$|Y|$ as points of an adic space

In the last talk we constructed for a primitive element of degree 1 $x \in W$ $(x = \pi - [x_0])$

a morphism θ:

$$\begin{tikzcd}
W \arrow[r, two heads, "\theta"] & A_x := W/xW \\
\mathcal{O}_F \arrow[u, "{[\cdot]}", "\text{Teichmüller lift}"'] \arrow[ur] &
\end{tikzcd}$$

and a discrete valuation w_x on A_x defined by $w_x(\theta[a]) := v_F(a)$

with corresponding norm $|w|_x := q^{-w_x(w)}$ $(w \in A_x)$

such that A_x is the valuation ring for w_x in $C_x := \mathrm{Frac}(A_x)$

We want to extend θ to a map $B \to C_x$.

Set $\rho_x := q^{-w_x(\pi)} = |\pi|_x$, choose a closed interval $I \subset (0,1)$ with $\rho_x \in I$, $I = [\rho_1, \rho_2]$

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[drr, "\theta"'] & B^{b+} = W[\frac{1}{\pi}] \arrow[r, hook] \arrow[dr] & B^{b} = B^{b+}[F^{\times}] \arrow[r, hook] \arrow[d, "(**)"] & B \arrow[r, hook] \arrow[dl] & B_I \arrow[dll, "(***)"] \\
 & & C_x = \mathrm{Frac}(A_x) \overset{(*)}{=} A_x[\frac{1}{\pi}] & &
\end{tikzcd}$$

(*) We have $w_x(\theta(\pi)) = w_x(\pi - [x_0]) = w_x([x_0]) = v(x_0)$ $(= v_{w_x(\pi)}(\pi)) > 0$

$\Leftrightarrow$ $|\theta(\pi)|_x = |\pi|_{\rho_x} < 1$

and C_x is obtained from A_x by inverting any element of valuation > 0

(**) $w_x(\theta([a])) = v(a) = v_r(a)$ $\forall r > 0$, $a \in \mathcal{O}_F$

In particular $\theta([a]) \neq 0$ if $a \neq 0$ $\Rightarrow$ extensions exists.

(***) We need to check that Θ is continuous with respect to $|\cdot|_x$ and $|\cdot|_I$.
Since $|b|_\rho \leq |b|_I$ for all $b \in B^b$ and all $\rho \in I$, it suffices to check that

$$\forall b \in B^b : \ |\Theta(b)|_x \leq |b|_{\rho_x}$$

Since $|\pi|_x = |\pi|_{\rho_x} \ (= |x_0|)$ (see (*))
and $|\Theta[a]|_x = |[a]|_{\rho_x} \ (= |a|)$ (see (**)) for $a \in F^\times$,
we can assume $b \in W$. Write $b = \sum_{n\geq 0} [b_n]\pi^n$. Then

$$|\Theta(b)|_x = \Big|\Theta\Big(\sum_{n\geq 0}[b_n]\pi^n\Big)\Big| \underset{x = \pi - [x_0]}{=} \Big|\Theta\Big(\sum_{n\geq 0}[b_n]\cdot[x_0]^n\Big)\Big| = \Big|\Theta\Big(\sum_{n\geq 0}[b_n x_0^n]\Big)\Big|$$

$$\leq \sup_{n\geq 0} |\Theta([b_n x_0^n])| = \sup_{n\geq 0} |b_n||x_0|^n = \sup_{n\geq 0} |b_n|\rho_x^n = |b|_{\rho_x}.$$

In all cases $\ker(\Theta)$ is the ideal generated by x and

$m_x := \ker(B \xrightarrow{\Theta} C_x)$ is a maximal ideal.

We obtain an injection

$$|Y| \hookrightarrow \mathrm{Spm}(B)$$
$$x \longmapsto m_x$$

We want to construct an adic space Y^{ad} with $|Y| = |Y^{ad}|$.
A first guess would be $\mathrm{Spa}(B, B^\circ)$ (power bounded elements), but this does not work well
(cannot prove that the structure presheaf is a sheaf)

<u>Theorem 1</u>: $Y_I^{ad} = \mathrm{Spa}(B_I, B_I^\circ)$ is an adic space, i.e. the structure presheaf is a sheaf.

