FFI

(1)

I Analytic functions on mixed characteristic.

$E/\mathbb{Q}_p$ finite, $k_E = \mathbb{F}_q$.

$$\begin{array}{ccc} \breve{E} := W(F)\otimes_{W(\mathbb{F}_q)} E & & F \\ \Big| & & \Big| \\ E & & \mathbb{F}_q \end{array}$$

F, complete wr v, perfect, $|\cdot| := q^{-v(\cdot)}$

$\breve{E}$: unique complete unram. ext of E s.t. $k_{\breve{E}} = F$.

π unif of $\breve{E}$, $[\cdot] : F \to \mathcal{O}_{\breve{E}}$ Teichmüller

$$\breve{E} = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \mid x_n \in F \right\} = W_{\mathcal{O}_E}(F)\left[\tfrac{1}{\pi}\right]$$

φ Frob $\circlearrowright$ $\breve{E}$ through $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

Definition 1:

$$B^b = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \in \breve{E} \mid \exists C : \forall n\ |x_n| \le C \right\}$$

$$B^{b,+} = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n \in \breve{E} \mid \forall n,\ x_n \in \mathcal{O}_F \right\} = W_{\mathcal{O}_E}(\mathcal{O}_F)\left[\tfrac{1}{\pi}\right]$$

Topology on these rings?

$x = \sum_n [x_n]\pi^n \in B^b$, $r \geq 0$ $\quad v_r := \inf_n \{ v(x_n) + nr \}$

$\rho := q^{-r} \in]0,1]$, $|x|_\rho = q^{-v_r(x)}$. We have

$$\boxed{v_r \text{ is a valuation on } B^b}$$

Moreover: (1) v_r does not depend on the choice of π

(2) $r \mapsto v_r(x)$ is a concave function

It means, if we write, for $\tilde{I} = [R_1, R_2]$, $I = [\underset{q^{-R_1}}{\rho_1}, \underset{q^{-R_2}}{\rho_2}] \subset]0,1[$

$$v_I(x) := \inf_{r \in [R_1,R_2]} v_r(x), \quad |x|_I = \sup_{\rho \in I} |x|_\rho ,$$

We have:

(i) $|x|_I = \mathrm{Inf}\{|x|_{\rho_1}, |x|_{\rho_2}\}$

(ii) $J \subset I \subset]0,1[\Rightarrow |x|_I \geq |x|_J$

Definition 2:

(1) $B_I^{(+)} :=$ completion of $B^{b,(+)}$ w.r.t. $|\cdot|_I$

(Banach alg. (i))

(2) $B^{(+)} = \varprojlim_I B_I$ (see (ii)) (Fréchet algebra)

. What about φ?

We have $|\varphi(x)|_{\rho^q} = |x|_\rho^q$ so $\varphi : B_{[\rho_1,\rho_2]} \xrightarrow{\sim} B_{[\rho_1^q,\rho_2^q]}$

and it extends (by continuity) to $B^{(+)}$.

. Special property of B^+: for $x \in B^{b,+}$ and $r \geq r' > 0$

$$v_{r'}(x) \geq \frac{r'}{r} v_r(x) \quad \text{so} \quad (B^+_\rho := B^+_{[\rho,\rho]})$$

$B^+_{[\rho,\rho']} = B^+_\rho \subset B^+_{\rho'}$. Therefore, for $\rho_0 \in]0,1[$

