# Perfectoid Spaces II (Mark Feldmann)

$K$ perfectoid field, $\varpi \in K$ such that $|p| \leq |\varpi| < 1$

Results on the almost category:

[GabRam] Remark 2.2.5.: $K^{oa}$ - Mod is a Grothendieck category satisfying (AB4*) and (AB6).

[Popescu] Corollary 8.13: (AB6) $\Rightarrow$ (AB5) $\Leftrightarrow$ $\varinjlim$ exact Functor.

[GabRam] $K^{oa}$ - Mod = localization of $K^{o}$ - Mod w.r.t. almost isomorphisms

[Schubert] $\overset{19.6.5.}{\Longrightarrow}$ $K^{oa}$ is a Grothendieck category with generator $K^{oa}$

[Roos] $\Rightarrow$ $\varprojlim$ exact on Mittag-Leffler sequences (i.e. transition maps are epimorphisms.) in $K^{oa}$ - Mod

$\overset{(\cdot)^a \text{ right-exact}}{\Longrightarrow}$ $\varpi$ - adic completion is exact in $K^{oa}$ - Mod

Let $(R, R^+)$ be a perfectoid affinoid $K$-algebra, $X := \mathrm{Spa}(R, R^+)$

Aims: • Finish proof of Theorem 6.3:

(iii) $\mathcal{O}_X, \mathcal{O}_{X^\flat}$ are sheaves

(iv) $H^i(X, \mathcal{O}_X^+)$ is $m$-torsion $\forall i > 0$.

• Category of perfectoid spaces has fiber products.

**Step 1:** **Assume: $(R, R^+)$ red. aff. $K$-alg. tft:**

**PROP 6.10:** $X := \mathrm{Spa}(S, S^+)$ with $(S, S^+)$ —"— .

(i) $S^+ := S^o \subseteq S$ open and bounded

(ii) $U \subseteq X$ rational $\Rightarrow$ $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ —"— .

(iii) $X = \bigcup_{i=1}^m U_i$ cov. by rat. subsets, obtain Čech-complex

$$0 \to \mathcal{O}_X(X)^o \to \prod_i \mathcal{O}_X(U_i)^o \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^o \to \dots$$

$$\Rightarrow \forall i \geq 1 \ \exists N \in \mathbb{N}: \ \varpi^N H^i(X, \mathcal{O}_X^+) = 0$$

Proof: Trick: Use results of [BGR] about category of aff. var. / $K$.

$A/K$ tft $\Rightarrow$ $K\langle T_1, \dots, T_n\rangle \twoheadrightarrow A$ cont. epim. $(n \in \mathbb{N})$.

$i: \mathrm{Max}(A) \to \mathrm{Spa}(A, A^o)$, $m \mapsto v_m$ $\left( v_m(a) = \begin{cases} 0, & a \in m \\ 1, & a \notin m \end{cases} \right)$

$\mathfrak{T} :=$ rat. subsets of $\mathrm{Spa}(A, A^o)$, $\mathfrak{T}_m :=$ rat. subsets of $\mathrm{Max}(A)$

Cov := finite coverings $\rightsquigarrow$ $i^{-1}: (\mathfrak{T}, \mathrm{Cov}) \to (\mathfrak{T}_m, \mathrm{Cov})$ equiv. of Grothendieck topologies $\rightsquigarrow$ $r$: (aff. var. / $K$) $\rightsquigarrow$ (aff. ad. sp. tft / $K$)

$(\mathrm{Max}(A) = X, \mathcal{O}_X) \mapsto (\mathrm{Spa}(A, A^o) = X, \mathcal{O}_{r(X)})$, $\mathcal{O}_{r(X)}(U) := \mathcal{O}_X(i^{-1}(U))$

for all $U \subseteq r(X)$ rational.

[BGR], 6.2.4. Theorem 1: $A$ $\boxed{\text{reduced}}$ $K$-affinoid algebra (tft) $\Rightarrow$ $\exists !$ complete norm on $A$ up to equivalence $\Rightarrow$ $S^\circ = \{s \in S \mid |s| \leq 1\} \subseteq S$ open and bounded. $\Rightarrow$ (i)

[BGR]: 7.3.2. Corollary 10: $\operatorname{Sp} B \cong V \subseteq T = \operatorname{Sp} A$ affinoid subdomain.

$A$ reduced $\Rightarrow$ $B$ reduced

"Translation" functor $r$ $\overset{V := i^{-1}(U)}{\rightsquigarrow}$ $(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \cong (B, B^\circ)$

$\Rightarrow$ $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ reduced & tft over $K$. $\Rightarrow$ (ii).

[BGR]: Tate's acyclicity theorem:

$X$ affinoid variety, $\mathcal{U} = \{U_i\}_{i=1,\dots,n}$ covering of $X$ by affinoid subdomains $U_i \subseteq X$ $\Rightarrow$ $\mathcal{U}$ is $\mathcal{O}_X$-acyclic, i.e.

$$0 \to \mathcal{O}_X(X) \xrightarrow{d_0} \prod_i \mathcal{O}_X(U_i) \xrightarrow{d_1} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j) \xrightarrow{d_2} \dots =: C^\cdot$$

is exact $(\Leftrightarrow H^q(\mathcal{U}, \mathcal{O}_X) = 0 \quad \forall q \neq 0)$.