Strategy of the proof: For $I \subsetneq (0,1)$, $I = [\rho_1, \rho_2]$ with $\rho_1 = |a|$, $\rho_2 = |b|$, $a, b \in O_F$ find a perfectoid field K/E with $B_I \hat{\otimes} K$ is perfectoid (and thus in particular $\mathrm{Spa}(B_I \hat{\otimes} K)$ is an adic space) and conclude that $\mathrm{Spa}(B_I, B_I^\circ)$ is an adic space.

For $n \geq 0$ set $E_n := E[\pi^{1/p^n}] := E[X]/(X^{p^n} - \pi)$ and let π_n denote the image of X in E_n, i.e. $\pi_n^{p^n} = \pi$.

Define $L := \bigcup_{n \geq 0} E_n$. Then $\hat{L}$ is a perfectoid field ($\hat{L}$ is π-adic completion of L)

$B_I \otimes_E L$ is equipped with the tensor product norm:

$$|x| := \inf \left\{ \sup_i |b_i|_I |\ell_i|_L \,\middle|\, x = \sum_i b_i \otimes \ell_i \right\}.$$

Denote its completion by $B_I \hat{\otimes} \hat{L}$ ($= B^b \hat{\otimes}_E L$, where B^b is equipped with the norm $|\cdot|_I$)

We want to show that $B_I \hat{\otimes} \hat{L}$ is perfectoid.

problem: The tensor product norm is not power-multiplicative, hence it is hard to show that the powerbounded elements $(B_I \hat{\otimes} \hat{L})^\circ$ are bounded and open

Theorem 2: There is a norm $\|\cdot\|$ on $B^b \otimes_E L$ equivalent to the tensor product norm $|\cdot|_I \otimes |\cdot|_L$, such that $(B^b \otimes_E L)^\wedge$ (completion with respect to $\|\cdot\|$) is perfectoid.

(Of course, since $|\cdot|_I \otimes |\cdot|_L$ and $\|\cdot\|$ are equivalent, we have $(B^b \otimes_E L)^\wedge \simeq B^b \hat{\otimes}_E \hat{L}$)

proof: Define $B_n^b := B_{E_n}^b = \left\{ \sum_{k \gg -\infty} [x_k] \pi_n^k \mid |x_k|_F \text{ are bounded} \right\}$

$W_n := W_{\mathcal{O}_{E_n}}(\mathcal{O}_F)$

$B_n^{b+} := W_n[\frac{1}{\pi}] = W_n[\frac{1}{\pi_n}]$

$I_n := [\rho_1^{1/p^n}, \rho_2^{1/p^n}]$, $\rho_1 = |a|$, $\rho_2 = |b|$, $a, b \in \mathcal{O}_F$

Then $B^b \otimes_E L = \varinjlim_n B^b \otimes_E E_n = \varinjlim_n B_n^b$

We want to show that the isomorphisms $B^b \otimes_E E_n \xrightarrow{\sim} B_n^b$ are topological, i.e. $|\cdot|_I \otimes |\cdot|_{E_n}$ and $|\cdot|_n$ are equivalent

We need the following lemma:

~~proof: Without loss of generality we may assume n~~

lemma: $A_n := W_n\left[\frac{[a^{1/p^n}]}{\pi_n}, \frac{\pi_n}{[b^{1/p^n}]}\right] \subseteq B_n^b$ is the set of powerbounded elements $B_n^{b\circ}$ (with respect to $|\cdot|_{I_n}$) = unit ball.

proof: Without loss of generality we may assume $n=0$ (replace E_n with E), $A := A_0$

$|\cdot|_I$ is power-multiplicative (as $|\cdot|_I = \max\{|\cdot|_{\rho_1}, |\cdot|_{\rho_2}\}$ and $|\cdot|_{\rho_i}$ are multiplicative)

