphism of functors. We say that X is a projective limit of the X_i and write

$$\varphi : X \sim \lim_i X_i$$

if the following conditions are satisfied:

(1) Denote by $\lim_i |X_i|$ the projective limit in the category of topological spaces. Then the natural mapping

$$\psi : |X| \to \lim_i |X_i|$$

induced by φ is a homeomorphism

(2) For every $x \in |X|$, there is an affinoid open neighborhood U of x such that the subring

$$\bigcup_{(i,V)} \mathrm{im}(\underline{\varphi}_i^* : \mathscr{O}_{\underline{X}_i}(V) \to \mathscr{O}_{\underline{X}}(U))$$

of $\mathscr{O}_{\underline{X}}(U)$ is dense where the union is over all pairs (i,V) with $i \in I$ and V an open subset of $\underline{X}_i$ with $\underline{\varphi}_i(U) \subseteq V$.

In this situation we have the following proposition ([Hub96], Proposition 2.4.4):

Proposition 5.1. *Let*

$$\tilde{\varphi} : \tilde{X}_{\acute{e}t} \times I \to (\tilde{X}_{i,\acute{e}t})_{i\in I}$$

be the morphism of topoi fibered over I which is induced by the $\tilde{\varphi}_i : \tilde{X}_{\acute{e}t} \to \tilde{X}_{i,\acute{e}t}$. Assume that $\varphi : X \sim \lim_i X_i$. Then $(\tilde{X}_{\acute{e}t}, \tilde{\varphi})$ is a projective limit of the fibered topos $(\tilde{X}_{i,\acute{e}t})_{i\in I}$.

In order to prove this proposition Huber proceeds as follows: For each $i \in I$ denote by $X_{i,\acute{e}t,f.p.}$ the restricted étale site, i.e. the site consisting of those objects in $X_{i,\acute{e}t}$ whose structure morphisms are quasi compact and quasi-separated ([Hub96], (2.3.12)). The topos associated with the projective limit site $\underrightarrow{X}$ of the fibered site $(X_{i,\acute{e}t,f.p.})_{i\in I}$ is isomorphic to the projective limit of the fibered topos $(\tilde{X}_{i,\acute{e}t})_{i\in I}$. Moreover, $(\tilde{X}_{\acute{e}t}, \tilde{\varphi})$ is isomorphic to the topos associated with the site $X_{\acute{e}t,g}$ which is defined as follows ([Hub96], Remark 2.3.4 ii)): The objects are the étale morphisms to X and the morphisms $Y \to Z$ are the equivalence classes of X-morphisms $Y' \to Z$ where Y' is an open subspace of Y with $|Y'| = |Y|$ and two morphisms are equivalent if they coincide on an open subspace V of Y with $|V| = |Y|$. There is a natural morphism of sites

$$\lambda : X_{\acute{e}t,g} \to \underrightarrow{X}$$

for which Huber proves that the conditions in the following proposition ([Hub96], Corollary A.5) are satisfied:

Proposition 5.2. *Let $f : C \to C'$ be a morphism of sites. The induced morphism of topoi $\tilde{f} : \tilde{C} \to \tilde{C}'$ is an equivalence if f satisfies the following conditions.*

(a) *In C' there exist finite projective limits and f^{-1} commutes with these.*
(b) *Every $X \in \mathrm{ob}(C)$ has a covering $(X_i \to X)_{i\in I}$ in C such that every $X_i \in \mathrm{ob}(C)$ lies in the image of the functor f^{-1}.*
(c) *A family $(X_i \to X)_{i\in I}$ of morphisms in C' is a covering in C' if $(f^{-1}(X_i) \to f^{-1}(X))$ is a covering in C.*
(d) *For every $X \in \mathrm{ob}(C)$, $Y \in \mathrm{ob}(C')$ and $(\varphi : X \to f^{-1}(Y)) \in \mathrm{mor}(C)$, there exist a covering $(\psi_i : X_i \to X)$ of X in C, and, for every $i \in I$ a $Y_i \in \mathrm{ob}(C')$, a $(\tau_i : Y_i \to Y) \in \mathrm{mor}(C')$ and a $(\varphi_i : X_i \to f^{-1}(Y_i)) \in \mathrm{mor}(C)$ such that, for every $i \in I$ the*

diagram in C

$$\begin{tikzcd} X_i \arrow[r, "\varphi_i"] \arrow[d, "\psi_i"'] & f^{-1}(Y_i) \arrow[d, "f^{-1}(\tau_i)"] \\ X \arrow[r, "\varphi"'] & f^{-1}(Y) \end{tikzcd}$$

commutes and $\varphi_i : X_i \to f^{-1}(Y_i)$ is an epimorphism and a covering of $f^{-1}(Y_i)$ in C.

We are now going to prove an analogue of Proposition 5.1 for the tame and the strongly étale topos:

Proposition 5.3. *In the situation of Proposition 5.1 the topos $(\tilde{X}_{s\acute{e}t}, \tilde{\varphi})$ is a projective limit of the fibered topos $(\tilde{X}_{i,s\acute{e}t})_{i\in I}$ and $(\tilde{X}_t, \tilde{\varphi})$ is a projective limit of the fibered topos $(\tilde{X}_{i,t})_{i\in I}$.*

Proof. We check that the strongly étale and tame analogues $\lambda_{s\acute{e}t}$ and λ_t of λ satisfy the conditions of Proposition 5.2:

(a) is true because $X_{s\acute{e}t}$ and X_t have fiber products and a terminal object.

(b). Let $Z \to X$ be strongly étale. In particular, it is étale. In the proof of Proposition 5.1 Huber constructs an open covering $Z = \bigcup_{j\in J} Z_j$ such that Z_j is X-isomorphic to an open subspace of $Y_i \times_{X_i} X$ for some $i \in I$ and $Y_i \to X_i$ in $X_{i,\acute{e}t,f.p.}$ with $|Z_j| = |Y_i|$. We have to find $k \to i$ in I such that

$$\psi_k : Y_k := Y_i \times_{X_i} X_k \to X_k$$

is strongly étale. By Corollary 4.3 for every $k \to i$ the set of points in $|Y_k|$ where ψ_k is *not* strongly étale is closed, hence compact in the constructible topology (note that $|Y_k|$ is locally constructible by the definition of a pseudo-adic space and quasi-compact as $|X_k|$ is quasi-compact and $Y_k \to X_k$ is qcqs). Therefore, its image D_k in $|X_k|$ is compact in the constructible topology of $|X_k|$. We write D_k^c for the set D_k equipped with the constructible topology. For $a : k \to k'$ denote by

$$u_a : \underline{X}_k \to \underline{X}_{k'}$$

the transition map and by

$$u_k : \underline{X} \to \underline{X}_k$$

the natural projection. Then u_a and u_k are continuous for the constructible topology by [Hub93b], Proposition 3.8 (iv). Since the property of being strongly étale is stable under base change,

$$u_a(D_k) \subseteq D_{k'}.$$

Furthermore, the assumption that $Z \to X$ is strongly étale implies that

$$\lim_{k\to i} D_k^c = \bigcap_{k\to i} u_k^{-1}(D_j^c) = \varnothing.$$

Since the projective limit of nonempty compact spaces is nonempty, there is $k \to i$ such that $D_k^c = \varnothing$. In other words $Y_k \to X_k$ is strongly étale. The proof for the tame topology is the same except for using Corollary 4.4 instead of Corollary 4.3.

(c) is obvious by the corresponding statement for the étale site and the proof for (d) is the same as for the étale site. □

Corollary 5.4. *In the situation of Proposition 5.1 assume that $i_0 \in I$ is a final object. Let $\mathcal{F}_0$ be a sheaf of abelian groups on $X_{i_0,s\acute{e}t}$. For $i \in I$ denote by $\mathcal{F}_i$ its pullback to $X_{i,s\acute{e}t}$ and by $\mathcal{F}$ its pullback to $X_{s\acute{e}t}$. Then the natural map*

$$\operatorname*{colim}_{i\in I} H^p_{s\acute{e}t}(X_i, \mathcal{F}_i) \longrightarrow H^p_{s\acute{e}t}(X, \mathcal{F})$$

is an isomorphism for all $p \geq 0$. Moreover, the analogous statement holds for the tame site.

Corollary 5.5. *Let S be an adic space and $\tau \in \{\acute{e}t, t, s\acute{e}t\}$ In the situation of Proposition 5.1 assume that X_i are adic spaces over S with compatible quasi-compact quasi-separated structure morphisms $g_i : X_i \to S$. We write $g : X \to S$ for the resulting morphism. For every $i \in I$ let $\mathcal{F}_i$ be an abelian sheaf on $(X_i)_\tau$ and for all $\alpha : i \to j$ let $\varphi_\alpha : \alpha^*\mathcal{F}_j \to \mathcal{F}_i$ be compatible transition morphisms. Denote by $\mathcal{F}$ the sheaf $\operatorname{colim}_I \varphi_i^*\mathcal{F}_i$. Then for all $p \geq 0$*

$$R^p g_* \mathcal{F} = \operatorname*{colim}_I R^p g_{i,*}\mathcal{F}_i.$$

6. Points of the strongly étale and tame topos

Definition 6.1. (i) A Huber pair (A, A^+) is *local* if A and A^+ are local, A^+ is the valuation subring of A associated with a valuation whose support is the maximal ideal of A, and the maximal ideal $\mathfrak{m}^+$ of A^+ is open.
(ii) (A, A^+) is *henselian* if it is local and A^+ is henselian.
(iii) (A, A^+) is *strongly henselian* if it is local and A^+ is strictly henselian.
(iv) A strongly henselian Huber pair (A, A^+) is *tamely henselian* if the value group of the associated valuation v is a $\mathbb{Z}[\frac{1}{p}]$-module, where p denotes the residue characteristic of v.

Lemma 6.2. *An adic space X is the spectrum of a local Huber pair if and only if X has a unique closed point x and any other point specializes to x.*

Proof. Suppose that every point of X specializes to x. Then every affinoid open neighborhood of x must contain all points of X. Hence $X = \operatorname{Spa}(A, A^+)$ for a complete Huber pair (A, A^+). Let $\mathfrak{m} \subseteq A$ denote the support of x. Suppose there is a maximal ideal $\mathfrak{m}' \subseteq A$ different from $\mathfrak{m}$. By [Hub94], Lemma 1.4 there is a point $y \in \operatorname{Spa}(A, A^+)$ whose support is $\mathfrak{m}'$. But y does not specialize to x, hence A is local with maximal ideal $\mathfrak{m}$.

Let a be an element of A which is not contained in A^+. We want to show that a is a unit in A and $1/a \in A^+$. Then we are done by [KZ02], Theorem I.2.5. Let A_a^+ denote the integral closure of $A^+[1/a]$ in A_a. Then

$$R(\frac{1}{a}) = \operatorname{Spa}(A_a, A_a^+)$$

is a rational subset of X. Since $a \notin A^+$, there is $y \in X$ with $|a(y)| > 1$. But y specializes to x and thus $|a(x)| > 1$. In particular, a is invertible in A as $a \notin \mathfrak{m} = \{b \in A \mid |b(x)| = 0\}$. This implies that $A_a = A$ and in particular, that (A_a, A_a^+) is complete. Moreover, x is contained in $\operatorname{Spa}(A_a, A_a^+)$. We conclude that $\operatorname{Spa}(A_a, A_a^+) = X$ and $1/a \in A^+$. □

In view of the lemma we say that a pseudo-adic space X is *local* if $\underline{X}$ is the adic spectrum of a local Huber pair and the closed point of $\underline{X}$ is contained in $|X|$.

Lemma 6.3. *For a pseudo-adic space X, the following conditions are equivalent:*

(i) *There is $x \in |X|$ such that for every strongly étale (tame) morphism of prepseudo-adic spaces $f : Y \to X$ and every $y \in |Y|$ with $f(y) = x$ there is an open neighborhood U of y such that f induces an isomorphism $U \to X$.*
(ii) *X is local and every strongly étale (tame) covering of X splits.*
(iii) *X is strongly (tamely) henselian.*

Proof. If (i) is true, x is the unique closed point of $\underline{X}$ as otherwise we get a contradiction by taking for f an open immersion which is not an isomorphism. Hence, X is local by Lemma 6.2. Moreover, it is clear by condition (i) that every covering of X splits. This shows that (i) implies (ii).

Assuming (ii), $\underline{X} = \mathrm{Spa}(A, A^+)$ for a local Huber pair (A, A^+) by Lemma 6.2. Let us show that A^+ is strictly henselian. Let $A^+ \to B^+$ be finite étale and set $B = B^+ \otimes_{A^+} A$. Then B^+ is integrally closed in B as this property is stable under smooth base change. Furthermore,

$$(A, A^+) \to (B, B^+)$$

is a finite strongly étale morphism of Huber pairs by Lemma 3.2. By assumption $\mathrm{Spa}(B, B^+)$ is a finite disjoint union of adic spaces isomorphic to X. This implies (iii) in the strongly étale case.

In the tame case it remains to show that the value group Γ of the valuation $|\cdot|$ corresponding to the closed point of X is divisible by all integers prime to the residue characteristic of A^+. Take $\gamma \in \Gamma$ and an integer m prime to the residue characteristic of A^+. We have to find $\gamma' \in \Gamma$ with $m\gamma' = \gamma$. We may assume that $\gamma \leq 1$. Otherwise we replace γ by its inverse. Take $a \in A$ with $|a| = \gamma$. Then $a \in A^\times \cap A^+$. Set

$$B^+ = A^+[T]/(T^m - a) \qquad \text{and} \qquad B = B^+ \otimes_{A^+} A = A[T]/(T^m - a).$$

We obtain a finite tame morphism $\varphi : (A, A^+) \to (B, B^+)$. As above, $\mathrm{Spa}(B, B^+)$ is a finite disjoint union of adic spaces isomorphic to $\mathrm{Spa}(A, A^+)$ via φ. Choose any connected component $\mathrm{Spa}(C, C^+)$ of $\mathrm{Spa}(B, B^+)$. The image of T in C corresponds via φ to an element of A with valuation equal to γ'.

In order to show that (iii) implies (i) assume that $\underline{X}$ equals the spectrum of a strongly (tamely) henselian Huber pair (A, A^+) and that the closed point x of $\underline{X}$ is contained in $|X|$. Let $f : Y \to X$ be a strongly étale (tame) morphism and $y \in |Y|$ with $f(y) = x$. Replacing Y by an open neighborhood of y we may assume that $\underline{Y}$ is affinoid and connected. By [Hub96], Corollary 1.7.3 iii), there is a Huber pair (B, B^+) of algebraically finite type over (A, A^+) such that $A \to B$ is étale and $\underline{Y} \cong \mathrm{Spa}(B, B^+)$. The closed point of $\mathrm{Spec}\, A$ is the support of x. Hence, the support of y provides a preimage of the closed point of $\mathrm{Spec}\, A$. As A is henselian and $\mathrm{Spec}\, B$ is connected, B is local and finite étale over A. Let C^+ be the integral closure of A^+ in B. We obtain a diagram

$$\begin{tikzcd}
\mathrm{Spa}(B, B^+) \arrow[rr, hook, "\circ"] \arrow[dr] & & \mathrm{Spa}(B, C^+) \arrow[dl] \\
& \mathrm{Spa}(A, A^+) &
\end{tikzcd}$$

As A^+ is henselian, C^+ is local. In the strongly étale case this implies already that C^+ is isomorphic to A^+. In the tame case this follows by Abhyankar's lemma. Since $\mathrm{Spa}(B, B^+)$ contains y, we conclude that $(A, A^+) = (B, B^+) = (B, C^+)$. □

Definition 6.4. A prepseudo-adic space X is called *strongly (tamely) local* if X satisfies the equivalent conditions of Lemma 6.3. A *strongly étale (tame) point* (in the category of prepseudo-adic spaces) is a strongly (tamely) local pseudo-adic space S such that $\underline{S}$ is the spectrum of an affinoid field and $|S| = \{s\}$ where s is the closed point of $\underline{S}$.

In [Hub96], Proposition 2.3.10 Huber proves the following:

Proposition 6.5. *Let X be an adic space and x a point of X. Let K be the henselization of $k(x)$ with respect to the valuation ring $k(x)^+$. Then the étale topos $(X,\{x\})^\sim_{\acute{e}t}$ of the pseudo-adic space $(X,\{x\})$ is naturally equivalent to the étale topos $(\operatorname{Spec} K)^\sim_{\acute{e}t}$.*

Restricting to the strongly étale and tame site, respectively, we obtain:

Corollary 6.6. *In the situation of Proposition 6.5 let K^+ be an extension of $k(x)^+$ to K. Let K_{nr} and K_t be the maximal extensions of K where K^+ is unramified and tamely ramified, respectively. Set $G_{nr} = \mathrm{Gal}(K_{nr}|K)$ and $G_t = \mathrm{Gal}(K_t|K)$. Then the strongly étale topos $(X,\{x\})^\sim_{s\acute{e}t}$ of $(X,\{x\})$ is naturally equivalent to the topos $(\operatorname{Spec} K^+_{\acute{e}t})^\sim$, which in turn is equivalent to the topos of G_{nr}-sets, and the tame topos $(X,\{x\})^\sim_t$ is naturally equivalent to the G_t-sets.*

Corollary 6.7. *For every strongly étale point S the global section functor*

$$\Gamma(S,-) : \tilde{S}_{s\acute{e}t} \to sets$$

is an equivalence of categories. Analogously for tame points.

