Pro-étale Topology I

<u>Question</u>

What does Weinstein's result have to do with fundamental groups?

<u>Rough idea</u>

- By the general approach of SGA 1 we can attach a fundamental group to $X_{\text{fét}}$ for any locally noetherian connected adic space X, including $\tilde{H}_C^{\mathrm{ad}} \setminus \{0\}$, a perfectoid space over the algebraically closed perfectoid field C, H_C the Lubin-Tate formal group for E
- The category $Z_{\text{fét}}$ for $Z = \tilde{H}_C^{\mathrm{ad}} \setminus \{0\}/E^\times$ will correspond to $E^\times$-$(\tilde{H}_C^{\mathrm{ad}} \setminus \{0\})_{\text{fét}}$
- Problem: Z is not an adic space, so we have to enlarge the category of perfectoid spaces
- Weinstein:
  - Z may be seen as a sheaf on Perf_C with respect to the pro-étale topology
  - This topology was defined in Scholze's p-adic Hodge theory paper

This is a blunt lie!

- Truth: A rough sketch of the definition only exists in Scholze's lecture notes. The approach differs very much from the Hodge theory article.

<u>Aims</u>

- Recall SGA 1
- Introduce pro-étale topology as in the Hodge theory article
- Compare it to the lecture notes version.

1. Fundamental groups

<u>Definition</u>

A Galois category is a category C with a functor $F: C \longrightarrow \mathrm{Sets}$ (fibre functor) such that

(G1) C has all finite fibre products
(G2) C has finite coproducts and quotients by finite groups of automorphisms
(G3) Every morphism factors as $X \xrightarrow{u'} Y' \xrightarrow{u''} Y$ such that

(a) u' is a strict epimorphism: for all Z

$$\mathrm{Hom}(Y', Z) \xrightarrow{\sim} \{ f \in \mathrm{Hom}(X, Z) \mid f \circ p_1 = f \circ p_2,\ X \times_Y X \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \}$$

(b) u'' is an isomorphism with a component of Y.

(G4) $\mathcal{F}$ is left exact (i.e. preserves right units and fibre products)
(G5) $\mathcal{F}$ preserves finite coproducts and quotients by finite groups of automorphisms
(G6) $\mathcal{F}$ is conservative: If $\mathcal{F}(u)$ is an isomorphism, then also u.

<u>Consequences</u>

(i) $\mathcal{F}$ is strictly pro-representable by Galois objects: There exists a filtered system (a fundamental pro-object)

$$P = (P_i)_{i \in I} \in \text{pro-}C$$

and elements $p_i \in \mathcal{F}(P_i)$ such that

(a) all transition maps ϕ_{ij} are epimorphisms
(b) $p_j = \mathcal{F}(\phi_{ij})(p_i)$
(c) $\mathrm{Aut}(P_i) \cong \mathcal{F}(P_i) \quad \sigma \mapsto \mathcal{F}(\sigma)(p_i)$
(d) $\varinjlim_{I} \mathrm{Hom}(P_i, Z) \xrightarrow{\sim} \mathcal{F}(Z) \quad P_i \xrightarrow{u} Z \mapsto \mathcal{F}(u)(p_i)$

(ii) Set $\pi = \mathrm{Hom}(P,P) = \varprojlim \mathrm{Aut}(P_i)$ (the fundamental group of C)
Then π acts on $\mathcal{F}(Z)$ for all $Z \in C$ and

$$\mathcal{F}: C \longrightarrow \pi\text{-fsets}$$

is an equivalence

(iii) $\mathcal{F}$ extends to an equivalence $\mathcal{F}: \text{pro-}C \longrightarrow \pi\text{-pfsets}$

<u>Example</u>

X a connected, loc. noeth. adic space, $x = \mathrm{Spa}(K, K^+) \longrightarrow X$ geometric point of X, $C = X_{f\acute{e}t}$, $\mathcal{F}(Z) = \mathrm{Hom}_X(x, Z)$

2. Sites

Let C be a category with all finite fibre products

<u>Definition</u>

A pretopology on C is given by a family of coverings $\mathrm{Cov}(Z)$ for each object $Z \in C$ such that

(P1) $Z \xrightarrow{\sim} Z$ is in $\mathrm{Cov}(Z)$
(P2) Stability under fibre products
(P3) Stability under composition

C together with a pretopology is a site. The associated topos is the category of sheaves on this site.

Example

π-pfsets with coverings $(f_i : U_i \to Z)_{i \in I}$ such that f_i is open, $Z = \bigcup_{i \in I} f_i(U_i)$.