$B^+_{\rho_0}$ is φ-stable and

$$B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{\rho_0}).$$

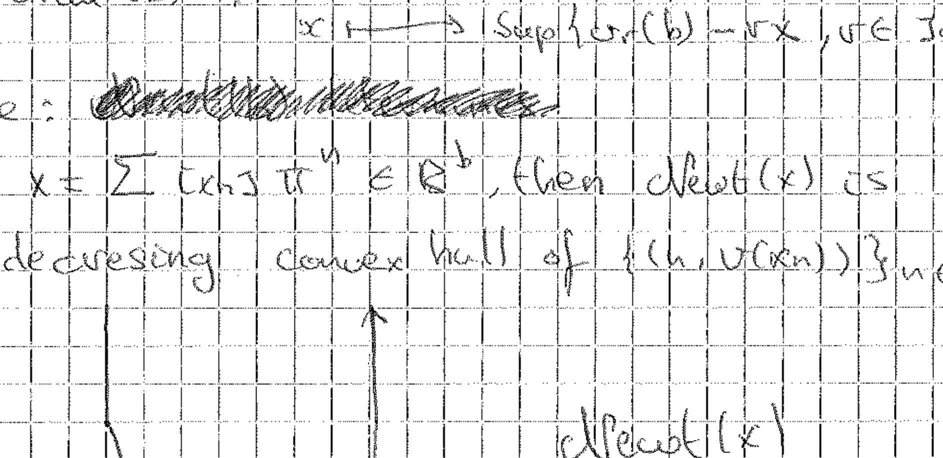
⚠ There is in general no way of writing an element of $B^{(+)}$ as a serie. But we have the Newton polygon: $(b \in B)$

$$\mathrm{Newt}(b) : \mathbb{R} \longrightarrow \mathbb{R}\cup\{+\infty\}$$
$$x \longmapsto \sup\{v_r(b) - rx,\ r \in]0,+\infty[\}$$

We have: ~~[illegible]~~

(1) If $x = \sum [x_n] \pi^n \in B^b$, then $\mathrm{Newt}(x)$ is the decreasing convex hull of $\{(n, v(x_n))\}_{n\in\mathbb{Z}}$

$(x \in B^{b,+})$



(2) If $x_n \to x$ in B, $\mathcal{N}ewt(x_n) \to \mathcal{N}ewt(x)$ converge compactly.

(3) $B^b = \{x \in B \mid \mathcal{N}ewt(x)$ is bounded below and $\exists A \mid \mathcal{N}ewt(x)_{]-\infty, A]} < +\infty \}$

$B^+ = \{x \in B \mid \mathcal{N}ewt(x) \geq 0\}$

(4) $B^\times = (B^b)^\times = \{x \in B^b \mid \mathcal{N}ewt(x)$ has 0 as its only non infinite slope $\}$

(5) $B^{\varphi = \pi^d} = 0$ for $d < 0$, $B^{\varphi = Id} = E$ and for $d \geq 0$ $B^{\varphi = \pi^d} = (B^+)^{\varphi = \pi^d}$.

Proof (of 5): "Lemma": • $\mathcal{N}ewt(\varphi(x)) = q\, \mathcal{N}ewt(x)$

• $\mathcal{N}ewt(\pi^d x)(t) = \mathcal{N}ewt(x)(t-d)$

For $d \neq 0$ use (2) and (3) $q\,\mathcal{N}ewt(x) = \mathcal{N}ewt(x)(t-d)$ then $(d \geq 0)$ (3).
or $\mathcal{N}ewt(x) = +\infty \Rightarrow x = 0$ $(d < 0)$

For $d = 0$ $\mathcal{N}ewt(x) = q\,\mathcal{N}ewt(x)$ shows (3) $x \in B^b$ and then (4) $x \in E$ ∎

II Primitive elements:

$$
\begin{tikzcd}
x = \sum_{n \geq 0} [x_n] \pi^n \arrow[r, mapsto] \arrow[dddd, mapsto] & \bar{x} = \sum_{n \geq 0} [\bar{x}_n] \pi^n \\
W_{\mathcal{O}_E}(\mathcal{O}_F) \arrow[r, two heads] \arrow[d] & W_{\mathcal{O}_E}(k_F) \\
W_{\mathcal{O}_E}(\mathcal{O}_F)/\pi & \\
\mathcal{O}_F \arrow[u, phantom, sloped, "\cong"] & \\
x_0 &
\end{tikzcd}
$$

⚠ Both rings here are domains.

Definition B: • $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ is called primitive if both projections are non zero (i.e. $x_0 \neq 0$ and $x_n \in \mathcal{O}_F^\times$ for some $n \geq 0$)

• The degree $\deg(x)$ of such an element is the smallest n s.t. $x_n \in \mathcal{O}_F^\times$.

We have: (1) $\deg(xy) = \deg(x) + \deg(y)$

(2) x primitive of deg $0 \Leftrightarrow x \in W_{\mathcal{O}_E}(\mathcal{O}_F)^\times$

(it's a special case of (4) for Newt pol)

Let x be primitive of degree 1.