Definition 6.8. For a strongly étale point $u : \xi \to X$ of a prepseudo-adic space X and a sheaf $\mathcal{F}$ on $\tilde{X}_{\acute{e}t}$ we define the *stalk* of $\mathcal{F}$ at ξ:

$$\mathcal{F}_\xi := \Gamma(\xi, u^*\mathcal{F}).$$

and for tame points and sheaves on $\tilde{X}_t$ accordingly.

For a strongly étale or tame point $u : \xi \to X$ of a prepseudo-adic space X we consider the category C_ξ of pairs (V,v) where V is an object of the strongly étale or tame site, respectively, and $v : \xi \to V$ is a morphism over X. The same argument as for the étale site (see [Hub96], Lemma 2.5.4) shows:

Lemma 6.9. *The category C_ξ is cofiltered. For every presheaf $\mathcal{P}$ on $X_{s\acute{e}t}$ or X_t, respectively, there is a functorial isomorphism*

$$(a\mathcal{P})_\xi \cong \operatorname*{colim}_{(V,v)\in C_\xi} \mathcal{P}(V),$$

where $a\mathcal{P}$ denotes the sheaf associated with $\mathcal{P}$.

For a strongly étale (tame) point ξ of the prepseudo-adic space X we define a strongly étale (tame) prepseudo-adic space X_ξ, the strong (tame) henselization of X at ξ: Set

$$\mathscr{O}_{X,\xi} := \lim_{(V,v)\in C_\xi} \mathscr{O}_{\underline{V}}(\underline{V}),$$

$$\mathscr{O}^+_{X,\xi} := \lim_{(V,v)\in C_\xi} \mathscr{O}^+_{\underline{V}}(\underline{V}).$$

and equip these rings with the following topology: Let (V,v) be an object of C_ξ with V affinoid. Choose an ideal of definition I of a ring of definition of $\mathscr{O}_{\underline{V}}(\underline{V})$ and take

$$\{I^n \cdot \mathscr{O}^+_{X,\xi} \mid n \in \mathbb{N}\}$$

to be a fundamental system of neighborhoods of zero. As in [Hub96], (2.5.9) this topology is independent of the choice of (V,v) and I and $(\mathscr{O}_{\underline{V}}(\underline{V}), \mathscr{O}^+_{\underline{V}}(\underline{V}))$ is a Huber pair. Put

$$\underline{X}_\xi := \operatorname{Spa}(\mathscr{O}_{\underline{V}}(\underline{V}), \mathscr{O}^+_{\underline{V}}(\underline{V}))$$

and

$$|X_\xi| := \bigcap_{(V,v)\in C_\xi} \underline{\varphi}^{-1}_{(V,v)}(|V|),$$

where $\underline{\varphi}_{(V,v)}$ is the natural morphism $\underline{X}_\xi \to \underline{V}$. We obtain a strongly (tamely) henselian prepseudo-adic space

$$X_\xi := (\underline{X}_\xi, |X_\xi|).$$

We call X_ξ the *strong (tame) localization* of X at ξ. Let D_ξ be the full (cofinal) subcategory of C_ξ consisting of those pairs (V,v) in C_ξ with affinoid $\underline{V}$ and quasi-compact $|V|$. Then X_ξ is a projective limit of the spaces V for $(V,v) \in D_\xi$ in the sense of [Hub96], (2.4.2). In particular, the results of Section 5 apply.

Over every point $x \in |X|$ we can choose a geometric point

$$\bar{x} := (\mathrm{Spa}(\bar{k}(x), \bar{k}(x)^+), \{s\})$$

such that $\bar{k}(x)$ is a separable closure of $k(x)$ (see [Hub96], (2.5.2)). Restricting to the maximal unramified and the maximal tamely ramified extension, respectively, yields a strongly étale and a tame point

$$x_{s\acute{e}t} = (\mathrm{Spa}(k_{nr}(x), k_{nr}(x)^+), \{s_{s\acute{e}t}\}), \quad x_t = (\mathrm{Spa}(k_t(x), k_t(x)^+), \{s_t\}),$$

where $k_{nr}(x)$ and $k_t(x)$ are the maximal unramified and maximal tamely ramified subextensions of $\bar{k}(x)|k(x)$. From Lemma 6.9 we conclude that these are enough points:

Corollary 6.10. *The families of functors*

$$(\tilde{X}_{s\acute{e}t} \to sets, \mathcal{F} \mapsto \mathcal{F}_{x_{s\acute{e}t}})_{x\in|X|} \quad \text{and} \quad (\tilde{X}_t \to sets, \mathcal{F} \mapsto \mathcal{F}_{x_t})_{x\in|X|}$$

are conservative.

Proof. Let $\mathcal{F}$ be a sheaf on $X_{s\acute{e}t}$ and assume that $\mathcal{F}_{x_{s\acute{e}t}} = 0$ for all $x \in |X|$. Take a strongly étale morphism $f : U \to X$ and an element $a \in \mathcal{F}(U)$. By Lemma 6.9 we find for each $u \in |U|$ a strongly étale neighborhood $U_u \to X$ of $f(u)_{s\acute{e}t}$ factoring through (U,u) such that $a|_{U_u} = 0$. The $U_u \to U$ comprise a covering of U, whence $a = 0$. ☐

Proposition 6.11. *Let X be a prepseudo-adic space, $\xi \to X$ a strongly étale (tame) point of X with support $x \in |X|$.*

(i) *Assume x is analytic. Consider the natural morphisms*

$$p_{s\acute{e}t} : \mathrm{Spa}(k_{nr}(x), k_{nr}(x)^+) \to \underline{X}, \quad p_t : \mathrm{Spa}(k_t(x), k_t(x)^+) \to \underline{X}.$$

Then

$$X_\xi \cong (\mathrm{Spa}(k_{nr}(x), k_{nr}(x)^+), p_{s\acute{e}t}^{-1}(|X|)) \quad or \quad X_\xi \cong (\mathrm{Spa}(k_t(x), k_t(x)^+), p_t^{-1}(|X|)),$$

according to whether ξ is a strongly étale or a tame point of X.

(ii) *Assume that x is non-analytic. Take an affinoid open neighborhood $U = \mathrm{Spa}(A, A^+)$ of x. Let (B, B^+) be the strong (tame) henselization of (A, A^+) and equip B with the $I \cdot B$-adic topology where I is an ideal of definition of a ring of definition of A. Then (B, B^+) is a Huber pair. Let p be the natural morphism $\mathrm{Spa}(B, B^+) \to \underline{X}$. Then*

$$X_\xi \cong (\mathrm{Spa}(B, B^+), p^{-1}(|X|)).$$

Proof. The argument is the same as the proof of the corresponding statement for the étale site ([Hub96], Proposition 2.5.13). ☐

7. Topological invariance

In this section we prove some assertions concerning the topological invariance of the tame cohomology. They are in analogy with the respective results concerning the étale topology.

Proposition 7.1. *Let $X \to Y$ be a morphism of adic spaces which induces an isomorphism on the underlying reduced adic spaces. Then*

$$U \mapsto U \times_Y X$$

defines an equivalence of categories $X_\tau \to Y_\tau$. In particular, the topoi $Sh(X_\tau)$ and $Sh(Y_\tau)$ are equivalent.

Proof. Without loss of generality we may assume that X and Y are affinoid. Moreover, it suffices to prove that $U \mapsto U \times_Y X$ defines an equivalence of the subcategories of affinoid spaces in X_τ and Y_τ, respectively. The general statement follows by glueing. Write $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$. By [Hub96], Corollary 1.7.3, the affinoid adic spaces that are étale over X are precisely the open subspaces of adic spaces of the form $\mathrm{Spa}(R, R^+)$ with R étale over A and R^+ the integral closure of A^+ in R and analogously for Y. By [EGA4.4], 18.1.2, the assignment $S \mapsto S \otimes_B A$ defines an equivalence of the categories of étale B-algebras and étale A-algebras. Moreover, for S étale over B and S^+ the integral closure of B^+ in S, the categories of open subspaces of $\mathrm{Spa}(S, S^+)$ and $\mathrm{Spa}((S, S^+) \otimes_{(B,B^+)} (A, A^+))$ are equivalent as they only depend on the underlying topological space. We conclude that $X_{ét}$ and $Y_{ét}$ are equivalent. In order to see that this is also true for the tame and strongly étale sites it suffices to note that the properties of being tame or strongly étale only depend on the underlying reduced subspaces. □

The following two results are analogs of the excision theorems in étale cohomology.

Lemma 7.2. *Consider the following diagram of adic spaces*

$$\begin{tikzcd} Z'_{red} \arrow[r, hook] \arrow[d, "\sim"] & X' \arrow[d, "\pi"] & U' \arrow[l, hook'] \arrow[d] \\ Z_{red} \arrow[r, hook] & X & U, \arrow[l, hook'] \end{tikzcd}$$

where $Z' \to X'$ and $Z \to X$ are closed immersions with open complements U' and U, respectively, π is a morphism in X_τ, and $Z'_{red} \to Z_{red}$ is an isomorphism. Then for any sheaf $\mathcal{F}$ on X_τ and any $i \geq 0$ we have

$$H^i_Z(X, \mathcal{F}) \xrightarrow{\sim} H^i_{Z'}(X', \mathcal{F}|_{X'}).$$

Proof. The proof is the same as for the étale topology on schemes (see [Fu15], Proposition 5.6.12). □

Proposition 7.3. *Let X be an adic space and $x \in X$ a Zariski-closed point (i.e., $x = \mathrm{Spa}(k(x), k(x))$ and $\mathrm{Spa}(k(x), k(x)) \to X$ is a closed immersion). Then for any sheaf $\mathcal{F}$ on X_τ and any $i \geq 0$ we have*

$$H^i_x(X, \mathcal{F}) = H^i_x(X^h_x, \mathcal{F}),$$

where X^h_x denotes the henselization of X at x.

Proof. As τ-cohomology commutes with limits by Corollary 5.4, we have

$$H^i_x(X^h_x, \mathcal{F}) = \operatorname*{colim}_{(Y,y)\to(X,x)} H^i_y(Y, \mathcal{F}),$$

where the colimit runs over all pointed étale morphisms $(Y, y) \to (X, x)$ such that $k(y) = k(x)$. We can as well restrict to pointed morphisms $(Y, y) \to (X, x)$ in X_τ as every étale morphism $(Y, y) \to (X, x)$ as above is strongly étale, hence tame, at y and the strongly étale locus is open (Corollary 4.3). For a pointed morphism $(Y, y) \to (X, x)$ in X_τ with $k(y) = k(x)$ we know by Lemma 7.2 that

$$H^i_y(Y, \mathcal{F}) = H^i_x(X, \mathcal{F}).$$

□

8. Comparison with étale cohomology

Lemma 8.1. *Let (A, A^+) be a henselian Huber pair. Denote by k the residue field of A and by k^+ the residue field of A^+. Choose a separable closure $\bar{k}$ of k and denote by $\bar{v}$ the continuation of the valuation of k corresponding to the closed point of $\mathrm{Spa}(A, A^+)$. This defines a geometric point $\xi \to \mathrm{Spa}(A, A^+)$ which we can also view as tame and strongly étale point. Write k^t for the maximal subextension of $\bar{k}|k$ where $\bar{v}$ is tamely ramified. Then for any abelian sheaf $\mathcal{F}$ on $\mathrm{Spa}(A, A^+)_{ét}$ and any $i \geq 0$*

$$H^i_{ét}(\mathrm{Spa}(A, A^+), \mathcal{F}) = H^i(G_k, \mathcal{F}_\xi),$$

for any sheaf $\mathcal{F}$ on $\mathrm{Spa}(A, A^+)_{sét}$ and any $i \geq 0$

$$H^i_{sét}(\mathrm{Spa}(A, A^+), \mathcal{F}) = H^i(G_{k^+}, \mathcal{F}_\xi),$$

and for any sheaf $\mathcal{F}$ on $\mathrm{Spa}(A, A^+)_t$ and any $i \geq 0$

$$H^i_t(\mathrm{Spa}(A, A^+), \mathcal{F}) = H^i(\mathrm{Gal}(k^t|k), \mathcal{F}_\xi).$$

Proof. This follows using the Hochschild-Serre spectral sequence for G_k, G_{k^+} (which can be identified with the Galois group of the maximal unramified subextension of $\bar{k}|k$) and $\mathrm{Gal}(k^t|k)$, respectively. □

For a prepseudo-adic space X we write $\mathrm{char}^+(X)$ for the set of characteristics of the residue fields of $\mathcal{O}^+_{X,x}$ for $x \in |X|$

Proposition 8.2. *Let X be a prepseudo-adic space and and $\mathcal{F}$ a torsion sheaf on $X_{ét}$ with torsion prime to $\mathrm{char}^+(X)$. Then the morphism of sites $\varphi : X_{ét} \to X_t$ induces isomorphisms*

$$H^i_t(X, \varphi_*\mathcal{F}) \xrightarrow{\sim} H^i_{ét}(X, \mathcal{F})$$

for all $i \geq 0$.

Proof. We have to show that for any tame henselian (A, A^+) and any torsion sheaf $\mathcal{G}$ on $(A, A^+)_{ét}$ with torsion prime to the residue characteristic p of A^+ the cohomology groups

$$H^i_{ét}(\mathrm{Spa}(A, A^+), \mathcal{G})$$

vanish for all $i \geq 1$. By Lemma 8.1 these cohomology groups equal

$$H^i(G_k, \mathcal{G}_\xi),$$

where k and ξ are defined as in Lemma 8.1. But G_k is a pro-p-group (see [EP05], Theorem 5.3.3) and $\mathcal{G}_\xi$ is a torsion G_k-module with torsion prime to p. Therefore, the above cohomology groups vanish. □

Lemma 8.3. *Let $X \to S$ be a morphism of schemes and $\mathcal{F}$ a torsion sheaf on $X_{\acute{e}t}$. Then the morphism of sites*

$$\psi : \mathrm{Spa}(X,S)_{\acute{e}t} \to X_{\acute{e}t}$$

induces isomorphisms

$$H^i_{\acute{e}t}(X, \psi^*\mathcal{F}) \xrightarrow{\sim} H^i_{\acute{e}t}(\mathrm{Spa}(X,S), \mathcal{F}).$$

for all $i \geq 0$.

Proof. If X and S are affine, the result is a special case of [Hub96], Theorem 3.3.3. Let us now assume that S is affine and X is arbitrary. By virtue of the Leray spectral sequence associated with ψ, it suffices to show

$$\psi_*\psi^*\mathcal{F} \xrightarrow{\sim} \mathcal{F} \qquad \text{and} \qquad (R^i\psi_*(\psi^*\mathcal{F}) = 0 \text{ for } i > 0.$$

These assertions are local on X. Hence, we are reduced to the affine case.

The next step is to only require S to be separated. We choose an open covering $\mathcal{U}$ of S by affine schemes S_i. It induces an open covering $\mathcal{V}$ of $\mathrm{Spa}(X,S)$ by the open subspaces

$$\mathrm{Spa}(X \times_S S_i, S_i) \subseteq \mathrm{Spa}(X,S).$$

We obtain a morphism of Čech-to-derived spectral sequences

$$\begin{tikzcd} H^i(\mathcal{U}, \mathcal{H}^j(\mathcal{F})) \arrow[r, Rightarrow] \arrow[d] & H^{i+j}(X, \mathcal{F}) \arrow[d] \\ H^i(\mathcal{V}, \mathcal{H}^j(\psi^*\mathcal{F})) \arrow[r, Rightarrow] & H^{i+j}(\mathrm{Spa}(X,S), \psi^*\mathcal{F}). \end{tikzcd}$$

The separatedness assumptions assures finite intersections of the S_i to be affine. Therefore, we can use the previous case to conclude that all vertical morphisms on the left are isomorphisms Hence, the right vertical morphism is an isomorphism. The general case follows from the case where S is separated by the same argument using a covering of S by separated open subschemes. □

Combining Lemma 8.3 with Proposition 8.2 we obtain:

Corollary 8.4. *Let $X \to S$ be a morphism of schemes and $\mathcal{F}$ a torsion sheaf on $X_{\acute{e}t}$ with torsion prime to the residue characteristics of S. Then the morphisms of sites*

$$\mathrm{Spa}(X,S)_t \xleftarrow{\varphi} \mathrm{Spa}(X,S)_{\acute{e}t} \xrightarrow{\psi} X_{\acute{e}t}$$

induce isomorphisms

$$H^i_t(\mathrm{Spa}(X,S), \varphi_*\psi^*\mathcal{F}) \cong H^i_{\acute{e}t}(X, \mathcal{F})$$

for all $i \geq 0$.

