Tobias Schmidt

Perfectoid algebras II

K perfectoid field, $K-\mathrm{Perf}$ category of perfectoid K-algebras

This talk has two parts:

(1) Deformation theory: Provide (C), (D) in the tilting equivalence

$$\begin{array}{ccccc} K-\mathrm{Perf} & \underset{(A)}{\cong} & K^{oa}-\mathrm{Perf} & \underset{(C)}{\cong} & K^{oa}/\varpi-\mathrm{Perf} \\ & & & & \parallel \\ K^{b}-\mathrm{Perf} & \underset{(B)}{\cong} & K^{boa}-\mathrm{Perf} & \underset{(D)}{\cong} & K^{boa}/\varpi^{b}-\mathrm{Perf} \end{array}$$

(A), (B) were explained in talk I (Elmar)

(2) Almost purity for finite étale covers (in char p):
given $R \in K^{b}-\mathrm{Perf}$, then (B) induces $R_{f\acute{e}t} \cong R^{oa}_{f\acute{e}t}$.

(1) Deformation theory

$R \to S$ morphism of K^{o}-algebras $\rightsquigarrow$ $\mathbb{L}_{S/R} \in D^{\leq 0}(S)$ cotangent complex

$\mathbb{L}^{a}_{S^{a}/R^{a}} :=$ image of $\mathbb{L}_{S/R}$ in $D^{\leq 0}(S^{a})$, well-defined, controls the def. th. of $R^{a} \to S^{a}$

<u>Lemma 1</u>: $\bar{A} \in K^{oa}/\varpi-\mathrm{Perf}$. Then $\mathbb{L}^{a}_{\bar{A}/(K^{oa}/\varpi)} \cong 0$.

Proof: Choose flat K^{o}/ϖ-algebra B with $B^{a} = \bar{A}$ & $B/\varpi^{1/p} \underset{\Phi}{\cong} B$
(e.g. $B = (\bar{A} \times K^{oa}/\varpi)_{!!}$)

$B^{k} := B[\{X_b\}_{b \in B^{k-1}}]$, $k \geq 0$ simplicial resolution of B
$(B^{0} \to B^{-1} := B, X_b \mapsto b)$

Extend Φ via $X_b \longmapsto X_b^p$ $\Longrightarrow$

$$K^\circ/\varpi \otimes_{K^\circ/\varpi^{1/p}} B^\bullet/\varpi^{1/p} \underset{\Phi}{\cong} B^\bullet \quad \text{augmented simplicial } K^\circ/\varpi\text{-algebras}$$

Pass to $\mathbb{L}$ (i.e. apply $\Omega^1_{(\cdot)^\circ/(K^\circ/\varpi)} \otimes_{(\cdot)^\bullet} (\cdot)^{-1}$) $\Longrightarrow$

$$K^\circ/\varpi \otimes \mathbb{L}_{(B/\varpi^{1/p})/(K^\circ/\varpi^{1/p})} \underset{d\Phi}{\cong} \mathbb{L}_{B/(K^\circ/\varpi)}$$

But this is the zero map: $d\Phi(X_b) = dX_b^p = pX_b^{p-1}dX_b \underset{pB=0}{=} 0$! $\square$

<u>Theorem I</u>: $A \to A/\varpi$ is an equivalence

$$K^{oa}\text{-Perf} \cong K^{oa}/\varpi\text{-Perf}.$$

Proof: Essentially surjective: let $\bar{A}_1 \in K^{oa}/\varpi\text{-Perf} \overset{\text{lemma}}{\Longrightarrow}$

$\exists!$ flat K^{oa}/ϖ^2-algebra $\bar{A}_2$ with $\bar{A}_2/\varpi \cong \bar{A}_1$.

Let $n \geq 2$ & suppose ex. unique flat K^{oa}/ϖ^k-algebras $\bar{A}_k$ with: $\bar{A}_k/\varpi^{k-1} \cong \bar{A}_{k-1}$ $\forall k = 1, \dots, n$,

$$\mathbb{L}^a_{\bar{A}_k/(K^{oa}/\varpi^k)} \cong 0 \quad \forall k = 1, \dots, n-1.$$

Have exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\varpi^{n-1}} \bar{A}_n \longrightarrow \bar{A}_{n-1} \longrightarrow 0$$

$$\bar{A} \cong \bar{A}_n/\varpi$$

$\overset{\text{apply}}{\Longrightarrow}$ get exact triangle in $D^{\leq}(\bar{A}_m)$

$\mathbb{L}^a_{\bar{A}_n/(K^{oa}/\bar{\omega}^n)} \otimes_{\bar{A}_m} (\cdot)$

$$\mathbb{L}^a_{\bar{A}/(K^{oa}/\bar{\omega})} \to \mathbb{L}^a_{\bar{A}_n/(K^{oa}/\bar{\omega}^n)} \to \mathbb{L}^a_{\bar{A}_{n-1}/(K^{oa}/\bar{\omega}^{n-1})} \to$$

Hypothesis $\Rightarrow$ $\mathbb{L}^a_{\bar{A}_m/(K^{oa}/\bar{\omega}^n)} \cong 0$ $\rightsquigarrow$ obtain $\bar{A}_{m+1}$ etc.

