This is the handout to the seminar talk on

Lie algebras

Main point of the talk is to introduce the Lie algebra associated to a Lie group. This will be done over left-invariant vector fields. The talk is divided into 3 parts with some examples tacked onto the end. In case it is important to the reader, the talk will be/is being/was held on the 17th of May 2016.

We start with some

Linear algebra

Def. A Lie algebra over a field $\mathbb{F}$ is a $\mathbb{F}$ vectorspace $\mathfrak{g}$ together with a bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ so that:

(1) $[\,,\,]$ is anti-symmetric

(2) $[\,,\,]$ satisfies the Jacobi identity, that is $\forall a, b, c \in \mathfrak{g}$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

Def. A linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras $\mathfrak{g}, \mathfrak{h}$ is called a Lie algebra homomorphism if

$$\Phi([a, b]_{\mathfrak{g}}) = [\Phi(a), \Phi(b)]_{\mathfrak{h}}$$

holds for all $a, b \in \mathfrak{g}$.

Remark Bijective Lie algebra homomorphisms are invertible.

Some simple examples of Lie algebras are:

(i) If V is a $\mathbb{F}$ vector space, then $\mathrm{End}(V)$ together with

$$\begin{aligned} [\,,\,] :\ & \mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V) \\ & (A, B) \mapsto [A, B] := AB - BA \end{aligned}$$

is a Lie algebra. Indeed bilinearity and anti-symmetry are clear. It may be instructive to verify the Jacobi identity.

(ii) Any subspace of $\mathrm{End}(V)$ that is closed under the application of $[\,,\,]$ is also a Lie algebra.

(iii) An example of a sub-Lie algebra of $\mathrm{End}(V)$ in the case of finite dimensional V would be the subspace of traceless linear maps $=: \mathfrak{sl}(V)$. Since the trace is cyclic and linear

$$\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0$$

always holds and $\mathfrak{sl}(V)$ is closed under $[\,,\,]$.

Indeed, every Lie algebra can be realised as the sub-algebra of an endomorphism space:

Prop. Any Lie algebra $\mathfrak{g}$ is isomorphic to a sub-algebra of $\mathrm{End}(V)$ for some vector space V.

Construction: Consider $\mathfrak{g}$ as a vector space. Since $[\,,\,]$ is bilinear, we have that for $x \in \mathfrak{g}$

$$\begin{aligned}\mathrm{ad}_x : \mathfrak{g} &\to \mathfrak{g} \\ y &\mapsto [x, y]\end{aligned}$$

Is an element of $\mathrm{End}(\mathfrak{g})$. From bilinearity we see again that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is also linear. One sees also that for all $z \in \mathfrak{g}$:

$$\begin{aligned}[\mathrm{ad}_x, \mathrm{ad}_y]_{\mathrm{End}(V)}(z) &= \mathrm{ad}_x \mathrm{ad}_y(z) - \mathrm{ad}_z \mathrm{ad}_x(z) = [x, [y, z]_\mathfrak{g}]_\mathfrak{g} - [y, [x, z]_\mathfrak{g}]_\mathfrak{g} \\ &= [x, [y, z]_\mathfrak{g}]_\mathfrak{g} + [y, [z, x]_\mathfrak{g}]_\mathfrak{g} \overset{*}{=} -[z, [x, y]_\mathfrak{g}]_\mathfrak{g} = \mathrm{ad}_{[x,y]_\mathfrak{g}}(z)\end{aligned}$$

In the equality denoted by $(*)$ the Jacobi identity was used. This means $[\mathrm{ad}_x, \mathrm{ad}_y]_{\mathrm{End}(V)} = \mathrm{ad}_{[x,y]_\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$. So ad is a Lie algebra homomorphism. It is called the **adjoint representation** of $\mathfrak{g}$.

But the adjoint representation does not need to be injective (and thus an isomorphism onto its image). The kernel $\{x \in \mathfrak{g} \mid \mathrm{ad}_x = 0\}$ is given by the center $Z(\mathfrak{g})$, that is the elements x of $\mathfrak{g}$ so that $[x, y] = 0\ \forall y \in \mathfrak{g}$. (The name center comes from $[x, y] = [y, x] \iff [x, y] = 0$ in characteristic $\neq 2$.)