We prove the following comparison of tame and strongly étale cohomology.

Proposition 8.5. *Let X be an adic space with $\mathrm{char}^+(X) = p > 0$ (i.e., for any point $x \in X$ the localization X_x is of the form $\mathrm{Spa}(A, A^+)$ such that the residue characteristic of A^+ is p). Then for any p-torsion sheaf $\mathcal{F}$ on X_t the natural morphism of sites*

$$\varphi : X_t \to X_{s\acute{e}t}$$

induces isomorphisms

$$H^i(X_{s\acute{e}t}, \varphi_*\mathcal{F}) \xrightarrow{\sim} H^i(X_t, \mathcal{F})$$

for every integer i.

Proof. We have to show that the stalks of the higher direct images $R^i\varphi_*\mathcal{F}$ vanish. Let $\bar{x}$ be a geometric point of $X_{sét}$. The strict localization $X^{sét}_{\bar{x}}$ is of the form $\mathrm{Spa}(A, A^+)$ with (A, A^+) local and A^+ strictly henselian. For the stalk of $R^i\varphi_*\mathcal{F}$ at $\bar{x}$ we get by Corollary 5.4 and Lemma 8.1

$$R^i\varphi_*\mathcal{F}_{\bar{x}} = H^i(\mathrm{Spa}(K, A)_t, \mathcal{F}) = H^i(\mathrm{Gal}(k^t|k), \mathcal{F}_{\bar{x}}),$$

where $k^t|k$ is the maximal tamely ramified extension of the residue field of A with respect to the valuation corresponding to $\bar{x}$. But by assumption $\mathcal{F}$ is a p-torsion sheaf and as A^+ is strictly henselian, $\mathrm{Gal}(k^t|k)$ has trivial p-Sylow subgroups. Therefore, the above cohomology group vanishes by [NSW08], Proposition 1.6.2. $\square$

The above proposition tells us that for p-torsion sheaves tame and strongly étale cohomology coincide. Moreover, by Proposition 8.2, for torsion sheaves with torsion invertible on X, tame cohomology coincides with étale cohomology. In that sense the tame topology is a bridge between étale and strongly étale cohomology.

Lemma 8.6. *Let $X \to S'$ be a morphism of schemes and $S' \to S$ a proper morphism of schemes. Then*

$$\mathrm{Spa}(X, S') \cong \mathrm{Spa}(X, S).$$

Proof. As $S' \to S$ is finitely generated and separated, the natural morphism $\mathrm{Spa}(X, S') \to \mathrm{Spa}(X, S)$ is an open immersion. In order to check surjectivity take a point $(x, R, \phi) \in \mathrm{Spa}(X, S)$. The morphism $\phi : \mathrm{Spec}\, R \to S$ lifts (uniquely) to a morphism $\phi' : \mathrm{Spec}\, R \to S'$ by the valuative criterion for properness. Hence, (x, R, ϕ') is a preimage in $\mathrm{Spa}(X.S')$ of (x, R, ϕ). $\square$

Lemma 8.7. *Let X be scheme and let $\tau \in \{t, sét, ét\}$ be one of the topologies. Then the center map $c : \mathrm{Spa}(X, X) \to X$ induces for every sheaf $\mathcal{F}$ on $(\mathrm{Spa}(X, X)_\tau$ isomorphisms*

$$H^i_{ét}(X, c_*\mathcal{F}) \xrightarrow{\sim} H^i_\tau(\mathrm{Spa}(X, X), \mathcal{F})$$

for all $i \geq 0$.

Proof. It is easy to check that c induces morphisms of cites $\mathrm{Spa}(X, X)_\tau \to X_{ét}$ by mapping an étale morphism $Y \to X$ to the strongly étale (and thus étale and tame) morphism $\mathrm{Spa}(Y, Y) \to \mathrm{Spa}(X, X)$. We need to check that the higher direct images of $\mathcal{F}$ vanish. In order to do so we may assume that X is strictly henselian. But then $\mathrm{Spa}(X, X)$ is strictly local (so in particular tamely and strongly local) and thus its cohomology groups vanish in degree greater than zero. $\square$

Combining Lemma 8.7 with Lemma 8.6 we obtain the following

Corollary 8.8. *Let $X \to S$ be a proper morphism of schemes and let $\tau \in \{t, sét, ét\}$ be one of the topologies. Then the center map $c : \mathrm{Spa}(X, S) = \mathrm{Spa}(X, X) \to X$ induces for every sheaf $\mathcal{F}$ on $(\mathrm{Spa}(X, S)_\tau$ isomorphisms*

$$H^i_{ét}(X, c_*\mathcal{F}) \xrightarrow{\sim} H^i_\tau(\mathrm{Spa}(X, S), \mathcal{F})$$

for all $i \geq 0$.

9. Comparison with the tame fundamental group

Let X be a regular scheme of finite type over some base scheme S. Suppose there is a compactification $\bar{X}$ of X over S such that the complement of X in $\bar{X}$ is the support of a strict normal crossing divisor D. Then, following [SGA1], Exp. VIII, § 2, we can study finite étale covers of X which are tamely ramified along D. This results in the definition of the tame fundamental group $\pi_1^t(X/S, \bar{x})$ for some geometric point $\bar{x}$ of X.

Under less favorable regularity assumptions, there are several approaches to define the tame fundamental group. We only state the two of these which we use in this section. Fix an integral, pure-dimensional, separated, and excellent base scheme S. In [Wie08] Wiesend introduces the notion of curve-tameness. It has been slightly extended by Kerz and Schmidt in [KS10] to the following definition: A curve over S is a scheme of finite type C over S which is integral and such that

$$\dim_S C := \mathit{trdeg}(k(C)|k(T)) + \dim_{\mathrm{Krull}} T = 1,$$

where T denotes the closure of the image of C in S. Any curve C has a canonical compactification $\bar{C}$ over S which is regular at the points in $\bar{C} - C$. Hence, we can define tameness over C as in [SGA1]: A finite étale cover $C' \to C$ by a connected, hence integral, curve C' is tame at a point $c \in \bar{C} - C$ if the corresponding valuation of the function field of C is tamely ramified in the extension of function fields $k(C')|k(C)$. For a general finite étale cover $C' \to C$ we require tameness for each connected component of C'. Given a scheme X of finite type over S, a finite étale cover $Y \to X$ is *curve-tame* if the base-change to any curve $C \to X$ is tamely ramified outside $C \times_X Y$.

Let us recall next the notion of valuation-tameness considered in [KS10]. A finite étale cover $Y \to X$ of connected, normal schemes of finite type over S is *valuation-tame* if every valuation of the function field $k(X)$ with center on S is tamely ramified in the finite, separable field extension $k(Y)|k(X)$.

This section is concerned with comparing the fundamental group of the tame site with the curve-tame and the valuation tame fundamental group. In order to do so we need to relate tame covers with torsors in the tame topos.

Lemma 9.1. *Let $\pi : Y \to X$ be a surjective étale morphism of discretely ringed adic spaces. Then π satisfies descent for finite morphisms.*

Proof. The same arguments as for schemes reduce us to the case where $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$ are affinoid. Then $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is a surjective étale morphism of schemes. Moreover, finite morphisms to X and Y correspond to finite A-algebras and B-algebras, respectively. Hence, we can apply descent theory for schemes ([SGA1], Exp. VIII, Théorème 2.1) to obtain the result. □

Corollary 9.2. *Let $\tau \in \{\acute{e}t, t, s\acute{e}t\}$ be one of the topologies on a discretely ringed adic space X. Let $\mathcal{F}$ be a torsor in $Sh(X_\tau)$ for some finite group G. Then $\mathcal{F}$ is represented by a finite Galois morphism $Y \to X$ in X_τ with Galois group G.*

Proof. Let $X' \to X$ be a covering of X such that $\mathcal{F}|_{X'}$ is trivial, hence represented by $\pi' : \coprod_G X' \to X'$. By Lemma 9.1 the morphism π' descends to a finite Galois morphism $\pi : Y \to X$ in X_τ representing $\mathcal{F}$. □

For a geometric point $\bar{x}$ of a connected, locally noetherian adic space X we want to define the fundamental group of the corresponding pointed site $(X_\tau, \bar{x})$ (for $\tau \in \{\acute{e}t, t, s\acute{e}t\}$). To be more precise, we want a pro-finite group $\pi_1^\tau(X, \bar{x})$ that classifies finite torsors, i.e.

for every finite group G the set of isomorphism classes of G-torsors in $Sh(X_\tau)$ should be given by
$$\mathrm{Hom}(\pi_1^\tau(X,\bar{x}),G).$$
In [AM69], §9 Artin and Mazur describe the construction of the fundamental pro-group of a locally connected site via the Verdier functor. By [AM69], Corollary 10.7, it classifies all torsors (not just finite). Taking the pro-finite completion we obtain a pro-finite group classifying finite torsors. In order to apply these results in our situation, we need to check that X_τ is locally connected. But this is true because the connected components of an affinoid noetherian adic space X are in one-to-one correspondence with the idempotents of the noetherian ring $\mathscr{O}_X(X)$. By descent (Corollary 9.2), the resulting fundamental group $\pi_1^\tau(X,\bar{x})$ not only classifies finite G-torsors in $Sh(X_\tau)$ but also finite Galois τ-covers.

Proposition 9.3. *Let $X \to S$ be a morphism of connected, noetherian schemes and $\bar{x}$ a geometric point of X. We can view $\bar{x}$ as a geometric point of* $\mathrm{Spa}(X,S)$ *by taking the trivial valuation on the residue field of $\bar{x}$. Then there is a natural isomorphism*
$$\pi_1^{\acute{e}t}(X,\bar{x}) \cong \pi_1^{\acute{e}t}(\mathrm{Spa}(X,S),\bar{x}).$$

Proof. By what we have just discussed, the étale fundamental group of $\mathrm{Spa}(X,S)$ classifies finite étale covers of $\mathrm{Spa}(X,S)$. Similarly, $\pi_1^{\acute{e}t}(X,\bar{x})$ classifies finite étale covers of X. Every finite étale cover $Y \to X$ induces a finite étale cover $\mathrm{Spa}(Y,S) \to \mathrm{Spa}(X,S)$. For two finite étale covers $Y \to X$ and $Y' \to X$ the natural homomorphism
$$\mathrm{Hom}_X(Y,Y') \longrightarrow \mathrm{Hom}_{\mathrm{Spa}(X,S)}(\mathrm{Spa}(Y,S),\mathrm{Spa}(Y',S))$$
is bijective, an inverse being given by assigning to a morphism $\mathrm{Spa}(Y,S) \to \mathrm{Spa}(Y',S)$ the corresponding morphism of supports $Y \to Y'$. It remains to show that every finite étale cover of $\mathrm{Spa}(X,S)$ comes from a finite étale cover of X.

Let $\varphi : Z \to \mathrm{Spa}(X,S)$ be a finite étale cover of adic spaces. We need to show that it comes from a finite étale cover of X as above. Let $\mathrm{Spa}(B,B^+)$ and $\mathrm{Spa}(A,A^+)$ be affinoid open subspaces of Z and $\mathrm{Spa}(X,S)$, respectively, such that $\varphi(\mathrm{Spa}(B,B^+)) \subseteq \mathrm{Spa}(A,A^+)$. By [Hub96], Corollary 1.7.3, we obtain a factorization
$$\begin{tikzcd}
\mathrm{Spa}(B,B^+) \arrow[rr, hook, "\circ"] \arrow[dr] & & \mathrm{Spa}(B,A^+) \arrow[dl] \\
& \mathrm{Spa}(A,A^+). &
\end{tikzcd}$$
and $A \to B$ is étale. Since we are working with discretely ringed adic spaces, this construction glues and we obtain a diagram
$$\begin{tikzcd}
Z \arrow[rr, hook, "\circ"] \arrow[dr, "\varphi"'] & & \mathrm{Spa}(Y,S) \arrow[dl] \\
& \mathrm{Spa}(X,S) &
\end{tikzcd}$$
with $Y \to X$ étale and Z dense in $\mathrm{Spa}(Y,S)$.

By assumption there is an étale covering $W \to \mathrm{Spa}(X,S)$ trivializing φ. Without loss of generality we may assume that W is a disjoint union of adic spaces of the form $\mathrm{Spa}(X_i,S_i)$. In particular, $\coprod_i X_i \to X$ is an étale covering of X. Moreover,
$$Z_i := Z \times_{\mathrm{Spa}(X,S)} \mathrm{Spa}(X_i,S_i) \cong \mathrm{Spa}(X_i,S_i) \otimes G$$

for some group G. Base changing the above diagram to $\operatorname{Spa}(X_i, S_i)$ we obtain

$$\begin{tikzcd} \operatorname{Spa}(X_i, S_i) \otimes G \arrow[rr, hook, "\circ"] \arrow[dr] & & \operatorname{Spa}(Y \times_X X_i, S_i) \arrow[dl] \\ & \operatorname{Spa}(X_i, S_i) & \end{tikzcd}$$

and $\operatorname{Spa}(X_i, S_i) \otimes G$ is open and dense in $\operatorname{Spa}(Y \times_X X_i, S_i)$. But $\operatorname{Spa}(X_i, S_i) \otimes G \to \operatorname{Spa}(X_i, S_i)$ satisfies the valuative criterion for properness and hence,

$$\operatorname{Spa}(X_i, S_i) \otimes G = \operatorname{Spa}(X_i \otimes G, S_i) = \operatorname{Spa}(Y \times_X X_i, S_i).$$

We conclude that $X_i \otimes G = Y \times_X X_i$. This shows that $Y \to X$ is a finite étale cover such that $Z = \operatorname{Spa}(Y, S)$. $\square$

Proposition 9.4. *Let X be a connected, regular scheme of finite type over S and $\bar{x}$ a geometric point of X. Then the valuation-tame fundamental group $\pi_1^{vt}(X/S, \bar{x})$ is canonically isomorphic to the fundamental group $\pi_1^t(\operatorname{Spa}(X, S), \bar{x})$ of the tame site $\operatorname{Spa}(X, S)_t$.*

Proof. By Proposition 9.3 we have to show that a finite étale cover $Y \to X$ is valuation-tame over S if and only if $\operatorname{Spa}(Y, S) \to \operatorname{Spa}(X, S)$ is tame. If the latter is true, it is clear that the former also holds. Suppose that $Y \to X$ is valuation-tame and pick a point $z = (x, R, \phi) \in \operatorname{Spa}(X, S)$. Since X is regular at x, we find a discrete valuation v (not necessarily of rank one) supported on the generic point $\eta = \operatorname{Spec} k(X)$ and a morphism $\psi : \operatorname{Spec} \mathscr{O}_v \to X$ mapping the closed point of $\operatorname{Spec} \mathscr{O}_v$ to x such that $k(v) = k(x)$. The concatenation of v with the valuation corresponding to R gives a valuation ring R' of $k(X)$ and ϕ and ψ determine a morphism $\alpha : \operatorname{Spec} R' \to S$. By assumption any point of $\operatorname{Spa}(Y, S)$ lying over (η, R', α) is tame over $\operatorname{Spa}(X, S)$. This implies that the same is true for any point lying over z. $\square$

Here is a stronger version but with some assumptions on resolutions of singularities:

Proposition 9.5. *Let S be an integral, excellent and pure-dimensional base scheme and X a connected scheme of finite type over S with a geometric point $\bar{x}$. Assume that every finite separable extension of every residue field of X admits a regular proper model. Then the curve-tame fundamental group $\pi_1^{ct}(X/S, \bar{x})$ is canonically isomorphic to $\pi_1^t(\operatorname{Spa}(X, S), \bar{x})$.*

Proof. By Proposition 9.3 we have to show that a finite étale cover $Y \to X$ is curve-tame over S if and only if $\operatorname{Spa}(Y, S) \to \operatorname{Spa}(X, S)$ is tame. Suppose $\operatorname{Spa}(Y, S) \to \operatorname{Spa}(X, S)$ is tame and let $C \to X$ be a curve mapping to X with compactification $\bar{C}$. Without loss of generality we may assume that $C \to X$ is a closed immersion. Let η_C be the generic point of C viewed as a point of X. A point $c \in \bar{C} - C$ corresponds to a valuation ring $\mathscr{O}_c \subseteq k(\eta_C)$ and comes naturally with a morphism $\phi_c : \operatorname{Spec} \mathscr{O}_c \to S$. This defines a point $(\eta_C, \mathscr{O}_c, \phi_c)$ of $\operatorname{Spa}(X, S)$. By assumption all points of $\operatorname{Spa}(Y, S)$ lying over $(\eta_C, \mathscr{O}_c, \phi_c)$ are tame over $\operatorname{Spa}(X, S)$. This translates to $C \times_X Y \to C$ being tamely ramified over c. We conclude that $Y \to X$ is curve-tame.