$\Rightarrow B^{b\circ} = \{x \in B^b \mid |x|_I \leq 1\}$

- For $x \in W$, $x = \sum_{k \geq 0} [x_k] \pi^k$ with $x_k \in \mathcal{O}_F$ we have

$$|x|_{\rho_i} = \sup_{k \geq 0} \underbrace{|x_k|}_{\leq 1} \underbrace{\rho_i^k}_{\leq 1} \leq 1 \Rightarrow W \subseteq B^{b\circ}$$

- $\left|\frac{[a]}{\pi}\right|_I = \max\{|a|\rho_1^{-1}, |a|\rho_2^{-1}\} = \max\{1, \frac{\rho_1}{\rho_2}\} = 1$
- $\left|\frac{\pi}{[b]}\right|_I = \max\{|b|^{-1}\rho_1, |b|^{-1}\rho_2\} = \max\{\frac{\rho_1}{\rho_2}, 1\} = 1$

$\Rightarrow A \subseteq B^{b\circ}$

Now let $x = \sum_{k \gg -\infty} [x_k] \pi^k \in B^{bo}$

$$\Rightarrow \quad 1 \geqslant |x|_I = \sup_{\substack{i=1,2 \\ k \gg -\infty}} |x_k| \rho_i^k$$

- $|x_k|$ bounded $\Rightarrow$ $\exists N \geqslant 0$ such that for $k \geqslant N$: $|x_k| \leqslant \rho_2^{-N}$

$$\Rightarrow \sum_{k \geqslant N} [x_k] \pi^k = \left(\frac{\pi}{[b]}\right)^N \sum_{k \geqslant N} \underbrace{[x_k b^N]}_{|x_k b^N| = |x_k| \rho_2^N \leqslant 1} \pi^{k-N} \in A$$

- for $k < 0$: $|x_k| \rho_1^k \leqslant 1 \Rightarrow \left|\frac{x_k}{a^{-k}}\right| \leqslant 1 \Rightarrow [x_k] \pi^k = \left(\frac{[a]}{\pi}\right)^k \left[\frac{x_k}{a^{-k}}\right] \in A$
- for $0 \leqslant k < N$: $|x_k| \rho_2^k \leqslant 1 \Rightarrow |x_k b^k| \leqslant 1 \Rightarrow [x_k] \pi^k = \left(\frac{\pi}{[b]}\right)^k [x_k b^k] \in A$

$\square$

$\Rightarrow x \in A$

<u>Continuation of the proof of theorem 2:</u>

The topology on $(B^b \otimes_E E_n, |\cdot|_I \otimes |\cdot|_{E_n})$ is generated by

$$B^{bo} \otimes_{O_E} O_{E_n} = A \otimes_{O_E} O_{E_n} = W \otimes_{O_E} O_{E_n} \left[\frac{[a]}{\pi}, \frac{\pi}{[b]}\right] = W_n \left[\frac{[a]}{\pi}, \frac{\pi}{[b]}\right].$$

The topology on $(B^b_n, |\cdot|_{I_n})$ is generated by

$$B^{bo}_n = A_n = W_n \left[\frac{[a^{1/p^n}]}{\pi_n}, \frac{\pi_n}{[b^{1/p^n}]}\right]$$

<u>claim</u>: $\pi B^{bo}_n \subseteq B^{bo} \otimes_{O_E} O_{E_n} \subseteq B^{bo}_n$

- $B^{bo} \otimes_{O_E} O_{E_n} \subseteq B^{bo}_n$ is obvious
- In order to see that $\pi B^{bo}_n \subseteq B^{bo} \otimes_{O_E} O_{E_n}$ we have to show that $\forall k \in \mathbb{N}$

$$\pi \left(\frac{[a^{1/p^n}]}{\pi_n}\right)^k \in B^{bo}_n \quad \text{and} \quad \pi \left(\frac{\pi_n}{[b^{1/p^n}]}\right)^k \in B^{bo}_n$$

Write $k = rp^n + s$ with $r \geq 0$ and $0 \leq s < p^n$

Then $$\pi \left(\frac{[a^{1/p^n}]}{\pi_n} \right)^k = \left(\frac{[a]}{\pi} \right)^r \cdot \pi_n^{p^n - s} [a^{s/p^n}] \in B_n^{bo}$$

$$\pi \left(\frac{\pi_n}{[b^{1/p^n}]} \right)^k = \left(\frac{\pi}{[b]} \right)^{r+1} \pi_n^s [b^{1 - s/p^n}] \in B_n^{bo}$$