$d_{i-1}: C^{i-1} \twoheadrightarrow \ker d_i$ is surjective, thus open by Banach's Open Mapping Theorem

⌜ [BGR], 2.8.1: $V, W$ Banach spaces, $\phi: V \to W$ bounded, surj.,
⌞ $K$-linear $\Rightarrow$ $\phi$ is open and $W$ carries quotient top. w.r.t. $\phi$ ⌟ } use THEO 6.16 in [Wed]. instead.

$(\ker d_i \subseteq C^i$ closed, $C^i$ Banach space $\Rightarrow$ $\ker d_i$ Banach space$)$

$\Rightarrow$ $\operatorname{im} d_{i-1} = \ker d_i \cong C^{i-1}/\ker d_{i-1}$ homeom.

differentials are morphisms $\Rightarrow$ $d_{i-1}((C^{i-1})^\circ) \subseteq (C^i)^\circ$, therefore bounded

$$\rightsquigarrow 0 \to \mathcal{O}_X(X)^\circ \xrightarrow{d_0^\circ} \prod_i \mathcal{O}_X(U_i)^\circ \xrightarrow{d_1^\circ} \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^\circ \xrightarrow{d_2^\circ} \dots =: C^\circ$$

$\Rightarrow$ $\{\varpi^n \cdot \operatorname{im}(d_{i-1}^\circ) \mid n \in \mathbb{Z}\}$ basis of open neighborhoods of $0$ in $\operatorname{im} d_{i-1}$

⌜ $d_{i-1}: C^{i-1} \to C^i$ cont., open, surj. $\rightsquigarrow$ $\operatorname{im} d_{i-1} \cong C^{i-1}/\ker d_{i-1} \overset{C^\cdot \text{ exakt}}{=} C^{i-1}/\operatorname{im} d_i$ ⌝

⌞ $\Rightarrow$ $\pi^{-1}(\varpi^n (C^{i-1})^\circ) = \varpi^n \pi^{-1}((C^{i-1})^\circ) = \varpi^n \operatorname{im}(d_{i-1}^\circ)$, $\hookrightarrow^{i}$ $C^{i-1}$ $\xrightarrow{\pi}$ factorizes. ⌟

$\Rightarrow$ $\{\varpi^n \cdot \ker(d_i^\circ) \mid n \in \mathbb{Z}\}$ basis of open neighborhoods of $0$ in $\ker d_i$

(since $\ker d_i \cap (C^i)^\circ = \ker d_i^\circ$)

$\Rightarrow$ $\exists N \in \mathbb{Z}$: $\varpi^N \ker d_i^\circ \subseteq \operatorname{im} d_{i-1}^\circ$

$\Rightarrow$ $\varpi^N H^i(\underbrace{\mathcal{O}_X(\cdot)^\circ}_{\overset{(i)}{=} \mathcal{O}_X^+}, \mathcal{U}) = \varpi^N \ker d_i^\circ / \operatorname{im} d_{i-1}^\circ = 0$ $\Rightarrow$ (iii). $\square$

char(K) = p.

**Step 2:** Proof under assumption: $(R,R^+)$ is compl. perfection of red. tft

DEF: Assume char$(K) = p$. A perfectoid affinoid K-algebra $(R,R^+)$ is called p-finite $:\Leftrightarrow$ $\exists$ reduced affinoid K-algebra $(S,S^+)$ tft such that $R^+ = \varpi$-adic completion of $\underbrace{\varinjlim_{\Phi} S^+}_{\text{call this perfection of } S^+}$

$R = R^+[\varpi^{-1}]$.

PROP 6.11: Assume char$(K) = p$, $(R,R^+)$ p-finite, given as the completed perfection of $(S,S^+)$.

(i) $X := \mathrm{Spa}(R,R^+) \overset{\psi}{\cong} \mathrm{Spa}(S,S^+) =: Y$ is a homeom. identifying rational subsets

(ii) $U \subseteq X$ rational, $V := \psi(U) \subseteq Y$

$\Rightarrow$ $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ = completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$

(iii) $X = \bigcup_{i \in I} U_i$ covering by rational subsets

$\Rightarrow$ $0 \to \mathcal{O}_X(X)^{\circ a} \to \prod_i \mathcal{O}_X(U_i)^{\circ a} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{\circ a} \to \dots$

is exact. In particular $H^i(X, \mathcal{O}_X^{\circ a}) = 0$ $\forall i \geq 1$.

REM: (iii) $\Leftrightarrow$ $\mathfrak{m} H^i(X, \mathcal{O}_X^+) = 0$ $\forall i > 0$.

Proof: (of Proposition)

$$\mathrm{Spa}(R,R^+) = \mathrm{Spa}\left(\left(\widehat{\varinjlim_{\Phi} S^+}\right)[\varpi^{-1}], \left(\widehat{\varinjlim_{\Phi} S^+}\right)\right) \underset{\substack{\uparrow \\ \text{completion doesn't change adic space}}}{\overset{\text{general fact}}{=}} \varprojlim \mathrm{Spa}(S,S^+)$$

[ general fact: $(A_i, A_i^+)_{i \in I}$ directed system of K-affinoid algebras $\Rightarrow$ $\mathrm{Spa}(\varinjlim_i (A_i, A_i^+)) \cong \varprojlim_i \mathrm{Spa}(A_i, A_i^+)$ as sets ] comp. Appendix.

But $\mathrm{Spa}(\Phi): \mathrm{Spa}(S,S^+) \to \mathrm{Spa}(S,S^+)$ is equal to the identity as a map of top. spaces since $\mathrm{Spa}(\Phi)([v]) = [v^p] = [v]$.