Suppose now that $Y \to X$ is curve-tame. Take a point $(x, R, \phi) \in \operatorname{Spa}(X, S)$. Let Z be the closed subset $\overline{\{x\}}$ of X with the reduced scheme structure. In order to show that $\operatorname{Spa}(Y, S) \to \operatorname{Spa}(X, S)$ is tame we may replace $Y \to X$ by its base change to Z. Note that $Z \times_X Y \to Z$ is still curve-tame. Hence, we may assume that X is integral with generic point x. Furthermore, by the same argument, we may replace X by a nonempty open subscheme. We may thus assume that X is regular. But now under our assumption on resolution of singularities $Y \to X$ is curve tame if and only if it is valuation-tame (see [KS10], Theorem 4.4). In particular, every point of $\mathrm{Spa}(Y, S)$ lying over (x, R, ϕ) is tame over $\mathrm{Spa}(X, S)$. □

10. Cohomology for discretely ringed adic spaces

Let S be a noetherian scheme. We say that *resolution of singularities holds over* S if for any reduced scheme X of finite type over S there is a locally projective birational morphism $X' \to X$ such that X' is regular and $X' \to X$ is an isomorphism over the regular locus of X. By [EGA4.2, IV, 7.9.5], this implies, in particular, that S is quasi-excellent.

In this section we compare the cohomology of the sheaf $\mathcal{O}_Z^+$ on the discretely ringed adic space $Z = \mathrm{Spa}(X, S)$ with the cohomology of the structure sheaf $\mathcal{O}_S$ of the scheme S. All cohomology groups in this section are sheaf cohomology groups on the underlying topological space of the scheme or adic space in question (not on the tame or étale site etc.).

Let $\pi : X \to S$ be a morphism of schemes. Recall that the structure sheaf $\mathcal{O}_Z$ on $Z = \mathrm{Spa}(X, S)$ is the pullback of the structure sheaf $\mathcal{O}_X$ on X via the support map. In particular,

$$\mathcal{O}_Z(Z) = \mathcal{O}_X(X).$$

Consider the center map

$$c : \mathrm{Spa}(X, S) \to S$$

sending $(x, R, \phi) \in \mathrm{Spa}(X, S)$ to the image of the closed point of $\mathrm{Spec}\, R$ under ϕ. It is continuous as the preimage of an open subset $S' \subseteq S$ is the open subset $\mathrm{Spa}(X \times_S S', S')$ of $\mathrm{Spa}(X, S)$. We have a natural identification of $c_*\mathcal{O}_Z$ with $\pi_*\mathcal{O}_X$. Hence, the homomorphism $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ induces a functorial homomorphism

$$\mathcal{O}_S \to c_*\mathcal{O}_Z.$$

Lemma 10.1. *The homomorphism $\mathcal{O}_S \to c_*\mathcal{O}_Z$ factors through $c_*\mathcal{O}_Z^+$.*

Proof. It is equivalent to show that the adjoint homomorphism $c^*\mathcal{O}_S \to \mathcal{O}_Z$ factors through $\mathcal{O}_Z^+$. It suffices to check this for affinoid opens $\mathrm{Spa}(A, A^+)$ of Z and the presheaf pullback $c^p\mathcal{O}_S$.

The sections $c^p\mathcal{O}_S(\mathrm{Spa}(A, A+))$ are given as the colimit of $\mathcal{O}_S(S')$ over all commutative diagrams

$$\begin{array}{ccc} \mathrm{Spa}(A, A^+) & \longrightarrow & S' \\ \big\downarrow & & \big\downarrow \\ Z = \mathrm{Spa}(X, S) & \xrightarrow{c} & S \end{array} \qquad (3)$$

with S' an open subscheme of S:

$$c^p\mathcal{O}_S(\mathrm{Spa}(A, A+)) = \operatorname*{colim}_{S'} \mathcal{O}_S(S').$$

The homomorphism $c^p\mathscr{O}_S(\mathrm{Spa}(A, A+)) \to \mathscr{O}_Z(\mathrm{Spa}(A, A+))$ is the limit of the homomorphisms

$$\begin{tikzcd} \mathscr{O}_S(S') \arrow[r] & \mathscr{O}_Z(\mathrm{Spa}(X \times_S S', S')) \arrow[r] \arrow[d, equal] & \mathscr{O}_Z(\mathrm{Spa}(A, A^+)) \arrow[d, equal] \\ & \mathscr{O}_X(X \times_S S') \arrow[r] & A. \end{tikzcd}$$

We want to show that $\mathscr{O}_S(S') \to A$ factors through

$$A^+ = \{a \in A \mid |a(z)| \le 1 \ \forall z \in \mathrm{Spa}(A, A^+)\}.$$

Let $z \in \mathrm{Spa}(A, A^+)$. By the commutativity of diagram (3), the valuation of A corresponding to z has center on S', which is equivalent to saying that $|b(z)| \le 1$ for all $b \in \mathscr{O}_S(S')$. This implies the claim. □

We denote the resulting homomorphism

$$\mathscr{O}_S \to c_*\mathscr{O}_Z^+$$

by c^+.

A morphism of schemes $X \to S$ is said to be a *pro-open immersion* if it is a limit of open immersions with affine transition morphisms. In this case we also say that X is pro-open in S. Examples are open subschemes of S and the localization of S at some point $s \in S$. A scheme X is *essentially of finite type* over S if there is a scheme T of finite type over S and a pro-open immersion $X \to T$ over S A *compactification* of a scheme X essentially of finite type over S is a proper S-scheme T together with a pro-open immersion $X \to T$ over S. By [Con07], if S is quasi-compact and quasi-separated and $X \to S$ is separated and essentially of finite type, a compactification exists.

Lemma 10.2. *Let $X \subseteq Y$ be pro-open in an integral normal scheme S. Set $Z' = \mathrm{Spa}(S, S)$. The restriction*

$$\rho : \mathscr{O}_{Z'}^+(\mathrm{Spa}(Y, S)) \to \mathscr{O}_{Z'}^+(\mathrm{Spa}(X, S))$$

is an isomorphism.

Proof. It suffices to prove the lemma for $Y = S$ and S affine. If $X = \mathrm{Spec}\, A$ is affine,

$$\mathrm{Spa}(X, S) = \mathrm{Spa}(A, A^+),$$

where A^+ is the integral closure of the image of $\mathscr{O}_S(S)$ in A. By our assumptions on S and X, we obtain

$$A^+ = \mathscr{O}_S(S)$$

and thus

$$\mathrm{Spa}(S, S) = \mathrm{Spa}(A^+, A^+).$$

The homomorphism ρ becomes the identity on A^+.

In the general case cover X by affine open subschemes X_i. We obtain an affinoid covering

$$\coprod_i \mathrm{Spa}(X_i, S) \to \mathrm{Spa}(X, S)$$

and thus a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}^+_{Z'}(\mathrm{Spa}(S,S)) \arrow[r] \arrow[d, "\rho"'] & \prod_i \mathscr{O}^+_{Z'}(\mathrm{Spa}(S,S)) \arrow[r] \arrow[d, "\sim"] & \prod_{ij} \mathscr{O}^+_{Z'}(\mathrm{Spa}(S,S) \arrow[d] \\
0 \arrow[r] & \mathscr{O}^+_{Z'}(\mathrm{Spa}(X,S)) \arrow[r] & \prod_i \mathscr{O}^+_{Z'}(\mathrm{Spa}(X_i,S)) \arrow[r] & \prod_{ij} \mathscr{O}^+_{Z'}(\mathrm{Spa}(X_i \cap X_j,S)).
\end{tikzcd}$$

Note that the assumptions of the lemma also hold for X_i or $X_i \cap X_j$ instead of X. Since the middle arrow is injective, ρ is injective. Applying the same reasoning to $\mathrm{Spa}(X_i \cap X_j, S)$ instead of $\mathrm{Spa}(X,S)$, we see that the right arrow is injective. This implies that ρ is surjective. □

Lemma 10.3. *Let X be pro-open in an integral normal scheme S. With the above notation the homomorphism*

$$c^+ : \mathscr{O}_S \to c_* \mathscr{O}^+_Z$$

is an isomorphism.

Proof. We can check this on open affines of S, i.e. we may assume that S is affine and have to show that

$$c^+(S) : \mathscr{O}_S(S) \to \mathscr{O}^+_Z(Z)$$

is an isomorphism. Denote by $c' : Z' = \mathrm{Spa}(S,S) \to S$ the center map. By functoriality we obtain a commutative diagram

$$\begin{tikzcd}
& \mathscr{O}_S(S) \arrow[dl, "(c')^+(S)"'] \arrow[dr, "c^+(S)"] & \\
\mathscr{O}^+_{Z'}(Z') \arrow[rr, "\rho"] & & \mathscr{O}^+_Z(Z).
\end{tikzcd}$$

Since ρ is an isomorphism by Lemma 10.2, it suffices to show that $(c')^+(S)$ is an isomorphism. But $(c')^+(S)$ is just the identity on $\mathscr{O}_S(S)$. □

For the rest of this section we assume that S is regular and connected and that X is dense pro-open in S. Denote by $\mathcal{B}$ the full subcategory of the category of open subspaces of $\mathrm{Spa}(X,S)$ of the form $\mathrm{Spa}(Y,T)$ coming from a commutative diagram of regular schemes

$$\begin{tikzcd}
Y \arrow[r, hook, "\circ"] \arrow[d, hook, "\circ"] & X \arrow[d, hook, "\circ"] \\
T \arrow[r] & S,
\end{tikzcd} \tag{4}$$

such that $Y \to X$ is an open immersion, $Y \to T$ is a pro-open immersion, and $T \to S$ is of finite type and locally quasi-projective. Since $X \to S$ is a pro-open immersion as well, $T \to S$ is birational.

Our assumption on resolution of singularities implies that the objects of $\mathcal{B}$ form a basis of neighborhoods of the topological space $\mathrm{Spa}(X,S)$. Indeed, if we start with an affinoid open $\mathrm{Spa}(A,A^+)$ we can first choose a projective compactification of $\mathrm{Spec}\, A$ over A^+ and then resolve the singularities of it to obtain a regular locally projective compactification Z. Then

$$\mathrm{Spa}(A,A^+) = \mathrm{Spa}(A,Z)$$

is an object of $\mathcal{B}$. In particular, all affinoid open subspaces are contained in $\mathcal{B}$.

Lemma 10.4. *The intersection of two objects in $\mathcal{B}$ is again an object of $\mathcal{B}$.*

Proof. Suppose we are given two objects $\operatorname{Spa}(Y_1, T_1)$ and $\operatorname{Spa}(Y_2, T_2)$ in $\mathcal{B}$. The intersection of $\operatorname{Spa}(Y_1, T_1)$ with $\operatorname{Spa}(Y_2, T_2)$ is the same as the intersection of $\operatorname{Spa}(Y_1 \cap Y_2, T_1)$ with $\operatorname{Spa}(Y_1, \cap Y_2, T_2)$. Hence, we may assume that $Y_1 = Y_2 =: Y$. Choose locally projective compactifications $\bar{T}_i$ of T_i over S. By elimination of indeterminacies and resolution of singularities, we find a locally projective birational morphism $T' \to S$ from a regular scheme T' dominating $\bar{T}_1$ and $\bar{T}_2$ which is an isomorphism over Y. We denote the preimages of T_1 and T_2 in T' by T_1' and T_2'. As $T_i' \to T_i$ is proper, we have

$$\operatorname{Spa}(Y, T_i) = \operatorname{Spa}(Y, T_i').$$

But then

$$\operatorname{Spa}(Y, T_1) \cap \operatorname{Spa}(Y, T_2) = \operatorname{Spa}(Y, T_1' \cap T_2'),$$

which is in $\mathcal{B}$. □

We equip $\mathcal{B}$ with the structure of a site by defining coverings in $\mathcal{B}$ to be surjective families.

Lemma 10.5. *The topoi associated with $\mathcal{B}$ and* $\operatorname{Spa}(X, S)$ *are equivalent.*

Proof. We have a natural morphism of sites $\varphi : \operatorname{Spa}(X, S)_\top \to \mathcal{B}$, where $\operatorname{Spa}(X, S)_\top$ denotes the site associated with the topological space $\operatorname{Spa}(X, S)$. The pullback φ^* is fully faithful and the topology on $\mathcal{B}$ is induced by the topology of $\operatorname{Spa}(X, S)$. In order to show that the corresponding morphism of topoi is an equivalence, it suffices to verify that the objects of $\mathcal{B}$ form a basis of the topology of $\operatorname{Spa}(X, S)$ (see [SGA4], Exposé III, Théorème 4.1). This is the case as we have seen above. □

Proposition 10.6. *Let X be dense and pro-open in a regular, connected scheme S and assume that resolution of singularities holds over S. Then the center map*

$$c : Z := \operatorname{Spa}(X, S) \to S$$

induces an isomorphism

$$H^i(S, \mathscr{O}_S) \cong H^i(Z, \mathscr{O}_Z^+)$$

for all $i \geq 0$.

Proof. Consider the Leray spectral sequence

$$H^i(S, R^j c_* \mathscr{O}_Z^+) \Rightarrow H^{i+j}(Z, \mathscr{O}_Z^+).$$

By Lemma 10.3,

$$c_* \mathscr{O}_Z^+ \cong \mathscr{O}_S.$$

In order to prove that $R^j c_* \mathscr{O}_Z^+ = 0$ for $j \geq 1$ it is enough to show that

$$H^j(\operatorname{Spa}(X \times_S S', S'), \mathscr{O}_Z^+)$$

vanishes for every open affine $S' \subseteq S$. Since S' and $X \times_S S'$ satisfy the assumptions of the proposition if S and X do, we are reduced to proving that

$$H^j(Z, \mathscr{O}_Z^+) = 0$$

in case S is affine.

Consider the site $\mathcal{B}$ defined before Lemma 10.5. By Lemma 10.5 we can compute the cohomology group $H^j(Z, \mathscr{O}_Z^+)$ in $\mathcal{B}$. We claim that the restriction of $\mathscr{O}_Z^+$ to $\mathcal{B}$ is flasque. Take an open covering

$$\operatorname{Spa}(Y,T) = \bigcup_{i\in I} \operatorname{Spa}(Y_i, T_i)$$

in $\mathcal{B}$ coming from commutative diagrams (4) as before and assume in addition that I is finite and that all T_i are affine. Every covering of $\operatorname{Spa}(Y,T)$ in $\mathcal{B}$ is dominated by one of this type. We want to examine the Čech complex

(5)

$$0 \to \mathscr{O}_Z^+\big(\operatorname{Spa}(Y,T)\big) \to \prod_i \mathscr{O}_Z^+\big(\operatorname{Spa}(Y_i,T_i)\big) \to \prod_{ij} \mathscr{O}_Z^+\big(\operatorname{Spa}(Y_i,T_i)\cap \operatorname{Spa}(Y_j,T_j)\big) \to \dots$$

By Lemma 10.2 this complex does not change if we replace Y and Y_i by $\bigcap_{i\in I} Y_i$. We may thus assume that $Y = Y_i$ for all $i \in I$. By the same argument as before, we may find a locally projective birational morphism $T' \to S$ with T' regular and open subschemes T_i' of T' such that the morphisms $\pi_i : T_i' \to S$ factor through locally projective birational morphisms $T_i' \to T_i$. Since the adic spaces $\operatorname{Spa}(Y, T_i')$ cover $\operatorname{Spa}(Y, T')$, it follows that the schemes T_i' cover T'. The following diagram summarizes the situation:

$$\begin{tikzcd}
 & Y \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \\
T_i' \arrow[r, "\pi_i"] \arrow[d, hook] & T_i \arrow[r, hook] & T \\
T', \arrow[urr, "\pi"] & &
\end{tikzcd}$$

where the morphisms π_i and π are locally projective birational and all schemes in the diagram are regular.