It is possible, beginning with the centre of a Lie algebra, to construct a faithful representation onto a endomorphism algebra that would preserve finite dimensionality. However the construction is involved, the theorem is known as **Ado's theorem**. In order to comply with the promise of representing any Lie algebra faithfully on an endomorphism space we instead construct the **enveloping algebra**.

Consider $T(\mathfrak{g}) := \oplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$, which with $a \cdot b := a \otimes b$ has the structure of an associative $\mathbb{F}$-algebra. Consider elements of the form $x \otimes y - y \otimes x - [x, y]_\mathfrak{g}$ for $x, y \in \mathfrak{g}$ and denote with I the smallest two-sided ideal in $T(\mathfrak{g})$ containing these elements. Denote now $U(\mathfrak{g}) := T(\mathfrak{g})/I$, this is the enveloping algebra.

The map $h : \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g}))$, $x \mapsto h_x$ with $h_x([a]) = [x \otimes a]$ is a well defined linear map. Forgetting for notational convenience the equivalence brackets, one finds

$$[h_x, h_y]_{U(\mathfrak{h})}(a) = x \otimes y \otimes a - y \otimes x \otimes a = (x \otimes y - y \otimes x) \otimes a = [x, y]_\mathfrak{g} \otimes a = h_{[x,y]_\mathfrak{g}}(a)$$

making it a Lie algebra homomorphism. If $h_x([a]) = 0\ \forall [a] \in U(\mathfrak{g})$, we have that $x \otimes a$ must lie in the ideal I. But then $x \otimes 1 = x$ would have to be in the ideal. This is impossible, as the only elements of the ideal that have parts in $\mathfrak{g}$ are sums of elements in the form $a \otimes b - b \otimes a - [a, b]$, but here we can never get rid of the $\mathfrak{g} \otimes \mathfrak{g}$ terms without also getting rid of the $\mathfrak{g}$ term. So the kernel of the representation is zero. $\square$

We may be interested in some **examples with more geometric content**

(iv) $\mathbb{R}^3$ with the cross product is a Lie algebra. The adjoint representation is injective and identifies the algebra with $\mathfrak{so}(3)$, the space of anti-symmetric 3×3 matrices.

(v) The space of smooth vector fields on a manifold with the Lie bracket form an infinite dimensional Lie algebra.

To investigate the last example we will now think about

Vector fields

Let M, N be smooth manifolds.

Reminder (vector fields) The tangent space at a point $p \in M$ is the vector space of derivations at p:

$$T_pM := \{x \in \mathcal{L}(C^\infty(M), \mathbb{R}) \mid x(f \cdot g) = x(f) \cdot g(p) + f(p) \cdot x(g)\}$$

At one point we will also use that this space can be identified with the space of *germs of paths through p*.

A vector field can be viewed as a map: $X : M \to \bigcup_{p \in M} T_pM$ so that $X_p \in T_pM$, ie assigning each point of M a tangent vector from that point. The vector field is called smooth if $X_p(f) : M \to \mathbb{R}$ is smooth ("in p") for all $f \in C^\infty(M)$. Alternatively describe a smooth vector field as a smooth section of the tangent bundle TM. The space of smooth vector fields on M has a vector space structure and is denoted $\Gamma(TM)$.

Prop. The Lie bracket $[X, Y]$ defined via $[X, Y]_p(f) := X_p(Y(f)) - Y_p(X(f))$ is a well defined map $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ and gives $\Gamma(TM)$ the structure of a Lie algebra.

Proof $[X, Y]_p$ is clearly a linear map from $C^\infty(M)$ to $\mathbb{R}$. Since X, Y are smooth vector fields, $Y(f), X(f)$ are smooth functions for all f and thus $X(Y(f))$ and $Y(X(f))$ are smooth functions, so $[X, Y](f)$ is a smooth function. For well definendness it remains to check the derivation property:

$$\begin{aligned}
[X, Y](fg) =& X(Y(fg)) - Y(X(fg)) = X(fY(g) + Y(f)g) - Y(fX(g) + X(f)g) \\
=& X(f)Y(g) + X(Y(g))f + X(Y(f))g + X(g)Y(f) \\
& - Y(f)X(g) - Y(X(g))f - Y(X(f))g - Y(g)X(f) \\
=& X(Y(g))f + X(Y(f))g - Y(X(g))f - Y(X(f))g \\
=& [X, Y](f)g + [X, Y](g)f
\end{aligned}$$

