A twisted topological trace formula for Hecke operators and liftings from symplectic to general linear groups

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Abstract

For the locally symmetric space X attached to an arithmetic subgroup of an algebraic group G of $\mathbb{Q}$-rank r we construct a compact manifold $\tilde{X}$ by glueing together 2^r copies of the Borel-Serre-compactification of X. We apply the classical Lefschetz fixed point formula to $\tilde{X}$ and get formulas for the traces of Hecke operators $\mathcal{H}$ acting on the cohomology of X. We allow twistings of $\mathcal{H}$ by outer automorphisms η of G. We stabilize this topological trace formula and compare it with the corresponding formula for an endoscopic group of the pair (G, η). As an application we deduce a weak lifting theorem for the lifting of automorphic representations from Siegel modular groups to general linear groups.

Introduction

0.1. Topological Trace Formula: The aim of this paper is to develop a topological trace formula for Hecke operators acting on the ordinary cohomology of locally symmetric domains X attached to congruence subgroups of an algebraic group $G/\mathbb{Q}$. We want to deal with the twisted case also where we allow the Hecke operators to be twisted by an outer automorphism of G. In the untwisted case such formulas have already been developped respectively applied by several authors: [Bew], [GKM1], [GKM2], [GM2], [Har2], [Har3], [KuS], [RoSp], [W1].

We will deduce our formula from a Lefschetz fixed point formula for compact manifolds, restated in 3.3. Since the spaces X are not compact, we have to use a trick for this reduction: We construct a compact manifold $\tilde{X}$, which is obtained by gluing together 2^r pieces of the Borel-Serre-compactification $\bar{X}$ ([BS]) along their boundary strata, where r denotes the $\mathbb{Q}$-rank of G. On $\tilde{X}$ we have an action of the group $S^\Delta := \{\pm 1\}^r$, such that the quotient $\tilde{X}/S^\Delta$ is isomorphic to $\bar{X}$. Under this isomorphism we can identify the ordinary cohomology of X with the S^Δ-invariant part of the cohomology $H(\tilde{X})$ and similarly the cohomology with compact supports of X with the χ_{-1}-eigenspace of $H(\tilde{X})$, where χ_{-1} :

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$S^{\Delta} \to \{\pm 1\}$ denotes the character $(\varepsilon_1, \ldots, \varepsilon_r) \mapsto \varepsilon_1 \cdots \varepsilon_r$. By twisting our (twisted) Hecke-correspondences with all elements $\sigma \in S^{\Delta}$ we thus get the correspondences, to which we can apply the simple fixed point formula for manifolds. By this method we avoid the application of intersection cohomology to a singular compactifications (e.g. the reductive Borel-Serre-compactification [GHM], [GT]).

It should be noted that a similar construction already appears in the work of Oshima [Osh]. But while she gives a compactification $\tilde{Y}_{Osh}$ of the symmetric space Y of $G(\mathbb{R})$ i.e. she makes a construction over $\mathbb{R}$, we want to construct a compactification of the locally symmetric quotient $\Gamma\backslash Y$, where Γ denotes some congruence subgroup in $G(\mathbb{Q})$, i.e. we have to introduce an arithmetic construction. In fact we will construct some extension (not a compactification) $\tilde{Y}$ of Y, such that the action of Γ can be continued to a proper discontinuous action on $\tilde{Y}$ (at least for some smaller neat congruence subgroup of Γ), such that $\tilde{X} \simeq \Gamma\backslash\tilde{Y}$. But the space $\tilde{Y}$ is topologically highly non trivial and has no relation to $\tilde{Y}_{Osh}$ apart from the fact that it contains 2^r copies of Y too.

0.2. The example SL_2. The upper half plane $\mathbb{H} = \mathbb{H}^+ \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is the symmetric space for $\mathrm{SL}_2(\mathbb{R})$. Then Oshimas construction just gives the complex projective line $\tilde{Y}_{Osh} = \mathbb{P}^1(\mathbb{C}) = \mathbb{H}^+ \cup \mathbb{H}^- \cup \mathbb{P}^1(\mathbb{R})$, but the action of Γ cannot be continued in a satisfactory way from $\mathbb{H}$ to $\mathbb{P}^1(\mathbb{C})$, so that we don't get a good compactification of $\Gamma\backslash Y$ in this way.

Our construction can be described as follows: We too can take $\mathbb{H}^+$ and $\mathbb{H}^-$ as the two copies of $\mathbb{H}$, but we embed them into the complex affine line $A = \mathbb{C}$ in the following way:

$$\iota : \mathbb{H}^+ \cup \mathbb{H}^- \quad \hookrightarrow \quad \mathbb{C}, \qquad\qquad -x + i \cdot y \quad \mapsto \quad x + i \cdot \frac{1}{y}.$$

We take a set of representatives $\{\delta\}_{\delta \in \Delta}$ for $\mathrm{SL}_2(\mathbb{Q})/B(\mathbb{Q}) \simeq \mathbb{P}^1(\mathbb{Q})$, where $B \subset G$ denotes the Borel subgroup of upper triangular matrices and define the embeddings

$$\iota_\delta : \mathbb{H}^+ \cup \mathbb{H}^- \quad \hookrightarrow \quad \mathbb{C}, \qquad\qquad z \mapsto \iota(\delta(z))$$

Now $\tilde{Y}$ is obtained by gluing together $\bigcup_{\delta \in \Delta} \mathbb{C}$ along their open subspaces $\mathbb{H}^+ \cup \mathbb{H}^-$, where each subspace is embedded via ι_δ into the component $\mathbb{C}$ which is indexed by δ. So we get for each rational cusp in $\mathbb{P}^1(\mathbb{Q})$ a real line which lies in the common closure of $\mathbb{H}^+$ to $\mathbb{H}^-$ and a homotopy class of paths from $\mathbb{H}^+$ to $\mathbb{H}^-$.

Let us illustrate the procedure of computing Euler characteristics $\chi(X)$ and Lefschetz numbers via the compactification procedure in some examples:

Example 0.3. Let X be a Riemann surface of genus g with $n \geqslant 1$ small disks removed. If one glues together 2 copies of X along the boundary ∂X which is the disjoint union of n copies of S^1, one gets a compact Riemann surface $\tilde{X}$ of genus $2g + n - 1$. One has $\chi(\tilde{X}) = 2 - 2(2g + n - 1)$, $\chi(\partial X) = 0$ and

$$\begin{aligned} \chi(X) \quad &= \quad \chi_c(X) \quad = \quad \frac{(2 - 2(2g + n - 1)) + 0}{2} \\ &= \quad 1 - (2g + n - 1) \quad = \quad h^0(X) - h^1(X). \end{aligned}$$

EXAMPLE 0.4. Let X be an open interval. Then ∂X consists of the two boundary points of X and $\tilde{X}$ is homoeomorphic to S^1, i.e. $\chi(\tilde{X}) = 0$ and $\chi(\partial X) = 2$. In this case we get

$$\chi(X) \;=\; 1 \;=\; \frac{\chi(\tilde{X}) + \chi(\partial X)}{2} \qquad\qquad \chi_c(X) \;=\; -1 \;=\; \frac{\chi(\tilde{X}) - \chi(\partial X)}{2}.$$

0.5. IN SECTION 1 we construct the spaces $\tilde{X}$ and $\tilde{Y}$ carrying an action of the group S^{Δ} in an adelic language. We avoid to refer to constructions in the paper of Borel and Serre [BS] and formulate our constructions in a more group theoretical language which gives the manifold structure of $\tilde{Y}$ immediately. It would be rather unnatural to start with manifolds with corners to get the manifold structure. The group theoretical description in an adelic language enables us to compute and describe the sets of fixed points.

0.6. IN SECTION 2 we compute the sets of fixed points of Hecke correspondences twisted by an outer automorphism η. This section uses well known methods ([Bew],[GM3]) and is of computational nature.

0.7. IN SECTION 3 we develop a general Lefschetz fixed point formula for η-twisted Hecke correspondences on locally symmetric spaces. At first we restate a more or less well known version of the Lefschetz fixed point formula for compact oriented manifolds. We do not assume that the correspondence has only isolated fixed points but allow higher dimensional submanifolds Y_j of fixed points, such that the correspondence is only transversal to the diagonal in the normal direction to Y_j.

We apply this fixed point formula to the η-twisted Hecke correspondences $\mathcal{H}$ twisted with elements $\sigma \in S^{\Delta}$ acting on $\tilde{X}$. Of course we have to prove that our modified transversality assumptions hold. The Lefschetz number of $\mathcal{H}$ on the cohomology (resp. cohomology with compact support)of X can then be obtained as linear combination of the Euler characteristics of different sets of fixed points. One has to stratify the sets of fixed points with respect to the different boundary strata of the Borel-Serre compactification. Fixed point strata on the boundary contribute several times to the fixed point formula. These contributions may cancel each other depending on the signs with which the fixed point components contribute to the trace formula. This corresponds to the theory of contracting and expanding fixed points in the work of Goresky and MacPherson ([GM1]) and of Bewersdorff ([Bew]). The Euler characteristics involved can be handled with the Gauss Bonnet formula of Harder ([Har1], [Leu]), so that we arrive at a first version of the trace formula involving orbital integrals.

0.8. IN SECTION 4 we stabilize this trace formula under certain conditions on the vanishing of the Galois cohomology of the group G, which are satisfied in the main applications we have in mind. We give a self contained version of this stabilization process independent of the general theory of [KoS], since the topological trace formula kills several difficulties of the general trace formula of Arthur and Selberg [Ar] but requires some additional considerations at the archimedean place.

0.9. IN SECTION 5 we compare two topological trace formulas for a group G with outer automorphism η and its stable endoscopic group G_1. We formulate a lemma which compares the traces of matching elements on the coefficient systems. We get that the Lefschetz numbers of matching (η-twisted for G) Hecke correspondences on the two locally symmetric spaces coincide. Using the work of Ngô and Waldspurger on the (twisted) fundamental lemma this implies that the cohomology of $\tilde{X}_G$ may be considered as the lift of the cohomology of $\tilde{X}_{G_1}$ modulo representations induced from $G(\mathbb{A}_f)$ to $G(\mathbb{A}_f) \rtimes \langle\eta\rangle$. We will formulate our final result for the lifting from Sp_{2n} to PGL_{2n+1} and for the lifting from GSpin_{2n+1} to $\mathrm{GL}_{2n} \times \mathrm{GL}_1$ over a totally real number field F. Remark that GSp_4 is GSpin_5, so that we get two liftings from symplectic groups of genus 2 to general linear groups. A lifting from PGSp_4 to PGL_4 has been obtained already by Flicker [Fl3] using a variant of Arthurs trace formula.

Our result depends on a naive definition of liftings of representations of the finite adele group: We have to assume, that the normalization of Haar measures on the centralizers of global elements is in such a way, that certain factors involving the infinity component agree. This will be sufficient to get weak lifting statements, but requires a more subtle analysis to get precise lifting statements including multiplicity formulas.

Details and applications of this result will be given in a forthcoming paper [Wes].

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1. The spaces

Levi and maximal compact subgroups

1.1. Reductive groups. Let $G/\mathbb{Q}$ be a connected reductive group, $G^{(1)}$ its derived group and $Z = Z_G$ its center. We fix a minimal parabolic $\mathbb{Q}$-subgroup P_0 and a maximal $\mathbb{Q}$-split torus $S_0 \subset P_0$. Let $\Phi = \Phi(G, S_0) \subset X^*(S_0)$ be the set of $\mathbb{Q}$-roots of G with respect to S_0, $\Phi^+ \subset \Phi$ the subset of positive roots with respect to P_0 and $\Delta \subset \Phi^+$ the set of simple roots.

1.2. Parabolics. The subsets J of Δ are in 1-1-correspondence with the $G(\mathbb{Q})$-conjugacy classes of rational parabolic subgroups. Each conjugacy class contains exactly one standard parabolic subgroup, denoted by P_J, i.e. satisfying $P_0 \subset P_J \subset G$. We define for $J \subset \Delta$:

$$\begin{array}{rcccl} S_J & = & \left(\bigcap_{\alpha\in J} \ker\alpha\right)^\circ & \subset & S_0 \\ M_J & = & Cent(S_J) & = & \text{centralizer of } S_J \text{ in } G \\ A_J & = & \left(S_J(\mathbb{R}) \cap G^{(1)}(\mathbb{R})\right)^\circ. & & \end{array}$$

As usual the upper index $^\circ$ describes the connected component of the identity (in the first line for the Zariski topology, in the last line for the real topology). We denote by U_J (resp. U_0) the unipotent radical of P_J (resp. P_0). Then we have

$$\begin{array}{rclrclrcl} P_J & = & M_J \cdot U_0 & = & M_J \ltimes U_J & & & & \\ S_\emptyset & = & S_0, & P_\emptyset & = & P_0, & P_\Delta & = & G \end{array}$$

and S_Δ is the maximal $\mathbb{Q}$-split torus in Z_G.

LEMMA 1.3. *(compare [Bol] 20.6(i), 11.23(ii))*

(a) *If $M \subset P_J$ is a Levi subgroup with $S_J \subset M$ then $M = M_J$.*

(b) *If $u^{-1} \cdot M_J \cdot u = M_J$ for some $u \in U_J(\bar{\mathbb{Q}})$ then $u = 1$.*

Proof: (a) Since M_J is a Levi subgroup of P_J and any two Levi subgroups of P_J are conjugate, there exists $u \in U_J(\bar{\mathbb{Q}})$ such that

$$M = u \cdot M_J \cdot u^{-1}. \tag{1}$$

This implies $u^{-1}S_J u \subset u^{-1}Mu = M_J = Cent(S_J)$, i.e. $(u^{-1}s_1 u) \cdot s_2 = s_2 \cdot (u^{-1}s_1 u)$ for all $s_1, s_2 \in S_J(\bar{\mathbb{Q}})$. We can rewrite this equation in the form (since S_J is abelian)

$$s_1 \cdot (s_2^{-1} u s_2 u^{-1}) \quad = \quad (s_2^{-1} u s_2 u^{-1}) \cdot s_1.$$

Since this is valid for all $s_1 \in S_J(\bar{\mathbb{Q}})$ we get:

$$(s_2^{-1} u s_2) \cdot u^{-1} \in M_J(\bar{\mathbb{Q}}) = Cent(S_J(\bar{\mathbb{Q}})).$$

On the other side we have $(s_2^{-1} u s_2) \cdot u^{-1} \in U_J(\bar{\mathbb{Q}})$, since s_2 normalizes U_J. Therefore $s_2^{-1} u s_2 \cdot u^{-1} \in M_J(\bar{\mathbb{Q}}) \cap U_J(\bar{\mathbb{Q}}) = \{1\}$, i.e. $us_2 = s_2 u$ for all $s_2 \in S_J(\bar{\mathbb{Q}})$, so that $u \in Cent(S_J(\bar{\mathbb{Q}})) \cap U_J(\bar{\mathbb{Q}}) = \{1\}$ and therefore $M = M_J$, which proves (a).

If we start with $M = M_J$ in (1) we arrive again at $u = 1$ with the same proof, i.e. we get the statement (b). $\square$

LEMMA 1.4. *There exists a maximal compact subgroup $K_\infty^m \subset G(\mathbb{R})$ such that*

$$M_J(\mathbb{R}) \cap K_\infty^m \quad = \quad P_J(\mathbb{R}) \cap K_\infty^m \qquad \textit{for all } J \subset \Delta.$$

Proof: Let K_1 be some maximal compact subgroup of $G(\mathbb{R})$. We denote by θ_1 the Cartan involution of $G/\mathbb{R}$ with respect to K_1 ([BS, 1.6]). The group $M_1 := P_0 \cap \theta_1(P_0)$ is the unique Levi subgroup of P_0 stable under θ_1 (apply [BS, 1.8] for $L = G$, $H = P_0$). We have $M_1 = u \cdot M_0(\mathbb{R}) \cdot u^{-1}$ for some $u \in U_0(\mathbb{R})$. Put $K_\infty^m := u^{-1}K_1 u$. Now $\theta_0 := int(u)^{-1} \circ \theta_1 \circ int(u)$ is the Cartan involution of $G/\mathbb{R}$ with respect to K_∞^m (This may be deduced easily from the characterization in [BS, 1.6].) We have $\theta_0(M_0) = int(u)^{-1}\theta_1(M_1) = int(u)^{-1}(M_1) = M_0$. For arbitrary $J \subset \Delta$ we get:

$$\theta_0(P_J) \cap P_J \;\supset\; \theta_0(P_0) \cap P_0 \;=\; u^{-1}\left(\theta_1(P_0) \cap P_0\right) u \;=\; u^{-1} M_1 u \;=\; M_0 \supset S_0 \supset S_J.$$

Again by [BS, 1.8] the left hand group is a Levi subgroup of P_J so that we get $M_J = \theta_0(P_J) \cap P_J$ by lemma 1.3(a). Now $P_J(\mathbb{R}) \cap K_\infty^m = \{p \in P_J(\mathbb{R}) | \theta_0(p) = p\} \subset P_J(\mathbb{R}) \cap \theta_0(P_J(\mathbb{R})) = M_J(\mathbb{R})$. Therefore $P_J(\mathbb{R}) \cap K_\infty^m = M_J(\mathbb{R}) \cap K_\infty^m$ for all $J \subset \Delta$. $\square$

LEMMA 1.5. *The family of simple roots $(\alpha)_{\alpha \in \Delta - J}$ induces an isomorphism of groups:*

$$A_J \;\xrightarrow{\sim}\; (\mathbb{R}^*_{>0})^{\Delta - J}.$$

Proof: (compare [BS] 4.2.(2)) The exact sequence of algebraic groups

$$1 \to S_\Delta \cap G^{(1)} \to S_J \cap G^{(1)} \to (\mathbb{G}_m)^{\Delta - J} \to 1$$

induces an exact sequence:

$$1 \to S_\Delta(\mathbb{R}) \cap G^{(1)}(\mathbb{R}) \to S_J(\mathbb{R}) \cap G^{(1)}(\mathbb{R}) \to (\mathbb{R}^*)^{\Delta - J} \to H^1(\mathbb{R}, S_\Delta \cap G^{(1)}) \to 1,$$

since $S_J \cap G^{(1)}$ is a split torus. Now the first and fourth term are finite groups, so that the middle map induces an isomorphism between the connected components of the identity of the second and third term. Since A_J is the connected component of the second term, the claim is now clear. $\square$

Multi-pushouts

1.6. The category $\mathcal{J}_\Delta$. For a set Δ we denote by $\mathcal{P}(\Delta)$ the set of its subsets. We define a category $\mathcal{J}_\Delta$ whose objects are pairs (I, J) with $I \subset J \subset \Delta$ i.e.

$$\begin{array}{rcll} Ob(\mathcal{J}_\Delta) & = & \{(I,J) \in \mathcal{P}(\Delta) \times \mathcal{P}(\Delta) \,|\, I \subset J\} & \text{and where} \\ Morph((I,J),(K,L)) & \begin{cases} \text{consists of one element } \Phi_{I,J}^{K,L} \\ = \quad \emptyset \end{cases} & & \begin{array}{l} \text{if } I \subset K \subset L \subset J \\ \text{else.} \end{array} \end{array}$$

There is a unique and obvious composition of morphisms.

If $\mathcal{C}$ is another category we denote by $\mathcal{C}^{\mathcal{J}_\Delta}$ the category of functors $F : \mathcal{J}_\Delta \to \mathcal{C}$. The category $\mathcal{C}$ may be embedded as a full subcategory into $\mathcal{C}^{\mathcal{J}_\Delta}$ if we associate to every $c \in Ob(\mathcal{C})$ the constant functor $F_c : (I, J) \mapsto c, \quad \Phi_{I,J}^{K,L} \mapsto id_c$.

For $F \in \mathcal{C}^{\mathcal{J}_\Delta}$ we denote by $\varinjlim_{\mathcal{J}_\Delta} F \in Ob(\mathcal{C})$ the direct limit of F (if it exists). This means

$$Hom_{\mathcal{C}^{\mathcal{J}_\Delta}}(F, F_c) = Hom_{\mathcal{C}}(\varinjlim_{\mathcal{J}_\Delta} F, c) \qquad \text{for all } c \in Ob(\mathcal{C}). \tag{2}$$

Example 1.7. If $\mathcal{C}$ is the category of sets, one can construct $\varinjlim F$ in the following way: Let $X = \dot{\bigcup}_{j \in Ob(\mathcal{J}_\Delta)} F(j)$ be the disjoint union of all $F(j)$. Define an equivalence relation $\sim$ by: For $x \in F(j)$ and $x' \in F(j')$ we have $x \sim x'$ if and only if there are sequences

$j = j_0, j_1, \ldots, j_{2n} = j'$ of objects in $\mathcal{J}_\Delta$,
$x_i \in F(j_i) \quad i = 0, 1, \ldots, 2n$ of elements and
$\phi_{2i+1} : j_{2i+1} \to j_{2i}, \quad \phi_{2i+2} : j_{2i+1} \to j_{2i+2}, \quad i = 0, 1, \ldots n-1$ of morphisms
such that $x = x_0, \quad x' = x_{2n}, \quad F(\phi_{2i+1})(x_{2i+1}) = x_{2i}, \quad F(\phi_{2i+2})(x_{2i+1}) = x_{2i+2}$.

Then it is obvious that $X/\sim$ satisfies the defining property (2) of the direct limit $\varinjlim_{\mathcal{J}_\Delta} F$.

Example 1.8. If $(I, J) \mapsto X_{I,J}$ is a functor from $\mathcal{C}^{\mathcal{J}_\Delta}$ to the category $\mathcal{T}$ of topological spaces, we may construct $X = \varinjlim X_{I,J}$ as follows: The set X is the limit in the category of sets; it carries the quotient topology with respect to the map $\dot{\bigcup} X_{I,J} \to X$. This means that a subset $U \subset X$ is open if and only if all $\Phi_{I,J}^{-1}(U) \subset X_{I,J}$ are open. Here we denote by $\Phi_{I,J} : X_{I,J} \to X$ the natural map.

EXAMPLE 1.9. If $\Delta = \{e\}$ consists of just one element, then $\varinjlim F$ is the pushout in the following diagram

$$\begin{array}{ccc} F((\emptyset, \{e\})) & \longrightarrow & F((\{e\}, \{e\})) \\ \big\downarrow & & \big\downarrow \\ F((\emptyset, \emptyset)) & \longrightarrow & \varinjlim F \end{array}$$

For general Δ we can think about $\varinjlim_{\mathcal{J}_\Delta} F$ as a multi-pushout.

EXAMPLE 1.10. *Assume that there exists $J_0 \subset \Delta$ such that F fulfills the following properties:*

$$\begin{array}{lccll} (3) & F(I,J) & = & \emptyset & \textit{(the initial object in the category } \mathcal{C}) \quad \textit{if } J \not\subseteq J_0, \\ (4) & \Phi : F(I,J) & \to & F(I,K) & \textit{is an isomorphism for } I \subset K \subset J \subset J_0. \end{array}$$

Then we have $\varinjlim F = F(J_0, J_0)$.

Proof: For $c \in Ob(\mathcal{C})$ consider the obvious map:

$$\Psi : \quad Hom_{\mathcal{C}^{\mathcal{J}_\Delta}}(F, F_c) \quad \longrightarrow \quad Hom_{\mathcal{C}}(F(J_0, J_0), c).$$

Conversely if $\varphi : F(J_0, J_0) \to c$ is given, we can associate to it the transformation $\varphi_\Delta : F \to F_c$ such that we have for $I \subset J \subset J_0$:

$$\varphi_\Delta(I,J) : \quad F(I,J) \xrightarrow{\left(\Phi_{I,J_0}^{I,J}\right)^{-1}} F(I,J_0) \xrightarrow{\Phi_{I,J_0}^{J_0,J_0}} F(J_0,J_0) \xrightarrow{\varphi} c$$

and such that $\varphi_\Delta(I,J)$ is the unique map from the initial object $\emptyset$ to c if $J \not\subseteq J_0$. It is easy to check that φ_Δ is an element of $Hom_{\mathcal{C}^{\mathcal{J}_\Delta}}(F, F_c)$ and the only one satisfying $\Psi(\varphi_\Delta) = \varphi$. Therefore Ψ is an isomorphism. $\square$

EXAMPLE 1.11. Let $\mathcal{C}$ be the category of sets and $\mathcal{C}_\Delta$ the category, whose objects are pairs (A, π), where A is a set and π is a map from A to $\mathcal{P}(\Delta)$, and where morphisms $\phi : (A, \pi_A) \to (B, \pi_B)$ are maps $\phi : A \to B$ such that $\pi_B \circ \phi = \pi_A$. If $F : \mathcal{J}_\Delta \to \mathcal{C}_\Delta$ is a functor then we get for every $J_0 \subset \Delta$ a functor $F_{J_0} : \mathcal{J}_\Delta \to \mathcal{C}$, such that $F_{J_0}(I,J)$ is the inverse image $\pi^{-1}(J_0)$ inside the first component of $F(I,J)$. If we assume that F_{J_0} satisfies (3) and (4) for every $J_0 \subset \Delta$ then we can describe the direct limit as follows:

$$\varinjlim_{\mathcal{J}_\Delta} F \quad \simeq \quad \left(\dot{\bigcup_{J_0 \subset \Delta}} F_{J_0}(J_0, J_0), \pi \right)$$

where the map π takes the value J_0 on the component $F_{J_0}(J_0, J_0)$.

Distance functions and reduction theory

1.12. Absolute values of characters. The natural inclusion $S_I \subset P_I$ induces a natural restriction map for characters $r : X^*(P_I) \to X^*(S_I)$ which becomes an isomorphism after tensoring with $\mathbb{Q}$:

$$r_{\mathbb{Q}} : \ X^*(P_I) \otimes \mathbb{Q} \quad \xrightarrow{\sim} \quad X^*(S_I) \otimes \mathbb{Q}, \qquad \text{i.e.}$$

for $\chi \in X^*(S_I)$ there exists $N \in \mathbb{N}$ and $\tilde{\chi} \in X^*(P_I)$ such that $\chi = r(\tilde{\chi})^N$. Then we denote by

$$(5) \qquad |\chi| : \quad P_I(\mathbb{A}) \quad \to \quad \mathbb{R}^*_{>0} \qquad \text{the character} \\ \underline{g} \mapsto |\tilde{\chi}(\underline{g})|^{(1/N)},$$

where $\tilde{\chi} : P_I(\mathbb{A}) \to \mathbb{A}^* = \mathbb{G}_m(\mathbb{A})$ and the absolute value denotes the idele norm.

Definition 1.13. Distance functions. Let $K = K_\infty K_f \subset G(\mathbb{A})$ be a compact subgroup such that $K_\infty \subset G(\mathbb{R})$ is maximal compact and $K_f \subset G(\mathbb{A}_f)$ is open. A *distance function* with respect to $I \subset \Delta$, to a character $\chi \in X^*(S_I)$ and to K is a map

$$d = d_\chi = d_{\chi,K} : \quad G(\mathbb{A}) \quad \longrightarrow \quad \mathbb{R}^*_{>0} \qquad \text{such that}$$
$$(6) \qquad d_\chi(\underline{p}\underline{g}\underline{k}) \quad = \quad |\chi|(\underline{p}) \cdot d_\chi(\underline{g}) \qquad \text{for } \underline{p} \in P_I(\mathbb{A}),\ \underline{k} \in K,\ \underline{g} \in G(\mathbb{A}).$$

1.14. The Iwasawa-decomposition $G(\mathbb{R}) = P_0(\mathbb{R}) \cdot K_\infty = P_I(\mathbb{R}) \cdot K_\infty$ implies the isomorphism of double coset spaces:

$$P_I(\mathbb{A})\backslash G(\mathbb{A})/K \cong P_I(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f.$$

The right hand side is finite since it is the set of (open!) K_f-orbits in the compact quotient space $P_I(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$ (K_f acting via right translations on this space). Let $\{\underline{g}_1, \ldots, \underline{g}_n\}$ be a set of representatives for $P_I(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K_f$. Then we have a bijection between the set of all distance functions d with respect to I, χ, K and $(\mathbb{R}^*_{>0})^n$ given by $d \mapsto (d(\underline{g}_i))_{1 \leqslant i \leqslant n}$: We get the injectivity of this map from the construction of the $\underline{g}_i$ together with the characterizing property (6) of distance functions. The surjectivity may be deduced from the fact, that an equation $\underline{p}\underline{g}_i\underline{k} = \underline{p}'\underline{g}_i\underline{k}'$ implies $\underline{p}^{-1} \cdot \underline{p}' \in P_I(\mathbb{A}_f) \cap \underline{g}_i K_f \underline{g}_i^{-1}$ and therefore $|\chi|(\underline{p}) = |\chi|(\underline{p}')$, since $\mathbb{R}^*_{>0}$ contains no nontrivial compact subgroups, so that the image of the compact group $P_I(\mathbb{A}_f) \cap \underline{g}_i K_f \underline{g}_i^{-1}$ under $|\chi|$ is trivial. This implies that one always gets via (6) well defined distance functions if one prescribes their values at the $\underline{g}_i$.

We observe that any two distance functions $d_\chi, \tilde{d}_\chi$ with respect to the same triple I, χ, K are equivalent in the sense that there exist $c_1, c_2 \in \mathbb{R}^*_{>0}$ such that

$$c_1 \cdot d_\chi(\underline{g}) \quad \leqslant \quad \tilde{d}_\chi(\underline{g}) \quad \leqslant \quad c_2 \cdot d_\chi(\underline{g}) \qquad \text{for all } \underline{g} \in G(\mathbb{A}_f).$$

In fact we can put $c_1 = \min_{1 \leqslant i \leqslant n} \tilde{d}_\chi(\underline{g}_i) \cdot d_\chi(\underline{g}_i)^{-1}$ and $c_2 = \max_{1 \leqslant i \leqslant n} \tilde{d}_\chi(\underline{g}_i) \cdot d_\chi(\underline{g}_i)^{-1}$.

Example 1.15. (compare [Har1]) Let

$$\chi_I = \chi_{P_I} = \sum_{\alpha \in \Phi^+} \alpha \cdot \dim\left(Lie(U_I)_\alpha\right) \quad \in X^*(P_I) \subset X^*(S_I) \subset X^*(S_0).$$

For $g_\infty \in G(\mathbb{R})$ we denote by θ_{g_∞} the Cartan involution with respect to the compact group $g_\infty K_\infty g_\infty^{-1}$, by B_{g_∞} the bilinear form $B_{g_\infty}(X,Y) = -B(X, \theta_{g_\infty} Y)$, where B is the Killing form on $\mathfrak{g} = Lie(G(\mathbb{R}))$, and by $d_{g_\infty} u_\infty$ the Haar measure on $U_I(\mathbb{R})$ which is induced by the restriction of B_{g_∞} to $Lie(U_I)$. Furthermore let $d_{g_f} u_f$ be the Haar measure on $U_I(\mathbb{A}_f)$ such that $U_I(\mathbb{A}_f) \cap g_f K_f g_f^{-1}$ has volume 1. Then

$$d_{\chi_I}(\underline{g}) \quad = \quad vol_{d_{g_\infty} u_\infty \cdot d_{g_f} u_f}(U_I(\mathbb{Q}) \backslash U_I(\mathbb{A}))$$

defines a distance function on $G(\mathbb{A})$ with respect to χ_I and K.

Now we fix K and distance functions d_α with respect to $\{\alpha\} \subset \Delta,\ \alpha \in X^*(S_{\{\alpha\}}) \subset X^*(S_0)$ and K.

We may state the main theorems of reduction theory in the following form:

Theorem 1.16. *For every $I \subset \Delta$ there exists $C_1 = C_1(I) > 0$ such that for every $\underline{g} \in G(\mathbb{A})$ there is $\delta \in P_I(\mathbb{Q})$ satisfying*

$$d_\alpha(\delta \underline{g}) \quad > \quad C_1 \qquad \textit{for all } \alpha \in I.$$

Remark: We may replace $C_1(I)$ by the constant $C_1 = \min_{J \subset \Delta} C_1(J)$, which is independent of I.

Proof: It is easy to see that it suffices to prove the theorem for one chosen K and a fixed family of distance functions $(d_\alpha)_{\alpha \in \Delta}$. In the case $I = \Delta$, i.e. $P_I = G$ the claim is an immediate consequence of Borels theorem as stated in [God, Théorème 7]. For arbitrary $I \subset \Delta$ let $(x_j)_{j \in J(I)}$ with $x_j \in G(\mathbb{A}_f) \subset G(\mathbb{A})$ be a finite set of representatives for the double cosets $P_I(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/K_f$. For $j \in J(I)$ define $d_\alpha^j(p) = d_\alpha(px_j)$ as a distance function on $M_I = P_I/U_I$ with respect to $\{\alpha\}$, $\alpha \in X^*(S_{\{\alpha\}}) \subset X^*(S_0)$ and $K_j = x_j K_f x_j^{-1} \cap P_I(\mathbb{A})$. Applying Borels theorem again we get constants $C_1^j > 0$ such that for every $\underline{p} \in P_I(\mathbb{A})$ there exists $\delta \in P_I(\mathbb{Q})$ satisfying $d_\alpha^j(\delta \underline{p}) \quad > \quad C_1$ for all $\alpha \in I$. In view of the double coset decomposition $G(\mathbb{A}) = \bigcup_{j \in J(I)} P_I(\mathbb{A}) x_j K$ we now get the claim with $C_1(I) = \min_{j \in J(I)} C_1^j$. □

Theorem 1.17. *For every $C_1 > 0$ there exists $C_2 > C_1$ such that we have for $I \subset \Delta$, $\delta \in G(\mathbb{Q})$, $\underline{g} \in G(\mathbb{A})$:*

$$\begin{array}{lll} \textit{If} & d_\alpha(\delta \underline{g}), d_\alpha(\underline{g}) > C_1 & \textit{for all } \alpha \in \Delta \\ \textit{and} & d_\alpha(\delta \underline{g}) > C_2 & \textit{for all } \alpha \in \Delta - I \\ \textit{then} & \delta \in P_I(\mathbb{Q}). & \end{array}$$

Proof: This is a reformulation of [Fr, Theorem 1(3)].