By Lemma 10.3, the above Čech complex (5) equals

$$0 \to \mathscr{O}_{T'}(T') \to \prod_i \mathscr{O}_{T'}(T_i') \to \prod_{i,j} \mathscr{O}_{T'}(T_i'\cap T_j') \to \dots$$

This is the Čech complex for the covering $T' = \bigcup_i T_i'$ and the structure sheaf $\mathscr{O}_{T'}$. By [CR15], Theorem 1.1, we know that for each i the higher direct images $R^j\pi_{i*}\mathscr{O}_{T_i'}$ vanish. Since T_i is affine, this implies

$$H^q(T_i', \mathscr{O}_{T'}) = 0 \quad \forall i,\ \forall q \geq 1.$$

Our Čech complex thus computes the cohomology groups $H^q(T', \mathscr{O}_{T'})$. Applying [CR15], Theorem 1.1 to $\pi : T' \to T$, we obtain that the corresponding higher direct images are trivial. Together with the fact that T is affine, this yields

$$H^q(T', \mathscr{O}_{T'}) \cong H^q(T, \mathscr{O}_T) = 0,\ \forall q \geq 1.$$

As a consequence the cohomology of the complex (5) is trivial. We conclude that $\mathscr{O}_Z^+$ is flasque on $\mathcal{B}$ and thus

$$H^q(Z, \mathscr{O}_Z^+) = 0,\ \forall q \geq 1. \qquad \square$$

11. Prüfer Huber pairs

For an affinoid adic space $X = \mathrm{Spa}(A, A^+)$ the cohomology of the structure sheaf $\mathscr{O}_X$ vanishes (see [Hub94], Theorem 2.2). For the sheaf $\mathscr{O}_X^+$, however, we can not expect in general that $H^i(X, \mathscr{O}_X^+) = 0$. Of course, if (A, A^+) is local, the cohomology of $\mathscr{O}_X^+$ vanishes. But the class of local adic spaces turns out to be too small to calculate cohomology groups as an étale covering of a local adic space does not necessarily admit a refinement by local adic spaces. In the following we investigate a broader class of Huber pairs containing the local Huber pairs: the Prüfer Huber pairs.

Definition 11.1. A Huber pair (A, A^+) is said to be Prüfer if $A^+ \subseteq A$ is a Prüfer extension, i.e. if $(A_{\mathfrak{m}^+}, A^+_{\mathfrak{m}^+})$ is local for every maximal ideal $\mathfrak{m}^+$ of A^+ (see [KZ02], Chapter I, § 5).

Recall that a ring homomorphism $A \to B$ is called *weakly surjective* if for any prime ideal $\mathfrak{p}$ of A with $\mathfrak{p}B \neq B$ the homomorphism $A_\mathfrak{p} \to B_\mathfrak{p}$ is surjective. Examples of weakly surjective ring homomorphisms are surjective ring homomorphisms and localizations. By [KZ02], Theorem I.5.2, (1) $\Leftrightarrow$ (2) a ring extension $A \to R$ is Prüfer if and only if A is weakly surjective in any R-overring of A.

It will turn out in Proposition 11.18 that if (A, A^+) is a complete Prüfer Huber pair and A is either a strongly noetherian Tate ring or noetherian with the discrete topology, then the cohomology of $\mathscr{O}_X^+$ vanishes on $X = \mathrm{Spa}(A, A^+)$.

Lemma 11.2. *Let (A, A^+) be a Prüfer Huber pair. Then its completion $(\hat{A}, \hat{A}^+)$ is Prüfer.*

Proof. We factor $(A, A^+) \to (\hat{A}, \hat{A}^+)$ as

$$(A, A^+) \to (\bar{A}, \bar{A}^+) \to (\hat{A}, \hat{A}^+)$$

such that $A \to \bar{A}$ is surjective and $\bar{A} \to \hat{A}$ is injective. Then $(\bar{A}, \bar{A}^+)$ is Prüfer by [Rho91], Proposition 3.1.1 (or [KZ02], Proposition I.5.8) and $(\hat{A}, \hat{A}^+)$ is the completion of $(\bar{A}, \bar{A}^+)$. We may therefore assume that the morphism $\iota : A \to \hat{A}$ is injective.

By [KZ02], Theorem I.5.2, (1) $\Leftrightarrow$ (2) a ring extension $B \hookrightarrow R$ is Prüfer if and only if every R-overring of B is integrally closed in R. We have mutually inverse bijections

$$\{\text{open subrings of } A\} \underset{C \cap A \mapsfrom C}{\overset{B \mapsto \hat{B}}{\rightleftarrows}} \{\text{open subrings of } \hat{A}\}.$$

The subsequent lemma shows that this correspondence restricts to a bijection of the open, integrally closed subrings of A with the open, integrally closed subrings of $\hat{A}$. Since A^+ is open and integrally closed in A, we obtain a bijection of the integrally closed A-overrings of A^+ with the integrally closed $\hat{A}$-overrings of $\hat{A}^+$. In particular, an $\hat{A}$-overring C of $\hat{A}^+$ is integrally closed in $\hat{A}$ if and only if $C \cap A$ is integrally closed in A. This finishes the proof as all A-overrings of A^+ are integrally closed in A by assumption. $\square$

Lemma 11.3. *For any linearly topologized ring A with completion $\sigma : A \to \hat{A}$ the mutually inverse bijections*

$$\{\textit{open subrings of } A\} \underset{\sigma^{-1}(C) \mapsfrom C}{\overset{B \mapsto \hat{B}}{\rightleftarrows}} \{\textit{open subrings of } \hat{A}\}.$$

establish a correspondence of the open, integrally closed subrings.

Proof. The argument is taken from the proof of Lemma 2.4.3 in [Hub93a]. The only nontrivial assertion we have to check is that the completion $\hat{B}$ of any open, integrally closed subring B of A is integrally closed. Denote by C the integral closure of $\hat{B}$ in $\hat{A}$. This is an open subring of $\hat{A}$. Take an element $c \in C$. In order to show that $c \in \hat{B}$ it suffices to check that for any open neighborhood U of c in C we have

$$U \cap \sigma(B) \neq \varnothing.$$

Since $\sigma(A)$ is dense in $\hat{A}$, we can find $a \in A$ with $\sigma(a) \in U$. Being contained in C the element $\sigma(a)$ satisfies an integral equation

$$\sigma(a)^n + \hat{b}_{n-1}\sigma(a)^{n-1} + \ldots + \hat{b}_0 = 0$$

with $\hat{b}_i \in \hat{B}$. As $\hat{B}$ is open, we can approximate the $\hat{b}_i$ by elements of the form $\sigma(b_i)$ with $b_i \in B$ such that

$$\sigma(a)^n + \sigma(b_{n-1})\sigma(a)^{n-1} + \ldots \sigma(b_0) \in \hat{B}.$$

Together with $B = \sigma^{-1}(\hat{B})$ this implies the existence of an element $b \in B$ such that

$$a^n + b_{n-1}a^{n-1} + \ldots + (b_0 - b) = 0$$

We conclude that $a \in B$ and thus $\sigma(a) \in U \cap \sigma(B)$. $\square$

11.1. **A flatness criterion.** For this subsection we fix a local Huber pair (A, A^+). We denote by $\mathfrak{m}$ the maximal ideal of A. It is contained in A^+ and $A^+/\mathfrak{m}$ is a valuation ring. Hence, every proper ideal of A is contained in A^+. We write $|\cdot|$ for the valuation of A corresponding to $A^+/\mathfrak{m}$.

We want to investigate whether an A^+-module M^+ is flat if its base change to A is flat. To this end we examine for an ideal $a^+ \subseteq A^+$ the vanishing of $\mathrm{Tor}_1^{A^+}(M^+, A^+/\mathfrak{a}^+)$.

Lemma 11.4. *Let $\mathfrak{a}$ be a proper ideal of A. Let M^+ be an A^+-module such that $M := M^+ \otimes_{A^+} A$ is a flat A-module. Then*

$$\mathrm{Tor}_1^{A^+}(M^+, A^+/\mathfrak{a}) = 0.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{a} \otimes_{A^+} M^+ \arrow[r] \arrow[d] & M^+ \arrow[d] \\ \mathfrak{a} \otimes_A M \arrow[r, hook] & M. \end{tikzcd} \tag{6}$$

The lower horizontal map is injective as M is a flat A-module. As $A^+ \to A$ is a localization, hence flat, the homomorphism

$$\mathfrak{a} \otimes_{A^+} A \to A$$

is injective. Its image is $A \cdot \mathfrak{a} = \mathfrak{a}$. We obtain an isomorphism $\mathfrak{a} \otimes_{A^+} A \to \mathfrak{a}$ whose inverse φ is given by $a \mapsto a \otimes 1$. Tensoring φ with M^+ yields the left vertical map in diagram (6), which is thus an isomorphism. We conclude that the upper horizontal map is injective. Hence,

$$\mathrm{Tor}_1^{A^+}(M^+, A^+/\mathfrak{a}) = \ker(\mathfrak{a} \otimes_{A^+} M^+ \to M^+) = 0.$$

$\square$

Lemma 11.5. *Let $\mathfrak{a}^+$ be an ideal of A^+. Let M^+ be an A^+-module such that $M := M^+ \otimes_{A^+} A$ is a flat A-module and $M^+/\mathfrak{m}M^+$ is torsion free over $A^+/\mathfrak{m}$. Then*

$$\operatorname{Tor}_1^{A^+}(M^+, A^+/(\mathfrak{m}^n + \mathfrak{a}^+)) = 0.$$

for all $n \geq 1$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{m}^n \otimes_{A^+} M^+ \arrow[r, "\sim"] \arrow[d, hook] & \mathfrak{m}^n M^+ \arrow[d, hook] \\
(\mathfrak{m}^n + \mathfrak{a}^+) \otimes_{A^+} M^+ \arrow[r] \arrow[d, two heads] & M^+ \arrow[d, two heads] \\
(\mathfrak{m}^n + \mathfrak{a}^+)/\mathfrak{m}^n \otimes_{A^+} M^+ \arrow[r, hook] & M^+/\mathfrak{m}^n M^+.
\end{tikzcd} \tag{7}$$

The upper horizontal map is an isomorphism by Lemma 11.4. This implies that the upper left vertical map is injective. Let us show that the lower horizontal map is injective. Since

$$(\mathfrak{m}^n + \mathfrak{a}^+)/\mathfrak{m}^n \otimes_{A^+} M^+ \to (\mathfrak{m}^n + \mathfrak{a}^+)/\mathfrak{m}^n \otimes_{A^+/\mathfrak{m}^n} M^+/\mathfrak{m}^n M^+$$

is an isomorphism, this comes down to showing that $M^+/\mathfrak{m}^n M^+$ is a flat $A^+/\mathfrak{m}^n$-module. If $n = 1$, this is true as $A^+/\mathfrak{m}$ is a valuation ring and $M^+/\mathfrak{m}M^+$ is torsion free, hence flat. The case $n > 1$ follows from the case $n = 1$ by [SP, Tag 051C]. Note that the assumption

$$\operatorname{Tor}_1^{A^+}(M^+, A^+/\mathfrak{m}) = 0$$

in [SP, Tag 051C] is satisfied by Lemma 11.4. We conclude that the lower horizontal map in diagram (7) is injective. A diagram chase now shows the injectivity of the middle horizontal map, which concludes the proof. □

The following lemma is a variant of the Artin-Rees lemma for local Huber pairs.

Lemma 11.6. *Assume that A is noetherian. Let $\mathfrak{a}$ be an ideal of A and $N^+ \subseteq M^+$ finite A^+-modules. Set $M := M^+ \otimes_{A^+} A$ and $N := N^+ \otimes_{A^+} A$ and assume that $M^+ \to M$ is injective. Then there is $K \in \mathbb{N}$ such that for all $n > K$*

$$\mathfrak{a}^n M^+ \cap N^+ = \mathfrak{a}^{n-K}(\mathfrak{a}^K M^+ \cap N^+) = \mathfrak{a}^n M \cap N = \mathfrak{a}^{n-K}(\mathfrak{a}^K M \cap N).$$

Proof. As $A^+ \to A$ is flat, the natural map $N \to M$ is injective and we view N, M^+ and N^+ as submodules of M. For positive integers $n > K$ consider the diagram

$$\begin{tikzcd}
\mathfrak{a}^{n-K}(\mathfrak{a}^K M^+ \cap N^+) \arrow[r, hook] \arrow[d, hook] & \mathfrak{a}^n M^+ \cap N^+ \arrow[d, hook] \\
\mathfrak{a}^{n-K}(\mathfrak{a}^K M \cap N) \arrow[r, hook] & \mathfrak{a}^n M \cap N
\end{tikzcd}$$

For K big enough the lower horizontal inclusion is the identity by the Artin-Rees lemma. Moreover, since $A^+ \to A$ is a localization and $\mathfrak{a}$ is an ideal not only of A^+ but of A, the left vertical map is the identity. This implies that the upper horizontal map and the right vertical map are the identity. □

Proposition 11.7. *Let (B, B^+) be a Prüfer Huber pair such that B is noetherian. Let M^+ be a torsion free B^+-module such that $M := M^+ \otimes_{B^+} B$ is flat over B. Then M^+ is flat.*

Proof. It suffices to show that $M^+_{\mathfrak{m}^+}$ is a flat $B^+_{\mathfrak{m}^+}$-module for every maximal ideal $\mathfrak{m}^+$ of B^+. By [KZ02], Proposition I.2.10, the pair $(B_{\mathfrak{m}^+}, B^+_{\mathfrak{m}^+})$ is a local Huber pair. In particular, $B_{\mathfrak{m}^+} = B_{\mathfrak{m}}$ for some prime ideal $\mathfrak{m}$ of B. As the assumptions are stable under localization, we may assume that (B, B^+) is local right away.

Using that $B^+/\mathfrak{m}$ is a valuation ring and that M^+ is torsion free, we see that $M^+/\mathfrak{m}M^+$ is torsion free over $B^+/\mathfrak{m}$. Let $\mathfrak{b}^+ \subseteq B^+$ be a finitely generated ideal. We have to show that

$$\mathfrak{b}^+ \otimes_{B^+} M^+ \to M^+$$

is injective. For $n \geq 1$ consider the following diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{b}^+ \cap \mathfrak{m}^n \arrow[r] \arrow[d, hook] & \mathfrak{b}^+ \oplus \mathfrak{m}^n \arrow[r] \arrow[d, hook] & \mathfrak{b}^+ + \mathfrak{m}^n \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & B^+ \arrow[r] & B^+ \oplus B^+ \arrow[r] & B^+ \arrow[r] & 0. \end{tikzcd}$$

Tensoring with M^+ we obtain

$$\begin{tikzcd} & (\mathfrak{b}^+ \cap \mathfrak{m}^n) \otimes_{B^+} M^+ \arrow[r] \arrow[d] & \mathfrak{b}^+ \otimes_{B^+} M^+ \oplus \mathfrak{m}^n \otimes_{B^+} M^+ \arrow[r] \arrow[d] & (\mathfrak{b}^+ + \mathfrak{m}) \otimes_{B^+} M^+ \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M^+ \arrow[r] & M^+ \oplus M^+ \arrow[r] & M^+ \arrow[r] & 0. \end{tikzcd}$$

Since $\mathfrak{m}^n \otimes_{B^+} M^+ \to M^+$ and $(\mathfrak{b}^+ + \mathfrak{m}^n) \otimes_{B^+} M^+ \to M^+$ are injective by Lemma 11.5, the snake lemma implies that

$$\ker\left((\mathfrak{b}^+ \cap \mathfrak{m}^n) \otimes_{B^+} M^+ \to M^+\right) \to \ker\left(\mathfrak{b}^+ \otimes_{B^+} M^+ \to M^+\right)$$

is surjective. We now apply Lemma 11.6 to the finite B^+-modules $\mathfrak{b}^+ \subseteq B^+$. Setting $\mathfrak{b} = \mathfrak{b}^+ \otimes_{B^+} B$ there is $N \in \mathbb{N}$ such that for all $n > N$

$$\mathfrak{m}^n \cap \mathfrak{b}^+ = \mathfrak{m}^{n-N}(\mathfrak{m}^N \cap \mathfrak{b}^+) = \mathfrak{m}^n \cap \mathfrak{b} = \mathfrak{m}^{n-N}(\mathfrak{m}^N \cap \mathfrak{b}).$$

The ideal $\mathfrak{m}^n \cap \mathfrak{b}^+$ of B^+ is thus also an ideal of B and by Lemma 11.4 we obtain

$$\ker\left((\mathfrak{b}^+ \cap \mathfrak{m}^n) \otimes_{B^+} M^+ \to M^+\right) = 0,$$

which implies that

$$\ker\left(\mathfrak{b}^+ \otimes_{B^+} M^+ \to M^+\right) = 0.$$

□

Remark 11.8. The flatness criterion Proposition 11.7 in case M^+ is a B^+-algebra resembles the one given in [Tem11], Lemma 2.3.1 (iii). However, in our application M^+ is not of finite type, in general. This impedes the application of Raynaud-Gruson flattening ([RG71]) in contrast to the situation in [Tem11].

11.2. **Cartesian coverings of Huber pairs.** Let (A, A^+) and (B, B^+) be Huber pairs with rings of definition $A_0 \subseteq A$ and $B_0 \subseteq B$ For a homomorphism

$$(A, A^+) \to (B, B^+)$$

of Huber pairs we equip the fiber product $B^+ \otimes_{A^+} A$ with the following topology: Let $I \subseteq B^+$ be an ideal of definition. Denote by C_0 the image of B^+ in $B^+ \otimes_{A^+} A$. We take C_0 to be a ring of definition of $B^+ \otimes_{A^+} A$ and IC_0 an ideal of definition Then $B^+ \otimes_{A^+} A$ is a Huber ring.