$\Rightarrow$ claim

The claim implies that the topologies on $B^{bo} \otimes_{\mathcal{O}_E} \mathcal{O}_{E_n} \simeq B_n^{bo}$ associated to $|\cdot|_I$ and to $|\cdot|_I \otimes |\cdot|_{E_n}$, respectively, coincide, i.e. $|\cdot|_I \otimes |\cdot|_{E_n}$ and $|\cdot|_I$ are equivalent

We want to combine these norms to norms on $B^b \otimes_E L$

- $|\cdot|_{E_n}$ are the norms induced by $|\cdot|_L$

  $\Rightarrow |\cdot|_I \otimes |\cdot|_{E_n}$ combine to the norm $|\cdot|_I \otimes |\cdot|_L$ on $B^b \otimes_E L$

- We have isometric embeddings

$$\iota_n : (B^b, |\cdot|_I) \longrightarrow (B_n^b, |\cdot|_{I_n})$$
$$\sum_k [x_k] \pi^k \longmapsto \sum_k [x_k] \pi_n^{kp^n}$$

(for any $\rho \in (0,1)$ we have

$$|\iota_n(x)|_{\rho^{1/p^n}} = \Big| \sum_k [x_k] \pi_n^{kp^n} \Big|_{\rho^{1/p^n}} = \sup_k |x_k| (\rho^{1/p^n})^{kp^n} = \sup_k |x_k| \rho^k = |x|_\rho$$

$$\Rightarrow |\iota_n(x)|_{I_n} = \max\{ |\iota_n(x)|_{\rho_1^{1/p^n}}, |\iota_n(x)|_{\rho_2^{1/p^n}} \} = \max\{ |x|_{\rho_1}, |x|_{\rho_2} \} = |x|_I \quad)$$

$\Rightarrow |\cdot|_{I_n}$ combine to a norm $\|\cdot\|$ on $B^b \otimes_{\mathcal{O}_E} L = \varinjlim_n B_n^b$, which is equivalent to $|\cdot|_I \otimes |\cdot|_L$

Moreover $|\cdot|_{I_n}$ are power-multiplicative and sub-multiplicative

Thus, the powerbounded element are given by the elements x of norm $\|x\| \leq 1$,

$$(B_I \hat{\otimes} \hat{L})^\circ = (\varinjlim_n B_n^{bo})^\wedge = (\varinjlim_n A_n)^\wedge$$

(Here completion is with respect to $\|\cdot\|$, which coincides with π-adic completion, because on the unit ball completion with respect to $\|\cdot\|$ is y-adic completion for any y with $0 < \|y\| < 1$)

We now show that $B_I \hat{\otimes}_E \hat{L}$ is perfectoid.

- powerbounded elements are open and bounded:

  $(B_I \hat{\otimes}_E \hat{L})^\circ = \{x \in B_I \hat{\otimes}_E \hat{L} \mid \|x\| \leq 1\}$ is the open unit ball and thus bounded and open.

- Frobenius is surjective on $(B_I \hat{\otimes} \hat{L})^\circ / \pi$:

$$(B_I \hat{\otimes} \hat{L})^\circ / \pi = (\varinjlim_n A_n)^\wedge / \pi \underbrace{=}_{\substack{\text{completion is} \\ \pi\text{-adic completion}}} (\varinjlim_n A_n) / \pi = \varinjlim_n (A_n / \pi)$$

$$A_n/\pi = W_n/\pi\,[x_n, y_n], \quad x_n \equiv \frac{[a^{1/p^n}]}{\pi_n} \bmod \pi, \quad y_n \equiv \frac{\pi_n}{[b^{1/p^n}]} \bmod \pi$$

$$= \mathcal{O}_F \otimes_{\mathbb{F}_q} \mathcal{O}_E/\pi\,[x_n, y_n]$$

The transition maps are given by

- $\mathcal{O}_F \otimes \mathcal{O}_{E_n}/\pi \longrightarrow \mathcal{O}_F \otimes_{\mathbb{F}_q} \mathcal{O}_{E_{n+1}}/\pi$ induced by inclusion $E_n \hookrightarrow E_{n+1}$
- $x_n \longmapsto x_{n+1}^p$
- $y_n \longmapsto y_{n+1}^p$