The components

1.18. The spaces $X_{I,J}$. Now we fix some maximal compact subgroup $K_\infty^m \subset G(\mathbb{R})$ satisfying the conditions of lemma 1.4 and some open normal subgroup $K_\infty \subset K_\infty^m$ satisfying $G(\mathbb{R}) = P_0(\mathbb{R}) \cdot K_\infty$.

Let Z_∞ be the connected component of the group of $\mathbb{R}$-valued points of the maximal $\mathbb{R}$-split subtorus of the center $Z_G/\mathbb{R}$.

For $J \subset \Delta$ we fix the notations

$$K_\infty^J \quad = \quad P_J(\mathbb{R}) \cap K_\infty \quad = \quad M_J(\mathbb{R}) \cap K_\infty.$$

Let the group (comp. 1.2, 1.5)

$$\begin{aligned} A_J &= \left(S_J(\mathbb{R}) \cap G^{(1)}(\mathbb{R})\right)^\circ \qquad \text{act on the space} \\ Y_J &:= \left\{ (e_\alpha)_{\alpha\in\Delta} \in \mathbb{R}^\Delta \middle| \, e_\alpha \in \{+1,-1\} \ \text{ for } \alpha \in J \right\} \subset \mathbb{R}^\Delta \qquad \text{via the roots} \\ a \cdot (e_\alpha)_{\alpha\in\Delta} &= (\alpha(a) \cdot e_\alpha)_{\alpha\in\Delta}\,. \end{aligned}$$

For $I \subset J$ the group A_J acts on the space $P_I(\mathbb{R})/K_\infty^I \cdot Z_\infty$ via right translations, since $A_J \subset S_J(\mathbb{R}) \subset S_I(\mathbb{R})$ centralizes $K_\infty^I \subset M_I(\mathbb{R})$. For $I \subset J$ we can form the quotient space

$$X_{I,J} \quad := \quad G(\mathbb{Q}) \ \times_{P_I(\mathbb{Q})} \ (P_I(\mathbb{R})/K_\infty^I \cdot Z_\infty) \ \times_{A_J} \ Y_J. \tag{7}$$

More precisely we consider the quotient of $G(\mathbb{Q}) \times P_I(\mathbb{R}) \times Y_J$ under the equivalence relation $(\gamma, p, y) \sim (\gamma', p', y')$ iff there exist $\delta \in P_I(\mathbb{Q}),\ a \in A_J,\ k \in K_\infty^I \cdot Z_\infty$ such that $\gamma' = \gamma\delta,\ p' = \delta^{-1} \cdot p \cdot k \cdot a,\ y' = a \cdot y$.

Lemma 1.19. *For $I \subset I'$ the canonical map*

$$P_I(\mathbb{R})/K_\infty^I \cdot Z_\infty \quad \to \quad P_{I'}(\mathbb{R})/K_\infty^{I'} \cdot Z_\infty$$

is an isomorphism.

Proof: The corresponding map with Z_∞ replaced by $\{1\}$ is injective by the definition of K_∞^I. Since the composite map $P_0(\mathbb{R}) \to P_0(\mathbb{R})/K_\infty^\emptyset \to G(\mathbb{R})/K_\infty$ is surjective by assumption, the claim is now clear for Z_∞ replaced by $\{1\}$ and then obviously also for the original Z_∞. $\square$

1.20. The manifold structure of $X_{I,J}$. By the above lemma we can replace $P_I(\mathbb{R})/K_\infty^I \cdot Z_\infty$ by the corresponding space $P_J(\mathbb{R})/K_\infty^J \cdot Z_\infty$ in (7). We denote by ${}^0P_J$ the intersection of the kernels of all χ^2, where χ ranges over all characters $\chi : P_J \to P_J/Z_G \to \mathbb{G}_m$. Then there is a unique decomposition $P_J(\mathbb{R}) = {}^0P_J(\mathbb{R}) \rtimes A_J$. We remark

$$\left(P_J(\mathbb{R})/K_\infty^J Z_\infty \times \mathbb{R}^{\Delta-J} \times \{\pm1\}^J\right)/A_J \quad \simeq \quad {}^0P_J(\mathbb{R})/K_\infty^J Z_\infty \times \mathbb{R}^{\Delta-J} \times \{\pm1\}^J\,.$$

Using a set of representatives for $G(\mathbb{Q})/P_I(\mathbb{Q})$ in $G(\mathbb{Q})$ we can thus identify

$$X_{I,J} \quad = \quad (G(\mathbb{Q})/P_I(\mathbb{Q})) \times \quad {}^0P_J(\mathbb{R})/K_\infty^J Z_\infty \times \mathbb{R}^{\Delta-J} \times \{\pm1\}^J\,.$$

Since ${}^0P_J(\mathbb{R})/K_\infty^J Z_\infty$ is a submanifold of the symmetric space $P_J(\mathbb{R})/K_\infty^J Z_\infty \simeq G(\mathbb{R})/K_\infty Z_\infty$, we get a structure of $X_{I,J}$ as a differentiable manifold, if we equip $G(\mathbb{Q})/P_I(\mathbb{Q})$ and $\{\pm1\}^J$ with the discrete topology, $\mathbb{R}^{\Delta-J}$ with the usual structure as a manifold and then take the product structure.

1.21. Functoriality for $X_{I,J}$. The isomorphism of lemma 1.19 induces surjective maps which are coverings in the category of differentiable manifolds:

$$\pi : \quad X_{I,J} \quad \twoheadrightarrow \quad X_{I',J} \qquad \text{for } I \subset I' \subset J.$$

If $I \subset J' \subset J$ we get an injective map (injective by the definition of A_J):

$$i: \quad X_{I,J} \quad \hookrightarrow \quad X_{I,J'}$$

which is induced from the inclusion $Y_J \hookrightarrow Y_{J'}$. For $I \subset I' \subset J' \subset J$ we get a commutative diagram:

$$\begin{array}{ccc} X_{I,J} & \longrightarrow & X_{I',J} \\ \downarrow & & \downarrow \\ X_{I,J'} & \longrightarrow & X_{I',J'} \end{array} \tag{8}$$

Consequently we get a functor $X_{..}$ from the category $\mathcal{J}_\Delta$ into the category of topological spaces. We denote by X the direct limit over all spaces $X_{I,J}$ where $I \subset J \subset \Delta$:

$$X \quad = \quad \varinjlim X_{I,J}.$$

1.22. The group $\mathcal{H}_\infty$. We introduce the group

$$\mathcal{H}_\infty \quad = \quad (K_\infty^m \cap P_0(\mathbb{R}))/K_\infty^\emptyset \quad = \quad (K_\infty^m \cap P_0(\mathbb{R}))/(K_\infty \cap P_0(\mathbb{R})).$$

For all $I \subset \Delta$ we have a canonical isomorphism $\iota_I : \mathcal{H}_\infty \tilde{\rightarrow} (K_\infty^m \cap P_I(\mathbb{R}))/K_\infty^I$: Injectivity of ι_I is implied by $K_\infty^I \cap P_0(\mathbb{R}) = K_\infty^\emptyset$. For the surjectivity observe that each $g_\infty \in K_\infty^m \cap P_I(\mathbb{R})$ can be written in the form $g_\infty = p_\infty \cdot k_\infty$ with $p_\infty \in P_0(\mathbb{R})$ and $k_\infty \in K_\infty$. But then also $p_\infty = g_\infty \cdot k_\infty^{-1} \in K_\infty^m$, i.e. $p_\infty \in K_\infty^m \cap P_0(\mathbb{R})$ and therefore $k_\infty = p_\infty^{-1} g_\infty \in P_I(\mathbb{R}) \cap K_\infty = K_\infty^I$.

Since each element in $K_\infty^m \cap P_0(\mathbb{R})$ normalizes the groups K_∞^I, Z_∞ and A_I the group $\mathcal{H}_\infty$ acts by right translations on the spaces $X_{I,J}$ and these actions are compatible with the maps π and i.

1.23. Sign maps. Next we introduce the sign space $\Sigma^\Delta = \{-1, 0, +1\}^\Delta$ and the sign map $sign : \mathbb{R}^\Delta \to \Sigma^\Delta$, which is component for component the usual sign map.

For $y = (y_\alpha)_{\alpha \in \Delta} \in \mathbb{R}^\Delta$ we call $supp(y) = \{\alpha \in \Delta | y_\alpha \neq 0\}$ its support. This definition also applies to the sign space $\Sigma^\Delta \subset \mathbb{R}^\Delta$, such that we have $supp(y) = supp(sign(y))$ for $y \in \mathbb{R}^\Delta$.

Since the action of $A_J \subset A_\emptyset$ on $\mathbb{R}^\Delta$ fixes the signs we get sign maps

$$sign: \quad X_{I,J} \quad \longrightarrow \quad \Sigma^\Delta \qquad \text{and} \qquad sign: \quad X \quad \longrightarrow \quad \Sigma^\Delta$$

For $I \subset J' \subset J$ we have

$$X_{I,J} \quad \cong \quad \left\{ x \in X_{I,J'} \;\middle|\; supp(x) \supset J \right\}.$$

We define for $J \subset \Delta$:

$$\begin{aligned} E_J \quad &:= \quad \{x \in X \mid supp(x) = J\}, \qquad \text{so that} \\ X \quad &= \quad \dot{\bigcup_{J \subset \Delta}} E_J. \qquad \text{We have} \end{aligned}$$

$$\begin{aligned} \{x \in X_{I,J} | \; supp(x) = J_0\} \quad &= \quad \emptyset \qquad \text{for } J \not\subset J_0 \qquad \text{and} \\ \{x \in X_{I,J} | \; supp(x) = J_0\} \quad &\cong \quad \{x \in X_{I,J_0} | \; supp(x) = J_0\} \qquad \text{for} \quad I \subset J \subset J_0 \end{aligned}$$

We consider $X_{I,J}$ as a set together with the support map to $\mathcal{P}(\Delta)$. The functor $(I,J) \mapsto (X_{I,J}, sign)$ satisfies the conditions of example 1.11 above. Then it is easy to see:

$$\begin{aligned} E_{J_0} &= \varinjlim_{I \subset J \subset J_0} \{x \in X_{I,J} |\ supp(x) = J_0\} \cong \{x \in X_{J_0,J_0} |\ supp(x) = J_0\} \\ &\cong G(\mathbb{Q}) \times_{P_{J_0}(\mathbb{Q})} P_{J_0}(\mathbb{R})/K_\infty^{J_0} \cdot A_{J_0} \cdot Z_\infty \times \{-1,+1\}^{J_0} \times \{0\}^{\Delta - J_0}. \end{aligned}$$

1.24. The sign group S^Δ. The set $S^\Delta = \{-1,+1\}^\Delta$ forms a group under componentwise multiplication. It acts on $\mathbb{R}^\Delta$, Σ^Δ, Y_J for all $J \subset \Delta$ by componentwise multiplication and therefore also on all $X_{I,J}$. We write the action of S^Δ as a right action. The sign map and all maps π, i are S^Δ-equivariant, so that S^Δ acts on X. S^Δ may be identified with the set of all subsets of Δ: For $J \subset \Delta$ we denote by $s_J = (r_\alpha)_{\alpha\in\Delta}$ the element with $r_\alpha = -1 \Leftrightarrow \alpha \in J$. It is rather obvious that

$$X^{s_J} = \{x \in X | x \cdot s_J = x\} = \dot{\bigcup_{I\cap J=\emptyset}} E_I.$$

1.25. The quotients $X_{I,J}(K_f)$ and $X(K_f)$. For a compact open subgroup $K_f \subset G(\mathbb{A}_f)$ we introduce the spaces:

$$\begin{aligned} X_{I,J}(K_f) &= G(\mathbb{Q})\backslash X_{I,J} \times G(\mathbb{A}_f)/K_f \qquad \text{and} \\ X(K_f) &= G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \varinjlim_{I,J} X_{I,J}(K_f). \end{aligned}$$

We have a canonical identification

$$X_{I,J}(K_f) = P_I(\mathbb{Q})\backslash \left(P_I(\mathbb{R})/K_\infty^I Z_\infty \times_{A_J} Y_J\right) \times G(\mathbb{A}_f)/K_f.$$

We fix an open compact subgroup $Z_f \subset Z_G(\mathbb{A}_f)$ (which will be assumed to be sufficiently small later). In the following we shall consider only such K_f, which satisfy

$$K_f \cap Z_G(\mathbb{A}_f) = Z_f. \tag{9}$$

The set of all K_f satisfying (9) is invariant under conjugation and under intersecting its members. If $K_f = K_f^1 \cdot Z_f$ for an open compact subgroup $K_f^1 \subset G^{(1)}(\mathbb{A}_f)$ then (9) is equivalent to the condition $K_f^1 \cap Z_G(\mathbb{A}_f) \subset Z_f$. In case $K_f^1 = \prod_p K_p^1$ and $Z_f = \prod_p Z_p$ the local conditions $K_p^1 \cap Z_G(\mathbb{Q}_p) \subset Z_p$ have to be checked only for those finitely many p where Z_p is not maximal compact in $Z_G(\mathbb{Q}_p)$. We define the group

$$\zeta = Z_G(\mathbb{Q}) \cap (K_\infty \cdot Z_\infty \times Z_f).$$

It acts trivially (from the left) on each $X_{I,J} \times G(\mathbb{A}_f)/K_f$ and on $X \times G(\mathbb{A}_f)/K_f$. We now assume

(Ass_{K_f}) For all $g_f \in G(\mathbb{A}_f)$, $g_\infty \in G(\mathbb{R})$ we have
$$\left(g_f K_f g_f^{-1} \cdot g_\infty K_\infty Z_\infty g_\infty^{-1}\right) \cap G(\mathbb{Q}) = \zeta \quad .$$

Lemma 1.26. *Each K_f satisfying (9) contains open subgroups satisfying (Ass_{K_f}).*

Proof: By shrinking K_f we may assume $K_f = K_f^1 \cdot Z_f$ for an open compact subgroup $K_f^1 = \prod_p K_p^1 \subset G^{(1)}(\mathbb{A}_f)$. We claim that we are done, if we replace some K_p^1 by an open pro-p-subgroup (which will be denoted by the same symbol): Let $\tilde{\zeta} = \left(g_f K_f g_f^{-1} \cdot g_\infty K_\infty Z_\infty g_\infty^{-1}\right) \cap G(\mathbb{Q})$. If n denotes the order of the finite algebraic group $G^{(1)} \cap Z_G$, then there exists an isogeny of tori $\omega : G/G^{(1)} \rightarrow Z_G$ such that $\tilde{\pi} \circ \omega$ is the multiplication by n, where $\pi : G \twoheadrightarrow G/G^{(1)}$ is the canonical projection and $\tilde{\pi} : Z_G \hookrightarrow G \twoheadrightarrow G/G^{(1)}$ the induced isogeny with kernel $G^{(1)} \cap Z_G$. For $\gamma \in \tilde{\zeta}$ we get $\gamma^n = \sigma \cdot \rho$ with $\sigma = \omega(\pi(\gamma)) \in Z_G(\mathbb{Q}) \cap \tilde{\zeta} = \zeta$ and $\rho \in G^{(1)}(\mathbb{Q}) \cap \tilde{\zeta}$. The rational element ρ is now of finite order, since its archimedian component lies in the compact group $g_\infty K_\infty g_\infty^{-1}$. But the p-component of ρ is contained in the product of the torsion free pro-p-group $g_p \cdot K_p^1 \cdot g_p^{-1}$ and a subgroup of the finite central group $(G^{(1)} \cap Z_G)(\mathbb{Q}_p)$. Therefore ρ must be central, i.e. $\rho \in Z_G(\mathbb{Q}) \cap G^{(1)}(\mathbb{Q}) \cap \tilde{\zeta} = \zeta$ and thus $\gamma^n \in \zeta$. Looking again at the p-component and using that $g_p \cdot K_p^1 \cdot g_p^{-1}$ is a pro-p-group we conclude that already γ must be central, i.e. $\gamma \in \zeta$. □

Lemma 1.27. *The action of $G(\mathbb{Q})/\zeta$ on each $X_{I,J} \times G(\mathbb{A}_f)/K_f$ and therefore on $X \times G(\mathbb{A}_f)/K_f$ is free of fixed points.*

Proof: Let $((\gamma, p, y), g_f)$ be a representative of an element of $X_{I,J} \times G(\mathbb{A}_f)/K_f$ which is a fixed point under $\delta \in G(\mathbb{Q})$. Then there exist $\rho \in P_I(\mathbb{Q})$, $k_\infty \in K_\infty^I$, $a \in A_I$, $z_\infty \in Z_\infty$, $k_f \in K_f$ such that

$$(\delta\gamma, p, y, \delta g_f) \quad = \quad (\gamma\rho, \rho^{-1} p k_\infty z_\infty a, ay, g_f k_f).$$

This means $\rho = \gamma^{-1}\delta\gamma = \gamma^{-1} g_f k_f g_f^{-1} \gamma \in \gamma^{-1} g_f K_f g_f^{-1} \gamma \cap P_I(\mathbb{A}_f)$. Since the latter is a compact subgroup of $P_I(\mathbb{A}_f)$, its image under the absolute value of each root $\alpha \in \Delta - I$ must be 1. Thus $|\alpha(\rho)|_\infty = |\alpha(\rho)|_f^{-1} = 1$. On the other side we have $a = z_\infty^{-1} k_\infty^{-1} p^{-1} \rho p$ and therefore $\alpha(a) = |\alpha(a)| = |\alpha(z_\infty)^{-1}| \cdot |\alpha(k_\infty)^{-1}| \cdot |\alpha(\rho)|_\infty = 1$ for all $\alpha \in \Delta - I$. Since we know this already for $\alpha \in J \supset I$ we get $a \in A_\Delta = \{1\}$. Now $\rho \in G(\mathbb{Q})$, $\rho \in \gamma^{-1} g_f K_f g_f^{-1} \gamma$ and $\rho \in p K_\infty Z_\infty p^{-1}$. Therefore $\rho \in \zeta$ by assumption Ass_{K_f}. Since ρ is central the equation $\delta\gamma = \gamma\rho$ implies $\delta = \rho \in \zeta$, i.e. δ represents the identity in $G(\mathbb{Q})/\zeta$. □

1.28. For each distance function $d_\alpha : G(\mathbb{A}) \rightarrow \mathbb{R}_{>0}^*$ associated to $\alpha \in \Delta$ we define a function $D_\alpha : X_{\emptyset,J} \rightarrow \mathbb{R}_{\geqslant 0}$ by

$$D_\alpha(\gamma, p_\infty, y) \quad = \quad d_\alpha((p_\infty, \gamma_f^{-1}))^{-1} \cdot |y_\alpha|.$$

This is well defined since we have $|\alpha|(\delta_\infty, \delta_f)| = 1$ for $\delta \in P_0(\mathbb{Q})$ by the product formula for the norm, so that

$$\begin{aligned} D_\alpha(\gamma\delta, \delta^{-1} p_\infty a, ay) &= d_\alpha(\delta_\infty^{-1} p_\infty a, \delta_f^{-1} \gamma_f^{-1})^{-1} \cdot |\alpha(a) \cdot y_\alpha| \\ &= |\alpha|(\delta_\infty, \delta_f)|^{-1} \cdot |\alpha|(a)^{-1} \cdot d_\alpha(p_\infty, \gamma_f^{-1})^{-1} \cdot |\alpha(a)| \cdot |y_\alpha| = D_\alpha(\gamma, p_\infty, y). \end{aligned}$$

In the same way we consider the function

$$\begin{aligned} D_\alpha : \quad X_{\emptyset,J}(K_f) &\quad = \quad G(\mathbb{Q}) \backslash (X_{\emptyset,J} \times G(\mathbb{A}_f)/K_f) \quad \rightarrow \mathbb{R}_{\geqslant 0} \qquad \text{defined by} \\ D_\alpha((\gamma, p_\infty, y), g_f) &\quad = \quad d_\alpha((p_\infty, \gamma_f^{-1} g_f))^{-1} \cdot |y_\alpha|. \end{aligned}$$

Gluing together

1.29. The neighborhoods $\mathcal{U}_{I,J}$ and $\mathcal{V}_{I,J}$. Let C_1 be a constant as in theorem 1.16 and $C_2 > C_1$ be an associated constant as in theorem 1.17. We define $\mathcal{U}_{I,J} \subset X_{\emptyset,J}$ by:

$$\mathcal{U}_{I,J} \quad = \quad \left\{x \in X_{\emptyset,J} |\ D_\alpha(x) < C_1^{-1} \text{ for } \alpha \in I, \quad D_\alpha(x) < C_2^{-1} \text{ for } \alpha \in \Delta - I\right\}$$

For $I \subset J$ we denote by $\mathcal{V}_{I,J} \subset X_{I,J}$ the image of $\mathcal{U}_{I,J}$ under the projection $X_{\emptyset,J} \to X_{I,J}$. We recall from 1.23:

$$X \quad = \quad \dot{\bigcup_{J_0 \subset \Delta}} \{x \in X_{J_0,J_0} |\ supp(x) = J_0\}$$

The relation $C_1^{-1} > C_2^{-1}$ implies $\mathcal{U}_{I,J} \subset \mathcal{U}_{K,J}$ for $I \subset K \subset L \subset J$. Together with the canonical inclusion $\mathcal{U}_{K,J} \subset \mathcal{U}_{K,L}$ this gives $\mathcal{U}_{I,J} \subset \mathcal{U}_{K,L}$ and induces a map:

$$\Phi_{I,J}^{K,L}: \quad \mathcal{V}_{I,J} \to \mathcal{V}_{K,L}.$$

Lemma 1.30. *The maps $\Phi_{I,J}^{K,L}$ are injective.*

Proof: Let $\Phi_{I,J}^{K,L}(x_1) = \Phi_{I,J}^{K,L}(x_2)$ where $x_1, x_2 \in \mathcal{V}_{I,J}$. Write $x_i = \Phi_{\emptyset,J}^{I,J}(\tilde{x}_i)$ where $\tilde{x}_i = (\gamma_i, p_i, y_i) \in \mathcal{U}_{I,J}$. Since $\Phi_{\emptyset,J}^{K,L}(\gamma_1, p_1, y_1) = \Phi_{\emptyset,J}^{K,L}(\gamma_2, p_2, y_2)$ there exists $\delta \in P_K(\mathbb{Q}),\ a \in A_L$ satisfying $\gamma_2 = \gamma_1 \cdot \delta^{-1},\ p_2 = \delta p_1 a,\ y_2 = a \cdot y_1$. Since the α-components of y_i equal ± 1 for $\alpha \in J$ we get $\alpha(a) = 1$ for $\alpha \in J$, i.e. $a \in A_J$. There exists $a_2 \in A_J$ such that $y_0 := a_2 \cdot y_1$ has components $-1, 0, +1$ and such that $d_\alpha(p_1 \cdot a_2, (\gamma_1)_f^{-1}) > C_2$ and $d_\alpha(p_2 \cdot a^{-1} \cdot a_2, (\gamma_2)_f^{-1}) > C_2$ for all α with $(y_1)_\alpha = 0 = (y_2)_\alpha$. Then we have $x_i = \Phi_{\emptyset,J}^{I,J}(x_i')$ where $x_1' = (\gamma_1, p_1 \cdot a_2, y_0) =: (\gamma_1, p_1', y_0)$ and $x_2' = (\gamma_2, p_2 \cdot a^{-1} \cdot a_2, y_0) =: (\gamma_2, p_2', y_0)$. We have $(p_2', (\gamma_2)_f^{-1}) = \delta \cdot \underline{g}$ for $\underline{g} = (p_1', (\gamma_1)_f^{-1})$ and $d_\alpha(\underline{g}), d_\alpha(\delta \underline{g}) > C_2$ for $\alpha \in \Delta - I$, $d_\alpha(\underline{g}), d_\alpha(\delta \underline{g}) > C_1$ for $\alpha \in I$. By theorem 1.17 we get $\delta \in P_I(\mathbb{Q})$. This means $x_1 = x_2$ in $X_{I,J}$. □

Lemma 1.31. *$\mathcal{V}_{J,J}$ contains $\{x \in X_{J,J} | supp(x) = J\}$.*

Proof: If $x = (\gamma, p, y) \in X_{J,J}$ has support J we can find by theorem 1.16 some $\delta \in P_J(\mathbb{Q})$ such that $d_\alpha(\delta \cdot \underline{g}) > C_1$ for all $\alpha \in J$ where $\underline{g} = (p, (\gamma_f)^{-1})$. Then $x' = (\gamma\delta^{-1}, \delta p, y) \in X_{\emptyset,J}$ lies in $\mathcal{U}_{J,J}$ and has x as its image in $X_{J,J}$ (observe $D_\alpha(x') = 0$ for $\alpha \notin J$ and $|y_\alpha| = 1$ for $\alpha \in J$). □

Lemma 1.32. *The composite map $\quad \mathcal{V}_{I,J} \xrightarrow{i} X_{I,J} \xrightarrow{\pi} X \quad$ is injective.*

Proof: The support of each $x \in \mathcal{V}_{I,J}$ contains J. Consider the following commutative diagram for $J \subset L$:

$$\begin{array}{ccc}
\{x \in \mathcal{V}_{I,L} | supp(x) = L\} & \longrightarrow & \{x \in \mathcal{V}_{L,L} | supp(x) = L\} = \{x \in X_{L,L} | supp(x) = L\} \\
\downarrow \wr & & \downarrow \wr \\
\{x \in \mathcal{V}_{I,J} | supp(x) = L\} & \longrightarrow & \{x \in X_{I,J} | supp(x) = L\} = \{x \in X | supp(x) = L\}
\end{array}$$

This implies the injectivity. $\square$

From now on we may and will identify $\mathcal{V}_{I,J}$ with its image in X.

Lemma 1.33.

$$\mathcal{V}_{I,J} \cap \mathcal{V}_{K,L} \quad = \quad \mathcal{V}_{I\cap K, J\cup L} \qquad \text{for } I \subset J,\ K \subset L$$

Proof: The inclusion $\supset$ being trivial we assume $x \in \mathcal{V}_{I,J} \cap \mathcal{V}_{K,L}$, i.e. there are $x_1 = (\gamma_1, p_1, y_1) \in \mathcal{U}_{I,J} \subset X_{\emptyset,J}$ and $x_2 = (\gamma_2, p_2, y_2) \in \mathcal{U}_{K,L} \subset X_{\emptyset,L}$ having the same image $x \in X$. If $S = supp(x_1) = supp(x_2)$ denotes the support of x, then by example 1.11 above x_1 and x_2 become equal in $X_{S,S}$, i.e. there exist $\delta \in P_S(\mathbb{Q})$ and $a \in A_S$ such that

$$\gamma_2 = \gamma_1 \cdot \delta^{-1}, \quad p_2 = \delta \cdot p_1 \cdot a, \quad y_2 = a \cdot y_1. \tag{10}$$

We may assume that $(y_1)_\alpha = (y_2)_\alpha = \pm 1$ for $\alpha \in S$. Since $J, L \subset S$ we have $A_S \subset A_J$ and may assume replacing x_1 by $(\gamma_1, p_1 \cdot a, a \cdot y_1)$ that we have $a = 1$ in (10). We put $\underline{g} = (p_1, (\gamma_1)_f^{-1})$. After modifying p_1 and p_2 by an element of A_S from the right we may assume $d_\alpha(\underline{g}) > C_2$, $d_\alpha(\delta\underline{g}) > C_2$ for $\alpha \notin S$. Then the assumption on x_1 and x_2 may be restated:

$$\begin{array}{llllllll} d_\alpha(\underline{g}) & > & C_1 & \text{for } \alpha \in I & d_\alpha(\underline{g}) & > & C_2 & \text{for } \alpha \in \Delta - I \\ d_\alpha(\delta\underline{g}) & > & C_1 & \text{for } \alpha \in K & d_\alpha(\delta\underline{g}) & > & C_2 & \text{for } \alpha \in \Delta - K. \end{array}$$

This implies $\delta \in P_K(\mathbb{Q})$, $\delta^{-1} \in P_I(\mathbb{Q})$ by theorem 1.17 a nd therefore $\delta \in P_I(\mathbb{Q}) \cap P_K(\mathbb{Q}) = P_{I\cap K}(\mathbb{Q})$. So we may assume $x_1 = x_2 \in \mathcal{U}_{I,J} \cap \mathcal{U}_{K,L} = \mathcal{U}_{I\cap K, J\cup L}$ and the claim is proven. $\square$

1.34. Continuation of example 1.11 For $X = \varinjlim X_{I,J}$ we denote by $\Phi_{I,J} : X_{I,J} \to X$ the canonical map. For a subset $\mathcal{U}_{I_0,J_0} \subset X_{I_0,J_0}$ we may compute the sets

$$\mathcal{U}^\infty_{I,J} \quad := \quad (\Phi_{I,J})^{-1}\left(\Phi_{I_0,J_0}(\mathcal{U}_{I_0,J_0})\right) \quad \subset \quad X_{I,J}$$

in the following way: We put

$$\mathcal{U}^0_{I,J} \quad = \quad \begin{cases} \emptyset & \text{for } (I,J) \neq (I_0,J_0), \\ \mathcal{U}_{I_0,J_0} & \text{for } (I,J) = (I_0,J_0), \end{cases}$$

and then inductively for $j \geqslant 0$:

$$\begin{aligned} \mathcal{U}^{2j+1}_{I,J} \quad &:= \quad \bigcup_{I\subset K\subset L\subset J} \left(\Phi^{K,L}_{I,J}\right)^{-1} \left(\mathcal{U}^{2j}_{K,L}\right) \\ \mathcal{U}^{2j+2}_{I,J} \quad &:= \quad \bigcup_{K\subset I,\, J\subset L} \Phi^{I,J}_{K,L} \left(\mathcal{U}^{2j+1}_{K,L}\right). \end{aligned}$$

Then we get:

$$\mathcal{U}^\infty_{I,J} \quad = \quad \bigcup_{j\geqslant 0} \mathcal{U}^j_{I,J}.$$

Recall from 1.8 the description of the topology on X, if $X_{\cdot,\cdot}$ is a functor to the category of topological spaces.

Lemma 1.35. *If the maps $\Phi_{I,J}^{K,L}$ are all open then the maps $\Phi_{I,J}$ are open too.*

We have to show that all $\mathcal{U}_{I,J}^{\infty} \subset X_{I,J}$ are open if $\mathcal{U}_{I_0,J_0} \subset X_{I_0,J_0}$ is open. But by induction all $\mathcal{U}_{I,J}^{j}$ are open for all $j \geqslant 0$ and so is their union $\mathcal{U}_{I,J}^{\infty}$. □

Now we associate to $X_{I,J}$ the quotient topology with respect to the actions of $P_I(\mathbb{Q}), Z_\infty$ and A_J where $P_I(\mathbb{Q}) \subset G(\mathbb{Q})$ carries the discrete topology and the other two factors the usual topology. Then it is obvious that the maps $\Phi_{I,J}^{K,L}$ are open.

We conclude from lemma 1.31 that the $\mathcal{V}_{I,J}$ form an open cover of X and that already the $\mathcal{V}_{I,I}$ form an open cover.

Lemma 1.36. *For $\tilde{x} \in \mathcal{U}_{I,J}$ and $\beta \in J$ there exists a constant $C_0 = C_0(I, \beta, \tilde{x}) > 0$ depending continuously on $\tilde{x}$ such that $D_\beta(\delta\tilde{x}) \geqslant C_0$ for all $\delta \in P_J(\mathbb{Q})$ with $\delta\tilde{x} \in \mathcal{U}_{J,J}$.*

Proof: Let $\tilde{x}$ be represented by (γ, p_∞, y). Put $\underline{g} = (p_\infty, \gamma_f^{-1}) \in P_0(\mathbb{R}) \times G(\mathbb{A}_f)$. After modifying the representative we may assume that $y_\alpha \in \{-1, 0, +1\}$ for all $\alpha \in \Delta$, especially $|y_\alpha| = 1$ for $\alpha \in J$, and that $d_\alpha(\underline{g}) > C_2$ for $\alpha \in \Delta - J$. We have to prove:

$$d_\beta(\delta\underline{g}) \leqslant C_0^{-1} \qquad \text{for all } \delta \in P_J(\mathbb{Q}) \text{ with } \delta\tilde{x} \in \mathcal{U}_{J,J}.$$

Let $\delta \in P_J(\mathbb{Q})$ with $\delta\tilde{x} \in \mathcal{U}_{J,J}$. We may assume $d_\alpha(\delta g) > C_2$ for all $\alpha \in \Delta - J$ by modifying p_∞ once more without changing $d_\alpha(\delta\underline{g})$ and $d_\alpha(\underline{g})$ for $\alpha \in J$: if $|y_\alpha| = 1$ then the condition $D_\alpha(\delta\tilde{x}) < C_2^{-1}$ is equivalent to $d_\alpha(\delta g) > C_2$, while for $y_\alpha = 0$ we can modify p_∞ by multiplication with a suitable element of A_J, which does not change the other values of distance functions.