The bracket is clearly anti-symmetric and bilinear, the Jacobi identity can be verified exactly as for $\mathrm{End}(V)$. □

Reminder (pushforward/differential) If $F : M \to N$ is a smooth map, then $F_{*,p} : T_pM \to T_{F(p)}N$, $x \mapsto F_{*,p}(x)$ with $[F_{*,p}(x)](f) := x(f \circ F)$ is a linear map between T_pM and $T_{F(p)}N$. $F_{*,p}$ is called the differential or pushforward of F at p.

Maybe one would like to extend this to get a linear map $F_* : \Gamma(TM) \to \Gamma(TN)$, $[F_*(X)]_{F(p)}(f) := X_p(f \circ F)$. But if F is not surjective we wouldn't know how to evaluate

$F_*(X)$ at points that do not lie in the image of F. If F is not injective we would not know what preimage of $F(p)$ to choose. So this can't work, instead we define the notion of two vector fields being F-related.

Def. If $F : M \to N$ is a smooth map, $X \in \Gamma(TM), Y \in \Gamma(TN)$ we say X, Y are F-related if $\forall p \in M$

$$F_{*,p}(X_p) = Y_{F(p)}$$

Note that this is equivalent to

$$X(f \circ F) = Y(f) \circ F$$

for all $f \in C^\infty(N)$. On the right $\circ F$ is composed with the *function* $Y(f)$.

Prop. If $F : M \to N$ is a diffeomorphism, then for each $X \in \Gamma(TM)$ there is one and only one $Y \in \Gamma(TN)$ that is F-related to X.

Cor. For Diffeomorphisms $F_* : \Gamma(TM) \to \Gamma(TN)$ is well defined and a vector space isomorphism.

Proofs Note that since diffeomorphisms are bijective, the equations $F_{*,p}(X)_{F(p)} = Y_{F(p)}$ uniquely determine a vector field on N. It is smooth since $Y(f) = X(f \circ F) \circ F^{-1}$ is smooth for all smooth f as a composition of smooth functions.

For the corollary note that linearity is clear and that $(F^{-1})_* = (F_*)^{-1}$. □

Lemma The Lie bracket preserves F-relation. Meaning if X_1, X_2 are F-related to Y_1, Y_2, then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

Proof Note

$$X_1(X_2(f \circ F)) \overset{X_2\ F\text{-rel}\ Y_2}{=} X_1(Y_2(f) \circ F) \overset{X_1\ F\text{-rel}\ Y_1}{=} Y_1(Y_2(f)) \circ F$$

Similarly $X_2(X_1(f \circ F)) = Y_2(Y_1(f)) \circ F$ and then $[X_1, X_2](f \circ F) = [Y_1, Y_2](f) \circ F$. □

Cor. If F is a diffeomorphism:

$$[F_*X, F_*Y]_{\Gamma(TN)} = F_* \left([X, Y]_{\Gamma(TM)}\right)$$

This last consideration is what we need to construct the

Lie algebra associated to a Lie group

Reminder (left translates) Let G be a Lie group. $\forall g \in G\ L_g : x \mapsto g \cdot x$ is a diffeomorphism and $L_g \circ L_h = L_{g \cdot h}$.

Denote with $\mathfrak{g}$ the set of vector fields of G that are L_g related to themselves for all $g \in G$. This means

$$(L_{g*,p}X_p)(f) = (L_{g*}X)_{g \cdot p}(f) \overset{!}{=} (X(f) \circ L_g)(p) = X_{g \cdot p}(f)$$

Which means $L_{g*}X = X$ as vector fields. This is why these vector fields are called **left invariant**.