$\Rightarrow$ all transition maps are homeom. $\Rightarrow$ $\varprojlim_{\mathrm{Spa}(\Phi)} \mathrm{Spa}(S,S^+) = \mathrm{Spa}(S,S^+)$.

The completed perfection of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$ is perfectoid aff. $K$-algebra

⌜ $\mathrm{char}(K) = p \rightsquigarrow R$ perfectoid $\Leftrightarrow R$ perfect complete (uniform) Tate ring ⌝

⌞ ⌟

⌜ Universal property of $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$: [PS, 2.14] $(X = \mathrm{Spa}(R,R^+))$ ⌝

$\forall$ morphisms $f: (R,R^+) \to (B,B^+)$

$\exists!$ morphisms $g: (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to (B,B^+)$ s.t.

$$\begin{tikzcd}
(R,R^+) \cong (\mathcal{O}_X(X), \mathcal{O}_X^+(X)) \arrow[r, "\mathrm{res}_{X,U} =: h"] \arrow[dr, "f"'] & (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \arrow[d, dashed] \\
& (B,B^+)
\end{tikzcd}$$

commutes. ⌞ ⌟

Assume w.l.o.g. $(S,S^+)$ complete, let $f: (R,R^+) \to (B,B^+)$ be morphism.

$$\begin{tikzcd}
(\mathcal{O}_Y(Y), \mathcal{O}_Y^+(Y)) \cong (S,S^+) \arrow[r, "\mathrm{res}_{Y,V}"] \arrow[d] \arrow[dr, phantom, "1"] & (\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \arrow[d, "j"] \arrow[ddl, dashed, bend left, "\exists!\, g'"] \\
(R,R^+) \arrow[r, "h"] \arrow[d, "f"] & (C,C^+) := \text{compl. perf. of } -''- \arrow[dl, dashed, "2"] \\
(B,B^+) &
\end{tikzcd}$$

w.l.o.g. we assume that the test ring is perfectoid $\rightarrow$ $(B,B^+)$

$\exists!\, g'$ by universal property of $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V))$

1 by functoriality of completed perfection

2 $\exists!\, g$ by universal property of completed perfection

abstract nonsense $\Rightarrow (C,C^+) \cong (\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \Rightarrow$ (ii).

Let $Y = \bigcup_i V_i$ denote cov. corr. to $X = \bigcup_i U_i$.

$$\rightsquigarrow \begin{cases}
0 \to \mathcal{O}_Y(Y)^\circ \xrightarrow{d_0^\circ} \prod_i \mathcal{O}_Y(V_i)^\circ \xrightarrow{d_1^\circ} \prod_{i,j} \mathcal{O}_Y(V_i \cap V_j)^\circ \to \dots \\
\quad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
0 \to (\mathcal{O}_Y(Y)^\circ)^{\mathrm{perf}} \xrightarrow[d_0^{\circ,\mathrm{perf}}]{} \prod_i (\mathcal{O}_Y(V_i)^\circ)^{\mathrm{perf}} \xrightarrow[d_1^{\circ,\mathrm{perf}}]{} \prod_{i,j} (\mathcal{O}_Y(V_i \cap V_j)^\circ)^{\mathrm{perf}} \to \dots
\end{cases}$$

Let $z \in \ker d_i^{\circ,\mathrm{perf}} = (\ker d_i^\circ)^{\mathrm{perf}} \Rightarrow \exists\, n \in \mathbb{N}, x \in \ker d_i^\circ : z = [x^{p^m}] \;\forall m \geq n$

6.10 $\Rightarrow \exists N \in \mathbb{N}: \varpi^N x^{p^m} \in \mathrm{im}\, d_{i-1}^\circ \quad \forall m \geq n \Rightarrow [\varpi^{N/p^m} x] \in (\mathrm{im}\, d_{i-1}^\circ)^{\mathrm{perf}} \;\forall m \geq n$

$\Rightarrow \varepsilon [x] \in (\mathrm{im}\, d_{i-1}^\circ)^{\mathrm{perf}} \quad \forall \varepsilon \in \mathfrak{m}$ $\qquad \overset{\|}{\varpi^{N/p^m} \cdot [x]}$

$\Rightarrow \ker d_i^{\circ,\mathrm{perf}} / \mathrm{im}\, d_{i-1}^{\circ,\mathrm{perf}}$ is $\mathfrak{m}$-torsion

analog argument for completion $\Rightarrow H^i(X, \mathcal{O}_X^\circ)$ is $\mathfrak{m}$-torsion $\forall i \geq 1$

$\Leftrightarrow H^i(X, \mathcal{O}_X^{\circ a}) = 0 \quad \forall i \geq 1. \Rightarrow$ (iii).

<u>Step 3</u>: $\mathrm{char}(K) = p$. $(R,R^+)$ perfectoid aff. $K$-alg. $\rightsquigarrow (R,R^+) =$ completion of filtered direct limit of $p$-finite perfectoid $K$-algebras.

**LEM 6.13:** Assume $\mathrm{char}(K) = p$.

(i) $(R,R^+)$ perfectoid affinoid $K$-algebra, $R^+$ $K^\circ$-algebra

$\Rightarrow$ $(R,R^+)$ = completion of filtered direct limit of $p$-finite perfectoid $K$-algebras $(R_i,R_i^+)$.