For $\beta \in J$ there exists a character $\chi_{J-\{\beta\},\beta} \in X^*(P_{J-\{\beta\}}) \otimes \mathbb{Q}$ whose restriction to $X^*(S_{J-\{\beta\}})$ coincides with the restriction of β. In $X^*(S_0) \otimes \mathbb{Q}$ we have a relation of the type

$$\chi_{J-\{\beta\},\beta} \quad = \quad \beta + \sum_{\alpha \in J-\{\beta\}} c_{J,\beta,\alpha} \cdot \alpha \qquad \text{with } c_{J,\beta,\alpha} \in \mathbb{Q}.$$

Assume $d_\beta(\delta\underline{g}) > C_2$. This implies $\delta \in P_{J-\{\beta\}}(\mathbb{Q})$ by theorem 1.17 and furthermore:

$$\begin{aligned}
d_{\chi_{J-\{\beta\},\beta}}(\underline{g}) &= d_{\chi_{J-\{\beta\},\beta}}(\delta\underline{g}) \qquad \text{which can be rewritten} \\
d_\beta(\delta\underline{g}) &= d_\beta(\underline{g}) \cdot \prod_{\alpha \in J-\{\beta\}} \left(\frac{d_\alpha(\delta\underline{g})}{d_\alpha(\underline{g})}\right)^{-c_{J,\beta,\alpha}} \\
&< C_1^{-\Sigma_{\alpha \in J-\{\beta\}} c_{J,\beta,\alpha}} \cdot d_\beta(\underline{g}) \cdot \prod_{\alpha \in J-\{\beta\}} d_\alpha(\underline{g})^{c_{J,\beta,\alpha}} \quad =: \quad C_3.
\end{aligned}$$

Thus we have proved $d_\beta(\delta\underline{g}) \leqslant \max(C_2, C_3)$. If we put $C_0 := (\max(C_2, C_3))^{-1}$ we get the claim. □

Proposition 1.37. *The space X is Hausdorff.*

Proof: Let us assume that $\tilde{x} \in \mathcal{U}_{I,I}$ maps to $x \in \mathcal{V}_{I,I}$ and $\tilde{y} \in \mathcal{U}_{J,J}$ maps to $y \in \mathcal{V}_{J,J}$, that $x \neq y$, $supp(x) = I$, $supp(y) = J$. If $I = J$ then we can use the fact that $\mathcal{V}_{I,I} \subset X_{I,I}$ is

Hausdorff, so let us assume that $I \neq J$, without loss of generality $\alpha \notin I$, $\alpha \in J$ for some $\alpha \in \Delta$. For $\varepsilon > 0$ define $\mathcal{U}_\varepsilon(y)$ to be the (topological) interior of the set

$$\left\{ z \in \mathcal{V}_{J,J} \,\middle|\, D_{\chi_{J-\{\alpha\},\alpha}}(\tilde{z}) > \varepsilon \text{ for all } \tilde{z} \in \mathcal{U}_{J,J} \text{ mapping to } z \right\}.$$

Let $U_1(\tilde{y})$ be an open neighborhood of $\tilde{y}$ lying relatively compact in some neighborhood $U_2(\tilde{y})$. Let $\varepsilon_0 > 0$ be half the maximum of the set of numbers $C_0(J,\alpha,\tilde{y}_0) \cdot \prod_{\beta \in J-\{\alpha\}} C_0(J,\beta,\tilde{y}_0)^{c_{J,\alpha,\beta}}$ where $\tilde{y}_0$ ranges over $\overline{U_1(\tilde{y})}$. Then $U_1(\tilde{y})$ maps into $\mathcal{U}_{\varepsilon_0}(y)$ via the projection map: Let $y_0 = p(\tilde{y}_0)$ be in the image of $U_1(\tilde{y})$. We have to prove $D_{\chi_{J-\{\alpha\},\alpha}}(\delta\tilde{y}_0) > \varepsilon_0$ for all $\delta \in P_J(\mathbb{Q})$ such that $\delta\tilde{y}_0 \in \mathcal{U}_{J,J}$. But this may be deduced from lemma 1.36.

Next we define $\mathcal{U}_{I,J}(\tilde{C}_2)$ and $\mathcal{V}_{I,J}(\tilde{C}_2)$ to be the sets obtained by replacing C_2 by $\tilde{C}_2 \geqslant C_2$ in the definitions of $\mathcal{U}_{I,J}$ and $\mathcal{V}_{I,J}$. We have $x \in \mathcal{V}_{I,I}(\tilde{C}_2)$ for all such $\tilde{C}_2$ since $supp(x) = I$ and since $\mathcal{V}_{I,I}(\tilde{C}_2)$ is an open neighborhood of x. We claim that $\mathcal{V}_{I,I}(\tilde{C}_2) \cap \mathcal{U}_{\varepsilon_0}(y) = \emptyset$ if $\tilde{C}_2$ is sufficiently large:

Let $z \in \mathcal{V}_{I,I}(\tilde{C}_2) \cap \mathcal{U}_{\varepsilon_0}(y) \subset \mathcal{V}_{I,I}(\tilde{C}_2) \cap \mathcal{V}_{J,J}(C_2) = \mathcal{V}_{I\cap J, I\cup J}(C_2,\tilde{C}_2)$, the latter being defined as the image under projection of

$$\mathcal{U}_{I\cap J,I\cup J}(C_2,\tilde{C}_2) = \left\{ x \in X_{\emptyset,I\cup J} \,\middle|\, \begin{array}{c} D_\alpha(x) < C_1^{-1} \text{ for } \alpha \in I \cap J, \\ D_\alpha(x) < C_2^{-1} \text{ for } \alpha \in I - (I\cap J), \\ D_\alpha(x) < \tilde{C}_2^{-1} \text{ for } \alpha \in \Delta - I \end{array} \right\}$$

We have a commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_{I,J} & \longleftarrow & \mathcal{U}_{I\cap J,I\cup J} & \longrightarrow & \mathcal{U}_{J,J} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{V}_{I,I} & \longleftarrow & \mathcal{V}_{I\cap J,I\cup J} & \longrightarrow & \mathcal{V}_{J,J} \end{array}$$

If $z \in \mathcal{V}_{I,I}(\tilde{C}_2) \cap \mathcal{U}_{\varepsilon_0}(y)$ is the image of some $\tilde{z} \in \mathcal{U}_{I\cap J,I\cup J}(C_2,\tilde{C}_2)$ then we have $D_{\chi_{J-\{\alpha\},\alpha}}(\tilde{z}) > \varepsilon_0$ by the definition of $\mathcal{U}_{\varepsilon_0}(y)$. On the other side

$$D_{\chi_{J-\{\alpha\},\alpha}}(\tilde{z}) = D_\alpha(\tilde{z}) \cdot \prod_{\beta \in J-\{\alpha\}} D_\beta(\tilde{z})^{c_{J-\{\alpha\},\alpha,\beta}} < \tilde{C}_2^{-1} \cdot C_1^{-\sum_\beta c_{J-\{\alpha\},\alpha,\beta}}$$

and this is $< \varepsilon_0$ if $\tilde{C}_2$ is sufficiently large. This contradiction proves $\mathcal{V}(\tilde{C}_2) \cap \mathcal{U}_{\varepsilon_0}(y) = \emptyset$. $\square$

PROPOSITION 1.38. *The action of $G(\mathbb{Q})/\zeta$ on $X \times G(\mathbb{A}_f)/K_f$ is properly discontinuous.*

Proof: In view of Proposition 1.37 this reduces to the same statement for the action of $G(\mathbb{Q})/\zeta$ on spaces of the form $\mathcal{V}_{I,I} \times G(\mathbb{A}_f)/K_f$, where the property is well known. $\square$

PROPOSITION 1.39. *The space $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$ is compact. It is a differentiable manifold, if K_f satisfies (Ass_{K_f}).*

Proof: The Hausdorff property of the quotient is a consequence of 1.37 and 1.38. To prove compactness it is thus sufficient to prove that the image of each $\mathcal{V}_{I,I} \times G(\mathbb{A}_f)/K_f$ under the quotient map is relatively compact for every $I \subset \Delta$. This may be deduced from ordinary reduction theory, resp. the properties of Siegel sets. The manifold property is a consequence of lemma 1.27. $\square$

1.40. We recall the sign map $sign : X \to \Sigma^\Delta$, where $\Sigma = \{-1, 0, 1\}$. We denote by X_{BS} the inverse image of $\{0,1\}^\Delta$ in X under the sign map and by $X_{sp} \simeq G(\mathbb{R})/K_\infty \cdot Z_\infty$ the inverse image of $\{1\}^\Delta$ under the sign map. Similarly we introduce the spaces: $X_{BS}(K_f) = G(\mathbb{Q})\backslash X_{BS} \times G(\mathbb{A}_f)/K_f$ and $X_{sp}(K_f) = G(\mathbb{Q})\backslash X_{sp} \times G(\mathbb{A}_f)/K_f$.

Proposition 1.41. *(a) The space X_{BS} is homeomorphic to the quotient space X/S^Δ under the canonical map $X_{BS} \hookrightarrow X \twoheadrightarrow X/S^\Delta$.*

(b) The space $X_{BS}(K_f)$ is homeomorphic to the quotient space $X(K_f)/S^\Delta$ for every open compact subgroup $K_f \subset G(\mathbb{A}_f)$.

(c) The space $X_{BS}(K_f)$ is the compactification of $X_{sp}(K_f)$ in the sense of Borel and Serre [BS].

Proof: (a) and (b) are clear. Since we do not use the original construction of Borel and Serre in this paper we leave the proof of (c) as an exercise to the interested reader. □

2. Sets of fixed points of Hecke correspondences

Normalizations of outer automorphisms

In the following technical subsection we introduce the quantities g_η and p_η attached to an automorphism of finite order η and derive some properties of them. The reader may skip these considerations, since we have $g_\eta = 1$ and $p_\eta = 1$ in several applications.

2.1. η and η_1. Let $\eta : G \to G$ be an automorphism of G, which is defined over $\mathbb{Q}$ and which is of finite order n. Since Z_∞ is by its definition an invariant subgroup, we have

$$\eta(Z_\infty) \quad = \quad Z_\infty.$$

Since all pairs (P, S), where P is a minimal $\mathbb{Q}$-parabolic and S is a maximal $\mathbb{Q}$-split torus lying in P, are conjugate by elements of $G(\mathbb{Q})$ there exists $g_\eta \in G(\mathbb{Q})$ such that

$$\eta(P_0) \quad = \quad g_\eta \cdot P_0 \cdot g_\eta^{-1} \qquad\qquad \eta(S_0) \quad = \quad g_\eta \cdot S_0 \cdot g_\eta^{-1}.$$

We may thus define the automorphism

$$\eta_1 : G \to G, \quad x \mapsto g_\eta^{-1} \cdot \eta(x) \cdot g_\eta.$$

Since $\eta_1(P_0) = P_0$ and $\eta_1(S_0) = S_0$ there must be a permutation of Δ, which we denote also by η, such that:

$$\begin{aligned} \alpha \circ \eta_1^{-1} &= \eta(\alpha) \qquad \text{for } \alpha \in \Delta,\ \alpha : S_0 \to \mathbb{G}_m \quad \text{and thus} \\ \eta_1(S_J) &= \{\eta_1(s) | \alpha(s) = 1 \ \text{ for all } \alpha \in J\}^\circ \\ &= \left\{s | \alpha(\eta_1^{-1}(s)) = 1 \ \text{ for all } \alpha \in J\right\}^\circ \\ &= S_{\eta(J)} \qquad\qquad \text{and therefore} \\ \eta_1(P_J) &= P_{\eta(J)}, \qquad\qquad \eta_1(M_J) = M_{\eta(J)} \qquad \text{i.e.} \\ \eta(P_J) &= g_\eta P_{\eta(J)} g_\eta^{-1}, \qquad \eta(M_J) = g_\eta M_{\eta(J)} g_\eta^{-1}. \end{aligned}$$

2.2. η_2. The finite group $\left\{1, \eta, \ldots, \eta^{n-1}\right\}$ has a common fixed point when acting (as group of isometries!) on the connected (!) symmetric space (of negative sectional curvature!) of maximal compact subgroups of $G(\mathbb{R})$ (compare [BGS, lemmma 6.3.]). Since all maximal compact subgroups of $G(\mathbb{R})$ are conjugate by elements of $P_0(\mathbb{R})$ there exists $b \in P_0(\mathbb{R})$ such that

$$\begin{aligned} \eta(b \cdot K_\infty^m \cdot b^{-1}) &= b \cdot K_\infty^m \cdot b^{-1} \qquad \text{or equivalently} \\ \eta(K_\infty^m) &= \eta(b)^{-1} b \cdot K_\infty^m \cdot b^{-1} \eta(b). \qquad \text{Write} \end{aligned}$$

$$g_\eta^{-1} \cdot \eta(b)^{-1} \cdot b = p_\eta \cdot k_\eta \qquad \text{with } p_\eta \in P_0(\mathbb{R}),\ k_\eta \in K_\infty. \tag{11}$$

Then $\eta_1(K_\infty^m) = p_\eta K_\infty^m p_\eta^{-1}$ and $\eta(K_\infty^m) = g_\eta p_\eta K_\infty^m p_\eta^{-1} g_\eta^{-1}$. Define

$$\eta_2 : G(\mathbb{R}) \to G(\mathbb{R}), \quad x \mapsto p_\eta^{-1} \eta_1(x) p_\eta = p_\eta^{-1} g_\eta^{-1} \eta(x) g_\eta p_\eta.$$

We have $\eta_2(K_\infty^m) = K_\infty^m$ and assume that (the assumption is automatically satisfied if K_∞ is an invariant subgroup of K_∞^m, e.g. if $K_\infty = (K_\infty^m)^\circ$):

$$\begin{aligned} \eta_2(K_\infty) &= K_\infty, \qquad \text{i.e.} \\ \eta(K_\infty) &= g_\eta p_\eta K_\infty p_\eta^{-1} g_\eta^{-1}. \end{aligned} \tag{Ass_K}$$

Since $\eta_2(K_\infty^m) = K_\infty^m$ the algebraic involution $\eta_2 \circ \theta_0 \circ \eta_2^{-1} : G(\mathbb{R}) \to G(\mathbb{R})$ fixes K_∞^m pointwise. By [BS, 1.6] it has to be the Cartan involution θ_0:

$$\eta_2 \circ \theta_0 = \theta_0 \circ \eta_2.$$

Since $p_\eta \in P_0(\mathbb{R})$ we have

$$\eta_2(P_0(\mathbb{R})) = P_0(\mathbb{R}). \tag{12}$$

Therefore $\eta_2(M_0(\mathbb{R})) = \eta_2(P_0(\mathbb{R}) \cap \theta_0(P_0(\mathbb{R}))) = \eta_2(P_0(\mathbb{R})) \cap \eta_2(\theta_0(P_0(\mathbb{R}))) = P_0(\mathbb{R}) \cap \theta_0(\eta_2(P_0(\mathbb{R}))) = M_0(\mathbb{R})$. Since $\eta_1(M_0(\mathbb{R})) = M_0(\mathbb{R})$ we get $p_\eta^{-1} M_0(\mathbb{R}) p_\eta = M_0(\mathbb{R})$. If we write $p_\eta = m_\eta \cdot u_\eta$ with $m_\eta \in M_0(\mathbb{R}),\ u_\eta \in U_0(\mathbb{R})$ we get $u_\eta^{-1} M_0(\mathbb{R}) u_\eta = M_0(\mathbb{R})$ which implies $u_\eta = 1$ by lemma 1.3. Therefore

$$p_\eta \in M_0(\mathbb{R}). \tag{13}$$

From this relation we conclude

$$\begin{aligned} \eta_2(P_I(\mathbb{R})) &= P_{\eta(I)}(\mathbb{R}) \\ \eta_2(M_I(\mathbb{R})) &= M_{\eta(I)}(\mathbb{R}) \\ \eta_2(K_\infty^I) &= K_\infty^{\eta(I)}. \end{aligned}$$

2.3. Norm maps. The (naive) norm map $\mathcal{N} = \mathcal{N}_0 : G \to G$ is defined by

$$\mathcal{N}(g) = \eta^{n-1}(g) \cdot \eta^{n-2}(g) \cdot \ldots \cdot \eta(g) \cdot g.$$

There are analogous maps $\mathcal{N}_1, \mathcal{N}_2 : G \to G$ defined by:

$$\mathcal{N}_i(g) = \eta_i^{n-1}(g) \cdot \eta_i^{n-2}(g) \cdot \ldots \cdot \eta_i(g) \cdot g.$$

The norm maps satisfy the following rules ($i = 0, 1, 2$; we put $\mathcal{N}_0 = \mathcal{N},\ \eta_0 = \eta$):

$$\mathcal{N}_i(\eta_i(x) \cdot g \cdot x^{-1}) = \eta_i^n(x) \cdot \mathcal{N}_i(g) \cdot x^{-1} \tag{14}$$

$$\mathcal{N}_i(\eta_i(x)^{-1} \cdot x \cdot g) = \eta_i^n(x)^{-1} \cdot \mathcal{N}_i(x g x^{-1}) \cdot x \tag{15}$$

and we remark

$$
\begin{aligned}
(16) \qquad x &= \eta^n(x) = \eta^{n-1}(g_\eta p_\eta \eta_2(x) p_\eta^{-1} g_\eta^{-1}) \\
&= \eta^{n-1}(g_\eta p_\eta) \cdot \eta^{n-2}(g_\eta p_\eta) \cdot \eta^{n-2}(\eta_2^2(x)) \cdot \eta^{n-2}(g_\eta p_\eta)^{-1} \cdot \eta^{n-1}(g_\eta p_\eta)^{-1} \\
&= \ldots = \mathcal{N}(g_\eta p_\eta) \cdot \eta_2^n(x) \cdot \mathcal{N}(g_\eta p_\eta)^{-1} \qquad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
(17) \qquad \eta_2(\mathcal{N}(g_\eta p_\eta)) &= p_\eta^{-1} g_\eta^{-1} \eta(\mathcal{N}(g_\eta p_\eta)) g_\eta p_\eta \\
&= p_\eta^{-1} g_\eta^{-1} \eta^n(g_\eta p_\eta) \mathcal{N}(g_\eta p_\eta) = \mathcal{N}(g_\eta p_\eta)
\end{aligned}
$$

Using (12) the equation (16) implies $P_0(\mathbb{R}) = \mathcal{N}(g_\eta p_\eta) \cdot P_0(\mathbb{R}) \cdot \mathcal{N}(g_\eta p_\eta)^{-1}$, and we conclude:

$$\mathcal{N}(g_\eta p_\eta) \in P_0(\mathbb{R}).$$

On the other side we reformulate (11):

$$
\begin{aligned}
g_\eta p_\eta &= \eta(b)^{-1} \cdot b \cdot k_\eta^{-1}. \qquad \text{This implies} \\
\mathcal{N}(g_\eta p_\eta) &= b^{-1} \cdot \mathcal{N}(b \cdot k_\eta^{-1} \cdot b^{-1}) \cdot b \\
&= (b^{-1}\eta^{n-1}(bk_\eta^{-1}b^{-1})b) \cdot (b^{-1}\eta^{n-2}(bk_\eta^{-1}b^{-1})b) \cdot \ldots \cdot (b^{-1}\eta(bk_\eta^{-1}b^{-1})b) \cdot k_\eta^{-1} \\
&= \mathcal{N}_3(k_\eta^{-1}),
\end{aligned}
$$

where $\mathcal{N}_3$ is the norm map associated to the automorphism $\eta_3 : G \to G,\ g \mapsto b^{-1}\eta(bgb^{-1})b$. Since $\eta_3(g) = (\eta(b)^{-1} \cdot b)^{-1} \cdot \eta(g) \cdot (\eta(b)^{-1} \cdot b) = k_\eta^{-1} \cdot (g_\eta p_\eta)^{-1} \cdot \eta(g) \cdot g_\eta p_\eta k_\eta = k_\eta^{-1} \cdot \eta_2(g) \cdot k_\eta$ we have $\eta_3(K_\infty) = k_\eta^{-1} \cdot K_\infty \cdot k_\eta = K_\infty$ and therefore $\mathcal{N}(g_\eta p_\eta) \in K_\infty$. This implies part (a) of the following:

Lemma 2.4. *(a)* $\mathcal{N}(g_\eta p_\eta) \in K_\infty^\emptyset = P_0(\mathbb{R}) \cap K_\infty = M_0(\mathbb{R}) \cap K_\infty$.

(b) $\mathcal{N}(g_\eta p_\eta g) = \mathcal{N}(g_\eta p_\eta) \cdot \mathcal{N}_2(g)$ *for* $g \in G(\mathbb{R})$.

(c) $\mathcal{N}(g_\eta g) = \mathcal{N}(g_\eta) \cdot \mathcal{N}_1(g)$ *for* $g \in G(\mathbb{R})$.

The proof of (b) is by induction on n (this may be done if we ignore the assumption that $\eta^n = id$ for the original n): Let $\mathcal{N}'$, $\mathcal{N}_2'$ be the norm maps with respect to the index $n-1$. Then

$$
\begin{aligned}
\mathcal{N}(g_\eta p_\eta g) &= \eta^{n-1}(g_\eta p_\eta g) \cdot \mathcal{N}'(g_\eta p_\eta g) = \eta^{n-1}(g_\eta p_\eta) \cdot \eta^{n-1}(g) \cdot \mathcal{N}'(g_\eta p_\eta) \cdot \mathcal{N}_2'(g) \\
&= \eta^{n-1}(g_\eta p_\eta) \cdot \mathcal{N}'(g_\eta p_\eta) \cdot \eta_2^{n-1}(g) \cdot \mathcal{N}'(g_\eta p_\eta)^{-1} \cdot \mathcal{N}'(g_\eta p_\eta) \cdot \mathcal{N}_2'(g) \\
&= \mathcal{N}(g_\eta p_\eta) \cdot \mathcal{N}_2(g).
\end{aligned}
$$

The proof of (c) is completely analogous. □

2.5. We remark that $\mathcal{N}(g_\eta\gamma) \in P_I(\mathbb{Q})$ if $\gamma \in P_I(\mathbb{Q})$ and $\eta(I) = I$. This is a consequence of $\mathcal{N}(g_\eta\gamma) = \mathcal{N}(g_\eta)\mathcal{N}_1(\gamma)$: We have $\mathcal{N}(g_\eta) \in P_0(\mathbb{Q}) \subset P_I(\mathbb{Q})$ since $P_0(\mathbb{Q}) = \eta^n(P_0(\mathbb{Q})) = \mathcal{N}(g_\eta) \cdot \eta_1^n(P_0(\mathbb{Q})) \cdot \mathcal{N}(g_\eta)^{-1} = \mathcal{N}(g_\eta) \cdot P_0(\mathbb{Q}) \cdot \mathcal{N}(g_\eta)^{-1}$ and $\mathcal{N}_1(\gamma) \in P_I(\mathbb{Q})$, since $\eta_1(P_I) = P_{\eta(I)} = P_I$.

Correspondences and fixed point sets

In this section we will define an action of η on the space $X(K_f)$ and will define a Hecke correspondence $\mathcal{H}$. In the rest of this and the next section we will compute the set of fixed points $F(\mathcal{H})$ of this correspondence: $F(\mathcal{H})$ will be the disjoint union of sets $F(\mathcal{H})_{I,\gamma,g_f}$

which are like locally symmetric spaces. The reader may read the summary 2.24 for more details.

2.6. The action of η on $X_{I,J}$. Let η act on the family of spaces $X_{I,J}$ as follows:

$$\begin{array}{rccc} \eta: & X_{I,J} & \to & X_{\eta(I),\eta(J)}, \\ & (\gamma, p, y) & \mapsto & (\eta(\gamma)\cdot g_\eta, \eta_1(p)\cdot p_\eta, \eta(y)) \end{array}$$

where $\gamma \in G(\mathbb{Q}),\ p \in P_I(\mathbb{R}),\ y \in Y_J$. If we interpret $y = (y_\alpha)_{\alpha\in\Delta}$ as a map $\Delta \to \mathbb{R}$, then $\eta(y)$ is defined to be the map $y \circ \eta^{-1} : \Delta \to \mathbb{R}$. This means $\eta(y) = (y_{\eta^{-1}(\alpha)})_{\alpha\in\Delta}$. The action η is well defined on the quotient $X_{I,J}$: If $\delta \in P_I(\mathbb{Q}),\ k \in K^I_\infty,\ a \in A_J$ then

$$\begin{aligned} \eta(\gamma\delta, \delta^{-1}pka, a(y_\alpha)_{\alpha\in\Delta}) &= \big(\eta(\gamma\delta)g_\eta, \eta_1(\delta)^{-1}\eta_1(p)\eta_1(k)\eta_1(a)\cdot p_\eta, \eta((\alpha(a)\cdot y_\alpha)_{\alpha\in\Delta})\big) \\ &= \big(\eta(\gamma)g_\eta\eta_1(\delta), \eta_1(\delta)^{-1}\cdot\eta_1(p)p_\eta\eta_2(k)\cdot\eta_1(a), (\eta^{-1}(\alpha)(a)\cdot y_{\eta^{-1}(\alpha)})_{\alpha\in\Delta}\big) \\ &\sim \big(\eta(\gamma)g_\eta, \eta_1(p)p_\eta, (y_{\eta^{-1}(\alpha)})_{\alpha\in\Delta})\big) = \eta(\gamma, p, (y_\alpha)_{\alpha\in\Delta}). \end{aligned}$$

Here we used $\eta^{-1}(\alpha)(a) = \alpha(\eta_1(a))$, which is an immediate consequence of the defining equation $\alpha \circ \eta_1^{-1} = \eta(\alpha)$. Observe that $p_\eta \in M_0(\mathbb{R})$ centralizes A_J for all J, so that $\eta_1(a) = \eta_2(a)$.

2.7. The action of η on $X(K_f)$. For K_f open compact we have the following map induced by η:

$$\begin{array}{rccc} \eta: & X_{I,J}(K_f) & \to & X_{\eta(I),\eta(J)}(\eta(K_f)) \\ & ((\gamma, p, y), g_f) & \mapsto & ((\eta(\gamma)g_\eta, \eta_1(p)\cdot p_\eta, \eta(y)), \eta(g_f)). \end{array}$$

This induces a map $\eta : X(K_f) \to X(\eta(K_f))$ in the obvious way. We may rewrite this map using the identification

$$X_{I,J}(K_f) \cong P_I(\mathbb{Q})\backslash \big((P_I(\mathbb{R})/K^I_\infty Z_\infty \times_{A_J} Y_J) \times G(\mathbb{A}_f)/K_f\big)$$

in the following form:

$$\begin{array}{rccc} \eta: & (p, y, g_f) & \mapsto & (\eta_1(p)\cdot p_\eta, \eta(y), g_\eta^{-1}\cdot\eta(g_f)) \\ & \| & & \| \\ & (1, p, y, g_f) & \mapsto & (\eta(1)\cdot g_\eta, \eta_1(p)\cdot p_\eta, \eta(y), \eta(g_f)) \end{array} \tag{18}$$

2.8. The Hecke correspondence. Now we take some $s_{J'} \in S^\Delta$, some $h_\infty \in K^m_\infty \cap M_0(\mathbb{R})$ and some $h_f \in G(\mathbb{A}_f)$. We consider the map

$$\begin{aligned} &\mathcal{H} = \mathcal{H}(s_{J'}) = (h_\infty, s_{J'}, h_f) \circ \eta : \\ &\qquad X(K_f) \to X(\eta(K_f)) \to X(h_f^{-1}\eta(K_f)h_f) \end{aligned}$$

induced by the maps

$$\begin{array}{l} X_{I,J}(K_f) \to X_{\eta(I),\eta(J)}(\eta(K_f)) \to X_{\eta(I),\eta(J)}(h_f^{-1}\eta(K_f)h_f) \\ (p, y, g_f) \mapsto (\eta_1(p)p_\eta, \eta(y), g_\eta^{-1}\eta(g_f)) \mapsto (\eta_1(p)p_\eta\cdot h_\infty, \eta(y)\cdot s_{J'}, g_\eta^{-1}\eta(g_f)h_f) \end{array}$$

We put $K'_f = K_f \cap \eta^{-1}(h_f K_f h_f^{-1})$. Then $\mathcal{H}$ maps:

$$\mathcal{H} : X(K'_f) \to X(h_f^{-1}\eta(K_f)h_f \cap K_f) = X(h_f^{-1}\eta(K'_f)h_f) \twoheadrightarrow X(K_f).$$

We will also consider the canonical projection induced from the inclusion $K'_f \hookrightarrow K_f$:

$$\varkappa : X(K'_f) \to X(K_f)$$

We finally make the assumption

$$(Ass_{Zf}) \qquad \eta(Z_f) \;=\; Z_f. \qquad \text{This implies } \eta(\zeta) = \zeta.$$

2.9. Set of fixed points: Sign conditions. We want to describe the set of fixed points:

$$F(\mathcal{H}) \quad = \quad \left\{ x \in X(K'_f) \middle| \, \varkappa(x) = \mathcal{H}(x) \right\}.$$

From $sign(\mathcal{H}(x)) = sign(\eta(x)) \cdot s_{J'}$ and $sign(\varkappa(x)) = sign(x)$ we get the following necessary condition for $x \in F(\mathcal{H})$:

$$\begin{aligned} (19) \qquad sign(x) \quad &= \quad (sign(x)) \circ \eta^{-1} \cdot s_{J'} \qquad \text{which implies for } I = supp(x) : \\ \eta(I) \quad &= \quad I \qquad \text{and} \\ &\# \left(J' \cap \{\alpha, \eta(\alpha), \dots \eta^{n-1}(\alpha)\} \right) \quad \text{is even for all } \alpha \in I. \end{aligned}$$

Conversely if the last two conditions are satisfied for some $I \subset \Delta$ one can construct an x such that $supp(x) = I$ and $sign(x)$ satisfies (19). The conditions imply specially:

$$supp(x)^\eta \cap J' \quad = \quad \emptyset.$$

2.10. Set of fixed points: Conditions. Now let $I = supp(x),\ x \in F(\mathcal{H})$. By the description of $E_I \subset X$ this means, if we write $x = (p, y, g_f)$ with $p \in P_I(\mathbb{R})$ and $y = sign(x)$:

$$\left(\eta_1(p)p_\eta h_\infty,\ \eta(y)s_{J'},\ g_\eta^{-1}\eta(g_f)h_f\right) \quad \sim \quad (p, y, g_f)$$

i.e. there exist $\gamma \in P_I(\mathbb{Q}),\ k_\infty \in K^I_\infty,\ z_\infty \in Z_\infty, a \in A_I, k_f \in K_f$ such that:

$$(1) \qquad g_\eta^{-1}\eta(p)g_\eta p_\eta h_\infty \quad = \quad \gamma p k_\infty^{-1} z_\infty^{-1} a^{-1}$$

$$(2) \qquad \eta(y)s_{J'} \quad = \quad a^{-1} y$$

$$(3) \qquad g_\eta^{-1}\eta(g_f)h_f \quad = \quad \gamma g_f k_f^{-1}.$$

The condition (2) is equivalent to $sign(\eta(x)) \cdot s_{J'} = sign(x)$, since we have $a \cdot y = y$ for $supp(y) = I$ and $a \in A_I$. As before this implies $\eta(I) = I$. We rewrite (1) and (3) as follows:

$$(1') \qquad \eta(p)^{-1}(g_\eta\gamma)p \quad = \quad g_\eta p_\eta h_\infty a z_\infty k_\infty$$

$$(3') \qquad \eta(g_f)^{-1}(g_\eta\gamma)g_f \quad = \quad h_f k_f.$$

The equation (1') implies by taking norms:

$$(1_{\mathcal{N}}) \qquad p^{-1} \cdot \mathcal{N}(g_\eta\gamma) \cdot p \quad = \quad \mathcal{N}(g_\eta p_\eta) \cdot \mathcal{N}_2(h_\infty \cdot a \cdot z_\infty \cdot k_\infty)$$

The map η_2 takes A_I, Z_∞ and K^I_∞ to themselves, and h_∞ normalizes $K^I_\infty Z_\infty A_I$. Therefore we have the following necessary condition, if we take lemma 2.4(a) into account:

$$(20) \qquad p^{-1} \cdot \mathcal{N}(g_\eta\gamma) \cdot p \quad \in \quad \mathcal{N}_2(h_\infty) \cdot K^I_\infty Z_\infty A_I.$$

2.11. Converse conditions. For some $I \subset \Delta$ with $\eta(I) = I$ and some fixed $\gamma \in P_I(\mathbb{Q})$ let us assume conversely that $\mathcal{N}(g_\eta\gamma)$ is conjugate in $P_I(\mathbb{R})$ to an element of $\mathcal{N}_2(h_\infty) \cdot$

$K_\infty^I Z_\infty A_I$, i.e. that (20) is satisfied with some $p_1 \in P_I(\mathbb{R})$ instead of p. We consider the map

$$\begin{aligned}\tilde\eta_{\gamma,h_\infty} : P_I(\mathbb{R}) \to P_I(\mathbb{R}),&\\ p \mapsto \gamma^{-1}\eta_1(p)p_\eta \cdot h_\infty = \gamma^{-1}g_\eta^{-1}\eta(p)g_\eta p_\eta h_\infty &= \gamma^{-1}\cdot p_\eta \cdot \eta_2(p)\cdot h_\infty.\end{aligned}$$

It is easy to calculate the n-th power of $\tilde\eta_{\gamma,h_\infty}$ (compare 2.4(b)):

$$\begin{aligned}(\tilde\eta_{\gamma,h_\infty})^n(p) &= \mathcal{N}(g_\eta\gamma)^{-1}\cdot \eta^n(p)\cdot \mathcal{N}(g_\eta p_\eta h_\infty)\\ &= \mathcal{N}(g_\eta\gamma)^{-1}\cdot p\cdot \mathcal{N}(g_\eta p_\eta)\cdot \mathcal{N}_2(h_\infty).\end{aligned} \tag{21}$$

For $k \in K_\infty^I Z_\infty A_I$ we get $\tilde\eta_{\gamma,h_\infty}(pk) = \tilde\eta_{\gamma,h_\infty}(p)\cdot h_\infty^{-1}\cdot \eta_2(k)\cdot h_\infty$ with $h_\infty^{-1}\eta_2(k)\cdot h_\infty \in K_\infty^I Z_\infty A_I$. Therefore $\tilde\eta_{\gamma,h_\infty}$ induces a map from $P_I(\mathbb{R})/K_\infty^I Z_\infty A_I$ to itself, which will be denoted by the same symbol. Let

$$F(g_\eta,\gamma) = \left\{p \in P_I(\mathbb{R}) \mid (\tilde\eta_{\gamma,h_\infty})^n(p) \in p\cdot K_\infty^I Z_\infty A_I\right\}.$$

Then this set is invariant under right translations by elements of $K_\infty^I Z_\infty A_I$ and the quotient space

$$\tilde F(g_\eta,\gamma) = F(g_\eta,\gamma)/K_\infty^I Z_\infty A_I$$

is the space of invariants of the n-th power map $(\tilde\eta_{\gamma,h_\infty})^n$ acting on $P_I(\mathbb{R})/K_\infty^I Z_\infty A_I$.

