J. Kohlhaase Proétale topology II May 7, 2015

K perfectoid field, $\mathrm{char}(K) = 0$

X loc. noetherian adic over $\mathrm{Spa}(K, K^+)$

$U \in X_{\text{proét}}$, i.e. $U = (U_i)_{i \in I}$ with
- $U_i \in X_{\text{ét}}$
- $U_j \to U_i$ fin. étale surj. for "$j > i$ large"

Def: (i) U affinoid perfectoid if $U_i = \mathrm{Spa}(R_i, R_i^+)$ can be chosen s.t. $(R, R^+) := \left((\varinjlim_i R_i^+)^\wedge [\tfrac{1}{p}], (\varinjlim_i R_i^+)^\wedge \right)$ is perfectoid over (K, K^+)

(ii) U perfectoid if it has an open cover by affinoid perfectoids

Remark on (ii): $W \subseteq |U|$ open qc. $\Rightarrow$ $W = |V|$ for a unique $V \xrightarrow{\text{ét}} U$ in $X_{\text{proét}}$

Example: $X = \mathbb{T}^n := \mathrm{Spa}(K\langle T^{\pm 1}\rangle, K^+\langle T^{\pm 1}\rangle)$ with $T = (T_1, \dots, T_n)$

$\widetilde{\mathbb{T}}^n = (\mathbb{T}^n_i)_{i \in \mathbb{N}}$ where $\mathbb{T}^n_i := \mathrm{Spa}(K\langle T^{\pm 1/p^i}\rangle, K^+\langle T^{\pm 1/p^i}\rangle)$

Def: for $U \in X_{\text{proét}}$ affinoid perfectoid as in (i) above set $\hat{U} := \mathrm{Spa}(R, R^+)$

Remark: $|\hat{U}| \cong \varprojlim_i |U_i|$ & $\hat{U}$ depends on $(U_i)_i$ only up to iso.

Lemma: (i) $U \mapsto \hat{U}$ is compatible with rational subsets & finite étale maps, i.e. if $V_i \to U_i$ rational or fin. ét., $V_j := V_i \times_{U_i} U_j$ for $j \geq i$, then $V = (V_j)_j$ is affinoid perfectoid and $\hat{V} \cong V_i \times_{U_i} \hat{U}$ as adic spaces

(ii) $\hat{U}$ can be defined for a general perfectoid $U \in X_{\text{proét}}$ by glueing

(iii) $U \in X_{\text{proét}}$ perfectoid, $V \to U$ proétale $\Rightarrow$ V perfectoid

Proof: (i): go through existence proof of fiber products for perfectoid spaces

(i) $\Rightarrow$ (ii);

(iii): part (i) reduces this to $V = (V_j)_j \xrightarrow{\text{fét}} U = (U_i)_i$ – use that the category of perfectoid algebras has complete direct limits. □

Prop.: Any $U \in X_{\text{proét}}$ has a proétale covering by affinoid perfectoids in $X_{\text{proét}}$

Proof 1 (X smooth): locally (analytic top.) $X \xrightarrow{\text{ét}} \mathbb{T}^n$

$\Rightarrow$ WLOG $X = \mathbb{T}^n$;

now $\widetilde{\mathbb{T}}^n \to \mathbb{T}^n$ proétale with $\widetilde{\mathbb{T}}^n$ affinoid perfectoid

$\Rightarrow$
- $U \times_{\mathbb{T}^n} \widetilde{\mathbb{T}}^n \to \widetilde{\mathbb{T}}^n$ proétale, hence perfectoid by the lemma
- $U \times_{\mathbb{T}^n} \widetilde{\mathbb{T}}^n \to U$ is a proétale cover

to see this consider

$$\begin{tikzcd} U_j \times_{\mathbb{T}^n} \mathbb{T}^n_\ell \arrow[r, "\text{fét}", "\text{surj.}"'] \arrow[d, "\text{fét. surj.}"'] & U_i \times_{\mathbb{T}^n} \mathbb{T}^n_\ell \arrow[r] & \mathbb{T}^n_\ell \\ U_j \times_{\mathbb{T}^n} \mathbb{T}^n_k \arrow[d, "\text{surj. because } |U_j \times_{\mathbb{T}^n} \mathbb{T}^n_k| \twoheadrightarrow |U_j| \times_{|\mathbb{T}^n|} |\mathbb{T}^n_k| \twoheadrightarrow |U_j|"] & & \\ U_j & & \end{tikzcd}$$