(ii) This induces homeom. $\mathrm{Spa}(R,R^+) \cong \varprojlim \mathrm{Spa}(R_i,R_i^+)$ and

$\bigcup_i \{U_i \subseteq X_i := \mathrm{Spa}(R_i,R_i^+) \text{ rat.}\} \twoheadrightarrow \{U \subseteq \mathrm{Spa}(R,R) \text{ rat.}\}$

$\pi_i^{-1}(U_i)$ $\quad U_i \longmapsto \pi_i^{-1}(U_i)$ is surjective.

(iii) Then $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ = completion of the filtered direct limit of the $(\mathcal{O}_{X_j}(U_j), \mathcal{O}^+_{X_j}(U_j))$ where $U_j = f_{ji}^{-1}(U_i)$ $\forall i \leq j$.

($f_{ji}: X_j \to X_i$ transition morphism.)

(iv) $U_i \subseteq X_i$ quasicomp. open subset, $\pi_i(X) \subseteq U_i$

$\Rightarrow \exists j: f_{ji}(X_j) \subseteq U_i$.

**Proof:** $I \subseteq R^+$ finite subset $\rightsquigarrow S_I := \mathrm{im}(K\langle T_i \mid i\in I\rangle \to R)$, $T_i \mapsto i$

$\Rightarrow (S_I \subseteq R)$ $S_I$ reduced ($R$ reduced $\overset{5.8}{\Leftarrow}$ $R$ perfectoid $K$-algebra)

Endow $S_I$ with quotient topology of $K\langle T_i \mid i\in I\rangle/\ker$

[BGR, Theo 5.2] $\quad S_I^+ := S_I^\circ \overset{\text{Tate's}}{\underset{\text{Theorem}}{\Rightarrow}} S_I^+ = \{f\in S_I \mid f \text{ integral over the image of } K^\circ\langle T_i \mid i\in I\rangle\}$

[Tate's Theo: $f \in A \in \mathcal{A}$: $f$ int. over a bounded $K^\circ$-subalgebra $A_0 \subseteq A$

$\Leftrightarrow$ $f$ is power bounded.]

$\Rightarrow S_I^+ \subseteq R^+$ (since image of $K^\circ\langle T_i \mid i\in I\rangle \subseteq R^+$ & $R^+$ int. closed)

$(R_I, R_I^+) :=$ completed perfection of $(S_I, S_I^+)$

Universal property:

$$\begin{tikzcd} (S_I, S_I^+) \arrow[r, "j"] \arrow[d, hook, "\text{incl.}"'] & (R_I, R_I^+) \arrow[dl, dashed, "\exists!\, g_I"] \\ (R,R^+) & \end{tikzcd}$$

Claim: $R^+/\varpi^n \cong \varinjlim_{I\in\mathcal{I}} R_I^+/\varpi^n$ $\quad (\mathcal{I} := \{I \subseteq R^+ \mid I \text{ finite}\})$.

$$\begin{tikzcd} R_I^+/\varpi^n \arrow[dr, "j_I"] \arrow[d, "\bar{f}_{I,J}"'] \arrow[rr, bend left, "\bar{g}_I"] & & \\ R_J^+/\varpi^n \arrow[r, "j_J"'] \arrow[rr, bend right, "\bar{g}_J"'] & \varinjlim_I R_I^+/\varpi^n \arrow[r, dashed, "\exists!\, \bar{j}"] & R^+/\varpi^n \end{tikzcd}$$

$I \subseteq J$:

Show: $\bar{j}$ isomorphism.

$\bar g$ surjective: $\bar x \in R^+/\varpi^n$, set $J := \{x\} \Rightarrow x \in S_J^+ \Rightarrow j(x) \in R_J^+$

$\Rightarrow \bar g(j_J(\overline{j(x)})) = \bar g_J(\overline{j(x)}) = \overline{(g_J \circ j)(x)} = \bar x$.

$\bar g$ injective: Take $[\bar x], [\bar y] \in \varinjlim_I R_I^+/\varpi^n$ and assume $\underbrace{\bar g([\bar x]) = \bar g([\bar y])}$

$\Rightarrow \exists I \in \mathcal{I}$, $\bar x, \bar y \in R_I^+/\varpi^n$ $\quad \Leftrightarrow \underbrace{g_I(x-y)}_{=: d \in R^+} = \varpi^n z$, $z \in R^+$

Set $J := I \cup \{z, d\} \Rightarrow d = g_J(d) \in \varpi^n R_J^+$

$\Rightarrow \bar f_{I,J}(\overline{x-y}) = \bar d = \bar 0 \Rightarrow [\bar x] - [\bar y] = [\overline{x-y}] = [\bar 0]$

$\Rightarrow \bar g$ injective. $\Rightarrow$ Claim.

$$\Rightarrow \varprojlim_n \left( \left(\varinjlim_I R_I^+\right) \Big/ \varpi^n \right) = \varprojlim_n \left( \varinjlim_I R_I^+ / \varpi^n \right) \overset{\text{Claim}}{=} \varprojlim_n \left( R^+ / \varpi^n \right) \overset{R^+\ \varpi\text{-adically compl.}}{=} R^+ \Rightarrow \text{(i)}.$$

Analog: $\varinjlim_I R_I \cong R$ (easier).

general fact $\Rightarrow \mathrm{Spa}(R,R^+) \overset{\alpha}{\cong} \varprojlim_I \mathrm{Spa}(R_I, R_I^+)$

Let $U \subseteq \mathrm{Spa}(R,R^+)$ be rational, for example $U = U\left(\frac{f_1,\dots,f_r}{s}\right)$