We have: $\cdot\ \varinjlim_n \mathcal{O}_{E_n}/\pi = \mathcal{O}_L/\pi$
$\cdot\ \mathcal{O}_F$ is perfect $\Big\}$ $\Rightarrow$ Frobenius is surjective on $\mathcal{O}_F \otimes \mathcal{O}_L/\pi$

$\Rightarrow$ Frobenius is surjective. $\square$

To conclude that $\mathrm{Spa}(B_I, B_I^\circ)$ is an adic space we need the following

<u>lemma:</u> Let K be a complete nonarchimedean discretely valued field, $C^\bullet$ a complex of K-Banach spaces and W a nonzero K-Banach space.
If $C^\bullet \hat{\otimes} W$ is exact, then $C^\bullet$ is exact.

<u>proof:</u> [Peter Schneider "Nonarchimedean Functional Analysis", Prop. 10.1] implies:
There is a set I, such that W is topologically isomorphic to

$$c_0(I, K) := \{\phi : I \to K \mid \forall \varepsilon > 0\ \#\{i \in I \mid |\phi(i)| \geq \varepsilon\} < \infty\}$$

with supremum norm $\|\cdot\|_\infty$

For any K-Banach space V there are canonical topological isomorphisms

$$V \hat{\otimes}_K c_0(I, K) \xrightarrow{\sim} c_0(I, V)$$

and

$$c_0(I, V) \xrightarrow{\sim} V \oplus c_0(I \setminus \{i\}, V)$$
$$\phi \longmapsto (\phi(i), \phi|_{I \setminus \{i\}})$$

Suppose that $\dots \to C^1 \hat{\otimes} W \to C^2 \hat{\otimes} W \to C^3 \hat{\otimes} W \to \dots$ is exact

$\overset{\text{choose any } i\in I}{\Longleftrightarrow}$ $\dots \to C^1 \oplus c_0(I\setminus\{i\}, C^1) \to C^2 \oplus c_0(I\setminus\{i\}, C^2) \to C^3 \oplus c_0(I\setminus\{i\}, C^3) \to \dots$

is exact

Note that for all $i \in I$: C^i maps to C^{i+1} and $c_0(I\setminus\{i\}, C^i)$ to $c_0(I\setminus\{i\}, C^{i+1})$

$\Rightarrow$ the sequence splits

$\Rightarrow$ $C^\bullet$ is exact

proof of theorem 1

Let $\mathcal{O}_I$ denote the structure presheaf on $\mathrm{Spa}(B_I, B_I^\circ)$

We need to show: for any rational $U \subseteq \mathrm{Spa}(B_I, B_I^\circ)$ and any cover by rationals $U = \bigcup_{j\in J} U_j$ the Čech complex

$$(*) \quad 0 \to \mathcal{O}_I(U) \to \prod_{j\in J} \mathcal{O}_I(U_j) \to \prod_{i,j\in J} \mathcal{O}_I(U_i \cap U_j)$$

is exact.

Tensoring with $\hat{L}$ yields

$$(*)\hat{\otimes}\hat{L} \quad 0 \to \mathcal{O}_I(U)\hat{\otimes}_E \hat{L} \to \prod_{j\in J} \mathcal{O}_I(U_j)\hat{\otimes}_E \hat{L} \to \prod_{i,j\in J} \mathcal{O}_I(U_i\cap U_j)\hat{\otimes}_E \hat{L}$$

For a rational subset $U\left(\frac{f_1, \dots, f_n}{g}\right)$ we have $\mathcal{O}_I\left(U\left(\frac{f_1,\dots,f_n}{g}\right)\right) = B_I\left\langle \frac{f_1, \dots, f_n}{g}\right\rangle$