The map $\tilde\eta_{\gamma,h_\infty}$ leaves $\tilde F(g_\eta,\gamma)$ and $F(g_\eta,\gamma)$ invariant.

By (21) we may describe $F(g_\eta,\gamma)$ as the set of $p \in P_I(\mathbb{R})$ satisfying $p^{-1}\mathcal{N}(g_\eta\gamma)p \in \mathcal{N}(g_\eta p_\eta)\cdot\mathcal{N}_2(h_\infty)\cdot K_\infty^I Z_\infty A_I$. But since $\mathcal{N}(g_\eta p_\eta) \in K_\infty^I$ and since $\mathcal{N}_2(h_\infty) \in (K_\infty^m)\cap M_0(\mathbb{R})$ normalizes $K_\infty^I Z_\infty A_I$ this condition may be rewritten in the following form:

$$F(g_\eta,\gamma) = \left\{p \in P_I(\mathbb{R}) \mid p^{-1}\mathcal{N}(g_\eta\gamma)p \in \mathcal{N}_2(h_\infty)\cdot K_\infty^I Z_\infty A_I\right\}$$

By assumption we have $F(g_\eta,\gamma) \neq \emptyset$.

2.12. Now fix some $p_1 \in F(g_\eta,\gamma)$, i.e. $p_1^{-1}\cdot\mathcal{N}(g_\eta\gamma)\cdot p_1 = \mathcal{N}_2(h_\infty)\cdot k_1$ with $k_1 \in K_\infty^I Z_\infty A_I$. We want to describe the set of connected components of $\tilde F(g_\eta,\gamma)$. Let $K_\infty^{I,m} = K_\infty^m \cap P_I(\mathbb{R})$ and let $\mathfrak{p}$ be a complement to $Lie(K_\infty^I Z_\infty A_I)$ in $Lie(P_I(\mathbb{R}))$ which is invariant under the adjoint action of $K_\infty^{I,m} Z_\infty A_I$.

Lemma 2.13. *Each $p \in F(g_\eta,\gamma)$ has a unique representation*

$$p = p_1\cdot\exp(\pi)\cdot k \qquad \text{where } k \in K_\infty^{I,m} Z_\infty A_I \quad \text{and} \quad \pi \in \mathfrak{p}^{Ad(\mathcal{N}_2(h_\infty)k_1)}. \tag{22}$$

Conversely each $p \in P_I(\mathbb{R})$ of the form (22) lies in $F(g_\eta,\gamma)$. Here $\mathfrak{p}^{Ad(\mathcal{N}_2(h_\infty)k_1)}$ denotes the set of elements in $\mathfrak{p}$ fixed by the adjoint action of $p_1^{-1}\mathcal{N}(g_\eta\gamma)p_1 = \mathcal{N}_2(h_\infty)k_1$.

Proof: Recall that $p_1^{-1}\cdot p$ has a unique Iwasawa decomposition

$$p_1^{-1}\cdot p = \exp(\pi)\cdot k \qquad \text{where } \pi \in \mathfrak{p} \text{ and } k \in K_\infty^{I,m} Z_\infty A_I,$$

and we have to prove $\pi \in \mathfrak{p}^{Ad(\mathcal{N}_2(h_\infty)k_1)}$ for $p \in F(g_\eta,\gamma)$. We calculate:

$$\begin{aligned}
p^{-1}\cdot\mathcal{N}(g_\eta\gamma)\cdot p &= k^{-1}\exp(\pi)^{-1}p_1^{-1}\mathcal{N}(g_\eta\gamma)p_1\exp(\pi)k\\
&= k^{-1}\exp(\pi)^{-1}\mathcal{N}_2(h_\infty)k_1\exp(\pi)k\\
&= k^{-1}\exp(\pi)^{-1}\exp(Ad(\mathcal{N}_2(h_\infty)k_1)\pi)\mathcal{N}_2(h_\infty)k_1k\\
&= \exp(Ad(k^{-1})\pi)^{-1}\cdot\exp(Ad(k^{-1}\mathcal{N}_2(h_\infty)k_1)\pi)\cdot k^{-1}\mathcal{N}_2(h_\infty)k_1k.
\end{aligned}$$

Now for $p \in F(g_\eta,\gamma)$ there exists $k_2 \in K_\infty^I Z_\infty A_I$ such that

$$p^{-1}\cdot\mathcal{N}(g_\eta\gamma)\cdot p \quad = \quad \mathcal{N}_2(h_\infty)\cdot k_2 \qquad . \tag{23}$$

The combination of the last two equations can be rewritten in the form

$$\exp\left(Ad(k^{-1})Ad(\mathcal{N}_2(h_\infty)k_1)\pi\right)\cdot k^{-1}\mathcal{N}_2(h_\infty)k_1k \quad = \quad \exp(Ad(k^{-1})\pi)\cdot\mathcal{N}_2(h_\infty)k_2$$

and by the uniqueness of the Iwasawa decomposition this is equivalent to the system of equations

$$\begin{aligned}
Ad(\mathcal{N}_2(h_\infty)k_1)\pi &= \pi \qquad\qquad \text{and}\\
k^{-1}\mathcal{N}_2(h_\infty)k_1k &= \mathcal{N}_2(h_\infty)k_2,
\end{aligned}\tag{24}$$

so that $\pi \in \mathfrak{p}^{Ad(\mathcal{N}_2(h_\infty)k_1)}$.

Conversely if p is of the form (22) we may define k_2 by the equation (24). But then k_2 lies automatically in $K_\infty^I Z_\infty A_I$ because k_1 does so and K_∞^I is a normal subgroup in $K_\infty^{I,m}$ with abelian quotient, so that

$$k^{-1}\mathcal{N}_2(h_\infty)K_\infty^I Z_\infty A_I k = \mathcal{N}_2(h_\infty)K_\infty^I Z_\infty A_I.$$

Reversing the above calculation then gives the equation (23), so that each p of the form (22) belongs to $F(g_\eta,\gamma)$.

2.14. Description of $\tilde{F}(g_\eta,\gamma)$. From $K_\infty^{I,m}\cap Z_\infty A_I = \{1\}$ we get an isomorphism of cosets $K_\infty^{I,m}Z_\infty A_I/K_\infty^I Z_\infty A_I \simeq K_\infty^{I,m}/K_\infty^I$. Now the preceding lemma implies that we get a bijection

$$\begin{aligned}
\tilde{F}(g_\eta,\gamma) \quad &\cong \quad \mathfrak{p}^{Ad(\mathcal{N}_2(h_\infty)k_1)}\times(K_\infty^{I,m}/K_\infty^I)\\
p_1\cdot\exp(\pi)\cdot k \quad &\leftarrow \quad (\pi, k \mod K_\infty^I)
\end{aligned}$$

Since the Iwasawa decomposition induces a homeomorphism this is a homeomorphism too. Thus we can read off immediately the description of the set of connected components of $\tilde{F}(g_\eta,\gamma)$ by the following isomorphism:

$$\begin{aligned}
\mathcal{P}_1: \quad K_\infty^{I,m}/K_\infty^I \quad &\tilde{\to} \quad \pi_0\left(\tilde{F}(g_\eta,\gamma)\right)\\
\text{class of } k \quad &\mapsto \quad \text{class of } p_1k
\end{aligned}\tag{25}$$

2.15. Fixed points of $\tilde{\eta}_{\gamma,h_\infty}$. Next we assume that $\tilde{\eta}_{\gamma,h_\infty}$ has a fixed point if acting on the finite set $\pi_0\left(\tilde{F}(g_\eta,\gamma)\right)$ of connected components. Then $\tilde{\eta}_{\gamma,h_\infty}$ induces an isometric automorphism of finite order of this connected component, which is a Riemannian manifold of negative curvature (i.e. the sectional curvature is $\leqslant 0$). By [Hel, I, thm. 13.5] or [BGS, 6.3.] it has a fixed point on this connected component. We may already assume that p_1 is this fixed point:

$$\gamma^{-1} g_\eta^{-1}\eta(p_1) g_\eta p_\eta h_\infty \quad = \quad p_1 \cdot k_0 \qquad \text{with } k_0 \in L_\infty^I = K_\infty^I Z_\infty A_I. \tag{26}$$

The map $\mathcal{P}_1$ satisfies $\mathcal{P}_1 \circ \eta_2 = \tilde{\eta}_{\gamma,h_\infty} \circ \mathcal{P}_1$ since we have

$$\tilde{\eta}_{\gamma,h_\infty}(p_1 k) \quad = \quad \tilde{\eta}_{\gamma,h_\infty}(p_1) \cdot h_\infty^{-1} \cdot \eta_2(k) \cdot h_\infty \quad = \quad p_1 k_0 h_\infty^{-1} \eta_2(k) h_\infty$$

and since $k \mapsto k_0 h_\infty^{-1} k h_\infty$ induces the identity on $K_\infty^{I,m}/K_\infty^I$. Therefore $\mathcal{P}_1$ induces an isomorphism

$$\left(K_\infty^{I,m}/K_\infty^I\right)^{\eta_2} \quad \tilde{\to} \quad \pi_0\left(\tilde{F}(g_\eta,\gamma)\right)^{\tilde{\eta}_{\gamma,h_\infty}}.$$

2.16. The centralizers $G^I_{\gamma,\eta}$. For $\gamma \in P_I(\mathbb{Q})$ we define the automorphism

$$\eta_\gamma : G \to G, \quad x \mapsto (g_\eta\gamma)^{-1} \cdot \eta(x) \cdot g_\eta \gamma = \gamma^{-1} \cdot \eta_1(x) \cdot \gamma$$

and the algebraic subgroup $G^I_{\gamma,\eta} = (P_I)^{\eta_\gamma}$ of η_γ-invariants, i.e.

$$G^I_{\gamma,\eta}(S) \quad = \quad \{x \in P_I(S) | \eta_\gamma(x) = x\} \quad = \quad \left\{x \in P_I(S) | \eta(x)^{-1} \cdot g_\eta\gamma \cdot x = g_\eta \gamma\right\}$$

for a $\mathbb{Q}$-algebra S. For $I = \Delta$ we will drop the index I, i.e. $G_{\gamma,\eta} = G^{\eta_\gamma}$

We introduce the notations

$$\begin{array}{rclcrcl} L_\infty^I & = & K_\infty^I Z_\infty A_I, & \quad & L_\infty^{I,m} & = & K_\infty^{I,m} Z_\infty A_I \\ \tilde{L} & = & p_1 \cdot L_\infty^I \cdot p_1^{-1}, & & \tilde{L}^m & = & p_1 \cdot L_\infty^{I,m} \cdot p_1^{-1}, \\ L_{\gamma,\eta} & = & \tilde{L} \cap G_{\gamma,\eta}(\mathbb{R}), & & L^m_{\gamma,\eta} & = & \tilde{L}^m \cap G_{\gamma,\eta}(\mathbb{R}). \end{array}$$

We have for $l \in L_\infty^{I,m}$ i.e. for $p_1 \cdot l \cdot p_1^{-1} \in \tilde{L}$:

$$\begin{aligned} \eta_\gamma(p_1 \cdot l \cdot p_1^{-1}) &= \gamma^{-1} g_\eta^{-1} \eta(p_1)\eta(l)\eta(p_1)^{-1} g_\eta \gamma \\ &= \gamma^{-1} g_\eta^{-1}\eta(p_1) g_\eta p_\eta \eta_2(l)(g_\eta p_\eta)^{-1}\eta(p_1)^{-1} g_\eta\gamma \\ &= p_1 \cdot k_0 h_\infty^{-1}\eta_2(l) h_\infty k_0^{-1} \cdot p_1^{-1}. \end{aligned}$$

Therefore $\eta_\gamma(\tilde{L}^m) = \tilde{L}^m$ and from $\eta_2(L_\infty^I) = L_\infty^I$ by (Ass_K) we conclude $\eta_\gamma(\tilde{L}) = \tilde{L}$. Furthermore the conjugation with p_1 intertwines the η_2-action on $K_\infty^{I,m}/K_\infty^I$ with the η_γ-action on $\tilde{L}^m/\tilde{L}$, since conjugation by $k_0 h_\infty^{-1}$ acts as identity on $\tilde{L}^m/\tilde{L}$.

2.17. The coset space $R^I_{\gamma,\eta}$. We introduce the coset space

$$R^I_{\gamma,\eta} \quad = \quad L^m_{\gamma,\eta} \backslash \left(\tilde{L}^m/\tilde{L}\right)^{\eta_\gamma}$$

and denote by

$$O_\eta^\infty(I,\gamma,h_\infty) \quad = \quad \# R^I_{\gamma,\eta}$$

its cardinality. Finally we choose and fix a representative $k_r \in \tilde{L}^m$ of each coset $r \in R^I_{\gamma,\eta}$.

Lemma 2.18. *The maps*

$$\begin{array}{rcl} \phi_1 : \left(G^I_{\gamma,\eta}(\mathbb{R})/L_{\gamma,\eta}\right) \times R^I_{\gamma,\eta} & \to & \left(P_I(\mathbb{R})/\tilde{L}\right)^{\eta_\gamma} \\ (x \text{ mod } L_{\gamma,\eta},\ \text{class of } k_r) & \mapsto & x \cdot k_r \text{ mod } \tilde{L} \end{array}$$

and

$$\begin{array}{rcl} \overline{\phi_2} : \left(P_I(\mathbb{R})/\tilde{L}\right)^{\eta_\gamma} & \to & \tilde{F}(g_\eta,\gamma)^{\tilde{\eta}_{\gamma,h_\infty}} \quad = \quad \left(P_I(\mathbb{R})/L^I_\infty\right)^{\tilde{\eta}_{\gamma,h_\infty}} \\ xk_r \text{ mod } \tilde{L} & \mapsto & xk_rp_1 \text{ mod } L^I_\infty \end{array}$$

are isomorphisms.

Proof: First observe that ϕ_1 is well defined since each k_r normalizes $\tilde{L}$. Then observe that

$$\begin{array}{rcl} \phi_2 : P_I(\mathbb{R})/\tilde{L} & \xrightarrow{\sim} & P_I(\mathbb{R})/L^I_\infty \\ x \mod \tilde{L} & \mapsto & xp_1 \mod L^I_\infty \end{array}$$

is an isomorphism and that the diagram

$$\begin{array}{ccc} P_I(\mathbb{R})/\tilde{L} & \xrightarrow[\phi_2]{} & P_I(\mathbb{R})/L^I_\infty \\ \downarrow{\scriptstyle \eta_\gamma} & & \downarrow{\scriptstyle \tilde{\eta}_{\gamma,h_\infty}} \\ P_I(\mathbb{R})/\tilde{L} & \xrightarrow[\phi_2]{} & P_I(\mathbb{R})/L^I_\infty \end{array}$$

commutes by a formal computation:

$$\begin{array}{rcl} \phi_2\left(\eta_\gamma(x)\right) & = & \gamma^{-1} \cdot g_\eta^{-1} \cdot \eta(x) \cdot g_\eta \cdot \gamma \cdot p_1 \\ \tilde{\eta}_{\gamma,h_\infty}\left(\phi_2(x)\right) & = & \gamma^{-1} \cdot g_\eta^{-1} \cdot \eta(x) \cdot \eta(p_1) \cdot g_\eta \cdot p_\eta \cdot h_\infty \\ & = & \gamma^{-1} \cdot g_\eta^{-1} \cdot \eta(x) \cdot g_\eta \cdot \gamma \cdot p_1 \cdot k_0 \\ & = & \phi_2\left(\eta_\gamma(x)\right) \cdot k_0, \end{array}$$

where k_0 is defined in (26). Therefore $\overline{\phi_2}$ is an isomorphism.

Next we prove that ϕ_1 is injective: If $x_1k_a = x_2k_b \cdot k$ with $k \in \tilde{L}$ and $x_1, x_2 \in G_{\gamma,\eta}(\mathbb{R})$ then $x_2^{-1}x_1 = k_bkk_a^{-1}$, but $x_2^{-1}x_1 \in G_{\gamma,\eta}(\mathbb{R}),\ k_bkk_a^{-1} \in \tilde{L}^m$. Therefore $k_bkk_a^{-1} \in L^m_{\gamma,\eta}$, so that k_a and $k_b = (k_bkk_a^{-1}) \cdot k_a \cdot k^{-1}$ lie in the same coset in $R^I_{\gamma,\eta}$. Since each coset has a unique representative we get $k_a = k_b$. But then $k_bkk_a^{-1} \in \tilde{L}$ since k_a normalizes $\tilde{L}$. This implies $x_1 \mod \tilde{L} = x_2 \mod \tilde{L}$.

To prove that ϕ_1 is surjective we reduce to the claim that the canonical map

$$\tilde{\phi}_1 : G^I_{\gamma,\eta}(\mathbb{R}) \quad \to \quad G^I_{\gamma,\eta}(\mathbb{R})/L^m_{\gamma,\eta} \quad \to \left(P_I(\mathbb{R})/\tilde{L}^m\right)^{\eta_\gamma}$$

is surjective: If $p \in P_I(\mathbb{R})$ with $\eta_\gamma(p\tilde{L}) = p\tilde{L}$ is given then $p\tilde{L}^m \in \left(P_I(\mathbb{R})/\tilde{L}^m\right)^{\eta_\gamma}$ and by assumption on $\tilde{\phi}_1$ there exists $x \in G^I_{\gamma,\eta}(\mathbb{R})$ with $p = xk,\ k \in \tilde{L}^m$. Then $xk\tilde{L} = p\tilde{L} = \eta_\gamma(p)\tilde{L} = \eta_\gamma(x)\eta_\gamma(k)\tilde{L} = x \cdot \eta_\gamma(k)\tilde{L}$, which implies $k\tilde{L} = \eta_\gamma(k)\tilde{L}$, i.e. $k\tilde{L} \in \left(\tilde{L}^m/\tilde{L}\right)^{\eta_\gamma}$. Therefore there exists $y \in L^m_{\gamma,\eta},\ k_1 \in \tilde{L}$ and $a \in R^I_{\gamma,\eta}$ such that $k = y \cdot k_a \cdot k_1$. Then $p = (xy) \cdot k_a \cdot k_1$, so $p \mod \tilde{L}$ is in the image of ϕ_1, since $xy \in G^I_{\gamma,\eta}(\mathbb{R})$.

To prove surjectivity of $\tilde{\phi}_1$ it is enough to show the existence of an η_γ-invariant subspace $\mathfrak{q}$ inside the Lie algebra $\mathfrak{p}_I$ of $P_I(\mathbb{R})$ such that the composite map $e : \mathfrak{q} \xrightarrow{exp} P(\mathbb{R}) \twoheadrightarrow P_I(\mathbb{R})/\tilde{L}^m$ is an η_γ-equivariant isomorphism. Then $\left(P_I(\mathbb{R})/\tilde{L}^m\right)^{\eta_\gamma} = e(\mathfrak{q}^{\eta_\gamma}) \subset e(\mathfrak{p}_I^{\eta_\gamma}) = e(Lie(G^I_{\gamma,\eta}(\mathbb{R})))$ and the claim follows.

We denote by $\tilde{\mathfrak{m}}_I$ the Lie algebra of the derived group of $p_1M_I(\mathbb{R})p_1^{-1}$. The Killing form is a non degenerate form on $\tilde{\mathfrak{m}}_I$. We take $\mathfrak{q}$ to be the sum of the following subspaces of $\mathfrak{p}_I$:

- the orthogonal complement $\mathfrak{c}_1$ of $Lie(p_1K^I_\infty p_1^{-1}) \cap \tilde{\mathfrak{m}}_I$ inside $\tilde{\mathfrak{m}}_I$;
- the Lie algebra $\mathfrak{u}_I$ of the unipotent radical $U_I(\mathbb{R})$ of $P_I(\mathbb{R})$;
- some η_γ invariant complement $\mathfrak{c}_2$ of $Lie(Z_\infty) + \left(Lie(p_1K^I_\infty p_1^{-1}) \cap Lie Z_G(\mathbb{R})\right)$ inside $Lie\,(Z_G(\mathbb{R}))$.

We observe that $p_1M_I(\mathbb{R})p_1^{-1}$ is η_γ-invariant: For $m \in M_I(\mathbb{R})$ we have

$$\begin{aligned}\eta_\gamma(p_1mp_1^{-1}) &= \gamma^{-1}g_\eta^{-1}\eta(p_1)g_\eta\eta_1(m)g_\eta^{-1}\eta(p_1)^{-1}g_\eta\gamma \\ &= p_1k_0h_\infty^{-1}p_\eta^{-1}\eta_1(m)\cdot p_\eta h_\infty k_0^{-1}p_1^{-1} \qquad \text{by (26).}\end{aligned}$$

Now $\eta_1(m) \in M_{\eta(I)}(\mathbb{R}) = M_I(\mathbb{R})$, $\quad p_\eta \in M_0(\mathbb{R}) \subset M_I(\mathbb{R})$, $\quad h_\infty, k_0 \in L^{I,m}_\infty \subset M_I(\mathbb{R})$ and therefore $\eta_\gamma(p_1mp_1^{-1}) = p_1m_1p_1^{-1}$ with $m_1 \in M_I(\mathbb{R})$. For $k \in K^I_\infty$ we conclude $\eta_\gamma(p_1kp_1^{-1}) = p_1k_0h_\infty^{-1}\eta_2(k)h_\infty k_0^{-1}p_1^{-1} \in p_1K^I_\infty p_1^{-1}$, since k_0 and h_∞ normalize K^I_∞. This implies that $\mathfrak{c}_1$ is η_γ-invariant.

Since η_γ acts as η on the center $Z_G(\mathbb{R})$ it acts as an automorphism of finite order on $Lie(Z_G(\mathbb{R}))$. Therefore $\mathfrak{c}_2$ exists.

Now observe that $\mathfrak{p}_I$ is the direct sum of $\tilde{\mathfrak{m}}_I$, of $\mathfrak{u}_I$ and of the Lie algebra of the center of $p_1M_I(\mathbb{R})p_1^{-1}$, which itself is the direct sum of $Lie\,(Z_G(\mathbb{R}))$ and $Lie\left(p_1A_Ip_1^{-1}\right)$. This implies that $\mathfrak{q}$ is an η_γ-invariant complement to $Lie(\tilde{L}^m_\infty)$ in $\mathfrak{p}_I$. We get the surjectivity of e by Iwasawa decomposition. This finishes the proof of lemma 2.18. □

2.19. A FIRST SUMMARY. We take $R^I_{\gamma,\eta}$ to be the empty set if $\pi_0\left(\tilde{F}(g_\eta,\gamma)\right)^{\tilde{\eta}_{\gamma,h_\infty}}$ is empty. We may summarize: Let $\gamma \in P_I(\mathbb{Q})$ be given. If the set

$$\begin{aligned}P_I^{\eta,\gamma} &= \left(P_I(\mathbb{R})/L^I_\infty\right)^{\tilde{\eta}_{\gamma,h_\infty}} \\ &= \left\{p \mod L^I_\infty \in P_I(\mathbb{R})/L^I_\infty \mid \eta(p)^{-1}(g_\eta\gamma)p \in g_\eta p_\eta h_\infty\cdot L^I_\infty\right\}\end{aligned}$$

is not empty, then $\mathcal{N}(g_\eta\gamma)$ is conjugate inside $P_I(\mathbb{R})$ to an element of $\mathcal{N}_2(h_\infty)\cdot K^I_\infty\mathcal{N}(Z_\infty)A_I$. If $\mathcal{N}(g_\eta\gamma)$ is conjugate to such an element, then we have an isomorphism:

$$\begin{aligned}\phi : \left(G^I_{\gamma,\eta}(\mathbb{R})/L_{\gamma,\eta}\right)\times R^I_{\gamma,\eta} &\to \left(P_I(\mathbb{R})/L^I_\infty\right)^{\tilde{\eta}_{\gamma,h_\infty}} \\ (x,k_a) &\mapsto xk_ap_1\end{aligned}$$

for some $p_1 \in P_I^{\eta,\gamma}$.

2.20. By lemma 1.27 the class of γ in $G(\mathbb{Q})/\zeta$ is uniquely determined by $x = (p,y,g_f)$ and the equations (1), (2), (3) in 2.10. Now let us take another representative $\tilde{x} = (\tilde{p},\tilde{y},\tilde{g}_f)$ for the class of x, where

$$\tilde{p} = \delta \cdot p \cdot \kappa_\infty \zeta_\infty b^{-1}, \quad \tilde{y} = by, \quad \tilde{g}_f = \delta g_f \kappa_f$$

with $\delta \in P_I(\mathbb{Q}),\ \kappa_\infty \in K^I_\infty,\ \zeta_\infty \in Z_\infty,\ b \in A_I,\ \kappa_f \in K_f$. Then the relation

$$\big(\eta_1(\tilde{p}) \cdot p_\eta h_\infty, \eta(\tilde{y}) s_{J'}, g_\eta^{-1} \eta(\tilde{g}_f) h_f\big) \quad \sim \quad (\tilde{p}, \tilde{y}, \tilde{g}_f)$$

is due to elements $\tilde{\gamma}, \tilde{k}_\infty, \tilde{z}_\infty, \tilde{a}, \tilde{k}_f$. Here we can take

$$\begin{array}{rclrcl}
\tilde{\gamma} & = & \eta_1(\delta)\gamma\delta^{-1}, & \tilde{k}_\infty & = & h_\infty^{-1}\eta_2(k_\infty^{-1})h_\infty \cdot k_\infty \cdot k_0, \\
\tilde{z}_\infty & = & \zeta_\infty \cdot \eta(\zeta_\infty) \cdot z_\infty, & \tilde{a} & = & b \cdot a \cdot \eta_2(b)^{-1}, \\
\tilde{k}_f & = & h_f^{-1}\eta(\kappa_f)^{-1} h_f \cdot k_f \cdot \kappa_f, & & &
\end{array}$$

since we have

$$\begin{array}{rcl}
g_\eta^{-1}\eta(\tilde{p}) g_\eta p_\eta h_\infty & = & g_\eta^{-1}\eta(\delta)\eta(p)\eta(\kappa_\infty\zeta_\infty b^{-1}) g_\eta p_\eta h_\infty \\
& = & \eta_1(\delta) \cdot g_\eta^{-1}\eta(p) \cdot g_\eta p_\eta h_\infty \cdot h_\infty^{-1}\eta_2(\kappa_\infty\zeta_\infty b^{-1})h_\infty \\
& = & \eta_1(\delta) \cdot \gamma p k_\infty^{-1} z_\infty^{-1} a^{-1} \cdot h_\infty^{-1}\eta_2(\kappa_\infty)\eta_2(\zeta_\infty)\eta_2(b^{-1})h_\infty \\
& = & (\eta_1(\delta)\gamma\delta^{-1}) \cdot \tilde{p} \cdot \big(a_\infty \zeta_\infty^{-1} k_\infty^{-1} z_\infty^{-1} a^{-1} \eta_2(\kappa_\infty)\eta_2(\zeta_\infty)\eta_2(b^{-1})h_\infty\big) \\
& = & \tilde{\gamma} \cdot \tilde{p} \cdot \tilde{k}_\infty^{-1} \cdot \tilde{z}_\infty^{-1} \cdot \tilde{a}^{-1} \qquad \text{and} \\
\eta(\tilde{y}) \cdot s_K & = & \eta_2(b) \cdot \eta(y) \cdot s_K \quad = \quad \eta_2(b) \cdot a^{-1} \cdot y \\
& = & \eta_2(b) \cdot a^{-1} \cdot b^{-1} \cdot \tilde{y} \quad = \quad \tilde{a}^{-1} \cdot \tilde{y}. \qquad \text{Furthermore:} \\
g_\eta^{-1}\eta(\tilde{g}_f)h_f & = & g_\eta^{-1}\eta(\delta)\eta(g_f)\eta(\kappa_f)h_f \quad = \quad \eta_1(\delta) \cdot g_\eta^{-1}\eta(g_f)h_f \cdot h_f^{-1}\eta(\kappa_f)h_f \\
& = & \eta_1(\delta) \cdot \gamma g_f k_f^{-1} \cdot h_f^{-1}\eta(\kappa_f)h_f \\
& = & \eta_1(\delta)\gamma\delta^{-1} \cdot \tilde{g}_f \cdot \kappa_f^{-1} k_f^{-1} h_f^{-1}\eta(\kappa_f)h_f \quad = \quad \tilde{\gamma} \cdot \tilde{g}_f \cdot \tilde{k}_f^{-1}.
\end{array}$$

The relation $\tilde{\gamma} = \eta_1(\delta) \cdot \gamma \cdot \delta^{-1}$ is equivalent to

$$g_\eta \tilde{\gamma} \quad = \quad \eta(\delta) \cdot g_\eta \gamma \cdot \delta^{-1}$$

Therefore we have to consider the elements $g_\eta\gamma$ up to η-conjugacy, i.e. the fixed point sets are indexed by the η-conjugacy classes of elements in $G(\mathbb{Q})/\zeta$.

Remark 2.21. We recall lemma 2.4(c): $\mathcal{N}(g_\eta\gamma) = \mathcal{N}(g_\eta) \cdot \mathcal{N}_1(\gamma)$. The construction of g_η implies:

$$\begin{array}{rcl}
\mathcal{N}(g_\eta) \cdot P_0 \cdot \mathcal{N}(g_\eta)^{-1} & = & \eta^{n-1}(g_\eta) \cdots \eta(g_\eta) \cdot g_\eta P_0 g_\eta^{-1} \cdot \eta(g_\eta)^{-1} \cdots \eta^{n-1}(g_\eta)^{-1} \\
& = & \eta^{n-1}(g_\eta) \cdots \eta(g_\eta) \cdot \eta(P_0) \cdot \eta(g_\eta)^{-1} \cdots \eta^{n-1}(g_\eta)^{-1} \\
& = & \eta\left(\eta^{n-2}(g_\eta) \cdots \eta(g_\eta) \cdot g_\eta P_0 g_\eta^{-1} \cdot \eta(g_\eta)^{-1} \cdots \eta^{n-2}(g_\eta)^{-1}\right) \\
& = & \ldots \quad = \quad \eta^{n-1}(g_\eta P_0 g_\eta^{-1}) \quad = \quad \eta^n(P_0) \quad = \quad P_0.
\end{array}$$

Using S_0 instead of P_0 we obtain by the same calculation: $\mathcal{N}(g_\eta) \cdot S_0 = S_0 \cdot \mathcal{N}(g_\eta)$, i.e. $\mathcal{N}(g_\eta)$ normalizes P_0 and S_0. But the normalizer of S_0 inside P_0 is the centralizer of S_0. This implies $\mathcal{N}(g_\eta) \in M_0(\mathbb{Q}) \subset M_I(\mathbb{Q}) \subset P_I(\mathbb{Q})$ for all I. Thus if $\gamma \in P_I(\mathbb{Q})$ and $\eta(I) = I$ we get $\mathcal{N}(g_\eta\gamma) \in P_I(\mathbb{Q})$, since we have $\eta_1(P_I(\mathbb{Q})) = P_{\eta(I)}(\mathbb{Q})$.