Recall:
$$\begin{tikzcd}
(S_I, S_I^+) \arrow[r, "j_I"] \arrow[d, phantom, "\cap"] & (R_I, R_I^+) \arrow[dl, dashed, "\exists!\, h_I"] \\
(R, R^+) &
\end{tikzcd}$$

$I := \{f_1, \dots, f_r, s\}$ (if $f_i \notin R^+$, replace by $\varpi^N f_i$, $\varpi^N s$, $N \gg 0$)

$\Rightarrow U_I := U\left(\frac{j_I(f_1), \dots, j_I(f_r)}{j_I(s)}\right) \subseteq \mathrm{Spa}(R_I, R_I^+)$

$\Rightarrow \alpha^{-1}(U_I) = U\left(\frac{f_1,\dots,f_r}{s}\right)$ since $h(j_I(f_i)) = f_i$ etc. $\Rightarrow$ (ii).

diagram chasing: Let $f: (R,R^+) \to (B,B^+)$ be a morphism.

$$\begin{tikzcd}
(R,R^+) \arrow[d, "\mathrm{res}"'] \arrow[ddddr, bend right=60, "f"'] & & (R_J,R_J^+) \arrow[ll, "h_J"'] \arrow[dd, "\mathrm{res}"] \arrow[ddddl, bend left=60, "f \circ h_J"] \\
(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \arrow[d, equal, "\text{for suitable } I \ \text{(ii)}"'] & & \\
(\mathcal{O}_{X_I}(U_I), \mathcal{O}_{X_I}^+(U_I)) \arrow[rr, "f_{IJ}"] \arrow[dr] \arrow[ddr, dashed, "(1)\ \exists!"'] & & (\mathcal{O}_{X_J}(U_J), \mathcal{O}_{X_J}^+(U_J)) \arrow[dl] \arrow[ddl, dashed, "(2)\ \exists!"] \\
& \varinjlim_{I \subseteq J} (\mathcal{O}_{X_J}(U_J), \mathcal{O}_{X_J}^+(U_J)) \arrow[d, dashed, "\exists!\ (3)"] & \\
& (B,B^+) &
\end{tikzcd}$$

abstract $\Rightarrow$ nonsense $\quad \varinjlim_{I \subseteq J} (\mathcal{O}_{X_J}(U_J), \mathcal{O}_{X_J}^+(U_J)) \cong (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. $\Rightarrow$ (iii).

(iv): Situation: Show.

$$\begin{tikzcd} U_i \arrow[r, phantom, "\overset{\text{qc.\&open}}{\subseteq}"] & X_i = \mathrm{Spa}(R_i,R_i^+) \arrow[rr, "\pi_i"] & & X = \mathrm{Spa}(R,R^+) = \varprojlim_i X_i \\ & X_j \arrow[u, "f_{ji}"] \arrow[ul, dotted] \arrow[urr, "\pi_j"] & & \end{tikzcd}$$

i.e. $f_{ji}(X_j) \subseteq U_i$

$A_i := X_i \setminus U_i \subseteq X_i$ closed and constructible

$\Rightarrow A_j := f_{ji}^{-1}(A_i) \subseteq X_j$ constructible $(\underbrace{\text{since } f_{ji} \text{ is spectral}}_{\text{(since it is induced by a ring homom., [H., Prop. 4.7])}})$

$\Rightarrow f_{jj'}|_{A_j}$ spectral $\forall j,j'$

$\overset{\text{[Wed., 3.27]}}{\Leftrightarrow}$ $f_{jj'}|_{A_j} : \underbrace{(A_j)_{cons}} \longrightarrow \underbrace{(A_{j'})_{cons}}$ continuous $\forall j,j'$

Hausdorff since $A_j$'s are quasi-separated [Wed., 3.13 (2)]

Claim: $\varprojlim_j A_j = \emptyset$

Assume: $\ni x = (x_j)_j$ such that $f_{jj'}(x_j) = x_{j'}$ $\forall j \leq j'$

but $f_{ji}(x_j) = x_i \in A_i = X \setminus U_i$ $\lightning$

$\pi_i(x) \in \pi_i(X) = U_i$.

[RibZal], Prop. 1.1.4

$\Rightarrow \exists j: A_j = \emptyset$

$f_{ji}^{-1}(A_i) = f_{ji}^{-1}(X_i \setminus U_i) \Rightarrow f_{ji}(X_j) \subseteq U_i$. $\square$

<u>Step 4</u>: Conclude Theorem in arbitrary Characteristic via Tilting.

PROP 6.14: $(R,R^+)$ perfectoid affinoid $K$-algebra

$\bigcup_{i \in I} U_i = X = \mathrm{Spa}(R,R^+)$ covering by rational subsets. Then

$$0 \to \mathcal{O}_X(X)^{oa} \to \prod_i \mathcal{O}_X(U_i)^{oa} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{oa} \to \ldots$$

is exact. In particular: $\mathcal{O}_X$ is a sheaf, $H^i(X,\mathcal{O}_X^{oa}) = 0$ $\forall i > 0$.

Proof: Assume $\mathrm{char}(K) = p$. W.l.o.g. assume $R^+$ is $K^\circ$-algebra.

$X \underset{\text{Lemma 6.11}}{\cong} \varprojlim_I X_I$ and $U \subseteq X$ rational $\underset{\text{Lemma 6.13 (ii)}}{\Rightarrow}$ $U = \pi_I^{-1}(U_I)$, $I \in \tilde{I}$ suitable.