Put $A := \varprojlim_m \bar{A}_m \in K^{oa}\text{-Perf}$ with $A/\bar{\omega}^m \cong \bar{A}_m$.

Similarly: unicity of lifts for morphisms $\Rightarrow$ fully faithful. $\square$

Check: The composite equivalence $K\text{-Perf} \cong K^b\text{-Perf}$

maps $R \mapsto R^b = \varprojlim_{x \mapsto x^p} R$, in particular have $R^b \xrightarrow{\#} R$ cts. multipl.

(key point: lifting in char p is explicit: if $R \in K\text{-Perf}$ and $A := R^{oa}$, then $A^b := \varprojlim_{\Phi} A/\bar{\omega}$ lifts $A/\bar{\omega} \in K^{boa}/\bar{\omega}\text{-Perf}$.)
$\in K^{boa}\text{-Perf}$

(2) Almost purity

As before $R \in K\text{-Perf}$, $A := R^{oa}$, $\bar{A} := R^{oa}/\bar{\omega}$.

$(\cdot)_{\text{fét}}$ category of finite étale covers

<u>Lemma 2</u>: $\bar{B}$ finite étale $\bar{A}$-algebra $\Rightarrow$ $\bar{B} \in K^{oa}/\varpi$-Perf

Proof: flat ✓ &

$$\begin{tikzcd} \bar{A}/\varpi^{1/p} \arrow[r] \arrow[d, "\cong"', "\Phi"] & \bar{B}/\varpi^{1/p} \arrow[d, "\Phi"] \\ \bar{A} \arrow[r] & \bar{B} \end{tikzcd}$$

is cocartesian by étaleness. □

Hence we have a commutative diagramm

$$\begin{tikzcd} A_{fét} \arrow[r, "\cong", "\text{mod } \varpi"'] \arrow[d, phantom, "\cap"] & \bar{A}_{fét} \arrow[d, phantom, "\cap \quad \text{Lemma}"] \\ K^{oa}\text{-Perf} \arrow[r, leftrightarrow, "\cong", "(C)"'] & K^{oa}/\varpi\text{-Perf} \end{tikzcd}$$

where the upper horizontal equivalence is 'idempotent lifting' (valid due to ϖ-adic completeness, cf. earlier talk)

<u>Recall</u>: $K\text{-Perf} \xleftarrow[(A)]{\cong} K^{oa}\text{-Perf}$ via $B \longmapsto B_*[\varpi^{-1}]$

In particular, $A_*[\varpi^{-1}] = R$.

<u>Lemma 3</u>: $B \in A_{fét} \Rightarrow B_*[\varpi^{-1}] \in R_{fét}$

Proof: • $\mathrm{Ext}^i_{A_*}(B_*, X) \overset{al.}{\cong} 0 \quad \forall i > 0, X$

• $B_* \otimes_{A_*} B_* \xrightarrow{al. \cong} (B \otimes_A B)_*$ $\ni e$ diagonal idempotent

- $\exists\, \varepsilon \in \mathfrak{m}$ & f.g. A_*-module N_ε w. $N_\varepsilon \xrightarrow{\varphi} B_*$ + has $\varepsilon \ker\varphi = \varepsilon \operatorname{coker}\varphi = 0$.

Now invert ϖ. $\square$.

<u>Theorem II</u>: $R_{\text{fét}} \underset{(A)}{\cong} A_{\text{fét}}$

Proof: Assume char $K = p$ (char $K = 0$ version in later talks)

Lemmas $\Rightarrow$ suffices to show: given $S \in R_{\text{fét}}$, then $S \in K$-Perf and S^{oa} is fét over $R^{oa} = A$.

(i) Topology on S: choose any R°-subalgebra $S_0 \subset S$, finite/R° module w. $S_0 \otimes K = S$ as "unitball" in S, top. is independent of choice of S_0.

