1/

→ a toy example : let $E = \mathbb{F}_q((X))$, $F = \mathbb{F}_q((Y))$ local fields char p

$\rightsquigarrow S = \mathrm{Spa}(\mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E, \mathcal{O}_F \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_E)$

$\swarrow$ $\mathrm{Spa}(\mathcal{O}_F, \mathcal{O}_F) = \{\eta_F, s_F\}$ $\searrow$ $\mathrm{Spa}(\mathcal{O}_E, \mathcal{O}_E)$

$\rightsquigarrow$ $\mathcal{O}_F \in S_{\eta_F}$ = open unit disc in var. $X / \mathrm{Spa}(F, \mathcal{O}_F)$ $= S \setminus V(Y)$

$V(X) \cap S_{\eta_F} = \mathcal{O}_F$

$V(Y) \cap S_{\eta_E} = \mathcal{O}_E \in S_{\eta_E}$ = — " ——— in var $Y / \mathrm{Spa}(E, \mathcal{O}_E)$ $= S \setminus V(X)$

note: $S_{\eta_F} \setminus \{\mathcal{O}_F\} = S_{\eta_E} \setminus \{\mathcal{O}_E\} = S \setminus V(X \cdot Y)$ as subsets of S

$\rightsquigarrow$ what happens if we replace E and F by perf'd fields (in char p) ?

<u>Recall</u>: let K be a perf'd field $\supseteq \mathcal{O}_K \ni \varpi \neq 0$ $|p| \leq |\varpi| < 1$.

<u>Prop A</u> S an $\mathcal{O}_K$-alg. complete wrt I fin. gen ideal of def'n

assume $\phi : S/\varpi \longrightarrow S/\varpi$ surj. and $\ker(\phi) \subseteq S/\varpi$ fin. gen

then $X_{\eta_K} = \varinjlim_n \mathrm{Spa}(K \otimes_{\mathcal{O}_K} \hat{S}_n, \hat{S}_n^+) = \mathrm{Spa}(S, S) \setminus V(\varpi) \subseteq \mathrm{Spa}(S, S) = X$

(as top spaces + presheaf + val's on stalks)

is a perf'd space $/ \mathrm{Spa}(K, \mathcal{O}_K) = \eta_K$

[note $\mathrm{Spa}(K \otimes_{\mathcal{O}_K} \hat{S}_n, \hat{S}_n^+) \longrightarrow \mathrm{Spa}(S, S)$ rat'l subset]

and $X_{\eta_K}^\flat \cong \mathrm{Spa}(S^\flat, S^\flat)_{\eta_{K^\flat}}$, $S^\flat = \varprojlim_\phi S/\varpi$

natural isom of perf'd spaces $/ K^\flat$ an $\mathcal{O}_{K^\flat}$-alg.

2/

Application: $C = \mathbb{C}_p^\flat = \widehat{\mathbb{F}_p((\varpi_C))}$ perf'd field char p

$K = \mathbb{F}_q((\varpi_K^{1/p^\infty}))$ — " —

$$X = \mathrm{Spa}(\mathcal{O}_C \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_K, \mathcal{O}_C \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_K)$$ as top space + pre-sheaf

$$\begin{array}{ccc} \cup\!\!| & & \cup\!\!| \\ X_{\eta_C} = X \setminus V(\varpi_C) & & X_{\eta_K} = X \setminus V(\varpi_K) \\ \downarrow & \text{perf'd spaces by Prop A} & \downarrow \\ \mathrm{Spa}(C, \mathcal{O}_C) & & \mathrm{Spa}(K, \mathcal{O}_K) \end{array}$$

Note: $X_{\eta_C} \setminus \{o_C\} = X_{\eta_K} \setminus \{o_K\} = X \setminus V(\varpi_C \cdot \varpi_K) = X_{\eta_C} \cap X_{\eta_K}$

perf'd. space over K and over C

where o_C = single pt in $V(\varpi_K) \cap X_{\eta_C}$

o_K = — " — $V(\varpi_C) \cap X_{\eta_K}$.

