Main Result & Functoriality

8.5.15
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$C/\mathbb{Q}_p$ non-arch cplt + alg closed. $\varpi \in \mathcal{O}_C$ pseudo-uniformizer s.t. $\varpi | p$.

Construction / Notation from Peter Schneider's talk:

$E/\mathbb{Q}_p$ finite, choose $\pi \in \mathcal{O}_E$ unif., $\phi \in \mathcal{O}_E[[t]]$ Frobenius PS.
res field $\mathbb{F}_q$

$\rightsquigarrow$ Lubin-Tate module $\mathrm{Spf}\,\mathcal{O}_E[[t]] =: H_E \supset \mathcal{O}_E^{\times}$

$$\widetilde{H}_E := \varprojlim_{[\pi]} H_E$$

$D_E := H_E^{ad} = \mathrm{Spa}(\mathcal{O}_E[[t]]) \setminus V(\pi)$ "rigid-analytic open unit disk $/E$"

$D_E^* := \mathrm{Spa}(\mathcal{O}_E[[t]]) \setminus V(\pi \cdot t)$ "$= D_E \setminus \{0\}$"

$\widetilde{D}_E := \widetilde{H}_E^{ad} = \varprojlim_{[\pi]} D_E$, $\widetilde{D}_E^{\times} = \varprojlim_{[\pi]} D_E^* = \widetilde{D}_E \setminus V(t)$

Subscript "C" := Base change to C resp. $\mathcal{O}_C$

Defn $Z_E := \widetilde{D}^{\times}_{E,C} / E^{\times}$

Main Result. $\pi_1^{et}(Z_E) \cong \mathrm{Gal}(\overline{E}/E)$

Rmk $H_{E,C}$ is formal π-div $\mathcal{O}_E$-module over $\mathrm{Spf}\,\mathcal{O}_C$ of ht 1 and dim 1. Such a module is unique up to iso, the iso being unique up to $\mathcal{O}_E^\times$.

$\longrightarrow$ Z_E is independent of choices of π, ϕ.

<u>Before Proof of Main Result, Recall some notation from Schneider:</u>

Let $E_n := E(H_E[\pi^n](\mathcal{O}_C))$, $E_\infty = \bigcup E_n$, $\hat{E}_\infty$ cplt

$z = (\ldots, z_3, z_2, z_1)$ comp family of generators of $H[\pi^n](\mathcal{O}_C)$

as $\mathcal{O}_E$ module, i.e. $[\pi](z_{n+1}) = z_n$.

Then $[\pi](z_{n+1}) = \phi(z_{n+1}) \equiv z_{n+1}^q \pmod{\pi}$ in $\mathcal{O}_{E_{n+1}}$,

so $z_{n+1}^q \equiv z_n \bmod \pi$

$\longrightarrow$ $z \in \hat{E}_\infty^\flat$

Define "field of norms" $\mathbb{E}(E) := \mathbb{F}_q((z)) \subset \hat{E}_\infty^\flat$

<u>Proof of Main Result</u>

The results of the previous talk have an obvious extension from $\mathbb{Q}_p$ to E. (Replace: K with $\hat{E}_\infty^\flat$, ω_K with z, $\mathbb{Z}_p^\times$ with $\mathcal{O}_E^\times$.)

There, everything was proved except for the existence of an iso

"$X_{\eta_K} \setminus \{0\}_K \cong (Y_{C,E} \hat{\otimes}_E \hat{E}_\infty)^\flat$, equivariant for

$E^\times \longrightarrow \phi^{\mathbb{Z}} \times \mathrm{Gal}(E/E)$."

Recall the definition: $Y_{C,E} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})) \setminus V(\pi[\varpi^\flat])$

By Prop A from prev talk, $Y_{C,E} \hat{\otimes} \hat{E}_\infty = \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \hat{\otimes}_{W(\mathbb{F}_q)} \mathcal{O}_{\hat{E}_\infty})$
$\setminus V(\pi[\varpi^\flat])$

is perfectoid

Lemma

$$\begin{array}{ccc} (\widehat{\mathbb{D}}^\times_{E,C})^\flat & \cong & (Y_{C,E} \hat{\otimes} \hat{E}_\infty)^\flat \\ \circlearrowleft & & \circlearrowleft \\ \mathcal{O}_E^\times & \xrightarrow[\sim]{LT} & \mathrm{Gal}(E_\infty / E) \\ \pi^{\mathbb{Z}} & = & (\phi^{-1} \circ \mathrm{Frob}_q)^{\mathbb{Z}} \end{array}$$

Proof: Schneider / Prev talk: $(\mathbb{D}^\times_{E,C})^\flat \cong \mathrm{Spa}(\mathcal{O}_{C^\flat} \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{\hat{E}_\infty^\flat}) \setminus V(z \cdot \varpi^\flat)$

equivariant as above. Left to identify RHS with tilt of FF-curve. By Prop A, can compute

$$(Y_{C,E} \hat{\otimes} \hat{E}_\infty)^\flat = \mathrm{Spa}\Big(\varprojlim_{\phi} \big(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \hat{\otimes}_{W(\mathbb{F}_q)} \mathcal{O}_{\hat{E}_\infty}\big)/z_q\Big) \setminus V(\pi^\flat[\varpi^\flat]^\flat)$$

$$= \mathrm{Spa}(\mathcal{O}_{C^\flat} \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_{\hat{E}_\infty^\flat}) \setminus V(z \cdot \varpi^\flat)$$