The lemma shows that $\mathfrak{g}$ is a Lie algebra

If $X, Y \in \mathfrak{g}$ then

$$L_{g*}[X,Y] = [L_{g*}X, L_{g*}Y] = [X,Y]$$

And $[X,Y]$ is also left invariant. Thus the left invariant vector fields are a sub-Lie algebra of the infinite dimensional Lie algebra of vector fields. However:

Theorem *If G is a finite dimensional Lie group, then the Lie algebra $\mathfrak{g}$ is a finite dimensional Lie algebra and its dimension is equal to the dimension of G.*

Proof We construct a canonical vector space isomorphism between $\mathfrak{g}$ and T_eG, where e is the unit of G. This isomorphism is of course more important than the theorem, as it allows us to view T_eG as a Lie algebra by pushing the bracket forward from $\mathfrak{g}$ via the isomorphism.

Let $X_e \in T_eG$. If we assume X_e comes from a left invariant vector field X, it is clear from $X_g = L_{g*,e}X_e$ that X is uniquely determined by X_e. Similarly the equation just given is actually a prescription to get a left invariant vector field from X_e. It must now be shown that the vector field constructed is smooth. This means $X_p(f) := L_{p*,e}X_e(f)$ must be smooth "in p"for all smooth functions $f \in C^\infty(M)$.

To do this identify X_e with a smooth path $\gamma : (-1,1) \to M$ with $\gamma(0) = e$ and $X_e(f) = \frac{d}{dt}f(\gamma(t))|_{t=0}$. Then $X_p(f) = L_{p*,e}X_e(f) = X_e(f \circ L_p) = \frac{d}{dt}f(p \cdot \gamma(t))|_{t=0}$. This is the same as $\frac{d}{dt}\varphi_f(p,t)|_t = 0$ with $\varphi_f : G \times (-1,1) \to G$, $(p,t) \mapsto f(p \cdot \gamma(t))$. Since φ_f is smooth as a composition of smooth functions so is its derivative and $X_p(f)$ is smooth "in p". $\square$

Remark Since $[X,Y]_p$ depends only on how X, Y look in a neighbourhood of p, the structure of a Lie algebra is determined by the behaviour of the group near e. From such a consideration one can see that the Lie algebra of a group is always the one of the subgroup given by the connected component of e.

Lemma Let G, H be Lie groups, $\mathfrak{g}, \mathfrak{h}$ the associated Lie algebras and $\Phi : G \to H$ a Lie group homomorphism. Then $\Phi_{*,e} : T_eG \to T_eH$ defines a Lie algebra homomorphism from $\mathfrak{g} \to \mathfrak{h}$.

Proof The only thing to prove is that the Lie bracket is conserved. Let X, Y be left invariant vector fields on G, H with $\Phi_{*,e}X_e = Y_e$. Since Φ is a Lie group homomorphism we have $\Phi \circ L_g = L_{\Phi(g)} \circ \Phi$ for all $g \in G$. Thus $L_{\Phi(g)*,e} \circ \Phi_{*,e} = \Phi_{*,g} \circ L_{g*,e}$ and for all $g \in G$:

$$\Phi_{*,g}X_g = \Phi_{*,g}(L_{g*,e}X_e) = L_{\Phi(g)*,e}(\Phi_{*,e}X_e) = L_{\Phi(g)*,e}(Y_e) = Y_{\Phi(g)}$$

Thus X and Y are Φ-related. It follows now from the naturality of the Lie bracket that if for $X^1, X^2 \in \mathfrak{g}, Y^1, Y^2 \in \mathfrak{h}$ with $\Phi_{*,e}X^i_e = Y^i_e$ that then $[X^1, X^2]$ and $[Y^1, Y^2]$ are

Φ-related. But Φ-relation at e is nothing other than

$$\Phi_{*,e}[X^1, X^2]_e = [Y^1, Y^2]_e$$

and thus we have a homomorphism.

Comments Since $(\Phi \circ \Psi)_{*,p} = \Phi_{*,\Psi(p)} \circ \Psi_{*,p}$ and a Lie algebra homomorphism is literally defined by the action of $F_{*,e}$ on T_e, it is clear that Lie algebra homomorphism associated to $\Phi \circ \Psi$ is $\Phi_{*,e} \circ \Psi_{*,e}$. Since diffeomorphisms F induce vector space isomorphisms $F_{*,e}$ it is also clear that two isomorphic Lie groups have isomorphic Lie algebras.

Now consider and actually compute some

Further Examples

We will look at the Lie algebras associated to the Lie groups $S^1, \mathbb{T}^n = \bigtimes^n S^1, \mathrm{GL_n}(\mathbb{R})$ and $O(n)$.