Moreover: $X_{\eta_C} = \widetilde{\mathbb{D}}_C \overset{\text{def}}{=} \mathrm{Spa}(\mathcal{O}_C[[\varpi_K^{1/p^\infty}]], \mathcal{O}_C[[\varpi_K^{1/p^\infty}]])_{\eta_C}$

and $X_{\eta_C} \setminus \{o_C\} = \widetilde{\mathbb{D}}_C^*$

Rem: $$X = X_{\eta_C} \cup X_{\eta_K} \cup \{\underbrace{\mathcal{O}_C \hat{\otimes} \mathcal{O}_K \to \overline{\mathbb{F}}_q \xrightarrow[\text{val}]{\text{triv}} \{0,1\}}_{\text{special point, which is the unique closed pt of } X}\}$$

⇝ can conclude from this that X is an adic space,

i.e. $\mathcal{O}_C \hat{\otimes}_{\mathbb{F}_q} \mathcal{O}_K$ is sheafy

(i) Note that X has two partial Frobenii: φ_C coming from Frob on $\mathcal{O}_C$

φ_K —"— $\mathcal{O}_K$

s.th. $\varphi_C \circ \varphi_K$ = absolute Frobenius

(ii) $K = (\widehat{\mathbb{Q}}_p^{cycl})^\flat$ with $\varpi_K = p^\flat$ $\quad (C = \mathbb{C}_p^\flat)$

<u>Prop.</u> $\exists$ isom $(\widetilde{\mathbb{D}}_{\mathbb{C}_p})^\flat \overset{\text{def}}{=} \big(\varprojlim_{T \mapsto (1+T)^p - 1} \mathbb{D}_{\mathbb{C}_p} \big)^\flat \cong \mathrm{Spa}(\mathcal{O}_C \hat{\otimes}_{\mathbb{F}_p} \mathcal{O}_K)_{\eta_c} = X_{\eta_c}$

of perf'd spaces over C inducing

$$(\widetilde{\mathbb{D}}^*_{\mathbb{C}_p})^\flat = (\widetilde{\mathbb{D}}_{\mathbb{C}_p} \setminus V(T))^\flat \cong (\widetilde{\mathbb{D}}_{\mathbb{C}_p})^\flat \setminus V(\varpi_K) = \widetilde{\mathbb{D}}^*_C$$

Moreover this isom is equivariant for the action of $\mathbb{Q}_p^\times$ where $\mathbb{Q}_p^\times$ acts

- on $\widetilde{\mathbb{D}}_{\mathbb{C}_p}$ via $\quad T \longmapsto (1+T)^a - 1 \quad a \in \mathbb{Q}_p^\times$
- on X_{η_K} via $\mathbb{Z}_p^\times \overset{\cong}{\longleftrightarrow} \mathrm{Gal}(\mathbb{Q}_p^{cycl}/\mathbb{Q}_p) \circlearrowright \mathcal{O}_{\widehat{\mathbb{Q}}_p^{cycl}}{}^\flat = \mathcal{O}_K$

and $p \in \mathbb{Q}_p^\times$ acts via φ_K (i.e. Frob on $\mathcal{O}_K$)

<u>Proof:</u> let $\hat{R} = \varprojlim_{T \mapsto (1+T)^p - 1} \mathcal{O}_{\mathbb{C}_p}[[T]]$ $\quad$ ((T,p)-adically complete)

$$\hat{R}^\flat = \varprojlim_{\varphi} \hat{R}/p = \varprojlim_{T \mapsto T^p} \mathcal{O}_C[[T]] = \mathcal{O}_C[[T^{1/p^\infty}]]$$

Prop A $\Rightarrow$ $(\widetilde{\mathbb{D}}_{\mathbb{C}_p})^\flat \cong \mathrm{Spa}(\mathcal{O}_C[[T^{1/p^\infty}]], \mathcal{O}_C[[T^{1/p^\infty}]])_{\eta_c}$

and $a \in \mathbb{Q}_p^\times$ acts on RHS via $T \mapsto (1+T)^a - 1$

(e.g. as $T \mapsto T^p$ for $a = p$)

we conclude by remarking that

- this isom maps $V(T)$ on LHS to $V(T)$ on RHS
- $\mathcal{O}_C[[T^{1/p^\infty}]] \cong \mathcal{O}_C \hat{\otimes}_{\mathbb{F}_p} \mathcal{O}_K$ via $T \leftrightarrow \varpi_K$

equivariant for $\mathbb{Q}_p^\times$-action.