Parametrization of fixed point sets

2.22. Let $g_\eta\gamma \in G(\mathbb{Q})$ be a representative of a fixed η-conjugacy class, where $\gamma \in P_I(\mathbb{Q})$. Define as a subset of $F(\mathcal{H})$:

$$F(\mathcal{H})_{I,\gamma} \quad = \quad \left\{ \begin{matrix} \text{class of} \\ x = (p, y, g_f) \end{matrix} \left| \begin{matrix} p \in P_I(\mathbb{R}),\ y = sign(x) \text{ such that there exist} \\ k_\infty \in K^I_\infty,\ z_\infty \in Z_\infty, a \in A_I,\ k_f \in K'_f \\ \text{satisfying (1),(2),(3) in (2.10) for this } \gamma \end{matrix} \right. \right\}$$

The condition (1) means that pL^I_∞ is invariant under $\tilde\eta_{\gamma,h_\infty}$ as an element of $P_I(\mathbb{R})/L^I_\infty$. We recall the condition (3'):

$$\eta(g_f)^{-1}(g_\eta\gamma)g_f \quad \in \quad h_f K_f.$$

In this condition we can replace g_f by $b_f g_f k_f$ for $k_f \in K'_f = K_f \cap \eta^{-1}(h_f K_f h_f^{-1})$ and $b_f \in G_{\gamma,\eta}(\mathbb{A}_f)$. Thus we can arrange with respect to the double cosets in $G_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K'_f$. Recall that $G_{\gamma,\eta} = G^{\eta_\gamma} = \left\{x \in G | \eta(x)^{-1}(g_\eta\gamma)x = g_\eta\gamma\right\}$ denotes the η-centralizer of $g_\eta\gamma$.

Now we fix some representative g_f of a double coset in $G_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K'_f$ satisfying $\eta(g_f)^{-1}(g_\eta\gamma)g_f \in h_f K_f$ and denote the corresponding set of fixed points $F(\mathcal{H})_{I,\gamma,g_f}$. By (2.18) we get a surjective map

$$\begin{array}{rcl} \left(G^I_{\gamma,\eta}(\mathbb{R})/L_{\gamma,\eta}\right) \times R^I_{\gamma,\eta} \times (\Sigma^\Delta)_{I,J'} \times G_{\gamma,\eta}(\mathbb{A}_f) & \to & F(\mathcal{H})_{I,\gamma,g_f} \\ (p, k_a, y, b_f) & \mapsto & (pk_a p_1, y, b_f g_f) \end{array}$$

where

$$(\Sigma^\Delta)_{I,J'} \quad = \quad \left\{y \in \Sigma^\Delta \text{ such that } supp(y) = I,\ \eta(y)\cdot s_{J'} = y\right\}$$

and p_1 is the element introduced in (2.12). We remark that (p, k_a, y, b_f) and (p', k_b, y', b'_f) have the same image in $F(\mathcal{H})_{I,\gamma,g_f}$ if and only if there exist $\delta \in P_I(\mathbb{Q}), \varkappa_\infty \in K^I_\infty,\ \zeta_\infty \in Z_\infty,\ a_\infty \in A_I,\ \varkappa_f \in K'_f$ such that

$$\begin{array}{rcl} pk_a p_1 & = & \delta \cdot p' k_b p_1 \cdot \varkappa_\infty^{-1}\zeta_\infty^{-1}a_\infty^{-1} \\ y & = & a_\infty \cdot y' \\ b_f g_f & = & \delta \cdot b'_f g_f \cdot \varkappa_f \end{array} \tag{27}$$

Observe that the second equation is equivalent to $y = y'$, since $a_\infty \in A_I$ and $supp(y) = supp(y') = I$.

As an equation in the coset space $P_I(\mathbb{R})/\tilde L$ the first equation can be restated as follows: $pk_a = \delta \cdot p'k_b$. Since $\eta_\gamma(\tilde L) = \tilde L$ and since we know from (2.18) that pk_a and $p'k_b$ are η_γ-invariant in the coset space we conclude that the following computation is valid in $P_I(\mathbb{R})/\tilde L$:

$$pk_a \;=\; \eta_\gamma(pk_a) \;=\; \eta_\gamma(\delta)\cdot\eta_\gamma(p'k_b) \;=\; \eta_\gamma(\delta)\cdot p'k_b \;=\; \eta_\gamma(\delta)\cdot\delta^{-1}\cdot pk_a.$$

Similarly we deduce from the third equation, thereby bearing in mind that $\eta(g_f)^{-1}\cdot g_\eta\gamma\cdot g_f = h_f\cdot k_f$ and $\eta(g_f\varkappa_f)^{-1}\cdot g_\eta\gamma\cdot g_f = h_f\cdot\tilde k_f$ with $k_f, \tilde k_f \in K_f$ so that $\eta_\gamma(g_f) =$

$g_f \cdot k_f^{-1} h_f^{-1} \cdot g_\eta \gamma$ and $\eta_\gamma(g_f \varkappa_f) = g_f \varkappa_f \tilde{k}_f^{-1} h_f^{-1} \cdot g_\eta \gamma$:

$$\begin{array}{rcll} b_f \cdot \eta_\gamma(g_f) & = & \eta_\gamma(b_f g_f) \quad = \quad \eta_\gamma(\delta \cdot b'_f \cdot g_f \cdot \varkappa_f) & \\ & = & \eta_\gamma(\delta) \cdot b'_f \cdot \eta_\gamma(g_f \varkappa_f) \qquad \text{and therefore} & \\ b_f g_f k_f^{-1} h_f^{-1} g_\eta \gamma & = & \eta_\gamma(\delta) \cdot b'_f \cdot g_f \varkappa_f \tilde{k}_f^{-1} h_f^{-1} \cdot g_\eta \gamma, \qquad \text{i.e.} & \\ b_f g_f & = & \eta_\gamma(\delta) \cdot b'_f \cdot g_f \varkappa_f \cdot \tilde{k}_f^{-1} \cdot k_f \quad = \quad \eta_\gamma(\delta) \cdot \delta^{-1} \cdot b_f g_f \cdot \tilde{k}_f^{-1} k_f. & \end{array}$$

Thus the element $\eta_\gamma(\delta) \cdot \delta^{-1}$ transforms the pair $(pk_a, b_f g_f)$ into itself as an element of $\left(P_I(\mathbb{R})/\tilde{L}\right) \times G(\mathbb{A}_f)/K_f$. By lemma 1.27 we deduce frow this:

$$c_1(\delta) \quad := \quad \eta_\gamma(\delta) \cdot \delta^{-1} \quad \in \zeta \quad \subset Z_G(\mathbb{Q}).$$

The element δ above is only unique up to elements of ζ. Since we have $\eta_\gamma(\varepsilon) = \eta(\varepsilon)$ for $\varepsilon \in \zeta$, we conclude: $c_1(\delta\varepsilon) = c_1(\delta) \cdot \eta(\varepsilon) \cdot \varepsilon^{-1}$. Furthermore $\mathcal{N}(c_1(\delta)) = \eta^{n-1}(c_1(\delta)) \cdots \eta(c_1(\delta)) \cdot c_1(\delta) = \eta_\gamma^{n-1}(c_1(\delta)) \cdots \eta_\gamma(c_1(\delta)) \cdot c_1(\delta) = \eta_\gamma^n(\delta) \cdot \delta^{-1}$. But we have

$$\begin{array}{rcl} \eta_\gamma^n(\delta) & = & (g_\eta\gamma)^{-1} \cdot \eta(g_\eta\gamma)^{-1} \cdots \eta^{n-1}(g_\eta\gamma)^{-1} \cdot \eta^n(\delta) \cdot \eta^{n-1}(g_\eta\gamma) \cdots (g_\eta\gamma) \\ & = & \mathcal{N}(g_\eta\gamma)^{-1} \cdot \delta \cdot \mathcal{N}(g_\eta\gamma). \end{array}$$

This means

$$\mathcal{N}\left(c_1(\delta)\right) \quad = \quad \mathcal{N}(g_\eta\gamma)^{-1} \cdot \delta \cdot \mathcal{N}(g_\eta\gamma) \cdot \delta^{-1}. \tag{28}$$

2.23. Now, if we assume conversely that $\eta_\gamma(\delta) \cdot \delta^{-1} \in \zeta$, it can easily be seen that $(pk_a p_1, y, b_f g_f) \in F(\mathcal{H})_{I,\gamma,g_f}$ implies $(\delta \cdot pk_a p_1, y, \delta \cdot b_f k_f) \in F(\mathcal{H})_{I,\gamma,g_f}$.

The condition (28) implies that $\mathcal{N}\left(c_1(\delta)\right)$ lies in the derived group $G^{(1)}$ of G. But the intersection $G^{(1)} \cap Z_G$ is finite. If we assume that K_f and therefore also ζ are sufficiently small, the following assumption is fulfilled:

$$\zeta \cap G^{(1)}(\mathbb{Q}) \quad = \quad \{1\}\,. \tag{$Ass_{\zeta,der}$}$$

The assumption implies $\mathcal{N}\left(c_1(\delta)\right) = 1$. If we identify 1-cocycles for the finite cyclic group $\langle\eta\rangle$ with their values at η, this means that $c_1(\delta)$ represents a class in $H^1(\langle\eta\rangle, \zeta)$.
We make the further assumption:

$$H^1(\langle\eta\rangle, \zeta) \quad = \quad 1. \tag{29}$$

This is satisfied for example if $\eta = id$ or if $\zeta = \{1\}$. If (29) is valid we can assume without loss of generality that $\eta_\gamma(\delta) = \delta$. Thus $\delta \in G^I_{\gamma,\eta}(\mathbb{Q})$. The third equation of (27) now implies

$$\begin{array}{rcl} \varkappa_f & = & g_f^{-1} \cdot ((b'_f)^{-1} \cdot \delta^{-1} \cdot b_f) \cdot g_f \in K_f \cap g_f^{-1} G_{\gamma,\eta}(\mathbb{A}_f) g_f \qquad \text{resp.} \\ g_f \varkappa_f g_f^{-1} & \in & G_{\gamma,\eta}(\mathbb{A}_f) \cap g_f K_f g_f^{-1}. \end{array}$$

2.24. Summary. Under the assumption $H^1(\langle\eta\rangle, \zeta) = 1$ the following map α is an isomorphism:

$$\begin{array}{rcl} \alpha : X^I_{\gamma,\eta}(g_f) \times R^I_{\gamma,\eta} \times (\Sigma^\Delta)_{I,J'} & \to & F(\mathcal{H})_{I,\gamma,g_f} \\ ((p, b_f), k_a, y) & \mapsto & (pk_a p_1, y, b_f g_f) \end{array}$$

where $X^I_{\gamma,\eta}(g_f) = G^I_{\gamma,\eta}(\mathbb{Q})\backslash \left(G^I_{\gamma,\eta}(\mathbb{R})/\tilde{L}_{\gamma,\eta} \times G_{\gamma,\eta}(\mathbb{A}_f)/(G_{\gamma,\eta}(\mathbb{A}_f)\cap g_f K'_f g_f^{-1})\right)$.

If the group $H^1(\langle\eta\rangle,\zeta)$ is not trivial, it is still finite and the map α is still surjective. By the considerations above α is a finite covering, and the degree $d^I_{\zeta,\gamma}$ of the covering is

$$d^I_{\zeta,\gamma} \quad = \quad \#\left\{x \in H^1(\langle\eta\rangle,\zeta) \mid x = \eta_\gamma(\delta)\cdot\delta^{-1} \text{ with } \delta \in P_I(\mathbb{Q})\right\}.$$

The set of fixed points $F(\mathcal{H})$ is stratified by the strata $F(\mathcal{H})_I$ for those $I \subset \Delta$ which satisfy $\eta(I) = I$. Each $F(\mathcal{H})_I$ is a union of $F(\mathcal{H})_{I,\gamma}$ over those η-conjugacy classes of elements γ in $G(\mathbb{Q})/\zeta$, for which $\mathcal{N}(g_\eta\gamma)$ is conjugate in $P_I(\mathbb{R})$ to an element of $\mathcal{N}_2(h_\infty)\cdot K^I_\infty Z_\infty A_I$.

Each $F(\mathcal{H})_{I,\gamma}$ itself is the union of $F(\mathcal{H})_{I,\gamma,g_f}$, where g_f runs over a set of representatives for those double cosets in $G_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K'_f$, which satisfy $\eta(g_f)^{-1}(g_\eta\gamma)g_f \in h_f K_f$.

3. The Lefschetz fixed point formula

A general fixed point formula for manifolds

3.1. Consider a pair of differentiable maps $f, g : X \to Y$ between compact oriented differentiable manifolds X and Y,such that g is locally a diffeomorphism. Let a local system $\mathcal{M}$ on Y be given and also a morphism

$$\varphi: \; f^*\mathcal{M} \quad \to g^!\mathcal{M}.$$

Denote by Γ_f, $\Gamma_g \subset X\times Y$ the graphs, and consider the decomposition

$$\Gamma_f\cap\Gamma_g \simeq F(f,g) := \{x\in X | f(x) = g(x)\} = \bigcup_{j\in J} F_j$$

of the set of fixed points $F(f,g)$ into connected components. We assume that the intersection of Γ_f and Γ_g is *transversal* in the following sense:

- each F_j is a differentiable submanifold of X and
- for each $x \in F_j$ we have the following relation between the tangent spaces in the point $(x,y) \in X\times Y$:

$$T_{(x,y)}\Gamma_f \quad \cap \quad T_{(x,y)}\Gamma_g \quad = \quad T_{(x,y)}\left(\Gamma_f\cap\Gamma_g\right).$$

The *global trace* of the correspondence (f,g,φ) is defined to be:

$$\begin{aligned} tr(g_*f^*) \quad &= \quad \sum_{i\geqslant 0}(-1)^i tr^i(g_*f^*) \qquad \text{where} \\ tr^i(g_*f^*) \quad &= \quad tr\left(H^i(Y,\mathcal{M}) \xrightarrow{f^*} H^i(X,f^*\mathcal{M}) \xrightarrow{\varphi} H^i(X,g^*\mathcal{M}) \xrightarrow{g_*} H^i(X,\mathcal{M})\right). \end{aligned}$$

For $x\in F_j$ we have an identification of the stalks $(f^*\mathcal{M})_x \simeq (g^!\mathcal{M})_x$ so that φ_x can be considered as an endomorphism of $(f^*\mathcal{M})_x \simeq \mathcal{M}_{f(x)}$ and thus has a trace. Since $\mathcal{M}$ is a local system, this trace is constant on each connected component F_j and is denoted by $tr(\varphi|F_j)$. We denote by

$$\chi(F_j) = \sum_{i\geqslant 0}(-1)^i\cdot\dim_{\mathbb{Q}}(H^i(F_j,\mathbb{Q}))$$

the Euler Poincaré characteristic of F_j. Let $N(F_j)$ denote the normal bundle of F_j inside X, i.e. $N_x(F_j) = T_xX/T_xF_j$ for $x \in F_j$. By the transversality assumption we have $\det(id - f_*g^*|N_xF_j) \neq 0$ for all $x \in F_j$. Since this real number depends continuously on x, we get a well defined sign

$$\varepsilon_j \quad = \quad sign\,(\det(id - f_*g^*|N(F_j))) \qquad \text{for each } j \in J.$$

Remark 3.2. The transversality assumptions imply that each fixed point component F_j has an open neighborhood $\mathcal{U}_j$ which meets no other fixed point component F_k. This implies that J is a finite set by the compactness of X. Therefore all sums occuring in the following are finite sums and we have no problems with convergence.

We can state the Lefschetz fixed point formula:

Theorem 3.3. *With the above notations and assumptions we have*

$$tr(g_*f^*) \quad = \quad \sum_{j\in J} \quad tr(\varphi|F_j) \cdot \chi(F_j) \cdot \varepsilon_j$$

Proof: The fixed point theorem is well known, if the F_j are isolated points. If F_j is a manifold of positive dimension, one reduces to this case by considering a vector field ξ_j on F_j, which has isolated and non degenerate zeros $\{x_i\}$, and extends ξ_j to a vectorfield ξ_j with support in an open tubular neighborhood $\mathcal{U}_j$ of F_j, such that $\overline{\mathcal{U}}_j$ meets no other $\overline{\mathcal{U}}_k$. If one modifies $f =: f_0$ to the homotopic $f_t = f_0 \circ \exp(t\xi_j)$ for a small enough $t > 0$, one does not change $tr(g_*f^*)$, but $F(f_t, g) \cap \mathcal{U}_j$ consists of a set of isolated fixed points $\{x_i\}$. Recall that $\chi(F_j)$ equals the number of x_i counted with an appropriate sign, We leave it as an exercise to the reader that the right hand side of the theorem does not change too. $\square$

The general setting

3.4. The local systems $\mathcal{M}$. Let M be a $(G(\mathbb{Q})/\zeta) \rtimes \langle\eta\rangle$-module. This gives rise to a local coefficient system $\mathcal{M}$ on $X(K_f)$ for each open compact K_f. We can obtain $\mathcal{M}$ as the quotient $\mathcal{M} = G(\mathbb{Q})\backslash M \times X \times G(\mathbb{A}_f)/K_f$, where we use the $G(\mathbb{Q})$-action on M and on X, together with the canonical projection to $X(K_f) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f$. Furthermore we consider the following sheaf on $X(K_f)$:

$$\mathcal{M}(U) \quad = \quad \left\{ \phi : \pi^{-1}(U) \to M \,\middle|\, \begin{matrix} \phi \text{ locally constant;} \quad \phi(\gamma x) = \gamma\phi(x) \\ \text{for all } \gamma \in G(\mathbb{Q}), x \in X \times G(\mathbb{A}_f)/K_f \end{matrix} \right\},$$

for $U \subset X(K_f)$ open, where $\pi : X \times G(\mathbb{A}_f)/K_f \to X(K_f)$ denotes the canonical projection. If the action of $G(\mathbb{Q})$ on $X \times G(\mathbb{A}_f)/K_f$ is free of fixed points then the sheaf $\mathcal{M}$ can be considered as the sheaf of local sections of the map from the space $\mathcal{M}$ to $X(K_f)$.

3.5. For $J \subset \Delta$ we denote the inverse image of $\{0,1\}^J \times \{1\}^{\Delta\backslash J}$ inside $X_{BS}(K_f)$ by $X^J_{BS}(K_f)$, and we denote the inclusion maps by $i^J : X_{sp}(K_f) \hookrightarrow X^J_{BS}(K_f)$ and $\tilde{i}^J : X^J_{BS}(K_f) \hookrightarrow X_{BS}(K_f)$, where the space called $X_{sp}(K_f)$ in 1.40 is $X^{\emptyset}_{BS}(K_f)$ in the new notation.

For a sheaf $\mathcal{M}$ as above we denote its restriction to the subspace $X_{sp}(K_f)$ by $\mathcal{M}_{sp}$. We introduce the sheaf $i_{*J!}\mathcal{M} := \tilde{i}^{\Delta-J}_! i^{\Delta-J}_* \mathcal{M}_{sp}$ on $X_{BS}(K_f)$.

If $\pi : X(K_f) \twoheadrightarrow X(K_f)/S^\Delta \simeq X_{BS}(K_f)$ denotes the canonical projection, then the sheaf $\pi_*\mathcal{M}$ on $X_{BS}(K_f)$ is a sheaf with an action of S^Δ. If multiplication by 2 is an automorphism of $\mathcal{M}$ then we may decompose $\pi_*\mathcal{M}$ into eigenspaces (eigensubsheaves) of the reflection group S^Δ.

The sign group $S^\Delta = \{-1, +1\}^\Delta$ may be identified with its dual group in such a way that $s_J \in S^\Delta$ may be identified with the character $S^\Delta \ni (r_\alpha)_{\alpha\in\Delta} \mapsto \prod_{\alpha\in J} r_\alpha$.

Lemma 3.6. *The eigensubsheaf of $\pi_*\mathcal{M}$ with respect to the character s_J of S^Δ is isomorphic to the sheaf $i_{*J!}\mathcal{M}_{sp}$.*

It is clear that the restriction of $\pi_*\mathcal{M}$ to $X_{sp}(K_f)$ is isomorphic to the tensor product of $\mathcal{M}_{sp}$ with the group ring $\mathbb{Z}[S^\Delta]$ such that S^Δ acts on the group ring. The eigensubsheaf of $\pi_*\mathcal{M}$ with respect to the character s_J is the subsheaf on which the reflection s_α act by -1 for $\alpha \in J$ and by $+1$ for $\alpha \notin J$. Then it becomes clear that the eigensubsheaf continues as direct image for the embedding $i^{\Delta-J}$, while it has to be continued by 0 for the embedding $\tilde{i}^{\Delta-J}$.

□

From the introduction we recall the notation χ_{-1} for the character $s_\Delta : (r_\alpha)_{\alpha\in\Delta} \mapsto \prod_{\alpha\in\Delta} r_\alpha$.

Proposition 3.7. *The Lefschetz number on the cohomology with compact support satifies*

$$\begin{aligned} & tr\left((h_\infty \times h_f) \circ \eta, H_c^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) \\ = \quad & 2^{-\#\Delta} \cdot \sum_{s_{J'} \in S^\Delta} \chi_{-1}(s_{J'}) \cdot tr\left(\mathcal{H}(s_{J'}), H^*(X(K_f), \mathcal{M})\right) \end{aligned}$$

Proof: We have an isomorphism which is equivariant with respect to the action of $(h_\infty \times h_f) \circ \eta$:

$$\begin{aligned} H_c^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right) &= H_c^*\left(X_{sp}(K_f), \mathcal{M}_{sp}\right) \\ &\cong H^*\left(X_{BS}(K_f), i_!^\Delta \mathcal{M}_{sp}\right). \end{aligned}$$

where we used the fact that the cohomology with compact support may be computed as the cohomology of the sheaf $i_!^\Delta\mathcal{M}_{sp}$ on the Borel-Serre compactification $X_{BS}(K_f)$. Observing $X_{BS}^\Delta(K_f) = X_{BS}(K_f)$ so that $i_{*\Delta!}\mathcal{M}_{sp} = i_!^\Delta\mathcal{M}_{sp}$ and the preceding lemma we thus get:

$$\begin{aligned} & tr\left((h_\infty \times h_f) \circ \eta, H_c^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) \\ = \quad & tr\left((h_\infty \times h_f) \circ \eta, H^*\left(X_{BS}(K_f), i_{*\Delta!}\mathcal{M}_{sp}\right)\right) \\ = \quad & tr\left((h_\infty \times h_f) \circ \eta, H^*\left(X_{BS}(K_f), (\pi_*\mathcal{M})^{\chi_{-1}}\right)\right) \\ = \quad & 2^{-\#\Delta} \cdot \sum_{s_{J'} \in S^\Delta} \chi_{-1}(s_{J'}) \cdot tr\left((h_\infty \times h_f) \circ \eta \times s_{J'}, H^*\left(X_{BS}(K_f), \pi_*\mathcal{M}\right)\right) \end{aligned}$$

where $s_{J'}$ only acts on the sheaf $\pi_*\mathcal{M}$ in the last line so that it commutes with the action of $(h_\infty \times h_f) \circ \eta$. Here we used the fact that the trace of an operator on an S^Δ eigenspace may be computed as the composition of the operator acting on the whole space with a projector onto this eigenspace which is $2^{-\#\Delta} \cdot \sum_{s_{J'} \in S^\Delta} \chi_{-1}(s_{J'}) \cdot s_{J'}$ in our case. Raising the action to the space $X(K_f)$ now gives

$$= \quad 2^{-\#\Delta} \cdot \sum_{s_{J'} \in S^\Delta} \chi_{-1}(s_{J'}) \cdot tr\left((h_\infty \times h_f \times s_{J'}) \circ \eta, H^*\left(X(K_f), \mathcal{M}\right)\right)$$

where $s_{J'}$ now acts on the space in the last line. The definition of $\mathcal{H}(s_{J'})$ in 2.8 now implies the claim. $\square$

Euler characteristics

3.8. We continue with the considerations of chapter 2. The Euler characteristic with compact support satisfies:

$$\chi_c\left(F(\mathcal{H})_{I,\gamma}\right) \quad = \sum_{\substack{g_f \in G^I_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)/K'_f \\ \eta(g_f)^{-1} \cdot g_\eta \gamma \cdot g_f \in h_f K_f}} \chi_c\left(F(\mathcal{H})_{I,\gamma,g_f}\right)$$

$$= \quad \frac{\# R^I_{\gamma,\eta} \cdot c_{I,J'}}{d^I_{\zeta,\gamma}} \cdot \sum_{g_f \text{ as above}} \chi_c\left(X^I_{\gamma,\eta}(g_f)\right)$$

where $X^I_{\gamma,\eta}(g_f) = G^I_{\gamma,\eta}(\mathbb{Q})\backslash \left(G^I_{\gamma,\eta}(\mathbb{R})/\tilde{L}^I_{\gamma,\eta} \times G^I_{\gamma,\eta}(\mathbb{A}_f)/(G^I_{\gamma,\eta}(\mathbb{A}_f) \cap g_f K'_f g_f^{-1})\right)$ and $c_{I,J'} = \#(\Sigma^\Delta)_{I,J'} = \#\left\{y \in \{-1,1\}^I \times \{0\}^{\Delta - I} \,|\eta(y) s_{J'} = y\right\}$.

Let dg_f be a Haar measure on $G(\mathbb{A}_f)$ and denote by $db = db_\infty \cdot db_f$ a Tamagawa measure on the group $\tilde{G} = G^I_{\gamma,\eta}$. Let $\tilde{h}$ denote the characteristic function of $h_f K_f$ multiplied with $(vol_{dg_f}(K'_f))^{-1}$. From the definition of a quotient measure we get immediately

$$\chi_c\left(F(\mathcal{H})_{I,\gamma}\right) \quad = \quad \frac{\# R_{\gamma,\eta} \cdot c_{I,J'}}{d^I_{\zeta,\gamma}} \cdot \tag{30}$$

$$\int_{G_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} \chi_c\left(X^I_{\gamma,\eta}(g_f)\right) \cdot vol_{db_f}(G_{\gamma,\eta}(\mathbb{A}_f) \cap g_f K'_f g_f^{-1}) \cdot \tilde{h}\left(\eta(g_f)^{-1}(g_\eta\gamma)g_f\right)\, db_f\backslash dg_f,$$

3.9. The Gauss-Bonnet formula. We furthermore put $\tilde{K}_f = G^I_{\gamma,\eta}(\mathbb{A}_f) \cap g_f K'_f g_f^{-1}$, $\tilde{K}_\infty = \tilde{L}^I_{\gamma,\eta}$. Now we are in the situation, where $\tilde{G} = G^I_{\gamma,\eta}$ is a linear algebraic group, $\tilde{K}_f \subset \tilde{G}(\mathbb{A}_f)$ is open, compact and sufficiently small and the connected component of $\tilde{K}_\infty \subset \tilde{G}(\mathbb{R})$ is the product of some maximal connected and compact subgroup with a connected subgroup $\tilde{Z}_\infty$ of the $\mathbb{R}$-split center $Z^{\mathbb{R}-split}_{\tilde{G}}$ such that $\tilde{Z}_\infty$ contains the connected component of the $\mathbb{R}$-split and $\mathbb{Q}$-anisotropic torus $\bigcap_\chi \ker\chi \cap Z^{\mathbb{R}-split}_{\tilde{G}}$, where $\chi \in X^*(Z_{\tilde{G}})$ runs over all $\mathbb{Q}$-rational characters of $Z_{\tilde{G}}$.

We furthermore put $\tilde{K} = \tilde{K}_\infty \cdot \tilde{K}_f$.

We make the assumption:

$$((Ass_{conn})) \qquad \tilde{G} \text{ is a connected group if it is reductive.}$$

$$
\begin{aligned}
D(\tilde{G}) &= 0 && \text{if } \tilde{G} \text{ is not reductive or does not have a Cartan,} \\
& && \text{which is compact modulo the center of } \tilde{G}(\mathbb{R}), \\
D(\tilde{G}) &= \frac{\#W(\tilde{G}/\mathbb{C}, T/\mathbb{C})}{\#N_{\tilde{G}(\mathbb{R})}(T)/T} && \text{if } \tilde{G} \text{ is reductive and } T \subset \tilde{K}_\infty \cdot Z_{\tilde{G}}(\mathbb{R}) \text{ is a} \\
& && \text{maximal torus, which is compact modulo } Z_{\tilde{G}}(\mathbb{R}).
\end{aligned}
$$

If $D(\tilde{G}) \neq 0$ then the adjoint group $\tilde{G}_{ad}$ has a compact Cartan, and we can denote by $\overline{G}$ the inner form of $\tilde{G}/\mathbb{R}$ which is compact modulo the center of $\tilde{G}$. We do not care about the definition of $\overline{G}$ if $D(\tilde{G}) = 0$.

The Haar measure db_∞ on $\tilde{G}(\mathbb{R})$ determines uniquely a Haar measure on $\overline{G}(\mathbb{R})$, which will be denoted by db_∞ also. The isomorphism between $\tilde{G}\times_{\mathbb{R}}\mathbb{C}$ and $\overline{G}\times_{\mathbb{R}}\mathbb{C}$ determines canonical isomorphisms over $\mathbb{R}$ between the centers $Z_{\tilde{G}}$ of $\tilde{G}$ and $Z_{\overline{G}}$ of $\overline{G}$ and also between the torus quotients $\tilde{G}/\tilde{G}^{(1)}$ and $\overline{G}/\overline{G}^{(1)}$. Each rational character $\chi \in X^*(\tilde{G}) : \tilde{G} \to \tilde{G}/\tilde{G}^{(1)} \to \mathbb{G}_m$ may thus be viewed as a character from $\overline{G} \to \overline{G}/\overline{G}^{(1)} \to \mathbb{G}_m$ and we may define $\overline{G}'$ to be the intersection of the kernels of these characters. Using some basis $\chi_1, \ldots, \chi_r$ of $X^*(\tilde{G})$ the Haar measure db_∞ may be written as the product of some Haar measure db'_∞ on $\overline{G}'(\mathbb{R})$ and the euclidean measure $\prod_{i=1}^r d^*x_i$ on $(\mathbb{R}^*)^r$, the image of $\overline{G}(\mathbb{R})$ under $(\chi_1, \ldots, \chi_r)$. Also we may view $\tilde{\zeta} = Z_{\tilde{G}}(\mathbb{Q}) \cap \tilde{K}$ as a subgroup of $\overline{G}'(\mathbb{R})$.

We denote by $\tau(\tilde{G})$ the Tamagawa number of $\tilde{G}/\mathbb{Q}$, by

$$q(\tilde{G}) = \dim\left(\tilde{G}^{(1)}(\mathbb{R})/(\tilde{L}^I_{\gamma,\eta} \cap \tilde{G}^{(1)}(\mathbb{R}))\right)$$

the dimension of the symmetric space associated to the derived group of $\tilde{G}$. Furthermore we consider the dimension

$$\Delta(\tilde{G}, \tilde{K}_\infty) = \dim(\tilde{G}(\mathbb{R})/\tilde{K}_\infty) - q(\tilde{G}) = \dim\left(Z_{\tilde{G}}^{\mathbb{R}-split}\right) - \dim \tilde{Z}_\infty.$$

Now we may state the following extension of Harder's Gauss-Bonnet formula ([Har1]) to reductive groups:

PROPOSITION 3.10. *If $\tilde{G}$ satisfies (Ass_{conn}) then*

$$\chi_c\left(\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K}\right) \cdot vol_{db_f}(\tilde{K}_f) = (-1)^{\Delta(\tilde{G},\tilde{K}_\infty)+\frac{1}{2}q(\tilde{G})} \cdot \frac{D(\tilde{G}) \cdot \tau(\tilde{G})}{vol_{db'_\infty}(\overline{G}'(\mathbb{R})/\tilde{\zeta})}$$

Proof: This is well known if $\tilde{G}$ is semisimple (compare [Roh, 3.3.]: his statement agrees with ours in the case that the torus quotient is anisotropic over $\mathbb{R}$. In the case that the central unit group $\tilde{\zeta}$ has positive rank the statement of Rohlfs simply reads $0 = 0$, since his symmetric space is a torus bundle, while our identity may be non trivial due to the fact that $\tilde{K}_\infty$ contains the connected component of the center of $G(\mathbb{R})$).

If the unipotent radical of $\tilde{G}$ is not trivial, then the Euler characteristic of the symmetric space vanishes, since it is a (topological) torus bundle, and the formula is clear from the definition of $D(\tilde{G})$.

If $\tilde{G}$ is a torus, then we have $q(\tilde{G}) = 0$, $D(\tilde{G}) = 1$, $\overline{G} = \tilde{G}$ and the symmetric space $\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K}$ is a disjoint union over the index set $\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{G}(\mathbb{R})^\circ\tilde{K}_f$ of affine spaces of the form $(\mathbb{R}^*_{>0})^{\Delta(\tilde{G},\tilde{K}_\infty)}$. The formula is thus equivalent to

$$\#\left(\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{G}(\mathbb{R})^\circ\tilde{K}_f\right)\cdot vol_{db_f}(\tilde{K}_f)\cdot vol_{db'_\infty}(\tilde{G}'(\mathbb{R})/\tilde{\zeta}) \quad = \quad \tau(\tilde{G}).$$

But if $t_1,\ldots,t_r \in \tilde{G}(\mathbb{A})$ denotes a set of representatives for the double coset space $\left(\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{G}(\mathbb{R})^\circ\tilde{K}_f\right)$, then we have an isomorphism

$$\begin{array}{ccc} \bigcup\limits_{i=1}^{r}(\tilde{G}'(\mathbb{R})/\tilde{\zeta}) & \longrightarrow & \left(\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K}_f\right)' \\ (g_\infty)_i & \mapsto & g_\infty\cdot t_i. \end{array}$$

The claim for tori is now clear from the definitions of measures.