Let $X = \bigcup_{i=1}^r U_i$ be a covering by rational subsets, ~~choose~~ then $U_i = \pi_I^{-1}(U_I)$ for a suitable index $I$ dependent on $i$. Choose the maximal index among the $I$'s and call it $J$. $V_i := f_{IJ}^{-1}(U_I)$

$\Rightarrow \pi_J^{-1}(V_i) = \pi_J^{-1} f_{IJ}^{-1}(U_I) = \pi_I^{-1}(U_I) = U_i$

⇒ Cover of $X$ is the pullback of the cover $X_j = \bigcup_{i=1}^{r} V_i$.

6.11. ⇒ $$0 \to \mathcal{O}_{X_j}(X_j)^{oa} \to \prod_i \mathcal{O}_{X_j}(V_i)^{oa} \to \prod_{i,j} \mathcal{O}_{X_j}(V_i \cap V_j)^{oa} \to \dots$$

is exact.

Lemma 6.13 ⇒ $$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = \left( \varinjlim_{j \in I} (\mathcal{O}_{X_j}(U_j), \mathcal{O}_{X_j}^+(U_j)) \right)^\wedge \quad (U = \pi_j^{-1}(U_j)).$$
and completion

Apply direct limit to sequence above:

⇒ $$0 \to \mathcal{O}_X(X)^{oa} \to \prod_i \mathcal{O}_X(U_i)^{oa} \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)^{oa} \to \dots$$

is exact in the category $K^{oa}$-Perf ⇒ $H^i(X, \mathcal{O}_X^{oa}) = 0 \ \forall i \geq 1$

⇒ $H^i(X, \mathcal{O}_X^\circ)$ is m-torsion $\forall i \geq 1$.

⇒ $H^i(X, \mathcal{O}_X) = 0 \ \forall i \geq 1$.

In particular: $\mathcal{O}_X$ is a sheaf.

General characteristic (char $K = 0$).

char$(K^\flat) = p$

⇒ $$0 \to \mathcal{O}_{X^\flat}(X^\flat)^{oa} \to \prod_i \mathcal{O}_{X^\flat}(U_i^\flat)^{oa} \to \prod_{i,j} \mathcal{O}_{X^\flat}(U_i^\flat \cap U_j^\flat) \to \dots$$

is exact.

⇝
$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \varpi^\flat K^{\flat oa} \otimes \mathcal{O}_{X^\flat}(X^\flat)^{oa} \arrow[r] \arrow[d] & \dots \arrow[r] & \dots \arrow[r] \arrow[d] & \dots \arrow[d] & \\
0 \arrow[r] & K^{\flat oa} \otimes \mathcal{O}_{X^\flat}(X^\flat)^{oa} \arrow[r] \arrow[d] & \dots \arrow[r] & \dots \arrow[r] \arrow[d] & \dots \arrow[d] & \\
0 \arrow[r] & K^{\flat oa}/\varpi^\flat \otimes \mathcal{O}_{X^\flat}(X^\flat)^{oa} \arrow[r] \arrow[d] & \dots \arrow[r] & \dots \arrow[r] \arrow[d] & \dots \arrow[d] \arrow[r] & \dots \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

exact by flatness of $K^{\flat oa}$ and $\varpi^\flat K^{\flat oa}$ (both torsion free)

is exact.

exact by flatness of $\mathcal{O}_{X^\flat}(X^\flat)^{oa}$ etc. (Theorem 6.3 (ii)).

( use $K^{\circ\circ}/\varpi^{n} = K^{\circ\circ}/\varpi$ )

$\rightsquigarrow$

$$\underbrace{\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \varpi K^{\circ\circ}/\varpi^{n} \otimes \mathcal{O}_X(X)^{\circ\circ} \arrow[r] \arrow[d] & \ldots \arrow[r] & \ldots \arrow[r] \arrow[d] & \ldots \arrow[d] & \\
0 \arrow[r] & K^{\circ\circ}/\varpi^{n} \otimes \mathcal{O}_X(X)^{\circ\circ} \arrow[r] \arrow[d] & \ldots \arrow[r] & \ldots \arrow[r] \arrow[d] & \ldots \arrow[d] & \\
0 \arrow[r] & K^{\circ\circ}/\varpi \otimes \mathcal{O}_X(X)^{\circ\circ} \arrow[r] \arrow[d] & \ldots \arrow[r] & \ldots \arrow[r] \arrow[d] & \ldots \arrow[d] & \\
 & 0 & & 0 & 0 &
\end{tikzcd}}_{\text{exact by flatness of } \mathcal{O}_X(U)}$$

$n=1$: all rows exact by first diagramm $\overset{\text{induction}}{\Longrightarrow}$ all rows exact $\forall n \geq 1$

[Roos] $\Rightarrow$ $0 \to \varprojlim_n \mathcal{O}_X(X)^{\circ\circ}/\varpi^n \to \ldots \to \ldots \to \ldots$ is exact

(since transition maps are epimorphisms, i.e. Mittag-Leffler sequences)

But $\mathcal{O}_X(X)^{\circ\circ} = \varprojlim_n \mathcal{O}_X(X)^{\circ\circ}/\varpi^n$ by completeness. $\square$

$\rightsquigarrow$ Theorem 6.3 is proven

**DEF:** Category of perfectoid spaces :=

Objects: Adic spaces, locally isomorphic to affinoid perfectoid spaces

Morphisms: morphisms of adic spaces.

Theorem $\rightsquigarrow$ equivalence: (perf. spaces $/K$) $\simeq$ (perf. spaces $/K^\flat$).