$\square$

Recall: $X_{q_C} \setminus \{0_C\} = X_{q_K} \setminus \{0_K\}$

Prop $\exists$ nat. bij.

$$\begin{pmatrix} \varphi_K\text{-equivar. fét covers} \\ \text{of } X_{q_C} \setminus \{0_C\} \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} \varphi_C\text{-equivariant fét covers} \\ \text{of } X_{q_K} \setminus \{0_K\} \end{pmatrix}$$

and
$$\begin{pmatrix} \varphi_K \text{ and } \mathbb{Z}_p^\times\text{-equivar.} \\ \text{fét covers of } X_{q_C} \setminus \{0_C\} \end{pmatrix} \xleftrightarrow{1:1} \begin{pmatrix} \varphi_C \text{ and } \mathbb{Z}_p^\times \text{ equivariant} \\ \text{fét covers of } X_{q_K} \setminus \{0_K\} \end{pmatrix}$$

Proof

$X_{q_C} \setminus \{0\}$ perf'd $\Rightarrow$ $Y \xrightarrow{\text{fét}} X_{q_C} \setminus \{0\}$ ~~perf~~

then Y is perf'd hence perfect

i.e. abs Frob is isom on Y

$\Rightarrow$ (φ_K lifts to Y $\Leftrightarrow$ φ_C lifts to Y)

$\square$

5/ **Thm.** $\exists$ nat. bij.

$$\left(\begin{matrix} \mathbb{Z}_p^* \text{ and } \varphi_c\text{-equivariant} \\ \text{fét-covers of } X_{\eta_K} \setminus \{a_c\} \end{matrix} \right) \xleftrightarrow{1:1} (\text{fin ext of } \mathbb{Q}_p)$$

proof will see in next talk:

$$X_{\eta_K} \setminus \{0\}_K \cong (Y_{C,\mathbb{Q}_p} \hat{\otimes} \hat{\mathbb{Q}}_p^{cyc})^\flat \quad \text{compatible w. } \varphi_c \text{ and } \mathbb{Z}_p^*\text{-action}$$

as perf'd spaces$/K$

$$\Rightarrow \text{LHS} \xhookrightarrow{1:1} \left(\begin{matrix} \mathbb{Z}_p^*\text{-equivariant} \\ \text{fét-covers of } X_{C,\mathbb{Q}_p} \hat{\otimes} \hat{\mathbb{Q}}_p^{cyc} \end{matrix} \right)$$

$$\xhookrightarrow[(*)]{1:1} 2\text{-}\varinjlim_n \left(\begin{matrix} \mathbb{Z}_p^*\text{-equivar fét covers} \\ \text{of } X_{C,\mathbb{Q}_p} \otimes \mathbb{Q}_p(\mu_{p^n}) \end{matrix} \right)$$

$$\xleftarrow[1:1]{} (\text{fin ext of } \mathbb{Q}_p)$$

ad $(*)$: apply $2\text{-}\varinjlim (A_i[\tfrac{1}{p}]_{fét}) \cong ((\varinjlim A_i)^\wedge [\tfrac{1}{p}])_{fét}$

for A_i henselian along p

to A_0 with $\mathrm{Spa}(A_0[\tfrac{1}{p}], A_0) \subseteq X_{C,\mathbb{Q}_p}$ aff'd open

$A_i = A_0 \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathbb{Q}_p(\mu_{p^i})}$