So it remains to prove the formula for a general connected reductive group $\tilde{G}$. We reduce the claim to the semisimple and to the torus case using an exact sequence

$$1 \to \tilde{G}^{(1)} \to \tilde{G} \xrightarrow{\nu} C \to 1,$$

where the derived group $\tilde{G}^{(1)}$ is semisimple and C is a torus. We have $q(\tilde{G}) = q(\tilde{G}^{(1)})$, $D(\tilde{G}) = D(\tilde{G}^{(1)})$ and $\Delta(\tilde{G},\tilde{K}_\infty) = \Delta(C,\nu(\tilde{K}_\infty))$. The role of $\tilde{K}$ for the torus C will be played by $\nu(\tilde{K})$. We may replace without loss of generality $\tilde{Z}_\infty$ by the connected component of $\tilde{Z}_\infty^{\mathbb{R}-split}$, since this operation multiplies both sides of the formula with $(-1)^{\Delta(\tilde{G},\tilde{K}_\infty)}$. Then ν induces a surjection to a finite set

$$\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K} \quad \xrightarrow{\nu} \quad \nu(\tilde{G}(\mathbb{A}))/\nu(\tilde{G}(\mathbb{Q}))\nu(\tilde{K}).$$

The fibre over the class of some $\nu(t) \in \nu(\tilde{G}(\mathbb{A}))$ is obviously the image of the map

$$\begin{array}{rcl} \epsilon_t : \tilde{G}^{(1)}(\mathbb{Q})\backslash\tilde{G}^{(1)}(\mathbb{A})/\tilde{K}_t^{(1)} & \to & \tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K} \\ \epsilon_t : g & \mapsto & gt \end{array}$$

with $\tilde{K}_t^{(1)} = \tilde{G}^{(1)}(\mathbb{A}) \cap t\tilde{K}t^{-1}$. But ϵ_t is in general not injective: From $g_1t = \gamma\cdot g_2t\cdot k$ with $g_1, g_2 \in \tilde{G}^{(1)}$, $\gamma \in \tilde{G}(\mathbb{Q})$ and $k \in \tilde{K}$ we conclude that $\nu(\gamma^{-1}) = \nu(k)$,i.e. $\nu(\gamma) \in \tilde{\zeta}_1 = \nu(\tilde{G}(\mathbb{Q})) \cap \nu(\tilde{K})$, but to modify γ to an element in $\tilde{G}^{(1)}(\mathbb{Q})$ it would be necessary to have $\nu(\gamma) \in \nu(\tilde{\zeta})$ (Recall that $\tilde{G}(\mathbb{Q}) \cap \tilde{K} = \tilde{\zeta}$, since K_f is assumed to be sufficiently small.) In fact it is easy to see, that ϵ_t is a covering with covering group $\tilde{\zeta}_1/\nu(\tilde{\zeta})$. Therefore

$$\chi_c\left(\tilde{G}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A})/\tilde{K}\right) \quad = \quad \sum_{t\in\nu(\tilde{G}(\mathbb{A}))/\nu(\tilde{G}(\mathbb{Q}))\nu(\tilde{K})} \frac{\chi_c\left(\tilde{G}^{(1)}(\mathbb{Q})\backslash\tilde{G}^{(1)}(\mathbb{A})/\tilde{K}_t^{(1)}\right)}{\#(\tilde{\zeta}_1/\nu(\tilde{\zeta}))}.$$

Now we may assume that the Tamagawa measure dc on the torus C is the quotient of the Tamagawa measures db on $\tilde{G}$ and of db^1 on $\tilde{G}^{(1)}$. From the semisimple case and the definition of a quotient measure we get:

$$\begin{aligned}
&\chi_c\left(\tilde G(\mathbb{Q})\backslash\tilde G(\mathbb{A})/\tilde K\right)\cdot vol_{db_f}(\tilde K_f)\\
&= \sum_{t\in\nu(\tilde G(\mathbb{A}))/\nu(\tilde G(\mathbb{Q}))\nu(\tilde K)} \frac{vol_{db_f}(\tilde K_f)}{vol_{db_f^1}((\tilde K_t^{(1)})_f)}\cdot\frac{(-1)^{\frac12 q(\tilde G^{(1)})}\cdot D(\tilde G^{(1)})\cdot\tau(\tilde G^{(1)})}{vol_{db^1_\infty}(\overline{G}^{(1)}(\mathbb{R}))\cdot\#(\tilde\zeta_1/\nu(\tilde\zeta))}\\
&= \#\left(\nu(\tilde G(\mathbb{A}))/\nu(\tilde G(\mathbb{Q}))\nu(\tilde K)\right)\cdot vol_{dc_f}(\nu(\tilde K_f))\cdot\frac{(-1)^{\frac12 q(\tilde G)}\cdot D(\tilde G)\cdot\tau(\tilde G^{(1)})}{vol_{db^1_\infty}(\overline{G}^{(1)}(\mathbb{R}))\cdot\#(\tilde\zeta_1/\nu(\tilde\zeta))}.
\end{aligned}$$

In the following commutative diagram the columns are exact and the map μ_K is surjective:

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d]\\
\nu(\tilde K)/\nu(\tilde K)\cap\nu(\tilde G(\mathbb{Q})) \arrow[r, "\mu_K"] \arrow[d] & \nu(\tilde K)/\nu(\tilde K)\cap C(\mathbb{Q}) \arrow[d]\\
\nu(\tilde G(\mathbb{A}))/\nu(\tilde G(\mathbb{Q})) \arrow[r, "\mu"] \arrow[d] & C(\mathbb{A})/C(\mathbb{Q}) \arrow[d]\\
\nu(\tilde G(\mathbb{A}))/\nu(\tilde G(\mathbb{Q}))\nu(\tilde K) \arrow[r, "\mu_{sp}"] \arrow[d] & C(\mathbb{A})/C(\mathbb{Q})\nu(\tilde K) \arrow[d]\\
1 & 1.
\end{tikzcd}$$

Using the notion of an index $\quad ind(\mu) := \#coker(\mu)/\#\ker(\mu) \quad$ we get

$$ind(\mu_{sp}) = ind(\mu)\cdot\#\ker(\mu_K),$$

where $\ker(\mu_K) = \left(\nu(\tilde K)\cap C(\mathbb{Q})\right)/\left(\nu(\tilde K)\cap\nu(\tilde G(\mathbb{Q}))\right) = \zeta_2/\tilde\zeta_1$ with $\zeta_2 = C(\mathbb{Q})\cap\nu(\tilde K)$. From the torus case we conclude:

$$\begin{aligned}
&\#\left(\nu(\tilde G(\mathbb{A}))/\nu(\tilde G(\mathbb{Q}))\nu(\tilde K)\right)\cdot vol_{dc_f}(\nu(\tilde K_f))\\
&\quad= \frac{\#C(\mathbb{A})/C(\mathbb{Q})\nu(\tilde K)}{ind(\mu_{sp})}\cdot vol_{dc_f}(\nu(\tilde K_f)) \quad = \quad \frac{\tau(C)}{ind(\mu_{sp})\cdot vol_{dc'_\infty}(C'(\mathbb{R})/\zeta_2)}.
\end{aligned}$$

Now using the Tamagawa number relation ([Sans, 10.4.])

$$\tau(\tilde G^{(1)})\cdot\tau(C) = \tau(\tilde G)\cdot ind(\mu)$$

we may summarize:

$$\begin{aligned}
&\chi_c\left(\tilde G(\mathbb{Q})\backslash\tilde G(\mathbb{A})/\tilde K\right)\cdot vol_{db_f}(\tilde K_f)\\
&\qquad\qquad = (-1)^{\frac12 q(\tilde G)}\cdot\frac{D(\tilde G)\cdot\tau(\tilde G)}{vol_{db^1_\infty}(\overline{G}^{(1)}(\mathbb{R}))\cdot\#(\zeta_2/\nu(\tilde\zeta))\cdot vol_{dc'_\infty}(C'(\mathbb{R})/\zeta_2)}
\end{aligned}$$

and the claim is implied by the relation

$$vol_{db'_\infty}(\overline{G}'(\mathbb{R})/\tilde\zeta) = vol_{db^1_\infty}(\overline{G}^{(1)}(\mathbb{R}))\cdot\#(\zeta_2/\nu(\tilde\zeta))\cdot vol_{dc'_\infty}(C'(\mathbb{R})/\zeta_2).$$

$\square$

3.11. If we introduce the (η-)twisted orbital integral

$$O_\eta(\gamma,\tilde{h}) \quad = \quad \int_{G^I_{\gamma,\eta}(\mathbb{A}_f)\backslash G(\mathbb{A}_f)} \tilde{h}\left(\eta(g_f)^{-1}(g_\eta\gamma)g_f\right)\ db_f\backslash dg_f$$

we can thus rewrite the equation (30):

$$(31)\ \ \chi_c\left(F(\mathcal{H})_{I,\gamma}\right) \quad = \quad \frac{\# R^I_{\gamma,\eta}\cdot c_{I,J'}}{d^I_{\zeta,\gamma}}\cdot O_\eta(\gamma,\tilde{h})\cdot\ (-1)^{\Delta(\tilde{G},L_{\gamma,\eta})+\frac{1}{2}q(\tilde{G})}\cdot\frac{D(\tilde{G})\cdot\tau(\tilde{G})}{vol_{db_\infty}(\overline{G}'(\mathbb{R})/\zeta)}$$

Local analysis

3.12. We recall the map

$$\mathcal{H}:(p,y,g_f)\mapsto(\eta_1(p)p_\eta\cdot h_\infty,\eta(y)\cdot s_{J'},g_\eta^{-1}\eta(g_f)h_f)$$

Let $x_0=(p_0,y_0,g_f)$ be a point in $F(\mathcal{H})_{I,\gamma}$, i.e. there exist $k_\infty\in K^I_\infty,\ z_\infty\in Z_\infty,\ a\in A_I,\ k\in K_f$ such that:

$$\begin{aligned}
&(1) & g_\eta^{-1}\eta(p_0)g_\eta p_\eta h_\infty &= \gamma p_0 k_\infty^{-1}z_\infty^{-1}a^{-1}\\
&(2) & \eta(y_0)s_{J'} &= a^{-1}y_0\\
&(3) & g_\eta^{-1}\eta(g_f)h_f &= \gamma g_f k_f^{-1}.
\end{aligned}$$

We want to analyze the effect of $\mathcal{H}$ in a neighborhood of x_0:

$$\begin{aligned}
\mathcal{H}(pp_0,y_0+y,g_f) &= \left(\eta_1(p)\eta_1(p_0)p_\eta h_\infty,\eta(y_0+y)\cdot s_{J'},g_\eta^{-1}\eta(g_f)h_f\right)\\
&= \left(\eta_1(p)\gamma p_0\cdot k_\infty^{-1}z_\infty^{-1}a^{-1},a^{-1}\cdot y_0+\eta(y)\cdot s_{J'},\gamma\cdot g_f\cdot k_f^{-1}\right)\\
&\sim \left(\eta_\gamma(p)\cdot p_0,y_0+a\cdot\eta(y)\cdot s_{J'},g_f\right)
\end{aligned}$$

As in 1.20 we denote by ${}^0P_I$ the intersection of the kernels of all χ^2, where χ ranges over all characters $\chi:P_I\to P_I/Z_G\to\mathbb{G}_m$. Then there is a unique decomposition $P_I(\mathbb{R})={}^0P_I(\mathbb{R})\rtimes A_I$. We can write each $p\in P_I(\mathbb{R})$ in the form

$$p \quad = \quad p^0\cdot p_0 a(p)p_0^{-1} \qquad \text{where}\ \ p^0\in{}^0P_I(\mathbb{R}),\ a(p)\in A_I$$

(Apply the above decomposition to $p_0^{-1}pp_0$ and observe that ${}^0P_I$ is a normal subgroup of P_I.) Now we can write

$$\mathcal{H}\left(pp_0,y_0+y,g_f\right) \quad \sim \quad \left(\eta_\gamma(p)^0\cdot p_0,a\left(\eta_\gamma(p)\right)^{-1}\cdot\left(y_0+a\cdot\eta(y)\cdot s_{J'}\right),g_f\right)$$

We remark

$$\left(P_I(\mathbb{R})/K^I_\infty Z_\infty\times\mathbb{R}^{\Delta-I}\times\{\pm1\}^I\right)/A_I \quad\simeq\quad {}^0P_I(\mathbb{R})/K^I_\infty Z_\infty\times\mathbb{R}^{\Delta-I}\times\{\pm1\}^I.$$

Since $supp(x_0)=I$ we can assume that $y_0\in\{0\}^{\Delta-I}\times\{\pm1\}^I$. Then our equation reads:

$$\mathcal{H}\left(pp_0,y_0+y,g_f\right) \quad\sim\quad \left(\eta_\gamma(p)^0\cdot p_0,y_0+a\cdot a(\eta_\gamma(p))^{-1}\cdot\eta(y)\cdot s_{J'},g_f\right)$$

We identify the tangent space of $X(K'_f)$ at x_0 with $Ad(p_0)Lie({}^0P_I(\mathbb{R})/K^I_\infty Z_\infty)\times\mathbb{R}^{\Delta-I}$. The tangent space of $X(K_f)$ at $\kappa(x_0)=\mathcal{H}(x_0)$ can be identified with the same vector space, such that the differential of the canonical projection $\kappa:X(K'_f)\to X(K_f)$ becomes the identity. Here we use the notation $Lie(G/H) = Lie(G)/Lie(H)$, if $H \subset G$ is a Lie subgroup.

Then the differential of the map $\mathcal{H}$ in the point $x_0 = (p_0, y_0, g_f)$, which is the differential of the map $(p, y) \mapsto \mathcal{H}(pp_0, y_0 + y, g_f)$ in $(p, y) = (1, 0)$ is

- the differential of the map $p \mapsto \eta_\gamma(p)^0$ in the neutral element, considered as an endomorphism of $Ad(p_0)Lie({}^0P_I(\mathbb{R})/K^I_\infty Z_\infty)$

times

- the linear map $l : \mathbb{R}^{\Delta-I} \to \mathbb{R}^{\Delta-I}$, $y \mapsto a \cdot \eta(y) \cdot s_{J'}$.

Observe that the differential of the map $p \mapsto a(\eta_\gamma(p))^{-1}$ at $p = 1$ does not come into the picture, since it has to be multiplied with $\eta(0) = 0$ by the product formula.

3.13. The map $\prod_{\alpha\in\Delta-I} \alpha$ induces an isomorphism between A_I equipped with the automorphism η_2 and the product $(\mathbb{R}^*_{>0})^{\Delta-I}$ equipped with the automorphism η. The logarithm map $\log^{\Delta-I}$ induces an η-equivariant isomorphism $(\mathbb{R}^*_{>0})^{\Delta-I} \simeq \mathbb{R}^{\Delta-I}$.

We conclude $H^1(\langle\eta_2\rangle, A_I) \simeq H^1\left(\langle\eta\rangle, (\mathbb{R}^*_{>0})^{\Delta-I}\right) \simeq H^1\left(\langle\eta\rangle, \mathbb{R}^{\Delta-I}\right) = 0$, since η is of finite order. This means that every $a \in A_I$ satisfying $\mathcal{N}_2(a) = 1$ is of the form $a = b \cdot \eta_2(b)^{-1}$.

If we replace p_0 by $p'_0 = p_0 \cdot b$ where $b \in A_I$ we get

$$\begin{aligned} g_\eta^{-1} \cdot \eta(p'_0) \cdot g_\eta p_\eta h_\infty &= g_\eta^{-1} \cdot \eta(p_0) \cdot g_\eta p_\eta \cdot \eta_2(b) h_\infty = \gamma p_0 k_\infty^{-1} z_\infty^{-1} a^{-1} \cdot \eta_2(b) \\ &= \gamma p'_0 k_\infty^{-1} z_\infty^{-1} (a')^{-1}, \qquad \text{where } a' = a \cdot b \cdot \eta_2(b)^{-1}. \end{aligned}$$

Thus the class of a modulo coboundaries is unique.

3.14. We decompose $\Delta - I$ into orbits under η and assume without loss of generality that $\{1, \ldots, m\} \subset \Delta - I$ is such an orbit, more precisely we may assume:

$$\eta(\alpha_i) = \alpha_{i+1}, \quad i = 1, \ldots, m-1, \qquad \eta(\alpha_m) = \alpha_1.$$

We write

$$\begin{aligned} a_i &= \alpha_i(a) \quad \text{for } i = 1, \ldots, m \ldots \\ s_{J'} &= (\epsilon_1, \ldots, \epsilon_m, \ldots) \qquad \text{where } \epsilon_i = \pm 1 \end{aligned}$$

Then $\mathbb{R}^m = \mathbb{R}^m \times \{0\} \subset \mathbb{R}^{\Delta-I}$ is an η- and S^Δ-stable factor of $\mathbb{R}^{\Delta-I}$, on which the map l is described as follows:

$$l : (y_1, \ldots, y_m) \mapsto (a_1 y_m \epsilon_1, a_2 y_1 \epsilon_2, \ldots, a_m y_{m-1} \epsilon_m)$$

The characteristic polynomial is $\det\left((T \cdot id - l)|_{\mathbb{R}^m}\right) = T^m - a_1 \cdots a_m \cdot \epsilon_1 \cdots \epsilon_m$. We remark $\alpha_i(\mathcal{N}_2(a)) = a_1 \cdot a_2 \cdots a_m$ for $i = 1, \ldots, m$.

3.15. The case $a_1 \cdots a_m = 1$ and $\epsilon_1 \cdots \epsilon_m = 1$. If $a_1 \cdots a_m = 1$ we can modify p_0 such that we get $a_1 = \ldots = a_m = 1$. But then we get from the definitions that $F(\mathcal{H})_{I,\gamma}$ is a component of the boundary of $F(\mathcal{H})_{I\cup\{1,\ldots,m\},\gamma}$ if additionally $\epsilon_1 \cdots \epsilon_m = 1$: The vector

$$v = (\epsilon_1, \epsilon_1\epsilon_2, \ldots, \epsilon_1 \cdots \epsilon_{m-1}, 1) \in \mathbb{R}^m$$

is an eigenvector of l with eigenvalue 1, such that the algebraic multiplicity of this eigenvalue is 1. Via the embeddings $\mathbb{R}^m \subset \mathbb{R}^{\Delta-I} \subset T_{x_0}X$ the vector v can be viewed as a tangent vector of the set of fixed points $F(\mathcal{H})_{I\cup\{1,\dots,m\},\gamma}$. More precisely, if we consider the map $\alpha = \alpha_I : X^I_{\gamma,\eta}(g_f) \times R_{\gamma,\eta} \times (\Sigma^\Delta)_{I,J'} \quad \to \quad F(\mathcal{H})_{I,\gamma,g_f}$ from 2.24, then $F(\mathcal{H})_{I,\gamma,g_f}$ lies in the boundary of $F(\mathcal{H})_{I\cup\{1,\dots,m\},\gamma,g_f}$ and the latter is the image of $X^{I\cup\{1,\dots,m\}}_{\gamma,\eta}(g_f) \times R_{\gamma,\eta} \times (\Sigma^\Delta)_{I\cup\{1,\dots,m\},J'}$ under $\alpha_{I\cup\{1,\dots,m\}}$. One gets the index set $(\Sigma^\Delta)_{I\cup\{1,\dots,m\},J'}$ from $(\Sigma^\Delta)_{I,J'}$ by replacing the part $(0,\dots,0) \in \mathbb{R}^m$ by the vectors $\pm v$.

The action of l on $\mathbb{R}^m/\langle v\rangle$ now gives a positive contribution to the expression $\det\left(id - d\mathcal{H}|_{Norm(F(\mathcal{H})_\gamma)}\right)$, where $Norm(F(\mathcal{H})_\gamma)$ is the normal bundle of $F(\mathcal{H})_\gamma$: One can easily see that the determinant in the part belonging to $\mathbb{R}^m/\langle v\rangle$ in the normal bundle is $m > 0$ using the formula $(T^m - 1) = (T-1)\cdot(T^{m-1} + \dots T + 1)$.

3.16. THE CASE $a_1\cdots a_m = 1$ AND $\epsilon_1\cdots\epsilon_m = -1$. If $a_1\cdots a_m = 1$ and $\epsilon_1\cdots\epsilon_m = -1$ the number 1 is not an eigenvalue of the linear map l and $\det((id - l)|_{\mathbb{R}^m}) = 2$ is also a positive contribution to the expression $sign\left(\det\left(id - d\mathcal{H}|_{Norm(F(\mathcal{H})_\gamma)}\right)\right)$.

We conclude

$$\sum_{\epsilon_1,\dots,\epsilon_m} \epsilon_1\cdots\epsilon_m \cdot sign\left(\det\left(id - d\mathcal{H})|_{Norm(F(\mathcal{H})_\gamma)}\right)\right) \quad = \quad 0 \tag{32}$$

if $a_1\cdots a_m = 1$.

3.17. IN THE CASE $a_1\cdots a_m \neq 1$ the number 1 is not an eigenvalue of the linear map l for all choices of ϵ_i, so that $sign\,(\det(id - l)|_{\mathbb{R}^m})$ is a factor of the expression $sign\left(\det\left(id - d\mathcal{H})|_{Norm(F(\mathcal{H})_\gamma)}\right)\right)$. We compute

$$\begin{aligned}\sum_{\epsilon_1,\dots,\epsilon_m} \epsilon_1\cdots\epsilon_m sign\,(\det(id - l)|_{\mathbb{R}^m}) &= \sum_{\epsilon_1,\dots,\epsilon_m} \epsilon_1\cdots\epsilon_m sign\,(1 - a_1\cdots a_m\cdot\epsilon_1\cdots\epsilon_m) \\ &= \begin{cases} 0 & \text{if } a_1\cdots a_m < 1 \\ -2^m & \text{if } a_1\cdots a_m > 1. \end{cases}\end{aligned} \tag{33}$$

LEMMA 3.18. *Assume $\prod_{j\in J} a_j > 1$ for all η-orbits J in $\Delta - I$. Then the eigenvalues of the differential of the map $p \mapsto \eta_\gamma(p)^0$ have absolute value $\leqslant 1$.*

Proof: For $\alpha \in \Delta - I$ there exists a positive integer $\epsilon_{I,\alpha}$ such that the restriction of $\epsilon_{I,\alpha}\cdot\alpha$ to A_I has a continuation to a rational character from P_I/Z_G to $\mathbb{G}_m$. Let $\chi_{I,\alpha}$ be the square of this character. Thus we have:

$$\chi_{I,\alpha}(a) \quad = \quad \alpha(a)^{2\epsilon_{I,\alpha}} \qquad \text{for all } a \in A_I.$$

If we apply $\chi_{I,\alpha}$ to equation $(1_\mathcal{N})$ in 2.10 we get

$$\chi_{I,\alpha}(\mathcal{N}(g_\eta\gamma)) \quad = \quad \alpha(\mathcal{N}_2(a))^{2\epsilon_{I,\alpha}} \qquad ,$$

since $\mathcal{N}(g_\eta p_\eta)$, $\mathcal{N}_2(h_\infty)$, $\mathcal{N}_2(z_\infty k_\infty) \in K^m_\infty \cap P_I(\mathbb{R})$ are all elements of $\ker(\chi_{I,\alpha})$.

The differential of the map $pp_0 \mapsto \eta_\gamma(p)p_0$, from the space $P^0_I(\mathbb{R})/K^I_\infty Z_\infty$ to itself is the same as that of the analogous endomorphism on $P_I(\mathbb{R})/L^I_\infty$. The n-th (iterated) power of this map is $pp_0 \mapsto \eta^n_\gamma(p)p_0 = \mathcal{N}(g_\eta\gamma)^{-1}\cdot p\cdot\mathcal{N}(g_\eta\gamma)\cdot p_0$. The claim about the eigenvalues of the differential of the original map is equivalent to the corresponding claim about the n-th composed map. But now we have

$$T_{p_0}\left(P_I(\mathbb{R})/L^I_\infty\right) \simeq Lie(M_I(\mathbb{R}))/Lie(L^I_\infty) \times Lie(U_I(\mathbb{R})).$$

Now the differential of the conjugation map $p \mapsto \mathcal{N}(g_\eta\gamma)^{-1} \cdot p \cdot \mathcal{N}(g_\eta\gamma)$ has eigenvalues of absolute value 1 on the first factor, since $\mathcal{N}(g_\eta\gamma) \in L^I_\infty = K^I_\infty Z_\infty A_I$, where $Z_\infty A_I$ centralizes the group $M_I(\mathbb{R})$ and K^I_∞ is compact. The effect of the map on $Lie(U_I(\mathbb{R}))$ on the other side is described by the inverses of the roots followed by a conjugation with something compact. Since the values of the roots are > 1 by assumption, the proof is complete. □

Proposition 3.19. *We may summarize the contribution of the I-component:*

$$2^{-\#\Delta} \cdot \sum_{s\in\{\pm1\}^\Delta} sign\left(\det(id - \mathcal{H}(s))|_{Norm(F(\mathcal{H})_{I,\gamma})}\right) \cdot c_{I,J'} \cdot \chi_{-1}(s)$$

$$= \begin{cases} 0 & \text{if } \chi_{I,\alpha}\left(\mathcal{N}(g_\eta\gamma)\right) \leqslant 1 \text{ for some } \alpha \in \Delta - I \\ & \\ (-1)^{\#((\Delta-I)/\eta)} & \text{else} \end{cases}$$

Proof: From (32) and (33) the vanishing in the first case is clear. If we have $\chi_{I,\alpha}\left(\mathcal{N}(g_\eta\gamma)\right) > 1$ for all $\alpha \in \Delta - I$ then the eigenvalues α_j of the map $p \mapsto \eta_\gamma(p)^0$ have absolute value $\leqslant 1$ by lemma 3.18. Since the non real of them appear in pairs of complex conjugates we conclude that $\prod_{j,\alpha_j\neq 1}(1-\alpha_j)$ is strictly positive. We furthermore may compute:

$$\sum_{\epsilon\in\{\pm1\}^I} c_{I,J'} \cdot \prod_{i\in I} \epsilon_i \quad = \quad \sum_{\epsilon\in\{\pm1\}^I} \prod_{i\in I} \epsilon_i \cdot \#\left\{y \in \{\pm1\}^I \times \{0\}^{\Delta-I} \,|\eta(y)\cdot s_{J'} = y\right\}$$

$$= \quad \sum_{y\in\{\pm1\}^I} \chi_{-1}\left(y\cdot\eta(y)^{-1}\right) \quad = \quad 2^{\#I},$$

since $\chi_{-1}\left(y\cdot\eta(y)^{-1}\right) = \chi_{-1}(y)\cdot\chi_{-1}\left(\eta(y)\right) = \chi_{-1}(y)^2 = 1$. Now we get the claim from this formula together with (33): The powers of 2 cancel against $2^{-\#\Delta}$ and from each η-orbit in $\Delta - I$ we get one minus sign. □

First version of the trace formula

3.20. The assumptions on Z_∞, Z_f, ζ. Recall that we fixed an open compact subgroup $Z_f \subset Z_G(\mathbb{A}_f)$ satisfying

(Ass_{Zf}) $\qquad \eta(Z_f) \;=\; Z_f. \qquad$ This implies $\eta(\zeta) = \zeta$.

We will consider only K_f satisfying

$$K_f \cap Z_G(\mathbb{A}_f) \quad = \quad Z_f.$$

The group $Z_\infty \subset Z_G(\mathbb{R})$ satisfies

(Ass_Z) $\qquad \eta(Z_\infty) \quad = \quad Z_\infty,$

since it is invariantly defined to be the connected component of the group of $\mathbb{R}$-valued points of the $\mathbb{R}$-split part of the center of G. Then the group

$$\zeta \quad = \quad Z_G(\mathbb{Q}) \,\cap\, (K_\infty \cdot Z_\infty \cdot A_\Delta \;\times\; Z_f)$$

is η-invariant and has to satisfy

(Ass_{K_f})

$$\left(g_f K_f g_f^{-1} \cdot g_\infty K_\infty Z_\infty A_\Delta g_\infty^{-1}\right) \cap G(\mathbb{Q}) = \zeta \quad \text{for all} \qquad g_f \in G(\mathbb{A}_f),\ g_\infty \in G(\mathbb{R}).$$

Finally Z_f and therefore also ζ are sufficiently small, in the sense that the following assumption is fulfilled:

$(Ass_{\zeta,der})$ $$\zeta \cap G^{(1)}(\mathbb{Q}) \quad = \quad \{1\}\,.$$

Recall from 2.17 the definition $O_\eta^\infty(\gamma, h_\infty) = \# R_{\gamma,\eta}$ of a substitute of an orbital integral at the infinit place. Finally we recall the assumtion on the twisted centralizers:

$((Ass_{conn}))$ $\qquad G^I_{\gamma,\eta}$ is a connected group if it is reductive.

Theorem 3.21. *Let h_f be a Schwartz-Bruhat function on $G(\mathbb{A}_f)$ which is right invariant under K_f, let M be a $G(\mathbb{Q}) \rtimes \langle\eta\rangle$-module and $h_\infty \in K^m_\infty \cap M_0(\mathbb{R})$. If all assumptions in 3.20 are fulfilled then we have*

$$tr\left((h_\infty \times h_f)\circ\eta, H_c^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) \quad =$$

$$\sum_{\substack{I\subset\Delta\\ I^\eta = I}} (-1)^{\#((\Delta-I)/\eta)} \cdot \sum_{\substack{\gamma\in(P_I(\mathbb{Q}))_\eta\\ \mathcal{N}(\gamma)\sim L^I_\infty\\ \chi_{I,\alpha}(\mathcal{N}(\gamma))>1\\ \text{for all }\alpha\in\Delta-I}} (-1)^{\Delta(\tilde G, L_{\gamma,\eta})+\frac12 q(\tilde G)} \cdot \frac{O_\eta^\infty(I,\gamma,h_\infty)}{d^I_{\zeta,\gamma}}$$

$$\cdot O_\eta(\gamma, h_f)\cdot tr(\gamma\circ\eta|M)\cdot \frac{D(G^I_{\gamma,\eta})\cdot\tau(G^I_{\gamma,\eta})}{vol_{db'_\infty}((\overline{G^I_{\gamma,\eta}})'/\zeta)}.$$

Remarks: The inner sum is formally over all η-conjugacy classes in $P_I(\mathbb{Q})$ which satisfy the two listed conditions, but the factor $D(G^I_{\gamma,\eta})$ encodes the further conditions, that $G^I_{\gamma,\eta}$ is reductive and contains a torus which is compact modulo the center at the archimedean prime. For the definition of $O_\eta^\infty(I,\gamma,h_\infty)$ we refer to (2.17).

Proof: First we use 3.7 and then we apply the general fixed point formula for compact manifolds 3.3 to each correspondence $\mathcal{H}(s_{J'})$. Then we use the additivity of the Euler characteristic with compact supports with respect to stratifications into locally closed manifolds. We get

$$\begin{aligned} &tr\left((h_\infty \times h_f)\circ\eta, H_c^*\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) \quad = \\ &2^{-\#\Delta}\cdot \sum_{s\in S^\Delta} \chi_{-1}(s)\cdot \sum_\gamma sign\left(\det(id-\mathcal{H}(s))|_{Norm(F(\mathcal{H})_\gamma)}\right)\cdot\chi_c(F(\mathcal{H})_{I,\gamma})\cdot tr(\gamma\circ\eta|M)\end{aligned}$$

Now we use 3.19 and (31) to get the claim. $\square$

4. Stabilization and Galois cohomology

Abelianized Galois cohomology

4.1. Let K be a perfect field. Recall the definition of abelianized Galois cohomology of Borovoi and Kottwitz [Bovo]: If G/K is a reductive group, let $G^{(1)} = G_{der}$ be its derived group and G_{sc} the simply connected cover of G_{der}. We denote by $Z \subset G$ the center, by $T \subset G$ some torus containing Z (in the applications T will be a maximal torus), and by $Z_{sc} = \rho^{-1}(Z)$ and $T_{sc} = \rho^{-1}(T)$ their inverse images in G_{sc} under the composite map $\rho : G_{sc} \twoheadrightarrow G_{der} \hookrightarrow G$. One defines $\mathbb{H}^1_{ab}(K,G)$ to be the Galois hypercohomology of the complex $1 \to Z_{sc} \to Z \to 1$, where Z_{sc} sits in degree -1 and Z in degree 0. Since this complex is quasiisomorphic to the complex $1 \to T_{sc} \to T \to 1$ we can as well define

$$\mathbb{H}^1_{ab}(K,G) \quad = \quad \mathbb{H}^1(K, 1 \to T_{sc} \to T \to 1).$$

There exists a canonical map $ab^1 : H^1(K,G) \to \mathbb{H}^1_{ab}(K,G)$: If $(\psi_\sigma) \in Z^1(K,G)$ denotes a cocycle, we may write $\psi_\sigma = \rho(\psi'_\sigma) \cdot \xi_\sigma$ for $\psi'_\sigma \in G_{sc}(\bar{K})$ and a cochain $\xi_\sigma \in Z(\bar{K})$. Then $\lambda_{\sigma,\tau} := \psi'_\sigma \cdot {}^\sigma\psi'_\tau \cdot (\psi'_{\sigma\tau})^{-1} \in Z_{sc}(\bar{K})$ and the pair $((\lambda_{\sigma,\tau}), (\xi_\sigma)) \in C^2(K, Z_{sc}) \times C^1(K,Z)$ defines a cocycle in the double complex which computes the hypercohomology $\mathbb{H}^1(K, 1 \to Z_{sc} \to Z \to 1)$. Then ab^1 of the class of (ψ_σ) is the class of this pair.