For equivariance: By construction in prev / Schneider's talk, $E^\times \supset \mathcal{O}_E \setminus \{0\}$ operates on these spaces via the Lubin-Tate law on $z \in \mathcal{O}_{\hat{E}_\infty^\flat}$. $\square$

$\square$ Main result by prev talk

§ Functoriality Now let E'/E be finite of degree d.

Prop There exists a morphism $Z_{E'} \xrightarrow{N} Z_E$, the "norm",

s.th.

$$\begin{array}{ccc} \pi_1(Z_{E'}) & \xrightarrow{N} & \pi_1(Z_E) \\ \wr\| & & \wr\| \\ \mathrm{Gal}(\bar{E}/E') & \hookrightarrow & \mathrm{Gal}(\bar{E}/E) \end{array}$$

commutes.

Construction of N: Choose $\pi', \pi, \phi', \phi \rightsquigarrow H', H$

LT groups for E', E.

As formal sch $/\mathrm{Spf}\, \mathcal{O}_{E'}$, $H_{E'} = \varinjlim_n H'[\pi'^n]$,

i.e. H' is the underlying formal sch of a π-div formal $\mathcal{O}_{E'}$-module of ht 1 and dim 1 over $\mathrm{Spf}\, \mathcal{O}_{E'}$.

⌜ $\Rightarrow$ H' is formal p-div grp $/\mathrm{Spf}\, \mathcal{O}_{E'}$ of dim 1

+ action $i: \mathcal{O}_{E'} \to \mathrm{End}(H')$ s.th. $\mathrm{Lie}(i(\alpha)) = \alpha \ \forall \alpha \in \mathcal{O}_{E'}$

s.th. $\mathrm{rk}_{\mathcal{O}_{E'}} H'[\pi'] = |\mathcal{O}_{E'}/\pi'|$ (i.e. ht $= 1$) ⌟

Restrict i to $\mathcal{O}_E$ $\rightsquigarrow$ H' formal π-div $\mathcal{O}_E$-module of dim 1 ht d.

Thm (9.2.36, Hedayatzadeh) (in our situation)

$\exists$ "determinant" $\bigwedge^d_{\mathcal{O}_E} H' / \mathrm{Spf}\, \mathcal{O}_{E'}$ i.e. a formal π-div $\mathcal{O}_E$-mod $\bigwedge^d_{\mathcal{O}_E} H'$ + $\lambda: (H')^d \to \bigwedge^d_{\mathcal{O}_E} H'$ universal alternating map.
$\mathcal{O}_E$-multilinear

s.th. $\forall\, S/\mathrm{Spf}\,\mathcal{O}_{E'}$, $\lambda(S)$ is the usual alternating map and

$$(\overset{d}{\Lambda} H')(S) \cancel{} = \overset{d}{\Lambda}(H'(S)).$$

Furthermore: $\overset{d}{\Lambda} H'$ has dim 1 and ht 1. $\square$

<u>Fact</u> A formal π-div $\mathcal{O}_E$-module of ht 1 and dim 1 over

$\mathrm{Spf}\,\mathcal{O}_C$ (or $\mathrm{Spf}\,\mathcal{O}_{E',nr}$) is unique up to isomorphism.

$$\Longrightarrow \overset{d}{\Lambda} H'_C \cong H_C.$$

Choose $\alpha_1, \dots, \alpha_d$ $\mathcal{O}_E$-basis of $\mathcal{O}_{E'}$ and define

$$\begin{aligned} N : H'_C &\longrightarrow \overset{d}{\Lambda} H'_C \xrightarrow{\sim} H_C \\ x &\longmapsto \lambda(\alpha_1 x, \dots, \alpha_d x) \end{aligned}$$

We check pointwise that $N(\alpha \cdot x) = N_{E'/E}(\alpha)\, N(x)$ $\forall\, \alpha \in \mathcal{O}_{E'}$.

(e.g. on Tate module of generic fibre, which is an $\mathcal{O}_{E'}$-module

free of rank 1.)

Thus we can pass to $\tilde{H}'_C$ and $\tilde{H}_C$ and get $N : Z_{E'} \to Z_E$.

Again, N is indep of all choices.