$\square$

$\rightarrow$ " $\tilde{\mathbb{D}}^*_C / \mathbb{Q}_p^\times$ is a diamond "

1) $\tilde{\mathbb{D}}^*_C / \varphi_K^{\mathbb{Z}}$ is a perf'd space$/C$ [recall: φ_K acts as $T \mapsto T^p$

proper discont + free

(similar to the case of $Y^{ad}_{C,\mathbb{B}}$)

$\rightarrow$ quotient ex as adic space]

2) let $\tilde{\mathbb{D}}^*_C / \underline{\mathbb{Z}}_p^\times$ denote the sheaf on Perf_C (for pro-étale top.)

that is given as the sheafification of the quotient $h_{\tilde{\mathbb{D}}^*_C}$ by the $\underline{\mathbb{Z}}_p^\times$-action

<u>claim</u> $\tilde{\mathbb{D}}^*_C \longrightarrow \tilde{\mathbb{D}}^* / \underline{\mathbb{Z}}_p^\times$ is a $\underline{\mathbb{Z}}_p^\times$-torsor, i.e.

$$\alpha : \tilde{\mathbb{D}}^*_C \times \underline{\mathbb{Z}}_p^\times \xrightarrow{(pr_1, \text{action})} \tilde{\mathbb{D}}^*_C \times_{\tilde{\mathbb{D}}^*_C / \underline{\mathbb{Z}}_p^\times} \tilde{\mathbb{D}}^*_C \quad \text{is an isom}$$

[isom of sheaves. But LHS is in fact representable:

$$\underline{\mathbb{Z}}_p^\times \overset{\text{def}}{=} \varprojlim_n \coprod_{(\mathbb{Z}/p^n\mathbb{Z})^\times} \mathrm{Spa}(C, \mathcal{O}_C) = \mathrm{Spa}(R, R^+)$$

$$R = \{ \text{cont. maps } \mathbb{Z}_p^\times \longrightarrow \mathcal{O}_C \} \;]$$

$\rightsquigarrow$ $\mathrm{Hom}_{\mathrm{Perf}}(S, \underline{\mathbb{Z}}_p^\times) = \mathrm{Hom}_{\mathrm{cont}}(S, \mathbb{Z}_p^\times)$

<u>proof of claim</u>: construct inverse.

giving $\mathrm{Spa}(R, R^+) \longrightarrow \tilde{\mathbb{D}}^*_C \times_{\tilde{\mathbb{D}}^*_C / \underline{\mathbb{Z}}_p^\times} \tilde{\mathbb{D}}^*_C$

is equiv to giving $f, g : \mathrm{Spa}(R, R^+) \longrightarrow \tilde{\mathbb{D}}^*_C$ (i.e. $f, g : \mathbb{F}_p[\![T]\!] \rightarrow R^+$)

s.th. $\exists$ $\mathrm{Spa}(\tilde{R}, \tilde{R}^+) \longrightarrow \mathrm{Spa}(R, R^+)$ pro-étale cover

and $\mathrm{Spa}(\tilde{R}, \tilde{R}^+) \xrightarrow{\gamma} \underline{\mathbb{Z}}_p^\times$ s.th.

$$f(T) = (1 + g(T))^\gamma - 1$$

in order to construct inverse to α need to define $\mathrm{Spa}(R, R^+) \longrightarrow \underline{\mathbb{Z}}_p^\times$

i.e. need to show γ factors over $\mathrm{Spa}(R, R^+)$

$\rightarrow$ enough to show: γ constant on fibers

$\rightsquigarrow$ may assume R alg closed field. If γ not const $\rightarrow \exists \gamma_0 \neq \gamma_1 \in \mathbb{Z}_p^\times$

s.th. $f(T) = (1 + g(T))^{\gamma_i} - 1$, but $\mathbb{Z}_p^\times \underset{\text{inj.}}{\hookrightarrow} \mathrm{Aut}(\mathbb{F}_p(\!(T^{1/p^\infty})\!))$