We denote by $\mathbb{X}_*$ the following complex of abelian groups with action of $Gal(\bar{K}/K)$ living in degrees -1 and 0:

$$\mathbb{X}_* : \qquad 0 \to X_*(T_{sc}) \to X_*(T) \to 0$$

Then we have $\mathbb{H}^1_{ab}(K,G) = \mathbb{H}^1(K, \mathbb{X}_* \otimes \overline{K}^*)$. We recall the definition of the algebraic fundamental group from [Bovo]:

$$\pi_1(G) \quad = \quad \mathbb{H}^0(\mathbb{X}_*) \quad = \quad X_*(T)/\rho_* X_*(T_{sc}).$$

4.2. Now let G be defined over $\mathbb{Q}$. Following [Bovo] the vanishing theorem of Kneser $H^1(\mathbb{Q}_p, G_{sc}) = 1$ and the Hasse principle for semisimple simply connected algebraic groups (Kneser, Harder and Chernousov) generalize to the statement that the following diagram is cartesian:

$$\begin{array}{ccc} H^1(\mathbb{Q},G) & \xrightarrow{ab^1} & \mathbb{H}^1_{ab}(\mathbb{Q},G) \\ \downarrow & & \downarrow \\ H^1(\mathbb{R},G) & \xrightarrow{ab^1} & \mathbb{H}^1_{ab}(\mathbb{R},G) \end{array}$$

(In the case $G = G_{sc}$ the groups $\mathbb{H}^1_{ab}(K,G)$ are trivial, and the diagram being cartesian just means, that the left arrow is a bijection.)
The short exact sequence $1 \to \overline{\mathbb{Q}}^* \to \mathbb{A}^*_{\overline{\mathbb{Q}}} \to \mathbb{A}^*_{\bar{\mathbb{Q}}}/\overline{\mathbb{Q}}^* \to 1$ gives rise to an exact sequence

$$\begin{array}{ccccc} \mathbb{H}^1(\mathbb{Q}, \mathbb{X}_* \otimes \overline{\mathbb{Q}}^*) & \longrightarrow & \mathbb{H}^1(\mathbb{Q}, \mathbb{X}_* \otimes \mathbb{A}^*_{\overline{\mathbb{Q}}}) & \longrightarrow & \mathbb{H}^1(\mathbb{Q}, \mathbb{X}_* \otimes \mathbb{A}^*_{\overline{\mathbb{Q}}}/\bar{\mathbb{Q}}^*) \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ \mathbb{H}^1_{ab}(\mathbb{Q},G) & \longrightarrow & \bigoplus_v \mathbb{H}^1(\mathbb{Q}_v, \mathbb{X}_* \otimes \overline{\mathbb{Q}}_v^*) & \longrightarrow & \hat{H}^{-1}(\mathbb{Q}, \pi_1(G)), \end{array}$$

where we have used the Tate-Nakayama isomorphism in the right column. Observe that $\hat{H}^{-1}(\mathbb{Q}, \pi_1(G)) = \left(\pi_1(G)_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}\right)_{tors}$. The local Tate-Nakayama map gives us an isomorphism:

$$\mathbb{H}^1_{ab}(\mathbb{R}, G) \simeq \hat{H}^{-1}(\mathbb{R}, \pi_1(G)) \simeq \left(\pi_1(G)_{Gal(\mathbb{C}/\mathbb{R})}\right)_{tors}.$$

4.3. The group of connected components of a real algebraic group. For $G/\mathbb{R}$ we consider the homomorphism $ab^0 : G(\mathbb{R}) \to \hat{\mathbb{H}}^0(\mathbb{R}, \mathbb{X}_* \otimes \mathbb{C}^*)$, which maps $g = \rho(s) \cdot z \in G(\mathbb{R})$ with $s \in G_{sc}(\mathbb{C})$ and $z \in Z_G(\mathbb{C})$ to the class of the 0-hypercocycle $(s \cdot \bar{s}^{-1}, z) \in \mathbb{Z}^0(\mathbb{R}, Z_{sc}(\mathbb{C}) \to Z_G(\mathbb{C}))$. Here $(a, b) \in Z_{sc}(\mathbb{C}) \times Z_G(\mathbb{C})$ is a 0-hypercocycle iff $\rho(a) = \bar{b} \cdot b^{-1}$ and $a \cdot \bar{a} = 1$. The hypercoboundaries are of the form $(\bar{c} \cdot c^{-1}, \rho(c) \cdot d\bar{d})$ for $c \in Z_{sc}(\mathbb{C}), d \in Z_G(\mathbb{C})$. We define the torus Z_G° to be the connected component of Z_G as an algebraic group.

Lemma 4.4. *(a) The kernel of ab^0 is the group $\rho(G_{sc}(\mathbb{R})) \cdot \left\{d_0\bar{d}_0 \mid d_0 \in Z_G^\circ(\mathbb{C})\right\}$.*
(b) The map ab^0 induces an injection $\pi_0(G(\mathbb{R})) \hookrightarrow \hat{\mathbb{H}}^0(\mathbb{R}, \mathbb{X}_* \otimes \mathbb{C}^*)$.

Proof: (a) If $ab^0(g) = 1$ with $g = \rho(s) \cdot z$ then $s \cdot c \in G_{sc}(\mathbb{R})$ and $g = \rho(s \cdot c) \cdot d\bar{d}$ with $c \in Z_{sc}(\mathbb{C})$ and $d \in Z_G(\mathbb{C})$. But since we can write $d = \rho(\delta) \cdot d_0$ with $\delta \in Z_{sc}(\mathbb{C})$ and d_0 in the torus $Z_G^\circ(\mathbb{C})$ we get the representation $g = \rho(sc\delta\bar{\delta}) \cdot d_0\bar{d}_0$ with $sc\delta\bar{\delta} \in G_{sc}(\mathbb{R})$. On the other side it is easy that each element of the form $g = \rho(s) \cdot d\bar{d}$ with $s \in G_{sc}(\mathbb{R})$ and $d \in Z_G^\circ(\mathbb{C})$ lies in the kernel of ab^0.

(b) Since $G_{sc}(\mathbb{R})$ and $Z_G^\circ(\mathbb{C})$ are connected as Lie groups, the same holds for their continuous images $\rho(G_{sc}(\mathbb{R}))$ and $\left\{d_0\bar{d}_0 \mid d_0 \in Z_G^\circ(\mathbb{C})\right\}$. Thus the kernel of ab^0 is connected. On the other side the kernel of ab^0 is an open subgroup of $G(\mathbb{R})$, since its Lie algebra coincides with the Lie algebra of $G(\mathbb{R})$. This implies the claim.

□

Stabilization

Definition 4.5. *We say that a pair (G, η), where $G/\mathbb{Q}$ is a reductive group and $\eta \in Aut(G)$ is of finite order, has trivial Galois cohomology, if all maps $H^1(F, G_{\gamma,\eta}) \to H^1(F, G)$ are trivial for $F = \mathbb{Q}$ and for all $F = \mathbb{Q}_v$, v an arbitrary valuation of $\mathbb{Q}$.*

Remark 4.6. The groups $G = \mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2g}, \mathrm{GSp}_{2g}$ have trivial H^1 over every field F. The pair $(\mathrm{PGL}_{2n+1}, \eta)$, where η is of the form $A \mapsto J \cdot {}^tA^{-1} \cdot J^{-1}$ also has trivial Galois cohomology, since every stabilizer $G_{\gamma,\eta}$ has a unique lift to the group SL_{2n+1} (compare the proof of [BWW, Prop. 6.5.]), so that $H^1(F, G_{\gamma,\eta}) \to H^1(F, G)$ factorizes over the trivial set $H^1(F, \mathrm{SL}_{2n+1})$.

Remark 4.7. If (G, η) has trivial Galois cohomology (which we will assume in the sequel), then it is well known that the conjugacy classes inside the η-stable conjugacy class of some $\gamma \in G(F)$ are parametrized by the elements in $H^1(F, G_{\gamma,\eta})$. In the following we will not distinguish between classes in $H^1(F, G_{\gamma,\eta})$ and representatives of conjugacy classes corresponding to them. This applies in the following definition, where we furthermore use the Kottwitz sign $e_v(G) \in \{\pm 1\}$ for an algebraic group $G/\mathbb{Q}_v$, if v is a place of $\mathbb{Q}$, as defined in [Ko1].

4.8. We introduce the local stable orbital integrals:

$$SO_\eta(\gamma_0, h_p) = \sum_{\gamma_p \in H^1\left(\mathbb{Q}_p, G_{\gamma_0,\eta}\right)} e_p\left(G_{\gamma_p,\eta}\right) \cdot O_\eta\left(\gamma_p, h_p\right)$$

and its analog in the finite adelic setting:

$$\begin{aligned} SO_\eta(\gamma_0, h_f) &= \prod_{p \text{ finite}} SO_\eta(\gamma_0, h_p) \qquad \text{if} \\ h_f &= \prod_{p \text{ finite}} h_p. \end{aligned}$$

We extend this definition by linearity to all Schwartz-Bruhat functions on $G(\mathbb{A}_f)$.

Theorem 4.9. *Assume that the pair (G,η) has only trivial Galois cohomology. For $I \subset \Delta$ and $\gamma_0 \in P_I(\mathbb{Q})$ assume that $\tilde{G} = G^I_{\gamma_0,\eta}$ is a connected reductive group, let $\tilde{G}_{qs}$ be the quasi-split inner form of $\tilde{G}$ and define $\Delta(\gamma_0,\eta) = \Delta(\tilde{G}, L_{\gamma_0,\eta}) + \frac{1}{2}q(\tilde{G}_{qs})$. Then we have*

$$\begin{aligned} tr\left((h_\infty \times h_f)\circ\eta, H^*_c\left(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) &= \\ \sum_{\substack{I\subset\Delta \\ I^\eta = I}} (-1)^{\#((\Delta-I)/\eta)} \cdot \sum_{\substack{\gamma_0\in(P_I(\mathbb{Q}))_{\eta-st} \\ \mathcal{N}(\gamma_0)\sim L^I_\infty \\ \chi_{I,\alpha}(\mathcal{N}(\gamma_0))>1 \\ \text{for all } \alpha\in\Delta-I}} \alpha_\infty(\gamma_0,h_\infty)\cdot SO_\eta(\gamma_0,h_f)\cdot tr(\gamma_0\circ\eta|\mathcal{M}), \end{aligned}$$

with

$$\alpha_\infty(\gamma_0, h_\infty) = \frac{O^\infty_\eta\left(I,\gamma_0,h_\infty\right)}{d^I_{\zeta,\gamma}} \cdot (-1)^{\Delta(\gamma_0,\eta)} \cdot \frac{\#H^1(\mathbb{R},T)}{vol_{db_\infty}\left((\overline{G^I_{\gamma_0,\eta}})'/\zeta\right)}.$$

Here γ_0 runs over the stable η-conjugacy classes inside $P_I(\mathbb{Q})$ satisfying the two listed conditions.

Proof: We start with a twisted conjugacy class γ_0 in $G(\mathbb{Q})$. Then all elements stably conjugate to γ are parametrized by the kernel of the map $H^1(\mathbb{Q},\tilde{G}) \to H^1(\mathbb{Q},G)$, where $\tilde{G} = G^I_{\gamma_0,\eta}$. Since (G,η) has trivial Galois cohomology, this kernel equals $H^1(\mathbb{Q},\tilde{G})$. Let us consider the following diagram, where the right column is exact and the left square is cartesian:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 & & & & & & Ш(\mathbb{Q},\tilde{G}) \\
 & & & & & & \downarrow \\
 & & H^1(\mathbb{Q},\tilde{G}) & \xrightarrow{ab^1} & \mathbb{H}^1_{ab}(\mathbb{Q},\tilde{G}) & = & \mathbb{H}^1_{ab}(\mathbb{Q},\tilde{G}) \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle loc} \\
H^1(\mathbb{R},T) & \xrightarrow{i_T} & H^1(\mathbb{R},\tilde{G}) & \xrightarrow{ab^1_{\mathbb{R}}} & \mathbb{H}^1_{ab}(\mathbb{R},\tilde{G}) & \longrightarrow & \bigoplus_v \mathbb{H}^1_{ab}(\mathbb{Q}_v,\tilde{G}) \\
 & & & & \sim\downarrow & & \downarrow \\
 & & & & \left(\pi_1(\tilde{G})_{Gal(\mathbb{C}/\mathbb{R})}\right)_{tors} & \xrightarrow{i_{\mathbb{R}}} & \left(\pi_1(\tilde{G})_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}\right)_{tors}
\end{array}$$

Remark that $i_{\mathbb{R}}$ is surjective if $\tilde{G} = G^I_{\gamma,\eta}$ and γ is an I-elliptic element. Furthermore if γ is I-elliptic, then we have the equality of the $\mathbb{Q}$-rank with the $\mathbb{R}$-rank of the torus $\tilde{G}/\tilde{G}_{der}$. Recall that the Kottwitz signs $e_v(\tilde{G})$ satisfy:

$$\begin{aligned}
e_p(\tilde{G}) &= (-1)^{rank_{\mathbb{Q}_p}(\tilde{G})-rank_{\mathbb{Q}_p}(\tilde{G}_{qs})} \qquad \text{for } p \text{ finite} \\
e_\infty(\tilde{G}) &= (-1)^{\frac{1}{2}q(\tilde{G}_{der})-\frac{1}{2}q((\tilde{G}_{qs})_{der})} \\
\prod_v e_v(\tilde{G}) &= 1.
\end{aligned}$$

Here $q(\tilde{G}_{der})$ denotes the dimension of the symmetric space associated to the derived group $\tilde{G}_{der}$. Thus we have

$$(-1)^{\frac{1}{2}q(\tilde{G}_{der})} = (-1)^{\frac{1}{2}q((\tilde{G}_{qs})_{der})} \cdot \prod_{p\ finite} e_p(\tilde{G}). \tag{34}$$

The Tamagawa numbers satisfy [Sans]:

$$\tau(\tilde{G}) = \frac{\#\left(\pi_1(\tilde{G})_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}\right)_{tors}}{\#Ш(\mathbb{Q},\tilde{G})} \cdot \tau(\tilde{G}_{sc}). \tag{35}$$

Recall that $\tau(\tilde{G}_{sc}) = 1$ by the main result of [Ko2].

Finally note that if $D(\tilde{G})$ does not vanish, it equals the order of the kernel of the map $H^1(\mathbb{R},T) \to H^1(\mathbb{R},\tilde{G})$. More precisely: If $\tilde{G}_\beta$ denotes the inner form of $\tilde{G}$ obtained by twisting $\tilde{G}/\mathbb{R}$ with $\beta \in H^1(\mathbb{R},\tilde{G})$, then $D(\tilde{G}_\beta)$ equals the cardinality of the inverse image of β in $H^1(\mathbb{R},T)$. (compare [She])

The process of stabilization now works as follows: The sum over all (η-twisted) conjugacy classes in the stable class of γ_0, which is a sum over $\gamma \in H^1(\mathbb{Q},\tilde{G})$ may be replaced by a sum over those pairs $(\alpha,\beta) \in \mathbb{H}^1_{ab}(\mathbb{Q},\tilde{G}) \times H^1(\mathbb{R},\tilde{G})$, which have the same image in $\mathbb{H}^1_{ab}(\mathbb{R},\tilde{G})$. This may be replaced by a sum over pairs $(\alpha,\delta) \in \mathbb{H}^1_{ab}(\mathbb{Q},\tilde{G}) \times H^1(\mathbb{R},T)$ having the same image in $\mathbb{H}^1_{ab}(\mathbb{R},\tilde{G})$, if we remove the factor $D(\tilde{G})$ from the trace formula. If we introduce an additional factor $\#\text{Ш}(\mathbb{Q},G)$ in the formula, we may replace the sum over (α,δ) by a sum over those $(\delta,\epsilon)\in H^1(\mathbb{R},T)\times\oplus_v\mathbb{H}^1_{ab}(\mathbb{Q}_v,\tilde{G})$, for which the image of ϵ in $\left(\pi_1(\tilde{G})_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}\right)_{tors}$ vanishes and for which the image of δ in $\mathbb{H}^1_{ab}(\mathbb{R},\tilde{G})$ is the archimedean component ϵ_∞. But since the maps $i_T, ab^1_{\mathbb{R}}$ and $i_{\mathbb{R}}$ are surjective, we may simply replace the sum over (δ,ϵ) by a sum over $\omega\in\bigoplus_{p\ finite}\mathbb{H}^1_{ab}(\mathbb{Q}_p,\tilde{G})$ after introducing an extra factor $\#\ker\left(H^1(\mathbb{R},T)\twoheadrightarrow\left(\pi_1(\tilde{G})_{Gal(\bar{\mathbb{Q}}/\mathbb{Q})}\right)_{tors}\right)$. But now the product of this last factor with $\#\text{Ш}(\mathbb{Q},G)$ equals $\#H^1(\mathbb{R},T)\cdot\tau(\tilde{G})^{-1}$ by (35). Now observe that $\mathbb{H}^1_{ab}(\mathbb{Q}_p,\tilde{G})\simeq H^1(\mathbb{Q}_p,\tilde{G})\simeq\ker\left(H^1(\mathbb{Q}_p,\tilde{G})\to H^1(\mathbb{Q}_p,G)\right)$ describes the local twisted conjugacy classes in the local stable twisted conjugacy class of γ_0. Putting everything together, especially (34), we get the claim. $\square$

5. Comparison of fixed point formulas

Twisted stable endoscopy

5.1. Split Groups with automorphism. Let G/R be a connected reductive split group scheme. We fix some "splitting" i.e. a triple $(B, T, \{X_\alpha\}_{\alpha\in\Delta})$ where T denotes a maximal split torus inside a rational Borel B, $\Delta = \Delta_G = \Delta(G, B, T) \subset \Phi(G, T) \subset X^*(T)$ the set of simple roots inside the system of roots and the X_α for the simple roots $\alpha \in \Delta$ are a system (nailing) of isomorphisms between the additive group scheme $\mathbb{G}_a$ and the unipotent root subgroups B_α. If R is a field we may think of the X_α as generators of the root spaces $\mathfrak{g}_\alpha$ in the Lie algebra. Here $X^*(T) = Hom(T, \mathbb{G}_m)$ denotes the character module of T, while $X_*(T) = Hom(\mathbb{G}_m, T)$ will denote the cocharacter module of T. Let $\eta \in Aut(G)$ be an automorphism of G which fixes the splitting, i.e. stabilizes B and T and permutes the X_α. We assume η to be of finite order l. We denote by

$$\tilde{G} = G \rtimes \langle\eta\rangle$$

the (nonconnected) semidirect product of G with η. η acts on the (co)character module via $X_*(T) \ni \alpha^\vee \mapsto \eta \circ \alpha^\vee$ resp. $X^*(T) \ni \alpha \mapsto \alpha \circ \eta^{-1}$.

5.2. The dual group. Let $\hat{G} = \hat{G}(\mathbb{C})$ be the dual group of G. By definition $\hat{G}$ has a triple $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ such that we have identifications $X^*(\hat{T}) = X_*(T)$, $X_*(\hat{T}) = X^*(T)$ which identifies the (simple) roots $\hat{\alpha} \in X^*(\hat{T})$ with the (simple) coroots $\alpha^\vee \in X_*(T)$, and the (simple) coroots $\hat{\alpha}^\vee \in X_*(\hat{T})$ with the (simple) roots $\alpha \in X^*(T)$. There exists a unique automorphism $\hat{\eta}$ of $\hat{G}$ which stabilizes $(\hat{B}, \hat{T}, \{\hat{X}_{\hat{\alpha}}\})$ and induces on $(X_*(\hat{T}), X^*(\hat{T}))$ the same automorphism as η on $(X^*(T), X_*(T))$.

5.3. The η-invariant subgroup in $\hat{G}$. Let $\hat{H} = (\hat{G}^{\hat{\eta}})^\circ$ be the connected component of the subgroup of $\hat{\eta}$-fixed elements in $\hat{G}$. It is a reductive split group with triple $(\hat{B}_H, \hat{T}_H, \{\hat{X}_{\hat{\beta}}\}_{\beta\in\Delta_{\hat{H}}})$, where $\hat{B}_H = \hat{B}^{\hat{\eta}}$, $\hat{T}_H = \hat{T}^{\hat{\eta}}$ and the $\hat{X}_{\hat{\beta}}$ are of the form $\hat{X}_{\hat{\beta}} = S_{\hat{\eta}}\hat{X}_{\hat{\alpha}}$ as elements of the Lie algebra $\hat{\mathfrak{g}}$, where map $S_{\hat{\eta}} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ will be explained soon.

We have the inclusion of cocharacter modules $X_*(\hat{T}_H) = X_*(\hat{T})^{\hat{\eta}} \subset X_*(\hat{T})$ and a projection for the character module

$$P_\eta : X^*(\hat{T}) \twoheadrightarrow (X^*(\hat{T})_{\hat{\eta}})_{free} = X^*(\hat{T}_H),$$

where $(X^*(\hat{T})_{\hat{\eta}})_{free}$ denotes the maximal free quotient of the coinvariant module $X^*(\hat{T})_{\hat{\eta}}$. For a $\mathbb{Z}[\eta]$-module X we define a map

$$S_\eta : \quad X \to X^\eta, \qquad x \mapsto \sum_{i=0}^{ord_x(\eta)-1} \eta^i(x)$$

where $ord_x(\eta) = \min\{i > 0 \mid \eta^i(x) = x\}$ is the length of the orbit $\langle\eta\rangle(x)$.

For the roots Φ and coroots $\Phi^\vee$ of a given root datum $(X^*, X_*, \Phi, \Phi^\vee)$ we have to introduce a modified map S'_η by

$$\begin{aligned} S'_\eta(\alpha) &= c(\alpha) \cdot S_\eta(\alpha) \qquad \text{where} \\ c(\alpha) &= \frac{2}{\langle\alpha^\vee, S_\eta(\alpha)\rangle} \end{aligned}$$

resp. by the formula where the roles of α and $\alpha^\vee$ are exchanged. For all simple root systems with automorphisms which are not of type A_{2n} we have $\langle\alpha^\vee,\eta^i(\alpha)\rangle = 0$ for $i=1,\dots,ord_\alpha(\eta)-1$ which implies $c(\alpha)=1$ i.e. $S'_\eta(\alpha)=S_\eta(\alpha)$. We furthermore introduce the subset of short-middle roots and the dual concept of long-middle coroots:

$$\begin{aligned}\Phi(\hat G,\hat T)^{sm} &= \left\{\alpha\in\Phi(\hat G,\hat T)\,\middle|\,\frac{1}{2}\cdot P_\eta(\alpha)\notin P_\eta(\Phi(\hat G,\hat T))\right\}\\ \Phi(G,T)^{lm} &= \Phi^\vee(\hat G,\hat T)^{lm} = \left\{\alpha^\vee\,\middle|\,\alpha\in\Phi(\hat G,\hat T)^{sm}\right\}\end{aligned}$$

Proposition 5.4. *With the above notations we have*

$$\begin{aligned}&(36) & \Phi(\hat H,\hat T_H) &= P_\eta(\Phi(\hat G,\hat T)^{sm}) && \textit{for the roots}\\ &(37) & \Phi^\vee(\hat H,\hat T_H) &= S'_\eta(\Phi^\vee(\hat G,\hat T)^{lm}) && \textit{for the coroots}\\ && \Delta^\vee_{\hat H}=\Delta^\vee(\hat H,\hat B_H,\hat T_H) &= S'_\eta(\Delta^\vee_{\hat G}) && \textit{for the simple coroots}\\ && \Delta_{\hat H}=\Delta(\hat H,\hat B_H,\hat T_H) &= P_\eta(\Delta_{\hat G}) && \textit{for the simple roots}\end{aligned}$$

Proof: This may be deduced from [St, 8.1]. □

Definition 5.5 stable η-endoscopic group. *In the above situation a connected reductive split group scheme H/R will be called a stable η-endoscopic group for (G,η) resp. $\tilde G$ if its dual group is together with the splitting isomorphic to the above $(\hat H,\hat B_H,\hat T_H,\{X_\beta\}_{\beta\in\Delta_{\hat H}})$.*

Remarks: Since H is unique up to isomorphism (up to unique isomorphism if we consider H together with a splitting) we can call H the stable η-endoscopic group for (G,η). For a maximal split torus $T_H\subset H$ we have:

$$\begin{aligned}&(38) & X_*(T_H) &= (X_*(T)_\eta)_{free} && \text{for the cocharacter module}\\ && X^*(T_H) &= X^*(T)^\eta && \text{for the character module}\end{aligned}$$

5.6. To get examples we use the following **notations**:

$diag(a_1,\dots,a_n)\in \mathrm{GL}_n$ denotes the diagonal matrix $(\delta_{i,j}\cdot a_i)_{ij}$ and

$antidiag(a_1,\dots,a_n)\in\mathrm{GL}_n$ the antidiagonal matrix $(\delta_{i,n+1-j}\cdot a_i)_{ij}$ with a_1 in the upper right corner. We introduce the following matrix

$$J=J_n=(\delta_{i,n+1-j}(-1)^{i-1})_{1\leqslant i,j\leqslant n}=antidiag(1,-1,\dots,(-1)^{n-1})\in\mathrm{GL}_n(R).$$

and its modification $J'_{2n}=antidiag(1,-1,1,\dots,(-1)^{n-1},(-1)^{n-1},\dots,1,-1,1)$. Since ${}^tJ_n=(-1)^{n-1}\cdot J_n$ and J'_{2n} is symmetric we can define the

standard symplectic group	Sp_{2n}	$=$	$\mathrm{Sp}(J_{2n})$
standard split odd orthogonal group	SO_{2n+1}	$=$	$\mathrm{SO}(J_{2n+1})$.
standard split even orthogonal group	SO_{2n}	$=$	$\mathrm{SO}(J'_{2n})$.

We consider the groups $\mathrm{GL}_n,\mathrm{SL}_n,\mathrm{PGL}_n,\mathrm{Sp}_{2n},\mathrm{SO}_n$ with the splittings consisting of the diagonal torus, the Borel consisting of upper triangular matrices and the standard nailing. We remark that the following map defines an involution of $\mathrm{GL}_n,\mathrm{SL}_n$ and PGL_n:

$$\eta=\eta_n: g\mapsto J_n\cdot{}^tg^{-1}\cdot J_n^{-1}.$$

Example 5.7 $A_{2n} \leftrightarrow C_n$.

$$\begin{array}{ccc} G = \mathrm{PGL}_{2n+1}, \quad \eta = \eta_{2n+1} & \text{has dual} & \hat{G} = \mathrm{SL}_{2n+1}(\mathbb{C}), \quad \hat{\eta} = \eta_{2n+1} \\ & & \bigcup \\ H = \mathrm{Sp}_{2n} & \text{has dual} & \hat{H} = \mathrm{SO}_{2n+1}(\mathbb{C}) \end{array}$$

Example 5.8 $A_{2n-1} \leftrightarrow B_n$. The group $G = \mathrm{GL}_{2n} \times \mathbb{G}_m$ has the automorphism

$$\eta : (g, a) \mapsto (\eta_{2n}(g), \det(g) \cdot a)$$

which is an involution since $\det(\eta_{2n}(g)) = \det g^{-1}$. The dual $\hat{\eta} \in Aut(\hat{G})$ satisfies

$$\hat{\eta}(g, b) = (\eta_{2n}(g) \cdot b, b), \qquad \text{so that we get}$$

$$\begin{array}{ccc} G = \mathrm{GL}_{2n} \times \mathbb{G}_m, \quad \eta & \text{has dual} & \hat{G} = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathbb{C}^\times, \quad \hat{\eta} \\ & & \bigcup \\ H = \mathrm{GSpin}_{2n+1} & \text{has dual} & \hat{H} = \mathrm{GSp}_{2n}(\mathbb{C}). \end{array}$$

Recall that GSpin_{2n+1} can be realized as the quotient $\left(\mathbb{G}_m \times \mathrm{Spin}_{2n+1}\right)/\mu_2$, where $\mu_2 \simeq \{\pm 1\}$ is embedded diagonally, so that we get an exact sequence

$$1 \to \mathrm{Spin}_{2n+1} \to \mathrm{GSpin}_{2n+1} \xrightarrow{\mu} \mathbb{G}_m \to 1,$$

where the "multiplier" map μ is induced by the projection to the $\mathbb{G}_m$ factor followed by squaring. Thus the derived group of GSpin_{2n+1} is Spin_{2n+1}, i.e. a connected, split and simply connected group.

Example 5.9 $A_{2n-1} \leftrightarrow B_n$ modified. In the example 5.8 the subtorus $Z_0 = \{(z \cdot Id_{2n}, z^{-n}) | z \in \mathbb{G}_m\} \subset Z$ is η-stable, in fact η acts by inverting elements of $\mathbb{G}_m \simeq Z_0$. Therefore the η-action descends to the quotient group $G' = G/Z_0$. We may identify

$$\begin{array}{rcl} G' & \simeq & \mathrm{GL}_{2n}/\mu_n \\ (A, b) mod\ R & \mapsto & A \cdot \sqrt[n]{b} \end{array}$$

The induced η-action reads $A \mod \mu_n \mapsto \eta_{2n}(A) \cdot \sqrt[n]{\det(A)}$.

We remark that η acts as identity on the center of G', which is $\mathbb{G}_m/\mu_n \simeq \mathbb{G}_m$. The group of η-invariants in the center is therefore a connected group.

The dual group $\hat{G}'$ is the following η-stable subgroup of $\hat{G}$:

$$\hat{G}' \quad = \quad \left\{ (A, b) \in \hat{G} | \det(A) = b^n \right\}.$$

Since $\hat{G}^{\hat{\eta}} \subset \hat{G}'$ we may consider $H = \mathrm{GSpin}_{2n+1}$ as a stable endoscopic group for (G', η').

Comparison of characters

5.10. Matching of finite dimensional representations. Let k be a field of characteristic 0. Let $M = M_\chi$ be the finite dimensional representation of G of highest weight $\chi \in X^*(T)^\eta$. We also denote by M_χ the extension of this representation to $\tilde{G} = G \rtimes \langle \eta \rangle$, such that η acts as identity on one (every) highest weight vector v_χ. Let $M_H = M_{H,\chi}$ be the corresponding representation of H where we now consider χ as a weight in

$X^*(T_H) = X^*(T)^\eta$. In this situation we say that the $\tilde{G}$-module M *matches* with the H-module M_H.

We can as well consider $M_{H,\chi}$ as an element in the Grothendieck-group $\mathcal{G}ro(H, alg)$ of finite dimensional algebraic representations of H and $M = M_\chi$ as an element of the quotient group $\mathcal{G}ro(G,\eta) = \mathcal{G}ro(\tilde{G}, alg)/Ind_G^{\tilde{G}}\mathcal{G}ro(G, alg)$. The correspondence $M_{H,\chi} \mapsto M_\chi$ induces an isomorphism between these groups (recall that the order of η is a prime). This isomorphism enables us to introduce the notion of matching on the level of Grothendieck groups.

5.11. Recall $\Phi(H,T_H) = \Phi^\vee(\hat{H},\hat{T}_H) = S'_\eta(\Phi^\vee(\hat{G},\hat{T})^{lm}) = S'_\eta(\Phi(G,T)^{lm})$ by (37) of Proposition 5.4. We may define $\Phi(G,T)^{sm}$ by the same formula as above using the projection $P_\eta : X^*(T) \twoheadrightarrow (X^*(T)_{\hat{\eta}})_{free}$. In the case of an irreducible root system each $\alpha_l \in \Phi(G,T) - \Phi(G,T)^{sm}$ (which exists only for type A_{2n} and η of order 2) is of the form $\alpha_l = \alpha_0 + \eta(\alpha_0)$ for some $\alpha_0 \in \Phi(G,T) - \Phi(G,T)^{lm}$ and vice versa. We have $c(\alpha_l) = 2$ and the η-orbit of α_0 is uniquely determined by α_l. Compare [Bal, 2.5.] for details.

Lemma 5.12. *Suppose the root system $\Phi(G,T)$ is irreducible. If $\alpha \in \Phi(G,T)^{sm}$, i.e. $\frac{1}{2}P_\eta(\alpha) \notin P_\eta(\Phi(G,T))$, then there exists a set of root vectors $\{X_\gamma \in \mathfrak{g}_\gamma\backslash\{0\} | \gamma \in \eta^{\mathbb{Z}}(\alpha)\}$, such that η acts by permutation on these root vectors.*
If α is such that $\frac{1}{2}P_\eta(\alpha) \in P_\eta(\Phi(G,T))$, then $\eta(\alpha) = \alpha$, η has order 2 and η acts as -1 on $\mathfrak{g}_\alpha$.

Proof: This is essentially [Bal, lemma 2.9.].

Proposition 5.13. *Let the finite dimensional irreducible representation M of $\tilde{G}$ match with the representation M_H of the stable endoscopic group H. Let $\gamma \in G(k)$ be η-semisimple and $\tau(\gamma)$ be a matching element in $H(k)$. Then we have:*

$$tr(\eta \circ \gamma, M) \quad = \quad tr(\tau(\gamma), M_H).$$

Proof: The proof is similar to a proof of the Weyl character formula (comp. [Hum, 24.3.]). In fact one can get the result by comparing a Weyl character formula for non-connected groups as in [Wen] with the formula for the endoscopic group.