(completion of $B_I\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$ with respect to the topology generated by $B_I^\circ\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$)

$$\mathcal{O}_I\left(U\left(\frac{f_1,\dots,f_n}{g}\right)\right)\hat{\otimes}_E \hat{L} = \left(B_I\left[\frac{f_1}{g},\dots,\frac{f_n}{g}\right]\otimes_E L\right)^\wedge$$

where the completion is taken with respect to the tensor product norm, i.e. the topology generated by the π-adic topology on

$$B_I^\circ\left[\frac{f_1}{g},\dots,\frac{f_n}{g}\right]\otimes_{\mathcal{O}_E}\mathcal{O}_L = \underbrace{B_I^\circ\otimes_{\mathcal{O}_E}\mathcal{O}_L}_{\substack{\text{generates the topology}\\ \text{of } B_I\otimes_E L}}\left[\frac{f_1\otimes 1}{g\otimes 1},\dots,\frac{f_n\otimes 1}{g\otimes 1}\right]$$

We thus obtain

$$\mathcal{O}_I\left(U\left(\frac{f_1,\dots,f_n}{g}\right)\right)\hat{\otimes}_E\hat{L} = \left(B_I\left[\frac{f_1}{g},\dots,\frac{f_n}{g}\right]\otimes_E L\right)^\wedge = \left(B_I\otimes_E L\left[\frac{f_1\otimes 1}{g\otimes 1},\dots,\frac{f_n\otimes 1}{g\otimes 1}\right]\right)^\wedge$$

$$= \mathcal{O}_{B_I\hat{\otimes}_E\hat{L}}\left(U\left(\frac{f_1\otimes 1,\dots,f_n\otimes 1}{g\otimes 1}\right)\right)$$

$\Rightarrow$ $(*)\hat{\otimes}\hat{L}$ is the Čech complex for some rational cover on $B_I\hat{\otimes}\hat{L}$

[Scholze "Perfectoid Spaces" Theorem 6.3 iii)] $\Rightarrow$ $(*)\hat{\otimes}\hat{L}$ is exact
(as $\mathcal{O}_{B_I\hat{\otimes}\hat{L}}$ is a sheaf)

lemma $\Rightarrow$ $(*)$ exact $\square$

<u>Corollary</u>: $Y^{ad} = \varinjlim_I Y_I^{ad}$ is an adic space

<u>Remarks</u>: 1) Instead of the explicit L we used, one can take any arithmetically profinite extension of E. The proof carries over with only minor modifications

2) In [Kedlaya "Noetherian properties of Fargues-Fontaine curves, Thm 4.10] it is shown that B_I is strongly noetherian, implying that $\mathrm{Spa}(B_I, B_I^\circ)$ is an adic space directly without using that $B_I\hat{\otimes}\hat{L}$ is perfectoid.

The adic Fargue - Fontaine curve

<u>Definition</u> : $X := \mathrm{Proj}\left(\bigoplus_{d\geq 0} B^{\varphi=\pi^d}\right)$ as a scheme over E

(we will examine this more closely in later talks)

X is called Fargue - Fontaine curve

X is not of finite type over E. It is thus not clear whether an adification exists, but we can construct an adic space as follows:

Frobenius ϕ acts properly discontinuously on Y^{ad}.

We want to define an adic space X^{ad} by

$$X^{ad} = \text{"} Y^{ad}/\phi^{\mathbb{Z}} \text{"}$$

This quotient exists in the category of adic spaces as ϕ acts properly discontinuously: Define $Y^{ad}/\phi^{\mathbb{Z}}$ as the topological quotient with the following structure sheaf: Let $\bar{y} \in Y^{ad}/\phi^{\mathbb{Z}}$ and $y \in Y^{ad}$ a preimage. There is an open neighborhood $U \ni y$ such that $\phi^n(U)$ are disjoint for all n. It thus makes sense to define $\mathcal{O}_{X^{ad}}(\bar{U}) = \mathcal{O}_{Y^{ad}}(U)$.

image of U in X^{ad}

This defines $\mathcal{O}_{X^{ad}}$ on a basis of the topology and one checks that $\mathcal{O}_{X^{ad}}$ is well defined.