Definition 11.9. The homomorphism

$$(A, A^+) \to (B, B^+)$$

of Huber pairs is called *Cartesian* if the natural homomorphism

$$B^+ \otimes_{A^+} A \to B$$

induces an isomorphism on completions. In this case we also say that $\mathrm{Spa}(B, B^+)$ is Cartesian over $\mathrm{Spa}(A, A^+)$. We say that a covering of $\mathrm{Spa}(A, A^+)$ by rational open subspaces $\mathrm{Spa}(B_i, B_i^+)$ (for i in some index set I) is Cartesian if for every $i \in I$ the homomorphism

$$(A, A^+) \to (\hat{B}_i, \hat{B}_i^+)$$

is Cartesian.

Proposition 11.10. *Let (A, A^+) be a complete Prüfer Huber pair. Let $Y \to \mathrm{Spa}(A, A^+)$ be a Cartesian, strongly étale morphism of affinoid adic spaces. Then Y is $\mathrm{Spa}(A, A^+)$-isomorphic to the adic spectrum of a Huber pair (B, B^+) with $A^+ \to B^+$ étale.*

Proof. By [Hub96], Corollary 1.7.3 iii), there is a Cartesian morphism $(A, A^+) \to (B, B^+)$ of algebraically finite type such that $A \to B$ is étale and Y is $\mathrm{Spa}(A, A^+)$-isomorphic to $\mathrm{Spa}(B, B^+)$. Let $\mathfrak{m}^+$ be a maximal ideal of A^+. In order to show that $A^+ \to B^+$ is étale at $\mathfrak{m}^+$ we can base change to $A^+_{\mathfrak{m}^+}$. As (A, A^+) is Prüfer, there is a unique point $x \in X := \mathrm{Spa}(A, A^+)$ such that $\mathscr{O}_{X,x} = A_{\mathfrak{m}^+}$ and $\mathscr{O}^+_{X,x} = A^+_{\mathfrak{m}^+}$. Therefore, base changing $Y \to X$ to X_x induces the base change of $A^+ \to B^+$ to $A^+_{\mathfrak{m}^+}$. We may thus assume that (A, A^+) is local such that $\mathfrak{m}^+$ is the maximal ideal of A^+. Denote by $\mathfrak{m}$ the maximal ideal of A.

By assumption $A \to B$ is étale and by Lemma 3.2 also $A^+/\mathfrak{m} \to B^+/\mathfrak{m}B^+$ is étale. In particular, both morphisms are flat and of finite presentation and thus [Tem11], Lemma 2.3.1 implies that $A^+ \to B^+$ is flat and of finite presentation (the flatness is a consequence of the flattening result by Raynaud and Gruson [RG71], Theorem 5.2.2). Let us show that $A^+ \to B^+$ is unramified, i.e. that $\Omega^1_{B^+/A^+} = 0$. Since $A^+/\mathfrak{m} \to B^+/\mathfrak{m}B^+$ is unramified, $\Omega^1_{B^+/A^+} \otimes_{A^+} A^+/\mathfrak{m} = 0$. It remains to show that $\mathfrak{m}\Omega^1_{B^+/A^+} = 0$. But the isomorphism $\mathfrak{m} \cong \mathfrak{m} \otimes_{A^+} A$ induces an isomorphism

$$\mathfrak{m}\Omega^1_{B^+/A^+} \cong \mathfrak{m}(\Omega_{B^+/A^+} \otimes_{A^+} A)$$

and $\Omega^1_{B^+/A^+} \otimes_{A^+} A = 0$ as $A \to B$ is unramified. □

Lemma 11.11. *Let (A, A^+) be a complete Prüfer Huber pair. Then, every integral morphism $(A, A^+) \to (B, B^+)$ is Cartesian and (B, B^+) is Prüfer.*

Proof. By definition $A \to B$ is integral and B^+ is the integral closure of A^+ in B. Hence, B is generated by B^+ and the image of A ([KZ02], Theorem I.5.9). By [KZ02], Proposition I.3.10, $B^+ \to B$ and $B^+ \to B^+ \otimes_{A^+} A$ are weakly surjective. Moreover, both are injective (the injectivity of $B^+ \to B^+ \otimes_{A^+} A$ follows from the injectivity of $B^+ \to B$). Therefore, by [KZ02], Corollary I.3.16 the surjective homomorphism $B^+ \otimes_{A^+} A \to B$ is injective. □

Lemma 11.12. *Let (A, A^+) be a Prüfer Huber pair with A is noetherian and*

$$(A, A^+) \to (B, B^+)$$

a Cartesian homomorphism such that $\mathrm{Spec}\, B$ is quasi-finite and essentially of finite type over $\mathrm{Spec}\, A$. Then (B, B^+) is Prüfer, too.

Proof. We may assume that (A, A^+) is complete and that $B^+ \otimes_{A^+} A \to B$ is an isomorphism (see Lemma 11.2). By Zariski's main theorem $A \to B$ factors as $A \to B_0 \to B$ with B_0/A finite and B/B_0 a localization. Denote by B_0^+ the integral closure of A^+ in B_0. Since B^+ is integrally closed in B, we obtain a diagram

$$\begin{tikzcd} B & B_0 \arrow[l, hook', "\text{loc.}"'] & A \arrow[l, hook', "\text{finite}", "\varphi"'] \\ B^+ \arrow[u, hook] & B_0^+ \arrow[l, hook'] \arrow[u, hook] & A^+. \arrow[l, hook', "\text{int.}", "\varphi^+"'] \arrow[u, hook] \end{tikzcd}$$

By Lemma 11.11 the Huber pair (B_0, B_0^+) is Prüfer and $A \otimes_{A^+} B_0^+ \to B_0$ is bijective. This implies that $(B_0, B_0^+) \to (B, B^+)$ is Cartesian.

If A is noetherian, so is B_0. Hence, Proposition 11.7 implies that $B_0^+ \to B^+$ is flat and thus weakly surjective by [KZ02], Proposition I.4.5. The result now follows from [KZ02], Theorem I.5.10. □

11.3. Laurent coverings and Zariski cohomology.

Definition 11.13. Let (A, A^+) be a Huber pair. A Laurent covering of $\mathrm{Spa}(A, A^+)$ is a covering by rational open subsets of the form

$$\mathrm{Spa}(A, A^+) = \bigcup_{\alpha_i \in \{\pm 1\}} R(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n})$$

with $f_1, \ldots, f_n \in A$.

Lemma 11.14. *Let (A, A^+) be a complete Huber pair. Every open covering of* $\mathrm{Spa}(A, A^+)$ *has a refinement which is a Laurent covering.*

Proof. By [Hub94], Lemma 2.6, every open covering of $\mathrm{Spa}(A, A^+)$ is dominated by a covering of the form

$$\mathrm{Spa}(A, A^+) = \bigcup_{j=1}^{m} R(\frac{g_1, \ldots, g_m}{g_j})$$

with $g_1, \ldots, g_m \in A$ such that $g_1 A + \ldots + g_m A = A$. By the reasoning of [BGR84], § 8.2.2 every such covering is dominated by a Laurent covering. □

Lemma 11.15. *Let (A, A^+) be a Huber pair such that $A^+ \to A$ is weakly surjective. Then for any $f \in A$ the Laurent covering*

$$R(\frac{f}{1}) \cup R(\frac{1}{f}) = \mathrm{Spa}(A, A^+)$$

is Cartesian.

Denote by $A^+[\frac{1}{f}]$ the subring of A_f generated by the image of A^+ and $1/f$. If in addition (A, A^+) is Prüfer and A is noetherian, $A^+[f]$ and $A^+[\frac{1}{f}]$ are integrally closed in A and A_f, respectively, i.e. $(A, A^+[f])$ and $(A_f, A^+[\frac{1}{f}])$ are Huber pairs and

$$R(\frac{f}{1}) = \mathrm{Spa}(A, A^+[f]), \qquad R(\frac{1}{f}) = \mathrm{Spa}(A_f, A^+[\frac{1}{f}]).$$

Proof. We only treat $R(\frac{1}{f})$. The examination of $R(\frac{f}{1})$ is similar (and even easier). We have

$$R(\frac{1}{f}) = \mathrm{Spa}(A_f, A_f^+),$$

where A_f^+ denotes the integral closure of $A^+[\frac{1}{f}]$. In order to show that $R(\frac{1}{f}) \to \mathrm{Spa}(A, A^+)$ is Cartesian it suffices to show that the natural homomorphism

$$\varphi : A \otimes_{A^+} A_f^+ \to A_f$$

is an isomorphism. The surjectivity of φ is obvious. Consider the diagram

$$\begin{tikzcd}
A_f & & \\
& A_f^+ \otimes_{A^+} A \arrow[ul, "\varphi"'] & A \arrow[l] \arrow[ull, bend right] \\
& A_f^+ \arrow[u, "\alpha'"] \arrow[uul, bend left, "\beta"] & A^+. \arrow[l] \arrow[u, "\alpha"']
\end{tikzcd}$$

As α is weakly surjective, so are α' and β (see [KZ02], Proposition I.3.10). Moreover, α' is injective because β is injective. We conclude by [KZ02], Corollary I.3.16 that φ is injective.

Assume now that (A, A^+) is Prüfer and A is noetherian. As the image of A^+ in A_f is Prüfer in the image of A in A_f by [KZ02], Proposition I.5.7, we may replace A^+ and A by their images in A_f and assume henceforth that $A \to A_f$ is injective. The same argument as above shows that

$$A \otimes_{A^+} A^+[\frac{1}{f}] \cong A_f.$$

By Proposition 11.7, $A^+ \to A^+[\frac{1}{f}]$ is flat. Moreover, $A^+ \to A \to A_f$ is weakly surjective. Hence, $A^+ \to A^+[\frac{1}{f}]$ is weakly surjective by [KZ02], Proposition I.4.5. Since A_f is generated by A and $A^+[\frac{1}{f}]$, [KZ02], Theorem I.5.10 implies that $A^+[\frac{1}{f}]$ is Prüfer in A_f. In particular, $A^+[\frac{1}{f}]$ is integrally closed in A_f. □

Corollary 11.16. *Let (A, A^+) be a complete Prüfer Huber pair. Then* $\mathrm{Spa}(A, A^+)$ *has a basis of Cartesian affinoid neighborhoods.*

Proof. By Lemma 11.14 there is a basis of neighborhoods of $\mathrm{Spa}(A, A^+)$ consisting of open subspaces of the form

$$R(f_1^{\alpha_1}, \ldots, f_n^{\alpha_n})$$

with $f_i \in A$ and $\alpha_i \in \{\pm 1\}$. By Lemma 11.15 these are Cartesian. □

Lemma 11.17. *Let (A, A^+) be a complete Prüfer Huber pair. Assume that either A is a strongly noetherian Tate ring or the topology of A is discrete and A is noetherian. Let $\mathcal{U}$ be a Laurent covering of $X = \mathrm{Spa}(A, A^+)$. Then the Čech cohomology groups*

$$\check{\mathrm{H}}^i(\mathcal{U}, \mathscr{O}_X^+)$$

vanish for $i \geq 1$.

Proof. Using [BGR84], 8.1.4 Corollary 4 and induction this comes down to showing that

$$0 \to A^+ \to \mathscr{O}_X^+(R(\frac{f}{1})) \oplus \mathscr{O}_X^+(R(\frac{1}{f})) \xrightarrow{\alpha} \mathscr{O}_X^+(R(\frac{f}{1}, \frac{1}{f})) \to 0$$

is exact for every $f \in A$. We know already that $\mathscr{O}_X^+$ is a sheaf. Hence, we are left with showing the surjectivity of α. By Lemma 11.15 we have

$$R(\frac{f}{1}) = \operatorname{Spa}(A, A^+[f]), \quad R(\frac{1}{f}) = \operatorname{Spa}(A_f, A^+[\frac{1}{f}]), \quad R(\frac{f}{1}, \frac{1}{f}) = \operatorname{Spa}(A_f, A^+[f, \frac{1}{f}]).$$

In case the topology of A is discrete the surjectivity of α is now obvious. In case A is a strongly noetherian Tate algebra we use the following identifications (see II.1 in the proof of Theorem 2.5 in [Hub94]):

$$A\langle\frac{f}{1}\rangle = A\langle X\rangle/(f - X), \quad A\langle\frac{1}{f}\rangle = A\langle Y\rangle/(1 - fY), \quad A\langle\frac{f}{1}, \frac{1}{f}\rangle = A\langle X, X^{-1}\rangle/(f - X).$$

Then $\mathscr{O}_X^+(R(\frac{f}{1}))$ is the closure of $A^+[f]$ in $A\langle X\rangle/(f - X)$, i.e. equal to

$$\{\sum_i b_i X^i \in A\langle X\rangle \mid b_i \in A^+\}/(f - X).$$

Similarly

$$\mathscr{O}_X^+(R(\frac{1}{f})) = \{\sum_i b_i Y^i \in A\langle Y\rangle \mid b_i \in A^+\}/(1 - fY)$$

$$\mathscr{O}_X^+(R(\frac{f}{1}, \frac{1}{f})) = \{\sum_i b_i X^i \in A\langle X, X^{-1}\rangle \mid b_i \in A^+\}/(f - X).$$

Now also in this case the surjectivity of α can be checked explicitly. □

Proposition 11.18. *Let (A, A^+) be a complete Prüfer Huber pair. Assume that either A is a strongly noetherian Tate ring or the topology of A is discrete and A is noetherian. Then, setting $X = \operatorname{Spa}(A, A^+)$,*

$$H^i(X, \mathscr{O}_X^+) = 0.$$

for all $i > 0$.

Proof. Let $\mathcal{B}$ be the category of Cartesian open immersions of affinoid adic spaces

$$\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+).$$

It has fiber products and becomes a site by defining coverings of $\operatorname{Spa}(B, B^+)$ to be the Laurent coverings

$$\operatorname{Spa}(B, B^+) = \bigcup_{\alpha_i \in \{\pm 1\}} R(f_1^{\alpha_1}, \dots, f_n^{\alpha_n})$$

of $\operatorname{Spa}(B, B^+)$ (with $f_1, \dots, f_n \in B$). Note that $R(f_1^{\alpha_1}, \dots, f_n^{\alpha_n})$ is contained in $\mathcal{B}$ by Lemma 11.15. By Lemma 11.14 we can compute cohomology groups in $\mathcal{B}$. But by Lemma 11.17 the sheaf $\mathscr{O}_X^+$ is flasque on $\mathcal{B}$. □

12. Strongly étale cohomology

If X is an analytic adic space, the additive group $\mathbb{G}_a$ is a sheaf for the étale site of X by [Hub96], (2.2.5). In case X is a discretely ringed adic space this follows from the corresponding statement for schemes. In particular, in both cases, $\mathbb{G}_a$ is a sheaf for the strongly étale and the tame site. Then, also the subpresheaf $\mathbb{G}_a^+$ of $\mathbb{G}_a$ defined by

$$(Y \to X) \mapsto \mathscr{O}_Y^+(Y)$$

is a sheaf.

In the following we say that an adic space X is *locally noetherian* if it is locally of the form $\mathrm{Spa}(A, A^+)$ such that the completion of A is noetherian. We say that X is noetherian if in addition X is quasi-compact and quasi-separated.

Lemma 12.1. *Let*

$$\varphi : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$

be an étale covering of the noetherian Prüfer affinoid adic space $\mathrm{Spa}(A, A^+)$*. Then there is a morphism*

$$\psi : \mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+),$$

which is a finite product of open immersions such that $\varphi \circ \psi$ is a Cartesian étale covering.

Proof. We may assume that φ is of finite presentation. Using Zariski's main theorem and [Hub96], Corollary 1.7.3 ii), we factor φ as

$$\mathrm{Spa}(B, B^+) \xrightarrow{\iota} \mathrm{Spa}(D, D^+) \xrightarrow{\pi} \mathrm{Spa}(A, A^+)$$

with an open immersion ι and a finite morphism π. Lemma 11.11 implies that π is Cartesian and (D, D^+) is Prüfer. Now it suffices to show that every point $x \in \mathrm{Spa}(B, B^+)$ has an open affinoid neighborhood $U \subseteq \mathrm{Spa}(B, B^+)$ such that $U \to \mathrm{Spa}(D, D^+)$ is Cartesian. This follows from Corollary 11.16. □

Corollary 12.2. *Every tame covering and every strongly étale covering of a noetherian Prüfer affinoid adic space* $\mathrm{Spa}(A, A^+)$ *has a Cartesian refinement.*

Proposition 12.3. *Let (A, A^+) be a Prüfer Huber pair such that A is noetherian and equipped with the discrete topology. Then*

$$H^i_{s\acute{e}t}(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+) = 0.$$

Proof. Let $\mathcal{B}$ be the category of Cartesian strongly étale morphisms of affinoid adic spaces

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+).$$

It has fiber products and becomes a site by defining coverings of $\mathrm{Spa}(B, B^+)$ to be the Cartesian strongly étale coverings of $\mathrm{Spa}(B, B^+)$. By Corollary 12.2 we can compute the cohomology groups $H^q_{s\acute{e}t}(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+)$ in $\mathcal{B}$.