(ii) S is perfect, so ~~remains to show~~ $S \in K$-Perf if S° open & bounded.

let $Y := $ int. closure of R° in S $\Rightarrow$
- $S_0 \subseteq Y$ so Y open
- $\operatorname{tr}_{S/R}(Y) \subseteq R^\circ$ $(*)$

<u>Claim</u>: Y bounded,

since: $\operatorname{Tr}: S \otimes_R S \longrightarrow R$, $(x,y) \longmapsto \operatorname{tr}_{S/R}(xy)$ non-degenerate

$\Rightarrow \exists \begin{matrix} x_1,\dots,x_n \\ y_1,\dots,y_n \end{matrix} \in S$ with $s \overset{(**)}{=} \sum_{i=1}^{n} \operatorname{Tr}(y_i, s)\, x_i \quad \forall s \in S$

$S_0^\perp := \{x \in S : \operatorname{Tr}(x, S_0) \subset R^\circ\}$ is bounded: choose N s.t. $\varpi^N y_i \in S_0$. Then $\varpi^N s \in \underset{\sum_i R^\circ x_i}{\cancel{S_0}}$ $\forall s \in S_0^\perp$ by $(**)$.

Claim follows since $Y \subseteq S_0^\perp$ by $(*)$.

Finally, $Y \overset{\text{claim}}{\subseteq} S^\circ \Rightarrow S^\circ$ open &

$s \in S^\circ \Rightarrow (\varpi s)^n \xrightarrow[n]{} 0$, hence $s \in \frac{1}{\varpi} Y \underset{\text{claim}}{\Rightarrow} S^\circ$ bounded

(iii) <u>Claim</u>: $\exists\, m \geq 1$ s.t. $\forall\, \varepsilon \in \mathfrak{m}$ (= maximal ideal $\subset K^\circ$)

have linear maps $\quad \iota_\varepsilon : S^\circ \longrightarrow R^{\circ n}$

$$p_\varepsilon : R^{\circ n} \longrightarrow S^\circ \quad \text{with } p_\varepsilon \circ \iota_\varepsilon = \varepsilon \cdot .$$

Since: $e \in S \otimes_R S$ diagonal idempotent $\Rightarrow N \gg 0$ s.t.

$\bar{\varpi}^N e \in S^\circ \otimes_{R^\circ} S^\circ$, say $= \sum_{i=1}^n a_i \otimes b_i$

$S \otimes_R S$ perfect $\Rightarrow \varpi^{N/p^m} e \in S^\circ \otimes_{R^\circ} S^\circ \quad \forall\, m$

$\mathfrak{m} = \bigcup_m K^\circ \bar{\varpi}^{N/p^m} \Rightarrow \varepsilon e \overset{\heartsuit}{=} \sum_{i=1}^n a_i^\varepsilon \otimes b_i^\varepsilon \in S^\circ \otimes_{R^\circ} S^\circ \quad \forall\, \varepsilon \in \mathfrak{m}$.

Put $\iota_\varepsilon(s) := (\mathrm{Tr}(s, b_1^\varepsilon), \dots, \mathrm{Tr}(s, b_n^\varepsilon))$, $p_\varepsilon(\underline{r}) := \sum_{i=1}^n r_i a_i^\varepsilon$.

(iv) • S° (uniformly) al. f.g. $/R^\circ$: $R^{\circ n} \xrightarrow[p_\varepsilon]{} S^\circ$ with $\varepsilon \operatorname{coker} p_\varepsilon = 0 \quad \forall\, \varepsilon$.

• $S^{\circ a}$ almost proj. $/R^{\circ a}$: $\quad \forall\, i > 0$

$$\begin{tikzcd} \mathrm{al\,Ext}^i_{R^{\circ a}}(S^{\circ a}, X) \arrow[r, "p_\varepsilon^a"'] \arrow[rr, bend right=30, "\varepsilon \cdot"'] & \mathrm{al\,Ext}^i_{R^{\circ a}}(R^{\circ a n}, X) \arrow[r, "\iota_\varepsilon^a"'] \arrow[d, equal] & \mathrm{al\,Ext}^i_{R^{\circ a}}(S^{\circ a}, X) \\ & 0 & \end{tikzcd}$$

• $S^{\circ a}$ unramified $/R^{\circ a}$: $e \in \mathrm{Hom}_{K^\circ}(\mathfrak{m}, S^\circ \otimes_{R^\circ} S^\circ) \overset{\text{def}}{=} (S^{\circ a} \otimes_{R^{\circ a}} S^{\circ a})_*$

(see $\heartsuit$) provides a diagonal idempotent.

Hence $S^{\circ a}/R^{\circ a}$ is finite étale. $\square$