Proof 2 (general): $X = \mathrm{Spa}(A, A^+)$ connected, K alg. closed

$A^{(1)} := A[\sqrt[p]{1 + A^{\circ\circ}}]$, $A^{(1),+} :=$ integral closure of A^+ in $A^{(1)}$

iterate $\longrightarrow$ $A^{(\infty)} = \varinjlim_n A^{(n)}$, $A^{(\infty),+} = \varinjlim_n A^{(n),+}$

set $R^+ := (A^{(\infty),+})^\wedge$, $R := R^+[\frac{1}{p}]$

Colmez: p-closed connected spectral K-Banach alg. are perfectoid

need: • $\hat{A} := \hat{A^+}[\frac{1}{p}]$ connected spectral

• $(1+\hat{A}^{\circ\circ})/(1+\hat{A}^{\circ\circ})^p \cong (1+A^{\circ\circ})/(1+A^{\circ\circ})^p$; what if A^+ is not p-adically separated, e.g. non-reduced?

Applications

1. finiteness thms. in étale cohom. (pass through $X_{\text{proét}}$ & cont. $\pi_1(X,x)$-group cohom.)

2. diamonds:

Perf := category of perfectoid spaces in char. p

big proétale site: $\{f_i : X_i \xrightarrow{\text{proét}} X\}_{i\in I}$ jointly surjective + qc. condition

here: $f_i : X_i \to X$ in Perf is proét. if (loc. on X_i) $X_i = \hat{U}$ with $U = (U_j)_j \in \text{pro-}X_{\text{ét}}$, $U_j \in$ Perf

note: no finiteness & surj. for U $\Longrightarrow$ f_i not always open + other difficulties

Def: (i) $X \in$ Perf $\longrightarrow$ $h_X := \mathrm{Hom}(\cdot, X)$ (pre) sheaf on Perf

(ii) diamond $\mathcal{D}$: sheaf on Perf with surjection $h_X \to \mathcal{D}$ and representable proétale pullbacks:

$\forall Y \in \text{Perf}\ \forall h_Y \to \mathcal{D}\ \exists Z \in \text{Perf}: h_Y \times_{\mathcal{D}} h_X \cong h_Z \xrightarrow{\text{proj} = h_f} h_Y$ and $f: Z \xrightarrow{\text{proét}} Y$

Example: • $\mathrm{Spd}\, \mathbb{Q}_p^{cyc} := h_{\mathbb{Q}_p^{cyc,\flat}} \circlearrowleft \mathbb{Z}_p^\times$

• $\mathrm{Spd}\, \mathbb{Q}_p := \mathrm{Spd}\, \mathbb{Q}_p^{cyc} / \mathbb{Z}_p^\times := \mathrm{Coeq}(\mathbb{Z}_p^\times \times \mathrm{Spd}\, \mathbb{Q}_p^{cyc} \rightrightarrows \mathrm{Spd}\, \mathbb{Q}_p^{cyc})$

get functor: $\{$analytic adic spaces over $\mathrm{Spa}\, \mathbb{Z}_p\} \longrightarrow \{$diamonds$\}$, $X \mapsto X^\diamond$,

$X^\diamond(Y) = \{(Y^\#, \iota) \mid Y^\#$ perfectoid, $Y^\# \to X$, $\iota: Y^{\#\flat} \xrightarrow{\sim} Y\}/\sim$;

Rem: $X \in \text{Perf}_K \Rightarrow X^\diamond \cong h_{X^\flat}$ on $\text{Perf}_{K^\flat}$ by tilting equivalence

Example: $(\tilde{\mathbb{D}}^*_{\mathbb{Q}_p})^\diamond / \mathbb{Z}_p^\times \cong \mathrm{Spd}\, \mathbb{Q}_p \times \mathrm{Spd}\, \mathbb{Q}_p$ coming from

$(\tilde{\mathbb{D}}^*_{\mathbb{Q}_p^{cyc}})^\flat \cong$ (Peter's talk) $\mathrm{Spa}\, \mathbb{Q}_p^{cyc,\flat} \times \mathrm{Spa}\, \mathbb{F}_p((t^{1/p^\infty})) \cong \mathrm{Spa}\, \mathbb{Q}_p^{cyc,\flat} \times \mathrm{Spa}\, \mathbb{Q}_p^{cyc,\flat}$

Def: morphism of diamonds $\mathcal{D}_1 \to \mathcal{D}_2$ is fin. étale if for all $h_Y \to \mathcal{D}_2$ there is $Z \in$ Perf with $h_Z \cong h_Y \times_{\mathcal{D}_2} \mathcal{D}_1 \xrightarrow{\text{fét}} h_Y$

$\mathcal{D}$ "connected" diamond $\rightsquigarrow$ Galois cat. $\mathcal{D}_{\text{fét}}$ $\longrightarrow$ étale fundamental grp. $\pi_1(\mathcal{D})$