$\boldsymbol{S^1}$ is super trivial, its a one dimensional Lie group and thus has a one dimensional Lie algebra. But all one dimensional Lie algebras have a trivial Lie bracket, which follows from anti-symmetry.

$\mathbb{T}^{\boldsymbol{n}}$ could be more interesting, but look at a coordinate chart around e where each S^1 is parametrised by some angle θ_i. This induces a standard basis on the tangent spaces via $e_{i,p} = \frac{\partial}{\partial \theta_i}$, because the group action on (a smaller neighbourhood of) this chart is just addition $L_{\boldsymbol{\varphi}}(\boldsymbol{\theta}) = \boldsymbol{\theta} + \boldsymbol{\varphi}$ you have $L_{\boldsymbol{\varphi}*}$ is given by the identity matrix. So in this coordinate chart the components of left invariant vector fields are constant. But then $[X, Y] = 0$ in the chart, and since the Lie algebra is determined by the local behaviour of the fields the Lie algebra is again trivial (isomorphic to $\mathbb{R}^n$ with bracket being the zero map).

$\mathbf{GL}_{\boldsymbol{n}}(\mathbb{R})$ does not have a trivial Lie algebra. Since $\mathrm{GL_n}(\mathbb{R})$ is an open subset of $\mathrm{Mat_{n\times n}}$, we can take the global coordinate chart given by the identity. $\mathrm{Mat_{n\times n}}$ is a (smooth) vector space and as such can be identified with all its tangent spaces. Since L_g acts on $\mathrm{Mat_{n\times n}}$ as a linear map, its differential in these coordinates is the same linear map meaning $L_{g*}(X) = gX$.

If we write $X = X_{ij}\partial_{ij}$, $Y = Y_{ij}\partial_{ij}$, we can calculate:

$$[L_{x*}X, L_{x*}Y] = [x^{ik}X_{kj}\partial_{ij}, x^{ml}Y_{ln}\partial_{mn}] = x^{mk}X_{kl}Y_{ln}\partial_{mn} - x^{il}Y_{lk}X_{kj}\partial_{ij} \equiv (x \cdot X \cdot Y - x \cdot Y \cdot X)$$

The dots denote matrix multiplication of components. Since this is supposed to be $L_{x*}[X, Y]$, we find that $[X, Y]$ is the same as the matrix commutator of X and Y in these coordinates, or that the Lie algebra is $\mathrm{Mat_{n\times n}}$. Since this part was done rather sloppily and the result is important it is recommended that one check the literature for a more detailed proof if one has not seen it before.

$\boldsymbol{O(n)}$ note that $O(n) \subset \mathrm{GL_n}(\mathbb{R})$. Let ι be the inclusion map. Then $\iota_{*,e}$ is injective and we get that its Lie algebra $\mathfrak{o}(n)$ can be identified with a sub-algebra of $\mathrm{Mat_{n\times n}}$. Note that with $\phi : \mathrm{GL_n}(\mathbb{R}) \to \mathrm{GL_n}(\mathbb{R}), \mathrm{A} \mapsto \mathrm{A^T A}$ we have $O(n) = \phi^{-1}(\{\mathbb{1}\})$.

So $\iota_{*,\mathbb{1}}(T_{\mathbb{1}}O(n))$ is the same as $\ker(\phi_{*,\mathbb{1}})$. To see this, note $\phi \circ \iota$ is constant, thus the image of $\iota_{*,\mathbb{1}}$ lies in the kernel of $\phi_{*,\mathbb{1}}$. On the other hand every path γ through $\mathbb{1}$ that is made constant by ϕ (ie whose corresponding tangent vector $[\gamma]$ lies in the kernel of $\phi_{*,\mathbb{1}}$) must lie in $\phi^{-1}(\{\mathbb{1}\})$, but that is just the domain of ι, so $\gamma = \iota(\gamma)$ and $[\gamma] = \iota_{*,e}([\gamma])$.

But $\phi_{*,\mathbb{1}}(a) = a^T + a$ as can be calculated by taking the standard coordinate chart. So $\mathfrak{o}(n)$ is the space of anti-symmetric $n \times n$ matrices.