We may assume that k is an algebraically closed field and therefore that $\gamma \in T(k)$ and $\tau(\gamma) \in T_H(k)$. We will work in the Grothendieck group $\mathcal{G}ro(\mathfrak{b}_-)$ of finitely generated $\mathfrak{b}_-$modules, where $\mathfrak{b}_- = \mathfrak{n}_- + \mathfrak{t}$ is the Borel subalgebra containing the negative roots in the decomposition $\mathfrak{g} = Lie(G) = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-$ and $\mathfrak{t} = Lie(T)$. For $\lambda \in X^*(T)$ we denote by Z_λ the Verma module

$$Z_\lambda \quad = \quad \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} k_\lambda \quad = \quad Ind_B^G \lambda \quad \simeq \quad \mathcal{U}(\mathfrak{b}_-) \otimes_{\mathcal{U}(\mathfrak{t})} k_\lambda.$$

Then we can write

$$M = M_\lambda \quad = \quad \sum_{w \in W(G,T)} sign_G(w) \cdot Z_{w(\lambda+\delta_G)-\delta_G},$$

where $\delta_G = \frac{1}{2}\sum_{\alpha\in\Phi(G,T)^+} \alpha$ is half the sum of the positive roots. Since $sign_G(\eta(w)) = sign_G(w)$ we may collect the Verma modules on the right hand side indexed by Weyl-group elements w in the same η-orbit to get $\tilde{G}$-modules on the right hand side. Here η acts as intertwining operator from $Z_{w(\lambda+\delta_G)-\delta_G}$ to $Z_{\eta(w)(\lambda+\delta_G)-\delta_G}$ in such a way that η acts by permutation on the set of some highest weight vectors $m_{w(\lambda+\delta_G)-\delta_G}$. Then the above identity becomes an identity in the Grothendieck group of $\tilde{G}$ modules. The computation of $tr(\eta\circ\gamma, M)$ reduces to the computation of the formal traces $tr(\eta\circ\gamma, Z_{w(\lambda+\delta_G)-\delta_G})$ for $w\in W(G,T)^\eta$, since the trace of $\eta\circ\gamma$ on a direct sum of $Z_{w(\lambda+\delta_G)-\delta_G}$ is obviously zero if w is not η-invariant.

To compute the formal trace we can view $Z_\lambda \simeq \mathcal{U}(\mathfrak{n}_-)$ as a symmetric algebra over $\mathfrak{n}_-$. We may take a basis $(X_\alpha)_{\alpha\in\Phi^-}$ of $\mathfrak{n}_-$ as in lemma 5.12 and view Z_λ as a polynomial algebra in this basis. Then the action of $\eta\circ\gamma$ respects the set of one dimensional monomial subspaces of Z_λ and only those monomials contribute to the trace, which contain all X_α in an η-orbit with the same exponent. If we have no α with $\frac{1}{2}P_\eta(\alpha)\in P_\eta(\Phi(G,T))$ then the formal trace may be written up to the factor $\lambda(\gamma)$ in the form

$$\begin{aligned}\prod_{\alpha_0\in\Phi(G,T)^-/\eta}\left(1-\prod_{\alpha\in\eta^{\mathbb{Z}}(\alpha_0)}\alpha(\gamma)\right)^{-1} &= \prod_{\alpha_0\in\Phi(G,T)^-/\eta}(1-(S_\eta(\alpha_0))(\gamma))^{-1}\\ &= \prod_{\alpha'\in\Phi(H,T_H)^-}\big(1-\alpha'(\tau(\gamma))\big)^{-1}.\end{aligned} \tag{39}$$

This coincides with the formal trace of $\tau(\gamma)$ acting on a Verma module for the endoscopic group H. If we have some α_l with $\frac{1}{2}P_\eta(\alpha_l)\in P_\eta(\Phi(G,T))$ then we have to replace $\Phi(G,T)^-$ in the above formula by $(\Phi(G,T)^{sm})^-$ and multiply with additional factors of the form (since η acts by -1 on X_{α_l} we get alternating signs in the geometric sum):

$$(1+\alpha_l(\gamma))^{-1} \quad = \quad 1-\alpha_l(\gamma)+\alpha_l(\gamma)^2-\dots$$

But each such α_l is of the form $\alpha_0+\eta(\alpha_0)=S_\eta(\alpha_0)$ and thus this factor may be multiplied with the corresponding factor $(1-S_\eta(\alpha_0)(\gamma))^{-1}$ to give the factor

$$\big(1-\alpha_l(\gamma)^2\big)^{-1}=\big(1-S'_\eta(\alpha_l)(\gamma)\big)^{-1},$$

since $S'_\eta(\alpha_l)=2S_\eta(\alpha_l)$ in this case. Now (37) of Proposition 5.4 tells us that we again arrive at the right hand side of (39).

From the above considerations we deduce moreover that $\delta_G=\delta_H$ as elements in $X^*(T_H)=X^*(T)^\eta$ so that $w(\lambda+\delta_G)-\delta_G$ may be identified with the corresponding element $w(\lambda+\delta_H)-\delta_H$ in $X^*(T_H)$ for $w\in W(G,T)^\eta=W(H,T_H)$. Reversing the computation for the group H we immediately get the claim.

□

Lemma 5.14. *In the notations of prop. 5.13 let $\mathfrak{n}$ be the unipotent radical of a standard parabolic subalgebra $\mathfrak{p}\subset\mathfrak{g}=Lie(G)$, let $\mathfrak{n}_H$ be the unipotent radical of the corresponding subalgebra $\mathfrak{p}_H\subset\mathfrak{h}=Lie(H)$. Let L resp. L_H denote the corresponding Levi groups. Then for every $w\in W(H)=W(G)^\eta$ we have that*

$$\begin{aligned}(-1)^{l_H(w)}\cdot H^{l_H(w)}\left(\mathfrak{n}_H, M_{H,\chi}\right)_{w(\chi+\delta)-\delta} &\in \mathcal{G}ro(L_H) \qquad \textit{matches with}\\ (-1)^{l_G(w)}\cdot H^{l_G(w)}\left(\mathfrak{n}, M_\chi\right)_{w(\chi+\delta)-\delta} &\in \mathcal{G}ro(L,\eta).\end{aligned}$$

Proof: Recall that $H^\nu\left(\mathfrak{n}, M_\chi\right)_\chi$ denotes the subspace of $H^\nu\left(\mathfrak{n}, M_\chi\right)$ which transforms under the action of L as the irreducible representation of highest weight χ. Recall from [Kos, theorem 5.14.] that the space $H^{l_G(w)}(\mathfrak{n}, M_\chi)_{w(\chi+\delta)-\delta}$ is an irreducible L-module if w is a Kostant representative for the coset space $W(G)/W(L)$. The theorem of Kostant furthermore tells us that the highest weight vector in $H^{l_G(w)}(\mathfrak{n}, M_\chi)_{w(\chi+\delta)-\delta}$ is the cohomology class having $e'_{-\Phi_w} \otimes m_{w\chi}$ as a representing cocycle, where $m_{w\chi} \in M_\chi$ is some weight vector for the extremal weight $w\chi$ and $\{e'_{-\Phi}\}$ for $\Phi \subset \Phi(\mathfrak{n})$ denotes the basis of $\Lambda^\cdot \mathfrak{n}'$ dual to the basis $\{e_\Phi\}$ of $\Lambda^\cdot \mathfrak{n}$, where

$$e_\Phi \quad = \quad e_{\phi_1} \wedge \ldots \wedge e_{\phi_\nu} \qquad \text{if } \Phi = \{\phi_1, \ldots, \phi_\nu\}$$

and the $e_\phi \in \mathfrak{n}$ are generators of the root spaces. From this description it is clear that the lemma is correct up to sign. At first recall from the existence of Steinberg representatives [Bal, lemma 2.7.] that there exists an η-invariant representative $\omega \in G(k)$ of w. We can take $m_{w\chi} = \omega(m_\chi)$ for some highest weight vector m_χ. Since η acts trivially on m_χ by the definition of M_χ as an $\tilde{G}$-module, we deduce that η acts as identity on $m_{w\chi}$.

Therefore it remains to prove that η acts as $(-1)^{l_G(w)-l_H(w)}$ on $e'_{-\Phi_w}$: Recall $\Phi_w = w(\Phi(G,T)^-) \cap \Phi(G,T)^+$ and $l_G(w) = \#\Phi_w$. We compare the contributions of the η-orbits of roots α to $l_G(w) - l_H(w)$. Let λ be the length of the η-orbit of α.

For $\alpha \in \Phi(G,T)^{sm} \cap \Phi(G,T)^{lm}$ the contribution is λ to $l_G(w) = \#\Phi_w$ and 1 to $l_H(w) = \#\Phi(H, T_H)_w$. By lemma 5.12 we can take basis elements e_ϕ for ϕ in the η-orbit of α, which are permuted by η. Now η acts by $(-1)^{\lambda-1}$ on the exterior product of these vectors, which gives the correct contribution.

If α is such that $2P_\eta(\alpha) \in P_\eta(\Phi(G,T))$, then there exists another root α' such that $2P_\eta(\alpha) = P_\eta(\alpha')$. In fact $\alpha' = \alpha + \eta(\alpha)$ and $\eta(\alpha') = \alpha'$, so that $\alpha' \in \Phi_w$ if $\alpha \in \Phi_w$. But the converse implication also holds: If $\alpha \notin \Phi_w$ then α lies in at least one of the halfsystems $w(\Phi(G,T)^+)$ and $\Phi(G,T)^-$. But since η stabilizes the decomposition in positive and negative roots and furthermore fixes w, we get that $\eta(\alpha)$ also lies in this halfsystem. Since the halfsystems are closed under addition of roots, we deduce that α' lies in one of them, i.e. $\alpha' \notin \Phi_w$. Thus we may compute the contribution of the η-orbit of α together with the contribution of α'. We conclude that we have a contribution $\lambda + 1$ to $l_G(w) = \#\Phi_w$. Only $S'_\eta(\alpha')$ contributes a 1 to $l_H(w)$, since $\alpha \notin \Phi(G,T)^{lm}$. By the same argument as above η acts by $(-1)^{\lambda-1}$ on the exterior product of the e_ϕ for ϕ in the η-orbit of α, but as -1 on $e_{\alpha'}$ (again by lemma 5.12), which gives the correct contribution $(-1)^\lambda$ to $e'_{-\Phi_w}$. This finishes the proof. □

Lifts

5.15. Let $G_1 = H/F$ be the stable endoscopic group of the pair (G, η), where $G/\mathcal{O}_F$ is a reductive connected split group over the ring of integers $\mathcal{O}_F$ of a number field F and η is an automorphism of finite order fixing some splitting of G. In the following definitions we denote by F either some local non-archimedean field $F_\mathfrak{p}$ or the ring of finite adeles $\mathbb{A}_f$.

While it does not matter in the following which Haar measures we take on the initial groups G and G_1 (we just have to multiply h_f resp. $h_{f,1}$ by a scalar), we have to be careful in using Haar measures on the (η-)centralizers of matching semisimple elements γ_0 and γ_1 when we define the matching of Schwartz-Bruhat functions in the sequel. If F is a local non archimedean field we normalize the Haar measures such that they give the measure 1 to the integral points of the connected component of the centralizer.

If $F = \mathbb{A}_f$ we take the Haar measures as finite parts of some Tamagawa measures $db = db_\infty \times db_f$ resp. $db_1 = db_{1,\infty} \times db_{1,f}$ which are normalized in such a way that the following identity holds:

$$|\alpha_\infty(\gamma_0, 1)| \quad = \quad |\alpha_\infty(\gamma_1, 1)|. \tag{40}$$

Recall from 4.9 that the definition of $\alpha_\infty(\gamma, 1)$ involves the infinity component of the Haar measure of the (η-)centralizer of γ.

Warning: We do not assume, that the product of the normalized local Haar measures at the finite places gives the Haar measure on the finite adeles. Therefore the results in the next subsection will need some careful analysis of the local factors $|\alpha_\infty(\gamma_0, 1)|$ (compare [W2]), before they can be used to get exact multiplicity statements in the lifting of representations (compare [Wes]).

Definition 5.16. *The Schwartz-Bruhat functions $h_f \in \mathcal{C}_c^\infty(G(F))$ and $h_{f,1} \in \mathcal{C}_c^\infty(G_1(F))$ are matching if they have matching stable orbital integrals i.e. if*

$$SO_\eta(\gamma, h_f) \quad = \quad SO(\gamma_1, h_{f,1})$$

for all matching semisimple elements $\gamma \in G(F)$ and $\gamma_1 \in G_1(F)$.

Recall that a distribution on $G(F)$ is called η-stable if it lies in the closure of the space of stable orbital integral distributions $h_f \mapsto SO_\eta(\gamma, h_f)$.

Definition 5.17. *The admissible representation $\pi \in Rep(G(F) \rtimes \eta)$ is a lift of $\pi_1 \in Rep(G_1(R))$, if $tr(h_f \cdot \eta|\pi) = tr(h_{f,1}|\pi_1)$ for all matching $h_f \in \mathcal{C}_c^\infty(G(F))$ and $h_{f,1} \in \mathcal{C}_c^\infty(G_1(F))$ and if furthermore the characters $\chi_\pi : h_f \mapsto tr(h_f \cdot \eta|\pi)$ and $\chi_{\pi_1} : h_{f,1} \mapsto tr(h_{f,1}|\pi_1)$ are (η-)stable distributions.*

Some virtual admissible representation $\Pi \in \mathcal{G}ro(G(F) \rtimes \eta)$ is the lift of $\Pi_1 \in \mathcal{G}ro(G_1(F))$ if we can write them in the form $\Pi = \pi - \pi'$ and $\Pi_1 = \pi_1 - \pi_1'$ such that the admissible representations $\pi, \pi' \in Rep(G(F) \rtimes \eta)$ are the respective lifts of $\pi_1, \pi_1' \in Rep(G_1(F))$.

5.18. Now we assume that we are in one of the following situations:

$$\begin{aligned}(G, \eta, G_1) \quad &= \quad (\mathrm{PGL}_{2n+1}, \eta, \mathrm{Sp}_{2n}) \\ (G, \eta, G_1) \quad &= \quad (\mathrm{GL}_{2n} \times \mathrm{GL}_1, \eta, \mathrm{GSpin}_{2n+1})\end{aligned}$$

In an earlier paper [BWW] we have shown that the twisted fundamental lemma for these situations can be reduced to a statement ("BC-conjecture") comparing stable orbital integrals on the groups Sp_{2n} and SO_{2n+1}, a phenomen which has been worked out by Waldspurger in more generality ([Wa3]). This statement has been proven by Ngô [Ngo, Théorème 2] in the case of positive characteristic, but the work of Waldspurger [Wa2] [Wa3] allows to reduce the case of p-adic fields to this fundamental result of Ngô. We remark that the cases $n = 1$ and $n = 2$ have been obtained earlier using explicit calculations of p-adic orbital integrals ([Fl1],[Fl2] and [BWW, 7.10]). We thus have:

THEOREM 5.19. *In the case that F is a local field with sufficiently large residue characteristic and (G, η, G_1) is as in 5.18 the characteristic functions of $G(\mathcal{O}_F)$ and $G_1(\mathcal{O}_F)$ match.*

REMARK 5.20. In the case that F is a local field it is well known that for each h_f there exists some matching $h_{f,1}$ and vice versa. This is elementary for functions having support in the set of (η-)regular elements and may be deduced in the above situations from [Wa1] (for the case $n = 2$ compare [Hal2]) and [Wa3] for all Schwartz-Bruhat functions. We conclude from this local matching property and the fundamental lemma that in the above situations the corresponding statement holds in the case $F = \mathbb{A}_f$ for sufficiently many functions to get weak lifting statements. Details will be explained elsewhere.

THEOREM 5.21. *In the case that F is a local field with residue characteristic not 2 and (G, η, G_1) is as in 5.18 then two elements of the Hecke algebra $f \in \mathcal{S}(G(F)//G(\mathcal{O}_F))$ and $f_1 \in \mathcal{S}(G_1(F)//G_1(\mathcal{O}_F))$ match, if f maps to f_1 under the Satake isomorphism.*

Proof: If the group Z_G^η is connected, this statement is reduced to the special case (5.19) in [W3], which is an extension of the results of [Hal1] to the twisted case. In the case $G = \mathrm{GL}_{2n} \times \mathbb{G}_n$ we may reduce to the situation $(G', \eta, G_1) = (\mathrm{GL}_{2n}/\mu_n, \eta', \mathrm{GSpin}_{2n+1})$ of example (5.9), where the η-invariants of the center form a connected group:

If $t \in T(F)$ maps to $t_1 \in T_1(F)$ under the norm map, we have to show that the characteristic functions f of $G(\mathcal{O}_F)tG(\mathcal{O}_F)$ and f_1 of $G_1(\mathcal{O}_F)t_1G_1(\mathcal{O}_F)$ match. This is equivalent to the same statement for G' and the characteristic function f' of $G'(\mathcal{O}_F) \cdot t' \cdot G'(\mathcal{O}_F)$, since we have the following identity between the stable orbital integrals: $O_\gamma^{st}(f, G) = O_{\gamma mod R}^{st}(f', G')$, compare [BWW, lemma 5.8.].

Lifting of cohomology

5.22. In the next theorem G will be defined over a totally real number field F.
As maximal connected and compact subgroups of $G(\mathbb{R})$ we choose the following: $K_\infty = \prod_{v|\infty} K_{\infty,v} \subset \overline{G}(\mathbb{R}) = \prod_{v|\infty} G(\mathbb{R})$ where $K_{\infty,v} = \mathrm{SO}_n(\mathbb{R})$ for $G = \mathrm{GL}_n, \mathrm{GL}_n \times \mathrm{GL}_1$ and in the case that n is odd also for $G = \mathrm{PGL}_n$, $K_{\infty,v} = U_n(\mathbb{R})$ for $G = \mathrm{GSp}_{2n}$ and for $G = \mathrm{Sp}_{2n}$.

THEOREM 5.23. *Let F be a totally real number field. Assume that $(G/F, \eta, G_1/F)$ is as in 5.18. For the groups $\overline{G} = Res_{F/\mathbb{Q}}G$ and $\overline{G_1} = Res_{F/\mathbb{Q}}G_1$ we have, if the $\overline{G}$-module M matches with the $\overline{G_1}$-module M_1:*

$$H_c^*\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty, \mathcal{M}\right) \in \mathcal{G}ro(\overline{G}(\mathbb{A}_f) \rtimes \eta) = \mathcal{G}ro(G(\mathbb{A}_{f,F}) \rtimes \eta)$$

is the lift of

$$H_c^*\left(\overline{G}_1(\mathbb{Q})\backslash\overline{G}_1(\mathbb{A})/K_{\infty,1} Z_\infty, \mathcal{M}_1\right) \in \mathcal{G}ro(\overline{G}_1(\mathbb{A}_f)) = \mathcal{G}ro(G_1(\mathbb{A}_{f,F})).$$

Proof: Let h_f and $h_{f,1}$ be matching Schwartz-Bruhat functions. We choose open compact subgroups K_f resp. $K_{f,1}$ of $G(\mathbb{A}_{f,F})$ resp. $G_1(\mathbb{A}_{f,F})$ such that h_f is right invariant under K_f and $h_{f,1}$ right invariant under $K_{f,1}$. Since we may make K_f smaller we can furthermore assume that Ass_{K_f} and $Ass_{\zeta,der}$ are satisfied. Replacing K_f by $K_f \cap \eta(K_f)$ we may furthermore assume that K_f is η-invariant, so that $Z_f = K_f \cap \overline{G}(\mathbb{A}_f)$ satisfies Ass_{Zf}.

We remark furthermore that Ass_{conn} is fulfilled in the cases under consideration: this is clear for the endoscopic groups since Sp_{2n} and the derived group of GSpin_{2n+1} are simply connected, which implies that the centralizer of a semisimple element is connected. Furthermore it is well known that the connected component of the centralizer of a non semisimple element is not reductive.

On the other hand it follows from the computations in [BWW] (compare Lemma 2.9. and Step 3 in the proof of Theorem 5.11) that the η-centralizer of an element in $\mathrm{GL}_{2n} \times \mathrm{GL}_1$ is a product of a symplectic group, a special orthogonal group, some centralizer inside a symplectic group and of $\mathbb{G}_m$. This implies that the centralizers $G^I_{\gamma,\eta}$ are connected. The case of PGL_{2n+1} reduces to the η-centralizers in SL_{2n+1} (Proof of Prop. 4.5. loc. cit.) and can be handled by the same argument.

Then we have to prove

$$\begin{aligned} & tr\left(\eta \circ h_f | H^*_c\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty \cdot K_f, \mathcal{M}\right)\right) \\ = \quad & tr\left(h_{f,1} | H^*_c\left(\overline{G}_1(\mathbb{Q})\backslash\overline{G}_1(\mathbb{A})/K_{\infty,1} Z_\infty \cdot K_{f,1}, \mathcal{M}_1\right)\right) \end{aligned}$$

Since the assumptions of the trace formula in 3.20 and the assumptions for the stabilization in §4 are satisfied , we may replace the traces by the right hand sides of 4.9.

First of all we note that the (stabilized) trace formula implies that the two virtual characters which are defined by the two sides of this equation are stable resp. η-stable distributions, so that the lifting claim makes sense.

We remark that the set Δ_1 of simple roots of G_1 can be identified with the set of η-orbits in the set of simple roots of Δ, i.e. we have a projection $\pi : \Delta \twoheadrightarrow \Delta/\eta \simeq \Delta_1$, so that we have a bijection between the set of η-invariant subsets $I \subset \Delta$ with the set of subsets $I_1 \subset \Delta_1$ given by $I \mapsto \pi(I)$ and $I_1 \mapsto \pi^{-1}(I_1)$. Since this bijection satisfies $(-1)^{\#((\Delta-I)/\eta)} = (-1)^{\#(\Delta_1-I_1}$ we are reduced to prove

$$\sum_{\substack{\gamma_0 \in (P_I(\mathbb{Q}))_{\eta-st} \\ \mathcal{N}(\gamma_0)\sim L^I_\infty \\ \chi_{I,\alpha}(\mathcal{N}(\gamma_0))>1 \\ \text{for all } \alpha\in\Delta-I}} \alpha_\infty(\gamma_0,1)\cdot SO_\eta(\gamma_0,h_f)\cdot tr(\gamma_0\circ\eta|\mathcal{M})$$

$$= \sum_{\substack{\gamma_1 \in \left(P_{I_1}(\mathbb{Q})\right)_{st} \\ \gamma_1\sim L^{I_1}_{\infty,1} \\ \chi_{I_1,\alpha_1}(\gamma_1))>1 \\ \text{for all } \alpha_1\in\Delta_1-I_1}} \alpha_\infty(\gamma_1,1)\cdot SO(\gamma_1,h_{f,1})\cdot tr(\gamma_1|\mathcal{M}_1)$$

We observe that M_{I_1} is the stable endoscopic group of (M_I,η). We remark that an element $\gamma_0 \in P_I(\mathbb{Q})$, such that $\mathcal{N}(\gamma_0)$ has a conjugate in L^I_∞, is η-semisimple, since L^I_∞ contains no unipotent elements. Thus its η-conjugacy class meets the Levi group $M_I(\mathbb{Q})$, so that we are reduced to consider elements $\gamma_0 \in M_I(\mathbb{Q})$. The definition of stable endoscopy implies that we have a bijection between η-semisimple η-conjugacy classes in $M_I(\bar{\mathbb{Q}})$ and semisimple conjugacy classes in the corresponding $M_{I_1}(\bar{\mathbb{Q}})$ such that this induces the projection $T(\mathbb{Q}) \mapsto T(\mathbb{Q})_\eta \simeq T_1(\mathbb{Q})$ on the diagonal tori. From Cor. 6.4. , Prop. 7.5(b) and Cor. 7.6. in [BWW] we deduce that "matching" defines a bijection between those (η-)stable (η-)conjugacy classes which have rational representatives $\gamma_0 \in M_I(\mathbb{Q})$ resp. $\gamma_1 \in M_{I_1}(\mathbb{Q})$. With these notations it remains to prove:

(a) $\chi_{I,\alpha}(\mathcal{N}(\gamma_0)) > 1$ for all $\alpha \in \Delta - I$ if and only if $\chi_{I_1,\alpha_1}(\gamma_1)) > 1$ for all $\alpha_1 \in \Delta_1 - I_1$,

(b) $\mathcal{N}(\gamma_0) \sim L^I_\infty \quad \Leftrightarrow \quad \gamma_1 \sim L^{I_1}_{\infty,1}$,

(c) $\alpha_\infty(\gamma_0, 1) \quad = \quad \alpha_\infty(\gamma_1, 1)$,

since we already know $SO_\eta(\gamma_0, h_f) = SO(\gamma_1, h_{f,1})$ by assumption and $tr(\gamma_0 \circ \eta|\mathcal{M}) = tr(\gamma_1|\mathcal{M}_1)$ by proposition 5.13.

5.24. To prove (a) we may replace γ_0 by an η-conjugate $\gamma_0' \in T(\bar{\mathbb{Q}})$ and γ_1 by a conjugate γ_1', such that γ_0' maps to γ_1' under the canonical projection $T(\bar{\mathbb{Q}}) \twoheadrightarrow T_1(\bar{\mathbb{Q}})$. The element $\mathcal{N}(\gamma_0)$ is then a conjugate of $\mathcal{N}(\gamma_0')$. But under the identification $X^*(T_1) = X^*(T)^\eta$ we can take χ_{I_1,α_1} to be a positive rational multiple of $\chi_{I,\alpha} \circ (id + \eta)$. The claim is now an immediate consequence of this.

5.25. To prove (b) we use γ_0' and γ_1' as in the proof of (a). Then γ_1 may be conjugated into $L^{I_1}_{\infty,1}$ if and only if $\tau(\alpha(\gamma_1'))$ has absolute value 1 for all embeddings $\tau : \mathbb{Q} \hookrightarrow \mathbb{C}$ and all roots $\alpha_1 \in I_1$ and if γ_1 satisfies a certain condition, which characterizes $L^{I_1}_{\infty,1}$ inside $L^{I_1,m}_{\infty,1}$. This condition is $\rho(\mu(\gamma_1)) > 0$ for all $\rho : F \hookrightarrow \mathbb{R}$ in the case $G_1 = \mathrm{GSpin}_{2n+1}$ and is the empty condition for $G_1 = \mathrm{Sp}_{2n}$. Similarly $\mathcal{N}(\gamma_0)$ may be conjugated into L^I_∞ if and only if $\tau(\alpha(\mathcal{N}(\gamma_0')))$ has absolute value 1 for all $\tau : \mathbb{Q} \hookrightarrow \mathbb{C}$ and if in the case $G = \mathrm{GL}_{2n} \times \mathrm{GL}_1$ we have $\rho(a^2 \cdot \det A) > 0$ for all $\rho : F \hookrightarrow \mathbb{R}$, where $\gamma_0 = (A, a)$. But since $\alpha \circ \mathcal{N} = \alpha \circ (id + \eta)$ is either a root or twice a root in I_1 and since the sign conditions correspond to each other under the identification $X^*(T_1) = X^*(T)^\eta$, (compare [BWW, 1.15]), the claim (b) is now clear.

5.26. The statement of (c)is up to sign just the assumption in normalizing the Haar measures on the centralizers made in (40) above. It remains to check that $\Delta(\gamma_0, \eta) = \Delta(\gamma_1, id)$ (at least modulo 2).

To prove that $q(\tilde{G}^I_{\gamma_0,\eta}) = q(\tilde{G}^{I_1}_{\gamma_1,id})$ for the quasisplit forms we remark that we may deduce from [BWW] that the centralizers of γ_0 and γ_1 have factorizations in factors which are either isogenous for the two groups or are of the shape that some SO_{2g+1} for one group corresponds to some Sp_{2g} for the other group. Since these two groups have no outer automorphism we have to take their split forms and then get

$$q(\mathrm{Sp}_{2g}) = \frac{g^2+g}{2} \text{ and } q(\mathrm{SO}_{2g+1}) = \frac{\dim(\mathrm{SO}_{2g+1}) - \dim(\mathrm{SO}_{g+1} \times \mathrm{SO}_g)}{2} = \frac{g^2+g}{2}.$$

The remaining summand $\Delta(\tilde{G}, \tilde{K}_\infty)$ is just the difference between the dimension of the maximal real split torus $Z^{\mathbb{R}-split}_{\tilde{G}}$ in the center of $\tilde{G}$ and the dimension of its intersection with the center of the original group. By the result already cited from [BWW] the centers of the two centralizers are isogenous, so the dimensions of their real split tori coincide. The dimensions of the intersections with the original centers also agree (they are 0 in the situation $G = \mathrm{PGL}_{2n+1}$ and $G_1 = Sp_{2n}$ and are 1, resp. the degree of the totally real ground field, for $G = \mathrm{GL}_{2n} \times \mathbb{G}_m$ and $G = \mathrm{GSpin}_{2n+1}$). The equality of the signs is proven. $\square$

COROLLARY 5.27. *Under the assumptions of theorem 5.23 we have:*

$$H^*\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty, \mathcal{M}\right) \in \mathcal{G}ro(\overline{G}(\mathbb{A}_f) \rtimes \eta) = \mathcal{G}ro(G(\mathbb{A}_{f,F}) \rtimes \eta)$$

is the lift of

$$H^*\left(\overline{G}_1(\mathbb{Q})\backslash\overline{G}_1(\mathbb{A})/K_{\infty,1} Z_\infty, \mathcal{M}_1\right) \in \mathcal{G}ro(\overline{G}_1(\mathbb{A}_f)) = \mathcal{G}ro(G_1(\mathbb{A}_{f,F}))$$

Proof: This may be deduced from the previous theorem by Poincaré duality: We have

$$H^i\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty, \mathcal{M}\right) \simeq Hom\left(H_c^{q(\bar{G})-i}\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty, \check{\mathcal{M}}\right), H_c^{q(\bar{G})}\left(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K_\infty Z_\infty, \mathbb{C}\right)\right),$$

and a similar relation holds for the group G_1. It is clear that the cohomology with compact support in the highest dimension lifts from the group G_1 to (G, η). $\square$

EXAMPLE 5.28. Let us consider the special case where $G = \mathrm{GL}_4/\mathbb{Q} \times \mathrm{GL}_1/\mathbb{Q}$ and $G_1 = \mathrm{GSp}_4/\mathbb{Q}$ and $\mathcal{M}$ and $\mathcal{M}_1$ are the constant sheaves. Furthermore let h_f resp. $h_{f,1}$ be the characteristic functions of the maximal compact subgroups $K_f = \mathrm{GL}_4(\hat{\mathbb{Z}}) \times \hat{\mathbb{Z}}^*$ and $K_{f,1} = \mathrm{GSp}_4(\hat{\mathbb{Z}})$. In this case the statement reduces to an identity which can be shown to be true by other methods: We have isomorphisms

$$\begin{array}{rcll} X & := & G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty Z_\infty \cdot K_f & \simeq \mathrm{SL}_4(\mathbb{Z})\backslash\mathrm{SL}_4(\mathbb{R})/\mathrm{SO}_4(\mathbb{R}) \quad \text{and} \\ X_1 & := & G_1(\mathbb{Q})\backslash G_1(\mathbb{A})/K_{\infty,1} Z_{\infty,1} \cdot K_{f,1} & \simeq \mathrm{Sp}_4(\mathbb{Z})\backslash\mathrm{Sp}_4(\mathbb{R})/\mathrm{U}_2(\mathbb{R}) \end{array}$$

and the formula states:

$$tr\left(\eta \circ h_f | H^*(X, \mathbb{C})\right) = tr\left(h_{f,1} | H^*(X_1, \mathbb{C})\right).$$

But the right hand side is just the Euler characteristic of X_1, which is known to be homeomorphic to $\mathbb{P}^3(\mathbb{C}) - \mathbb{P}^1(\mathbb{C})$, i.e. the Betti numbers are $b_i(X_1) = 1$ for $i = 0, 2$ and $b_i(X_1) = 0$ else. Thus the right hand side equals 2. The left hand side is the Lefschetz number of the involution η acting on X. It is known [LeeS, Theorem 2] that $H^i(X, \mathbb{C})$ is one dimensional for $i = 0, 3$ and is zero for all other values of i. The fact that the left hand side also equals 2 is thus equivalent to the assertion that η acts by -1 on $H^3(X, \mathbb{C})$. Since the antidiagonal matrix J_4 lies in $K_\infty \times K_f$ the involution η on X may be written in the form: $\eta_0 : A \mapsto {}^tA^{-1}$.

By Poincaré duality (which holds for coefficient domains in characteristic 0, since X is a quotient of a manifold by a finite group) we get isomorphisms $H^i(X, \mathbb{C}) \simeq H_{9-i}(\bar{X}, \partial\bar{X}, \mathbb{C})$, where $\bar{X}$ denotes the Voronoi compactification of X and $\partial\bar{X} = \bar{X} - X$ the complement (compare [LeeS]). Now $H_9(\bar{X}, \partial\bar{X}, \mathbb{C})$ is generated by the relative fundamental class c of X, and η_0 acts on it by -1, since the action on the tangent space $sl_4(\mathbb{R})/so_4(\mathbb{R})$, which may be identified with the space of real symmetric matrices, is minus the identity and since $\dim(X) = 9$ is odd. A generator of $H_6(\bar{X}, \partial X, \mathbb{C})$ which is called σ_4^6 in the notation of [LeeS, 3.2.] is easily seen to be the image of the relative fundamental class of the locally symmetric space $S = \mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{GL}_3(\mathbb{R})^+/\mathrm{SO}_3(\mathbb{R})$ under the embedding of spaces, which is induced from the embedding of groups $\iota : A \mapsto diag(A, \det(A)^{-1})$. One checks immediately that η_0 acts by -1 on the 6-dimensional tangent space, so that $H_6(\bar{X}, \partial X, \mathbb{C})$ is η_0-invariant. Since Poincaré duality is induced by cap product with c we deduce that η_0 acts by -1 on $H^3(X, \mathbb{C})$. $\square$

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