Note

Any open $f : U \to Z$ factors as $U \overset{f'}{\twoheadrightarrow} Z' \subset Z$ where f' is open surjection and $Z' \subset Z$ is open and closed.

Lemma

$f : U \to Z$ is an open surjection in π-pfsets if and only if there exists $(f_i : U_i \to Z_i)$ such that $U_i, Z_i \in \pi$-fsets and

(i) $f_0 : U_0 \to Z_0$ is surjection
(ii) $U_j \twoheadrightarrow U_i \times_{Z_i} Z_j$ is surjective for $j \leq i$

(iii) $Z = \varprojlim Z_i \quad U = \varprojlim_{i,j} U_i \times_{Z_i} Z_j \quad f = \varprojlim f_i$

Proof

(⇐) $\begin{tikzcd} \varprojlim_i U_i \arrow[r, "(ii)(iii)"] \arrow[rr, bend right, "id"'] & U \arrow[r] & \varprojlim U_j \end{tikzcd} \Rightarrow U = \varprojlim U_j$

By (i), (iii) and since surjective maps in π-pfsets are universally surjective:

$U_i \times_{Z_i} Z \twoheadrightarrow Z$

Since $U_i \times_{Z_i} Z = \bigsqcup_{u \in U_i} \varphi_i^{-1}(f_i(u))$, $\varphi_i : Z \to Z_i$, this is clearly an open map.

$U = \varprojlim U_j \xrightarrow{(ii)} \varprojlim_j U_i \times_{Z_i} Z_j = U_i \times_{Z_i} Z$

Since $U = \varprojlim_i U_i \times_{Z_i} Z$, this map is open, as well.

$\rightsquigarrow f : U \twoheadrightarrow U_i \times_{Z_i} Z \twoheadrightarrow Z$ is open surjective.

(⇒) - Let A be the set of partitions of U into finitely many open and closed subsets compatible with the π-operation

$\rightsquigarrow U = \varprojlim_{P \in A} P$

- Let $P \in A \rightsquigarrow \{f(x) \mid x \in P\}$ is an open and closed cover of Z which may be refined into a partition by taking all possible intersections and complements. Call this partition Z_P (it is still compatible with the π-operation)

- Set $U_P = \{x \cap f^{-1}(y) \mid x \in P,\ y \in Z_P\ y \subset f(x)\}$, the common refinement of P and $f^{-1}(Z_P)$. Note that $y \subset f(x)$ holds by construction if and only if $x \cap f^{-1}(y) \neq \emptyset$.

$\rightsquigarrow U_P \to Z_P \quad x \cap f^{-1}(y) \mapsto y$

- If $Q \leq P$ then clearly, $Z_Q \leq Z_P$ and $U_Q \leq U_P$. Moreover,

$$U_Q \twoheadrightarrow U_P \times_{Z_P} Z_Q = \{(x \cap f^{-1}(y), z) \mid x \in P, y \in Z_P, z \in Z_Q, z \subset y \subset f(x)\}$$
$$u \cap f^{-1}(z) \mapsto (x \cap f^{-1}(y), z) \text{ with } x, y \text{ determined by } u \subset x,\ z \subset y$$

- $(U_P)_{P \in A}$ and $(Z_P)_{P \in A}$ are clearly cofinal among all finite partitions of U and Z

$$\rightsquigarrow Z = \varprojlim Z_P \qquad U = \varprojlim U_P = \varprojlim_{P,Q} U_P \times_{Z_P} Z_Q.$$

qed.

<u>Consequence</u>

For all Galois categories (C, F) we get a pretopology on pro-C with coverings $(U_k \xrightarrow{f_k} Z)$ such that

(i) (f_k) is an epimorphic family in pro-C

(ii) For each f_k there exists $(f_{ki} : U_{ki} \to Z_i)$ $U_{ki}, Z_i \in C$ such that

$$U_j \twoheadrightarrow U_i \times_{Z_i} Z_j \text{ for } j \leq i \in O, \quad Z = \varprojlim Z_i \quad U = \varprojlim_{i,j} U_j \times_{Z_j} Z_i$$

<u>Remark</u>

- A conservative family of points for π-pfsets is given by sets with a free π-action, i.e. sets of the form $\pi \times X$ with trivial π-action on X
- Useful for computing continuous group cohomology: For any topological π-module M set

$$\mathcal{F}_M(S) = \mathrm{Hom}_{cont,\pi}(S, M) \rightsquigarrow H^i(pt, \mathcal{F}_M) = H^i_{cont}(\pi, M)$$

3. Pro-étale morphisms – version 2012

<u>Idea</u>

Extend the site pro-$X_{fét}$ to some parts of pro-$X_{ét}$.