**PROP:** Fiber products exist in (perf. spaces$/K$).

Proof: $X := \mathrm{Spa}(A,A^+)$, $Y = \mathrm{Spa}(B,B^+)$, $Z = \mathrm{Spa}(C,C^+)$, s.t. $X \to Y \leftarrow Z$

$D := A \hat{\otimes}_B C$, $D^+ :=$ completion of int. closure of $\mathrm{im}(A^+ \otimes_{B^+} C^+) \subseteq D$

($I \subseteq B_0 \subseteq B$ pair of definition $\rightsquigarrow$ $I D_0' \subseteq D_0' := A_0 \otimes_{B_0} C_0 \subseteq D' := A \otimes_B C$)

$\rightsquigarrow$ $D :=$ $I$-adic completion of $D'$.

Clear: If $\underbrace{\mathrm{Spa}(D,D^+)}_{=:W}$ exists in category of perfectoid spaces $\Rightarrow W = X \times_Y Z$ (general construction in category of adic spaces)

Claim: $A \hat{\otimes}_B C$ perfectoid $K$-algebra

$$\Updownarrow$$

$$\left.\begin{array}{l}(A\hat{\otimes}_B C)^{oa} \text{ perfectoid } K^{oa}\text{-algebra} \\ \quad \| \\ (A^o \hat{\otimes}_{B^o} C^o)^a = A^{oa} \hat{\otimes}_{B^{oa}} C^{oa}\end{array}\right\} (*)$$

$$\Updownarrow$$

$\left(A^{oa} \hat{\otimes}_{B^{oa}} C^{oa}\right)/\varpi$ perfectoid $K^{oa}/\varpi$-algebra

$$\|$$

$$\underbrace{\left(A^{oa} \otimes_{B^{oa}} C^{oa}\right)/\varpi = A^{oa}/\varpi \otimes_{B^{oa}/\varpi} C^{oa}/\varpi}_{\text{Enough to proof that in } \mathrm{char}(K)=p \text{ via tilting}} \overset{(**)}{\underset{\Phi_A \otimes \Phi_C}{\cong}} A^{oa}/\varpi^{1/p} \otimes_{B^{oa}/\varpi^{1/p}} C^{oa}/\varpi^{1/p} = \left(A^{oa} \hat{\otimes}_{B^{oa}} C^{oa}\right)/\varpi^{1/p}$$

$\rightsquigarrow$ Proof $(*)$ in $\mathrm{char}(K)=p$ - case: $(**) \Rightarrow$ it suffices to check flatness.

$A^o, B^o, C^o$ perfect $\Rightarrow A^o \otimes_{B^o} C^o$ perfect.

$x \in A^o \otimes_{B^o} C^o$ such that $\varpi x = 0 \Rightarrow \varpi^{\frac{1}{p}} x^{\frac{1}{p}} = 0 = \varpi^{\frac{1}{p}} x$

$\Rightarrow \varpi^{\frac{1}{p^m}} x = 0 \ \forall m \geq 0 \Rightarrow \varepsilon x = 0 \ \forall \varepsilon \in K^{oo} = \mathfrak{m}$ ($(|\varpi^{\frac{1}{p^m}}|)_m$ is zero-sequence)

$\Rightarrow$ $\varpi$-torsion of $A^o \otimes_{B^o} C^o$ is almost zero

$\Rightarrow$ —"— $A^{oa} \otimes_{B^{oa}} C^{oa}$ is zero $\overset{(\cdot)_* \text{ left ex.}}{\underset{5.3(i)}{\Rightarrow}}$ $(A^{oa} \otimes_{B^{oa}} C^{oa})_* \xrightarrow{\varpi} (A^{oa} \otimes_{B^{oa}} C^{oa})_*$

is injective $\Leftrightarrow A^{oa} \otimes_{B^{oa}} C^{oa}$ is flat over $K^{oa}$

$\overset{\text{Roos}}{\Rightarrow} A^{oa} \hat{\otimes}_{B^{oa}} C^{oa}$ —"— $\Rightarrow$ Claim. $\square$

Appendix: General fact: $\varprojlim \mathrm{Spa}(\cdot) = \mathrm{Spa}(\varinjlim \cdot)$

Let $A$ be a ring and $v$ be a valuation on $A$ (multiplicative).

We define a relation: $a | b :\Leftrightarrow v(b) \leq v(a) \quad \forall a,b \in A$.

This relation has the following properties:

(1) $\forall a,b \in A: \ a|b \vee b|a$

(2) $\forall a,b,c \in A: \ a|b \wedge b|c \Rightarrow a|c$

(3) $\forall a,b,c \in A: \ a|b \wedge a|c \Rightarrow a|b+c$ (since $v(a+b) \leq \max(v(a), v(b))$)

(4) $\forall a,b,c \in A: \ a|b \Rightarrow ac|bc$ (since value group $\Gamma_v$ is a totally ordered group in sense of [Wed].)

(5) $\forall a,b,c \in A: \ ac|bc \wedge 0 \nmid c \Rightarrow a|b$

(6) $0 \nmid 1$.

We denote the relation induced by a valuation $v$ by $|_v$.

Then we have: $v \sim \mu \Leftrightarrow |_v = |_\mu$ for valuation $v, \mu$ on $A$.