Then $x = [x_0] - \pi a$, $x_0 \neq 0$, $a \in W_{\mathcal{O}_E}(\mathcal{O}_F)^{\times}$.

$$\Theta([\cdot]) : \mathcal{O}_F \xrightarrow{[\cdot]} W \xrightarrow{\Theta} A_x := W/_{xW} \quad \left(\begin{array}{c} \text{mult. con}(g) \\ \text{map} \end{array} \right)$$

Note that $W/_{\pi W}$ is a domain $\Rightarrow$ A_x is $\Theta(\pi)$-torsionfree

$$A_x/_{\Theta(\pi) A_x} \cong W/_{xW + \pi W} \cong \mathcal{O}_F/_{x_0 \mathcal{O}_F}$$

$$\Theta(g) \longmapsto g_0 \mod x_0$$

in part.

$$\Theta([a]) \longmapsto a \mod x_0$$

Therefore we have the

<u>Lemma 1</u>: For $a, b \in \mathcal{O}_F$:

(1) $\Theta([a]) = 0 \Rightarrow a = 0$ (2) $\Theta([a]) = \Theta([b]) \Rightarrow v(a) = v(b)$.

Therefore (1) $\Rightarrow$ $\Theta([\cdot])$ is an injection

(2) $v_x : A_x \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a valuation on A_x

$$\Theta([a]) \longmapsto v(a)$$

because ... (For now on, F is <u>algeb. closed</u>!)

<u>Theorem 1</u>: $\Theta([\cdot]) : \mathcal{O}_F \longrightarrow A_x$ is surjective.

"<u>Proof</u>": (1) We use the tilting equivalence:

$$\pi\text{-adic } \mathcal{O}_E\text{-algebras} \underset{W_{\mathcal{O}_E}}{\overset{R}{\rightleftarrows}} \text{perfect } \mathbb{F}_q\text{-algebras}$$

and we have: $\left(R(A) = \varprojlim_{x \mapsto x^q} A \right)$

$$\mathcal{O}_F \underset{\mathcal{O}_F \text{ perf.}}{\cong} R(\mathcal{O}_F) \cong R\left(\mathcal{O}_F/_{x_0 \mathcal{O}_F}\right) \underset{(1)}{\cong} R\left(A_x/_{\Theta(\pi) A_x}\right) \cong R(A_x)$$

$$x \longmapsto \left(\Theta[x^{q^{-n}}] \right)_{n}$$

$$\mathcal{O}_F \xrightarrow{\;\;} A_x, \quad x \longmapsto \Theta([x]) \overset{!}{=} \Theta([x])$$

(2) They then prove (via explicit computations) that every element in A_x has a u-th root of unity. (5)

<u>Corollary 1:</u> A_x is a domain and C_x is a perfectoid field for w_x with valuation ring A_x.

Proof: • A_x is a domain follows from mult and surj of $\Theta([\cdot])$

• C_x is perfectoid: $A_x/_{\Theta(\pi)A_x} \cong O_F/_{x_0 O_F}$ and $\mathbb{F}$ perf.

and $w_x(A_x) = v(O_F)$ is non discrete (F alg closed) ∎

Therefore for A_x, we have "$R = \flat$"
F is alg. closed and multiplicity of (1) shows then that $\mathbb{F}_q(\mu_\infty) \subset C_x^\flat$

$$\begin{tikzcd} C_x \arrow[d, dash] & & C_x^\flat \arrow[ll, dotted, "\text{Scholze}"'] \arrow[d, dash] \\ E(\mu_\infty) & & \mathbb{F}_q(\mu_\infty) \end{tikzcd}$$

But, under these conditions, Kummer theory and "higher ramification theory" shows

Theorem 2: • C_x is a complete alg. closed ext of E
• $C_x^\flat = F$

we also have that if L/E is perfectoid s.t. $L^\flat \cong F$

$$W_{O_E}(O_F) \cong W_{O_E}(R(O_L)) \twoheadrightarrow O_L$$

is surjective (same proof) and has kernel generated by a deg 1 primitive element; in part. $L \cong C_x$.

<u>Last remark:</u> We have the injective map (A_x is a domain)

$$|Y| = \{\text{prim deg 1 elt}\}/W^\times \longrightarrow \mathrm{Spec}(W)$$
$$x \longmapsto W_x$$