<u>Here</u>: $X_{ét}$ étale site of a locally noetherian adic space X

<u>Essential Properties</u>

(i) étale maps are open with finite fibres, stable under base change and compositions

(ii) $|U \times_V W| \twoheadrightarrow |U| \times_{|V|} |W|$ (with finite fibres)

(iii) $f : X \to Y$ étale. Equivalent

(a) f is an epimorphism in $X_{ét}$

(b) f is a universal strict epimorphism in $X_{ét}$

(c) $|f| : |X| \to |Y|$ is surjective

(iv) the diagonal is open and locally closed
(v) locally, X is connected.

Notation

For $U = (U_i)$ a pro-$X_{ét}$ set $|U| = \varprojlim |U_i|$

Definition

Let $U, V \in$ pro-$X_{ét}$, $f : U \to V$

(i) f is (finite) étale if $U = U_0 \times_{V_0} V \to V$ for $U_0 \to V_0$ (finite) étale $U_0, V_0 \in X_{ét}$.

(ii) f is pro-étale $U = \varprojlim U_i$ with $U_i \in$ pro-$X_{ét}$ étale over V and $U_i \to U_j$ finite étale, $|U_i| \to |U_j|$ surjective for $i \leq j \leq 0$

(iii) $X_{proét}$ is the full subcategory of pro-$X_{ét}$ of objects pro-étale over X.
A covering is a family $(f_i : U_i \to U)$ such that

(i) f_i is pro-étale
(ii) $|U| = \bigcup_i f_i(|U_i|)$

Properties

(i) the base change of a (finite/pro-) étale map is (finite/pro-) étale and

$$|U \times_V W| \twoheadrightarrow |U| \times_{|V|} |W|$$

(use that the fibres of $|U_i \times_{V_i} W_i| \to |U_i| \times_{|V_i|} |W_i|$ are finite)

(ii) Composition of (finite) étale maps are (finite) étale

(iii) Every quasi-compact open subset W of $|U|$ is the image of an étale map. If $U \in X_{proét}$, there exists a minimal étale map in $X_{proét}$ with image W.

(iv) Any pro-étale map is open.

Proof

Write $U = \varprojlim U_i \to V$. $W \subset |U|$ open. Wlog W quasi-compact and $W = \varphi_i^{-1}(W_i)$ with $\varphi_i : |U| \to |U_i|$ surjective, $W_i \in |U_i|$ open and quasi-compact, $U_i \to V$ étale.

Write W_i as the image of an étale map according to (iii) $\rightsquigarrow$ $Y \to V$ étale with same image.

Write $Y = Y_0 \times_{V_0} V \to V$

$\rightsquigarrow$ $|Y| \twoheadrightarrow |Y_0| \times_{|V_0|} |V| \xrightarrow{\text{open}} |V|$

$\rightsquigarrow$ the image of Y is open.

(v) Any surjective (finite) étale map $U \to V$ with $V \in X_{proét}$ is the base change of an epimorphism in $X_{ét}$.

(vi) if $U \xrightarrow{f} V \xrightarrow{g} W$ is the composition of two pro-étale morphisms with W in $X_{proét}$ then $U, V \in X_{proét}$ and $g \circ f$ is proétale

<u>Proof</u>

Case (a) f étale. With $V = \varprojlim V_i \to W$

$$\rightsquigarrow U = U_0 \times_{V_0} V = \varprojlim U_0 \times_{V_0} V_i \to W \text{ is pro-étale}$$

Case (b) f profinite étale surjective : clear.

(vii) All finite limits exist in $X_{proét}$

<u>Proof</u>

Finite products: see (i)
Equalizers:

We show: $U \subset V = (V_i)$ intersection of open and closed subsets $\Rightarrow U$ is in $X_{proét}$
Apply this to the intersection of two graphs of morphisms.

Write $U = (U_i)$ with U_i image of U in V_i. Assume X is connected
Then each V_i has only finitely many components $\Rightarrow U_i$ is open and closed and $U_j \to U_i$ is finite étale surjective for $0 \le i \le 0$.

(viii) If X is connected, then pro-$X_{fét} \to X_{proét}$ is a morphism of sites.

(ix) The objects $(U_i) \in X_{proét}$ with U_i affinoid are quasi-compact for the pro-étale topology. These objects form a generating family.

(x) If $|X|$ is coherent, then so is $X_{proét}$

(xi) $X_{proét}$ has enough points, given by profinite covers of geometric points