Obtain injective map: $\mathrm{Spv}\, A \hookrightarrow \mathcal{P}(A \times A)$

$$v \longmapsto \{(a,b) \in A \times A \mid a |_v b\}$$

Endow $\{0,1\}$ with the discrete topology and endow $\mathcal{P}(A\times A) = \{0,1\}^{A \times A}$ with the product topology

$\Rightarrow \mathcal{P}(A \times A)$ is a compact Hausdorff space (Tychonoff)

$\Rightarrow \mathrm{Spv}\, A \subseteq \mathcal{P}(A \times A)$ closed and the subspace topology on $\mathrm{Spv}(A)$

[Huber] is the constructible topology $\mathcal{T}_{cons}$.

Take $V \in \mathrm{Spv}\, A \subseteq \mathcal{P}(A \times A)$. Then: (Assume $A$ top. Ring)

(C) $\big((0,c) \notin V \Rightarrow \{a \in A \mid (a,c) \notin V\} \subseteq A$ offen $\big)$

$\Leftrightarrow V \in \mathrm{Cont}\, A$ = set of continuous valuations on $A$.

(Assume $A$ is a $K$-algebra). Then:

(K) $\big(\forall x,y \in K: \ (x,y) \in V_K \Leftrightarrow (x,y) \in V\big)$ $\quad V_K \subseteq \mathcal{P}(K \times K)$

$\Leftrightarrow$ valuation $v \in \mathrm{Spv}\, A$ is compatible with $v|_K \in \mathrm{Spv}\, K$

(Assume $(A, A^+)$ is affinoid). Then:

(+) $\big(a \in A^+ \Rightarrow (1,a) \in V\big)$

$\Rightarrow$ {Elements of $Spa(A,A^+)$} $\overset{1:1}{\longleftrightarrow}$ $\{V \in \mathcal{P}(A\times A)$ satisfying (1)-(6), (k), (c), (+)$\}$

$$v \longmapsto V := \{ (a,b) \in A\times A \mid v(b) \leq v(a) \}.$$

$$\overset{1:1}{\longleftrightarrow} \{ m : A\times A \to \{0,1\} \text{ \& analog axioms} \}$$

$$V \longmapsto m_V : A\times A \to \{0,1\}$$

$$(a,b) \longmapsto \begin{cases} 0, & (a,b)\notin V \\ 1, & (a,b) \in V \end{cases}$$

Assume $(A,A^+)$ is filtered direct limit of $\underbrace{(A_i,A_i^+)}_{(K,K^0)\text{-algebra with } \pi\text{-adic topology on } K^0}$,

i.e. $A^+ = \underbrace{\varinjlim_i A_i^+}_{\text{inductive limit in the category of } K^0\text{-algebras}}$ and $A := A^+[\pi^{-1}]$, w.l.o.g. assume $A^+ = \bigcup_i A_i^+$

$$\Rightarrow Map(A^+\times A^+, \{0,1\}) = Map\left(\varinjlim_i (A_i^+\times A_i^+), \{0,1\}\right) = \varprojlim_i Map(A_i^+\times A_i^+, \{0,1\})$$

in category of sets

$$m \longmapsto (m|_{A_i^+\times A_i^+})_i$$

$m$ given by $m(a,b) := m_i(a,b)$ for $(a,b)\in A_i^+\times A_i^+$ $\longleftarrow (m_i)_i$

and this bijection is compatible with axioms $\overbrace{(1)-(6), (k), (c), (+)}^{\text{all clear besides (c).}}$

$\Rightarrow$ bijection restricts to $Spa(A,A^+) = \varprojlim_i Spa(A_i,A_i^+)$.

Now assume that $(A,A^+)$ and $\{(A_i,A_i^+) \mid i\}$ perfectoid.

Check (c): Let $m$ have (c). Let $c\in A_i^+$ and $m|_{A_i^+\times A_i^+}(0,c)=0$

$\Rightarrow \exists n\in\mathbb{N}: \pi^n A^+ \subseteq \{a\in A \mid m(a,c)=0\}$

$\Rightarrow \pi^n A_i^+ \subseteq \pi^n A^+ \cap A_i^+ \subseteq \{a\in A_i^+ \mid m|_{A_i^+\times A_i^+}(a,c)=0\} \overset{\text{open}}{\subseteq} A_i^+$.

Converse: Let $c\in A^+ \Rightarrow \exists i: c\in A_i^+ \Rightarrow R_i := \{a\in A_i^+ \mid m_{|A_i^+\times A_i^+}(a,c)=0\} \overset{\text{open}}{\subseteq} A_i^+$

$\Rightarrow \exists n\in\mathbb{N}: \pi^n A_i^+ \subseteq R_i \Leftrightarrow m_i(\pi^n,c)=0 \Rightarrow m_j(\pi^n,c)=0 \;\forall j\geq i$

$$\Leftrightarrow \pi^n A_j \subseteq R_j$$

$$\Rightarrow \underbrace{\bigcup_j \pi^n A_j^+}_{= \pi^n A^+} \subseteq \bigcup_j R_j = \{a\in A \mid m(a,c)=0\} \subseteq A^+$$

$\Rightarrow \{a\in A \mid m(a,c)=0\} \overset{\text{open}}{\subseteq} A^+ \Rightarrow$ (c) is compatible.

Literature:

[BGR], Bosch, Güntzer, Remmert: Non-Archimedian Analysis.

[Wed]: Torsten Wedhorn: Online Script: Adic Spaces

[Roos]: Jan Erik Roos, 'Sur les functors dérivés de $\varprojlim$. Applications' & Journal of the London Math. Soc. 2006, 73, 01.

[GabRam] Gabber, Ramero: Almost Ring Theory

[Popescu] Nicolae Popescu, Abelian Categories with Applications to Rings & Modules.

[Schubert] Horst Schubert, Categories.