We show that $\mathbb{G}_a^+$ is flasque on $\mathcal{B}$. In order to do so we prove that for every covering

$$\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$$

in $\mathcal{B}$ the associated Čech complex for the sheaf $\mathbb{G}_a^+$ is exact. The fact that $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ is Cartesian implies that the diagram

$$\begin{tikzcd} C \otimes_B \ldots \otimes_B C & B \arrow[l] \\ C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+ \arrow[u] & B^+ \arrow[l] \arrow[u] \end{tikzcd}$$

is Cartesian. Since $\operatorname{Spec} C^+ \to \operatorname{Spec} B^+$ is an étale covering by Proposition 11.10, so is $\operatorname{Spec} C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+ \to \operatorname{Spec} B^+$. In particular, it is flat and thus the left vertical arrow is injective. Moreover, taking integral closures commutes with étale base change. Therefore, $C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+$ is integrally closed in $C \otimes_B \ldots \otimes_B C$. By construction of the fiber product for adic spaces, this is equivalent to saying that

$$\operatorname{Spa}(C, C^+) \times_{\operatorname{Spa}(B,B^+)} \ldots \times_{\operatorname{Spa}(B,B^+)} \operatorname{Spa}(C, C^+) = \operatorname{Spa}(C \otimes_B \ldots \otimes_B C, C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+).$$

The Čech complex for $\mathbb{G}_a^+$ thus equals the Amitsur complex

$$0 \longrightarrow B^+ \longrightarrow C^+ \longrightarrow C^+ \otimes_{B^+} C^+ \longrightarrow C^+ \otimes_{B^+} C^+ \otimes_{B^+} C^+ \longrightarrow \ldots$$

This complex is exact as $B^+ \to C^+$ is faithfully flat. Hence, $\mathbb{G}_a^+$ is flasque on $\mathcal{B}$. In particular,

$$H^i_{s\acute{e}t}(\operatorname{Spa}(A, A^+), \mathbb{G}_a^+) = 0.$$

□

Proposition 12.4. *Let (A, A^+) be a complete Prüfer Huber pair such that A is a non-Archimedean field. Then*

$$H^i_{s\acute{e}t}(\operatorname{Spa}(A, A^+), \mathbb{G}_a^+) = 0.$$

for all $i \geq 1$.

Proof. Set $X = \operatorname{Spa}(A, A^+)$. Note first that (A, A°) (where A° denotes the power bounded elements) is henselian by Hensel's lemma for non-Archimedean fields and that $\operatorname{Spa}(A, A^\circ)$ consists of a single point. Consider an étale morphism $Y \to X$ with Y affinoid. The base change of Y to $\operatorname{Spa}(A, A^\circ)$ is a disjoint union of affinoid adic spaces of the form (B, B°) such that B is a finite separable extension of A. Since the set of generalizations of an analytic point of an adic space is totally ordered by specialization, every connected component of Y is of the form (B, B^+) with B as above. In particular, B is a complete, non-Archimedean field. Furthermore, B^+ is a B-overring of the integral closure of A^+ in B, hence Prüfer.

Let $\mathcal{B}$ be the full subcategory of $X_{s\acute{e}t}$ whose objects are the strongly étale morphisms $Y \to X$ such that Y is affine. We can compute the cohomology of X in $\mathcal{B}$. We show that $\mathbb{G}_a^+$ is flasque on $\mathcal{B}$.

Let $Y \to X$ be in $\mathcal{B}$ and $Z \to Y$ a covering of Y. We may assume that Y is the adic spectrum of a complete Prüfer Huber pair (B, B^+) such that B is a non-Archimedean field. Then $Z = \operatorname{Spa}(C, C^+)$ with C finite étale over B and C^+ flat over B^+ (as any torsion free module over a Prüfer domain is flat). Since $(B, B^+) \to (C, C^+)$ is strongly étale, $B^+ \to C^+$ is even étale by Lemma 3.2. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & B^+ \arrow[r] \arrow[d, hook] & C^+ \arrow[r] \arrow[d, hook] & C^+ \otimes_{B^+} C^+ \arrow[r] \arrow[d, hook] & C^+ \otimes_{B^+} C^+ \otimes_{B^+} C^+ \arrow[r] \arrow[d, hook] & \ldots \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] & C \otimes_B C \arrow[r] & C \otimes_B C \otimes_B C \arrow[r] & \ldots
\end{tikzcd}$$

of exact Amitsur complexes. As integral closure commutes with étale base change, $C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+$ is integrally closed in $C \otimes_B \ldots \otimes_B C$. Moreover, being a finite B-module, $C \otimes_B \ldots \otimes_B C$ is complete and $C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+$ is an open subring. Therefore,

$$\mathbb{G}_a^+(Z \times_Y \ldots \times_Y Z) = C^+ \otimes_{B^+} \ldots \otimes_{B^+} C^+$$

and the lower row of the above diagram is the Čech complex of $\mathbb{G}_a^+$ associated with the covering $Z \to Y$. $\square$

Corollary 12.5. *Let Z be a locally noetherian adic space. Assume that Z is either discretely ringed or analytic. The canonical homomorphism*

$$H^i(Z, \mathbb{G}_a^+) \xrightarrow{\sim} H^i_{s\acute{e}t}(Z, \mathbb{G}_a^+)$$

is an isomorphism for all $i \geq 0$.

Proof. Consider the Leray spectral sequence associated with the morphism of sites

$$\varphi : Z_{s\acute{e}t} \to Z$$

We have to show that

$$R^q \varphi_* \mathbb{G}_a^+ = 0.$$

Put differently, for every local Huber pair (A, A^+) such that either A is discrete and noetherian or a non-Archimedean field we have to show that

$$H^q_{s\acute{e}t}(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+) = 0.$$

But every local Huber pair is Prüfer and thus the result follows from Proposition 12.3 and Proposition 12.4. $\square$

13. Tame cohomology

In this section we compute the tame cohomology of $\mathbb{G}_a^+$. The main problem we face is that for a Cartesian tame morphism $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ the image of $B^+ \otimes_{A^+} B^+$ in $B \otimes_A B$ is not necessarily integrally closed. But it turns out that the tameness condition makes the integral closure tractable.

13.1. **Computation of integral closures.** We fix a local, Cartesian, tame homomorphism $(A, A^+) \to (B, B^+)$ of strongly henselian, local, complete, Huber pairs. Assume moreover that A is noetherian. Since A and B are henselian, the extension B/A is finite étale. Let $|\cdot|$ be the valuation of B corresponding to the closed point of $\mathrm{Spa}(B, B^+)$. We denote by Γ_B the value group of $|\cdot|$ and by Γ_A the value group of the restriction of $|\cdot|$ to A. As A^+ and B^+ are strictly henselian and $(A, A^+) \to (B, B^+)$ is a tame morphism of complete, local Huber pairs, we can choose a presentation

$$B = A[T_1, \ldots, T_r]/(T_1^{m_1} - \alpha_1, \ldots, T_r^{m_r} - \alpha_r)$$

with $\alpha_i \in A^\times$ and $m_i > 1$ prime to the residue characteristic of A^+. It induces an isomorphism

$$\mathbb{Z}/m_1\mathbb{Z} \times \ldots \mathbb{Z}/m_r\mathbb{Z} \to \Gamma_B/\Gamma_A, \quad (i_1, \ldots, i_r) \mapsto |T_1^{i_1} \cdot \ldots \cdot T_r^{i_r}|.$$

For $\gamma \in \Gamma_B/\Gamma_A$ we set

$$e_\gamma = T_1^{i_1} \cdot \ldots \cdot T_r^{i_r}$$

with $0 \leq i_k \leq m_k - 1$ and $|T^{i_1} \cdot \ldots \cdot T^{i_r}| \equiv \gamma \mod \Gamma_A$. We denote the Galois group of B/A by G.

We write B_n for the n-fold tensor product of B over A:

$$B_n = B \otimes_A \ldots \otimes_A B.$$

Then $\{e_{\gamma_1} \otimes \ldots \otimes e_{\gamma_n}\}_{\gamma_1, \ldots, \gamma_n \in \Gamma_B/\Gamma_A}$ is a basis of B_n over A. As $(A, A^+) \to (B, B^+)$ is Cartesian and B^+ is flat over A^+ by Proposition 11.7, the natural homomorphism $B^+ \otimes_{A^+} \ldots \otimes_{A^+} B^+ \to B_n$ is injective. We view $B^+ \otimes_{A^+} \ldots \otimes_{A^+} B^+$ as a subring of B_n

and denote its integral closure by B_n^+. Then (B_n, B_n^+) is complete and $\mathrm{Spa}(B_n, B_n^+)$ is the n-fold fiber product of $\mathrm{Spa}(B, B^+)$ over $\mathrm{Spa}(A, A^+)$. This subsection is concerned with describing B_n^+ more explicitly.

Proposition 13.1. *For an element $b = \sum_{\gamma_1,\dots,\gamma_n} a_{\gamma_1,\dots,\gamma_n} e_{\gamma_1} \otimes \dots \otimes e_{\gamma_n}$ of B_n and $\delta \in \Gamma_B$ the following are equivalent:*

(i) $|b(x)| \le \delta$ *for all* $x \in \mathrm{Spa}(B_n, B_n^+)$.
(ii) $|a_{\gamma_1,\dots,\gamma_n}| \le \delta |e_{\gamma_1} \cdot \ldots \cdot e_{\gamma_n}|^{-1}$ *for all* $\gamma_1, \dots, \gamma_n \in \Gamma_B/\Gamma_A$.

Proof. For an $(n-1)$-tuple $\underline{\sigma} = (\sigma_1, \dots, \sigma_{n-1})$ of elements of G we define a homomorphism $m_{\underline{\sigma}} : B_n \to B$ by setting

$$m_{\underline{\sigma}}(b_1 \otimes \dots \otimes b_n) = \sigma_1 b_1 \cdot \ldots \cdot \sigma_{n-1} b_{n-1} \cdot b_n.$$

Consider the isomorphism

$$\varphi : B_n \longrightarrow \prod_{\underline{\sigma} \in G^{n-1}} B$$
$$b \mapsto (m_{\underline{\sigma}}(b))_{\underline{\sigma}}.$$

Via φ the elements of $\mathrm{Spa}(B_n, B_n^+)$ correspond to the valuations of $\prod_{\underline{\sigma} \in G^{n-1}} B$ of the form

$$|(b_{\underline{\sigma}})_{\underline{\sigma}}|' = |b_{\underline{\sigma}^0}(y)|$$

for fixed $\underline{\sigma}^0$ and a valuation $y \in \mathrm{Spa}(B, B^+)$. As $\mathrm{Spa}(B, B^+)$ is local with closed point corresponding to $|\cdot|$, it suffices to test condition (i) for valuations as above with $|\cdot(y)| = |\cdot|$. For an element of B_n of the form $b_1 \otimes \dots \otimes b_n$ and any $\underline{\sigma} \in G^{n-1}$ we have

$$|m_{\underline{\sigma}}(b_1 \otimes \dots \otimes b_n)| = |b_1| \cdot \ldots \cdot |b_n|$$

because B is henselian. Together with the triangle inequality this proves that (ii) implies (i).

Set

$$C = A[T_1, \dots, T_{r-1}]/(T_1^{m_1} - \alpha_1, \dots, T_{r-1}^{m_{r-1}} - \alpha_{r-1}).$$

This is an intermediate extension of B/A and $B = C[T_r]/(T_r^{m_r} - \alpha_r)$. By flatness we can view $C_n = C \otimes_A \dots \otimes_A C$ as a subalgebra of B_n. Denote by Γ_C the value groups of the restriction of $|\cdot|$ to C. Then e_γ for $\gamma \in \Gamma_C/\Gamma_A \subset \Gamma_B/\Gamma_A$ form a basis of C_n/A. Moreover,

$$\{T_r^{i_1} \otimes \dots \otimes T_r^{i_n} \mid 0 \le i_1, \dots, i_n \le m - 1\}$$

constitutes a basis of B_n over C_n. Taking all combinations of products

$$e_\gamma \cdot (T_r^{i_1} \otimes \dots \otimes T_r^{i_n})$$

with $i_j \in \{0, \dots, m_r - 1\}$ and $\gamma \in \Gamma_C/\Gamma_A$ yields the basis $\{e_\gamma\}_{\gamma \in \Gamma_B/\Gamma_A}$. Fix a primitive m_r-th root of unity $\zeta \in A^+$ and denote by σ the element of G which maps T_r to ζT_r and leaves C invariant. Every element of G can be written in the form $\tau\sigma^j$ for $0 \le j \le m_r - 1$ and $\tau \in G$ with $\tau\zeta = \zeta$. For an $(n-1)$-tuple $\underline{\sigma} = (\tau_1\sigma^{j_1}, \dots, \tau_{n-1}\sigma^{j_{n-1}})$ in G^{n-1} and an element $b = \sum_{i_1,\dots,i_n=0}^{m_r-1} a_{i_1,\dots,i_n} T_r^{i_1} \otimes \dots \otimes T_r^{i_n}$ of B_n we have

$$m_{\underline{\sigma}}(b) = \sum_{i_1,\dots,i_n=0}^{m_r-1} m_{\underline{\tau}}(a_{i_1,\dots,i_n}) \zeta^{i_1 j_1 + \dots i_{n-1} j_{n-1}} T_r^{i_1} \otimes \dots \otimes T_r^{i_n}.$$

As $|T_r|^k$ for $k = 0, \ldots, m_r - 1$ represent the m_r distinct elements of Γ_B/Γ_C, we obtain

$$|m_{\underline{\sigma}}(b)| = \max_{0\leq k\leq m_r-1} |\sum_{i_1+\ldots+i_n\equiv k \mod m_r} m_{\underline{\tau}}(a_{i_1,\ldots,i_n}\alpha_r^{(i_1+\ldots+i_n-k)/m_r})\zeta^{i_1j_1+\ldots+i_{n-1}j_{n-1}}| \cdot |T_r|^k.$$

Suppose $|b(x)| \leq \delta$ for all $x \in \mathrm{Spa}(B_n, B_n^+)$. Then in particular,

$$|m_{\underline{\sigma}}(b)| \leq \delta$$

for all $\underline{\sigma} \in G^{n-1}$. By the above this is equivalent to

$$(8) \qquad |\sum_{i_1+\ldots+i_n\equiv k \mod m_r} m_{\underline{\tau}}(a_{i_1,\ldots,i_n}\alpha_r^{(i_1+\ldots+i_n-k)/m_r})\zeta^{i_1j_1+\ldots+i_{n-1}j_{n-1}}| \leq \delta|T_r|^{-k}$$

for all $\underline{\sigma}$ and all $k = 0, \ldots, m_r - 1$. The following Lemma 13.2 shows that the matrix $(\zeta^{i_1j_1+\ldots+i_{n-1}j_{n-1}})$ is invertible in A^+. Therefore, inequality (8) holds for all $j_1, \ldots, j_{n-1} = 0, \ldots, m_r - 1$ if and only if

$$|m_{\underline{\tau}}(a_{i_1,\ldots,i_n}\alpha_r^{(i_1+\ldots+i_n-k)/m_r})| \leq \delta|T_r|^{-k}$$

for all $i_1, \ldots, i_{n-1} = 0, \ldots, m_r - 1$. The result now follows by induction on r. □

Lemma 13.2. *Consider the $m_r^{n-1} \times m_r^{n-1}$-matrix V_n whose rows are indexed by the $(n-1)$-tuples $(i_1, \ldots, i_{n-1}) \in \{0, \ldots, m_r - 1\}^{n-1}$ and whose columns by $(j_1, \ldots, j_{n-1}) \in \{0, \ldots, m_r - 1\}^{n-1}$ (both provided with the lexicographical ordering) and whose entry at $(i_1, \ldots, i_{n-1}, j_1, \ldots, j_{n-1})$ is $\zeta^{i_1j_1+\ldots+i_{n-1}j_{n-1}}$. Then, considered as a matrix with coefficients in A^+, V_n is invertible.*

Proof. We have

$$V_n = \begin{pmatrix} V_{n-1} & V_{n-1} & V_{n-1} & \ldots & V_{n-1} \\ V_{n-1} & \zeta V_{n-1} & \zeta^2 V_{n-1} & \ldots & \zeta^{m_r-1}V_{n-1} \\ V_{n-1} & \zeta^2 V_{n-1} & \zeta^4 V_{n-1} & \ldots & \zeta^{2(m_r-1)}V_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_{n-1} & \zeta^{m_r-1}V_{n-1} & \zeta^{2(m_r-1)}V_{n-1} & \ldots & \zeta^{(m_r-1)^2}V_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{1} & \mathbb{1} & \mathbb{1} & \ldots & \mathbb{1} \\ \mathbb{1} & \zeta\mathbb{1} & \zeta^2\mathbb{1} & \ldots & \zeta^{m_r-1}\mathbb{1} \\ \mathbb{1} & \zeta^2\mathbb{1} & \zeta^4\mathbb{1} & \ldots & \zeta^{2(m_r-1)}\mathbb{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{1} & \zeta^{m_r-1}\mathbb{1} & \zeta^{2(m_r-1)}\mathbb{1} & \ldots & \zeta^{(m_r-1)^2}\mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} V_{n-1} & 0 & 0 & \ldots & 0 \\ 0 & V_{n-1} & 0 & \ldots & 0 \\ 0 & 0 & V_{n-1} & \ldots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & V_{n-1} \end{pmatrix}.$$

The left hand matrix is a Vandermonde matrix over the ring of $m_r^{n-2} \times m_r^{n-2}$-matrices with coefficients in A^+. Its determinant is

$$\prod_{0\leq i<j\leq m_r-1} (\zeta^j - \zeta^i)\mathbb{1},$$

which is a unit since $(\zeta^j - \zeta^i)$ divides m_r and m_r is invertible in A^+. Therefore the left hand matrix is invertible. The right hand matrix is invertible by induction. □

Corollary 13.3. *The integral closure B_n^+ of $B^+ \otimes_{A^+} \ldots \otimes_{A^+} B^+$ in B_n is the subring generated by*

$$\{b_1 \otimes \ldots \otimes b_n \in B_n \mid \prod_{i=1}^n |b_i| \leq 1\}.$$

An element $\sum_{\gamma_1,\ldots,\gamma_n} a_{\gamma_1,\ldots,\gamma_n} e_{\gamma_1} \otimes \ldots \otimes e_{\gamma_n}$ *is integral over* $B^+ \otimes_{A^+} \ldots \otimes_{A^+} B^+$ *if and only if*

$$|a_{\gamma_1,\ldots,\gamma_n}| \leq |e_{\gamma_1} \cdot \ldots \cdot e_{\gamma_n}|^{-1}$$

for all $\gamma_1, \ldots, \gamma_n \in \Gamma_B/\Gamma_A$.