<u>Proof of Prop</u> Crucial point is that the following diagram "exists" &

commutes:

$$\begin{array}{ccc} \{E^\times\text{-cov of } \tilde{D}^\times_{E,C}\} & \xrightarrow{N^*} & \{E'^\times\text{-cov of } \tilde{D}^\times_{E',C}\} \\ \wr\Vert & & \wr\Vert \\ \{(\phi^{-1}\circ \mathrm{Frob}_q)\times \mathcal{O}_E^\times\text{-cov of} & \overset{\exists}{\dashrightarrow} & \{(\phi^{-1}\circ \mathrm{Frob}_{q'})\times \mathcal{O}_{E'}^\times\text{-cov of} \\ (Y_{C,E} \hat{\otimes}_E \hat{E}_\infty)^\flat\} & & (Y_{C,E'} \hat{\otimes}_{E'} \hat{E}'_\infty)^\flat\} \end{array}$$

The problem is that $\hat{E}_\infty$ need not embed into $\hat{E}'_\infty$.

But the isomorphism $\hat{\Lambda}^d H' \cong H$ is already defined over E'^{nr}.

So $E^{nr}(H[\pi^n]) \subset E'^{nr}(\hat{\Lambda}^d H'[\pi^n]) \subset E'^{nr}(H'[\pi^n])$ and

we get compatible towers $\hat{E}^{nr}_n \subset \hat{E}'^{nr}_n$, $\hat{E}^{nr}_\infty \subset \hat{E}'^{nr}_\infty$.

The diagram

$$\begin{tikzcd}
\mathcal{O}_{E'}^* \arrow[r] \arrow[d, "N_{E'/E}"'] \arrow[dr, phantom, "@"] & \mathrm{Gal}(E'^{nr}_\infty / E'^{nr}) \arrow[d] \\
\mathcal{O}_E^* \arrow[r] & \mathrm{Gal}(E^{nr}_\infty / E^{nr})
\end{tikzcd}$$

commutes which expresses the "N-linearity" of $N: H' \to H$.

Now note that z defines an isomorphism
(my recall)

$$\mathrm{Spf}\, \mathcal{O}_{E^{nr}_n}/z_n \cong H[\pi^{n-1}] \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}, \quad \bar{\mathbb{F}} = \mathbb{F}_q^{alg}$$

which is equivariant for $\mathcal{O}_E^* \to \mathrm{Gal}(E^{nr}_\infty / E^{nr})$

This iso is behind the result of Schneider: $(\tilde{D}^*_{E,C})^\flat \cong D^*_{C^\flat} \hat{\otimes}_{\mathbb{F}_q((t))} \hat{E}^\flat_\infty$.

We need the compatibility

$$\begin{tikzcd}
\mathrm{Spec}\, \mathcal{O}_{E'^{nr}_n}/z'_n \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dr, phantom, "C"] & H'[\pi^{n-1}] \otimes_{\mathcal{O}_{E'}} \bar{\mathbb{F}} \arrow[d, "N"] \\
\mathrm{Spec}\, \mathcal{O}_{E^{nr}_n}/z_n \arrow[r, phantom, "\cong"] & H[\pi^{n-1}] \otimes_{\mathcal{O}_E} \bar{\mathbb{F}}
\end{tikzcd} \qquad (\dagger)$$

which follows by definition of the maps $E^{nr}_n \subset E'^{nr}_n$ in the towers above.

So we get, equivariant for diagram ⓐ.

$$\begin{array}{ccccc}
(\tilde{\mathbb{D}}^{\circ}_{E',C})^{b} & \simeq & \mathrm{Spa}(\mathcal{O}_{C^b} \hat{\otimes}_{\mathbb{F}} \mathcal{O}_{E'^{ur,b}_{\infty}}) \setminus V(z'\cdot\varpi^b) & = & (Y_{C,E'} \otimes \hat{E}'_{\infty})^{b} \\
\downarrow N & \text{L} & \downarrow \text{natural} & \text{R} & \vdots \ \text{(Not induced from map before } b) \\
(\tilde{\mathbb{D}}^{\circ}_{E,C})^{b} & \simeq & \mathrm{Spa}(\mathcal{O}_{C^b} \hat{\otimes}_{\mathbb{F}} \mathcal{O}_{E^{ur,b}_{\infty}}) \setminus V(z,\varpi^b) & = & (Y_{C,E} \otimes \hat{E}_{\infty})^{b}
\end{array}$$

Only left to show equivariance for

$$\begin{array}{ccc}
(\pi')^{\mathbb{Z}} & \xrightarrow{\sim} & (\mathrm{id} \otimes (z' \mapsto (z')^{q'}))^{\mathbb{Z}} \\
N \downarrow & & \downarrow \text{natural} \\
(N\pi')^{\mathbb{Z}} & \xrightarrow{\sim} & \cancel{(\mathrm{id}\otimes(z\mapsto z^{q'}))^{\mathbb{Z}}} \ (\mathrm{id} \otimes (z \mapsto [N\pi'](z)))^{\mathbb{Z}}.
\end{array}$$

q' residue cardinality of E'

But again, this follows from diagram (+). $\square$

Rmk Eventually, the curve Y did not enter here.

We already have the horizontal arrow in R, and we know the induced action on the two middle terms.

Then we have to see that N induces an equivariant diagram L. But this follows since (+) is a diagram of $\mathbb{Q}_E^{\circ}$-Spaces in the sense of ⓐ.