Proof. By [Hub93b] an element b of B_n is contained in B_n^+ if and only if $|b(x)| \leq 1$ for all $x \in \mathrm{Spa}(B_n, B_n^+)$. The result thus follows by Proposition 13.1 with $\delta = 1$. □

Assume that A is noetherian. Since B is faithfully flat over A and B^+ is faithfully flat over A^+ by Proposition 11.7, we obtain a diagram of exact Amitsur complexes

$$\begin{tikzcd}
0 \arrow[r] & A^+ \arrow[r] \arrow[d, hook] & B^+ \arrow[r] \arrow[d, hook] & B^+ \otimes_{A^+} B^+ \arrow[r] \arrow[d, hook] & B^+ \otimes_{A^+} B^+ \otimes_{A^+} B^+ \arrow[r] \arrow[d, hook] & \ldots \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & B \otimes_A B \arrow[r] & B \otimes_A B \otimes_A B \arrow[r] & \ldots
\end{tikzcd}$$

As the image of an integral element is integral, the diagram factors as

$$\begin{tikzcd}
0 \arrow[r] & A^+ \arrow[r] \arrow[d, equal] & B^+ \arrow[r] \arrow[d, equal] & B^+ \otimes_{A^+} B^+ \arrow[r] \arrow[d, hook] & B^+ \otimes_{A^+} B^+ \otimes_{A^+} B^+ \arrow[r] \arrow[d, hook] & \ldots \\
0 \arrow[r] & A^+ \arrow[r] \arrow[d, hook] & B^+ \arrow[r] \arrow[d, hook] & (B \otimes_A B)^+ \arrow[r] \arrow[d, hook] & (B \otimes_A B \otimes_A B)^+ \arrow[r] \arrow[d, hook] & \ldots \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & B \otimes_A B \arrow[r] & B \otimes_A B \otimes_A B \arrow[r] & \ldots
\end{tikzcd}$$

Proposition 13.4. *Let* $(A, A^+) \to (B, B^+)$ *be a local, Cartesian, tame homomorphism of strongly henselian, local, complete, Huber pairs. Assume moreover that* A *is noetherian. Then the complex*

$$0 \longrightarrow A^+ \longrightarrow B^+ \longrightarrow (B \otimes_A B)^+ \longrightarrow (B \otimes_A B \otimes_A B)^+ \longrightarrow \ldots$$

is exact.

Proof. Consider the section s of the inclusion $A \hookrightarrow B$ sending an element $\sum_\gamma a_\gamma e_\gamma$ of B to the coefficient a_1 of $e_1 = 1$. Mapping $b_1 \otimes \ldots \otimes b_n$ to $s(b_1) \cdot \ldots \cdot s(b_n)$, s induces a morphism Φ of complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B \otimes_A B \arrow[r] \arrow[d] & B \otimes_A B \otimes_A B \arrow[r] \arrow[d] & \ldots \\
0 \arrow[r] & A \arrow[r, "\mathrm{id}"] & A \arrow[r, "0"] & A \arrow[r, "\mathrm{id}"] & A \arrow[r, "0"] & \ldots
\end{tikzcd}$$

It is well known that Φ is a homotopy equivalence whose inverse is the natural inclusion I of the lower complex in the upper one. Namely, $\Phi \circ I = \mathrm{id}$ and $I \circ \Phi$ is homotopic to the identity by the homotopy given by

$$\begin{array}{rcl} D_i : B_n & \longrightarrow & B_n \\ (c_1 \otimes \ldots \otimes c_n) & \mapsto & s(c_1) \otimes \ldots \otimes s(c_{i-1}) \otimes c_i \otimes \ldots \otimes c_n. \end{array}$$

In order to show that the complex in the statement of the proposition is exact, it suffices to show that Φ restricts to homomorphisms $B_n^+ \to A^+$ and D_i to a homomorphism $B_n^+ \to B_n^+$.

Writing D_i in terms of the basis $\{e_\gamma\}_\gamma$ we obtain:

$$D_i(\sum_{\gamma_1,\dots,\gamma_n} a_{\gamma_1,\dots,\gamma_n} e_{\gamma_1}\otimes\dots\otimes e_{\gamma_n}) = \sum_{\gamma_i,\dots,\gamma_n} a_{1,\dots,1,\gamma_i,\dots\gamma_n} 1\otimes\dots\otimes 1\otimes e_{\gamma_i}\otimes\dots\otimes e_{\gamma_n}.$$

Therefore, Corollary 13.3 assures that D_i maps B_n^+ to B_n^+. The argument for Φ is the same. □

13.2. **Computation of tame cohomology.**

Proposition 13.5. *Let (A, A^+) be a strongly henselian Huber pair where A is either a strongly noetherian Tate ring or noetherian and discrete. Then*

$$H_t^i(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+) = 0$$

for all $i \geq 1$.

Proof. Let $\mathcal{B}$ be the category of Cartesian tame morphisms of affinoid adic spaces

$$\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+).$$

It has fiber products and becomes a site by defining coverings of $\mathrm{Spa}(B, B^+)$ to be the Cartesian tame coverings of $\mathrm{Spa}(B, B^+)$. By Corollary 12.2 we can compute the cohomology groups $H_t^q(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+)$ in $\mathcal{B}$.

We show that $\mathbb{G}_a^+$ is flasque on $\mathcal{B}$. Let

$$\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$$

be a covering in $\mathcal{B}$. We need to show that the Čech complex for $\mathbb{G}_a^+$ associated with this covering is exact. Using the notation of Section 13.1 we have

$$\mathrm{Spa}(C, C^+) \times_{\mathrm{Spa}(B,B^+)} \dots \times_{\mathrm{Spa}(B,B^+)} \mathrm{Spa}(C, C^+) = \mathrm{Spa}(C_n, C_n^+).$$

Note that since B is henselian, C_n is finite over B, hence complete. Therefore,

$$\mathbb{G}_a^+(\mathrm{Spa}(C_n, C_n^+)) = C_n^+$$

and the Čech complex for the covering $\mathrm{Spa}(C, C^+) \to \mathrm{Spa}(B, B^+)$ equals

$$0 \to B^+ \to C^+ \to C_2^+ \to C_3^+ \to \dots$$

This complex is exact by Proposition 13.4. □

Corollary 13.6. *Let Z be a locally noetherian adic space. Assume that Z is either discretely ringed or analytic. The canonical homomorphism*

$$H^i_{s\acute{e}t}(Z, \mathbb{G}_a^+) \to H_t^i(Z, \mathbb{G}_a^+)$$

is an isomorphism for all $i \geq 0$.

Proof. Consider the Leray spectral sequence associated with the morphism of sites

$$\varphi : Z_t \to Z_{s\acute{e}t}.$$

We have to show that

$$R^q\varphi_*\mathbb{G}_a^+ = 0.$$

Put differently, for every strongly henselian Huber pair (A, A^+) where A is either a strongly noetherian Tate ring or noetherian and discrete we have to show that

$$H_t^q(\mathrm{Spa}(A, A^+), \mathbb{G}_a^+) = 0.$$

This is true by Proposition 13.5. □

Combining Corollary 12.5, Corollary 13.6 and Proposition 10.6 we obtain:

Theorem 13.7. *Let X be pro-open in an essentially smooth scheme S over k such that X is dense in S. Assume that resolution of singularities holds over k. There is a natural isomorphism*

$$H^i(S, \mathscr{O}_S) \cong H^i_t(\mathrm{Spa}(X, S), \mathbb{G}_a^+)$$

for all $i \geq 0$.

14. The Artin Schreier sequence

Let Z be an adic space with $\mathrm{char}(Z) = \{p\}$. There is an Artin Schreier sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a^+ \xrightarrow{F-1} \mathbb{G}_a^+ \to 0$$

on Z_t and on $Z_{s\acute{e}t}$, where $F-1$ is the homomorphism $x \mapsto x^p - x$. We can check exactness on stalks. Let (A, A^+) be strongly henselian. Then

$$F - 1 : A^+ \to A^+$$

is surjective as A^+ is strictly henselian.

Proposition 14.1. *Let (A, A^+) be a complete Prüfer Huber pair such that A is of characteristic $p > 0$ and is either noetherian with the discrete topology or a strongly noetherian Tate ring. If $\mathrm{Spa}(A, A^+)$ is connected,*

$$H^i_t(\mathrm{Spa}(A, A^+), \mathbb{Z}/p\mathbb{Z}) \cong H^i_{s\acute{e}t}(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, \\ A^+/(F-1)A^+ & i = 1, \\ 0 & i \geq 2. \end{cases}$$

Proof. This follows from Proposition 12.3, Proposition 12.4 and Corollary 13.6 via the Artin Schreier sequence. □

Corollary 14.2. *Let Z be a locally noetherian adic space with $\mathrm{char}(Z) = \{p\}$ which is either analytic or discretely ringed. Then the Leray spectral sequence associated with $Z_t \to Z_{s\acute{e}t}$ induces isomorphisms*

$$H^i_t(Z, \mathbb{Z}/p\mathbb{Z}) \cong H^i_{s\acute{e}t}(Z, \mathbb{Z}/p\mathbb{Z})$$

for all $i \geq 0$.

Proposition 14.3. *Let S be an affine, regular, and integral scheme of characteristic $p > 0$ and X dense and pro-open in S. Assume that resolution of singularities holds over S. Then we have*

$$H^i_t(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z}) \cong H^i_{s\acute{e}t}(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, \\ \mathscr{O}_S(S)/(F-1)\mathscr{O}_S(S) & i = 1, \\ 0 & i \geq 2. \end{cases}$$

Proof. This follows from Theorem 13.7 via the Artin Schreier sequence. □

Corollary 14.4. *Let S be a regular integral scheme of characteristic $p > 0$ and X dense and pro-open in S. Assume that resolution of singularities holds over S. The Leray spectral sequences associated with the morphisms of sites $\mathrm{Spa}(X, S)_t \to \mathrm{Spa}(X, S)_{s\acute{e}t}$ and $\mathrm{Spa}(X, S)_{s\acute{e}t} \to S_{\acute{e}t}$ induce natural isomorphisms*

$$H^i_{\acute{e}t}(S, \mathbb{Z}/p\mathbb{Z}) \cong H^i_{s\acute{e}t}(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z}) \cong H^i_t(\mathrm{Spa}(X, S), \mathbb{Z}/p\mathbb{Z})$$

for all $i \geq 0$.

Proof. It suffices to show that

$$H^i_t(\mathrm{Spa}(X,S),\mathbb{Z}/p\mathbb{Z}) = H^i_{s\acute{e}t}(\mathrm{Spa}(X,S),\mathbb{Z}/p\mathbb{Z}) = 0$$

for $i>0$ in case S is strictly henselian. This follows directly from the description given in Proposition 14.3. $\square$

Corollary 14.5 (Purity)**.** *Let S be a noetherian scheme of characteristic $p>0$ and X a regular scheme which is separated and essentially of finite type over S. Assume that resolution of singularities holds over S. Then for any pro-open dense subscheme $U\subseteq X$ we have*

$$H^i_t(\mathrm{Spa}(U,S),\mathbb{Z}/p\mathbb{Z}) \cong H^i_t(\mathrm{Spa}(X,S),\mathbb{Z}/p\mathbb{Z}).$$

Proof. Let $\bar{X}$ be a regular compactification of X over S. Then $\bar{X}$ is also a compactification of U over S. Hence, by Corollary 14.4 both cohomology groups equal $H^i_{\acute{e}t}(\bar{X},\mathbb{Z}/p\mathbb{Z})$. $\square$

Corollary 14.6 (Homotopy invariance)**.** *Let S be a noetherian scheme of characteristic $p>0$ and X a regular scheme which is essentially of finite type over S. Assume that resolution of singularities holds over S. Then*

$$H^i_t(\mathrm{Spa}(X,S),\mathbb{Z}/p\mathbb{Z}) \cong H^i_t(\mathrm{Spa}(\mathbb{A}^1_X,S),\mathbb{Z}/p\mathbb{Z}).$$

Proof. Consider the Leray spectral sequence associated with $\mathrm{Spa}(\mathbb{A}^1_X,S)\to\mathrm{Spa}(X,S)$. It suffices to show that there is a basis $\mathcal{B}$ of the topology of $\mathrm{Spa}(X,S)_t$ consisting of spaces of the form $\mathrm{Spa}(U,T)$ such that for every cohomology class

$$\xi\in H^n_t(\mathrm{Spa}(\mathbb{A}^1_U,T),\mathbb{Z}/p\mathbb{Z})$$

there is a covering $\{\mathrm{Spa}(U_i,T_i)\}\to\mathrm{Spa}(U,T)$ such that ξ restricted to $\mathrm{Spa}(\mathbb{A}^1_{U_i},T_i)$ vanishes for all i.

By our assumption on resolution of singularities, there is a basis $\mathcal{B}$ of the topology of $\mathrm{Spa}(X,S)_t$ consisting of adic spaces $\mathrm{Spa}(U,T)$ where T is regular and U is pro-open in T. Fix an object $\mathrm{Spa}(U,T)$ in $\mathcal{B}$. Since $\mathrm{Spa}(\mathbb{A}^1_U,T)=\mathrm{Spa}(\mathbb{A}^1_U,\mathbb{P}^1_T)$ and $\mathbb{A}^1_U$ is pro-open in the regular scheme $\mathbb{P}^1_T$, Corollary 14.4 tells us that the cohomology group $H^n_t(\mathrm{Spa}(\mathbb{A}^1_U,T),\mathbb{Z}/p\mathbb{Z})$ equals

$$H^n_{\acute{e}t}(\mathbb{P}^1_T,\mathbb{Z}/p\mathbb{Z}).$$

(Remember that T has characteristic p). Using the Leray spectral sequence associated with $\mathbb{P}^1_T\to T$ we find that this is isomorphic to $H^n_{\acute{e}t}(T,\mathbb{Z}/p\mathbb{Z})$. For every class ζ in $H^n_{\acute{e}t}(T,\mathbb{Z}/p\mathbb{Z})$ there is an étale covering $\{T_i\}\to T$ such that $\zeta|_{T_i}$ vanishes. But then the corresponding class ξ in $H^n_t(\mathrm{Spa}(\mathbb{A}^1_U,T),\mathbb{Z}/p\mathbb{Z})$ vanishes when restricted to $\mathrm{Spa}(\mathbb{A}^1_{U\times_T T_i},T_i)$. Since the family

$$\{\mathrm{Spa}(\mathbb{A}^1_{U\times_T T_i},T_i)\}\to\mathrm{Spa}(\mathbb{A}^1_U,T)$$

is a covering family, this finishes the proof. $\square$

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