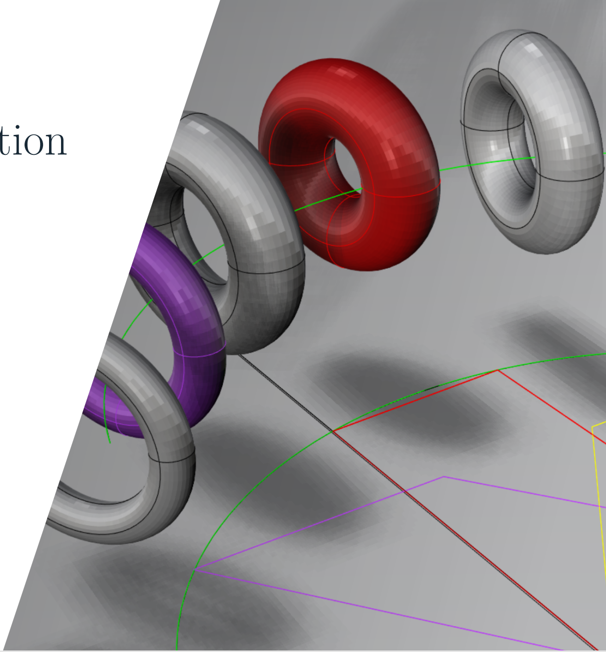
$\mathfrak{g}$-Opers and the Hitchin Section (Part1)



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Talk in JSG, June 17, 2020

Set Up

- Fix a compact Riemann surface $(X, \mathcal{O}_X)$ of genus $g \geq 2$, where $\mathcal{O}_X$ is the sheaf of germs of holomorphic functions. I.e. we see X as the analytification of a complex projective algebraic variety.

Abelian Hodge Theory

Classify flat line bundles over X:

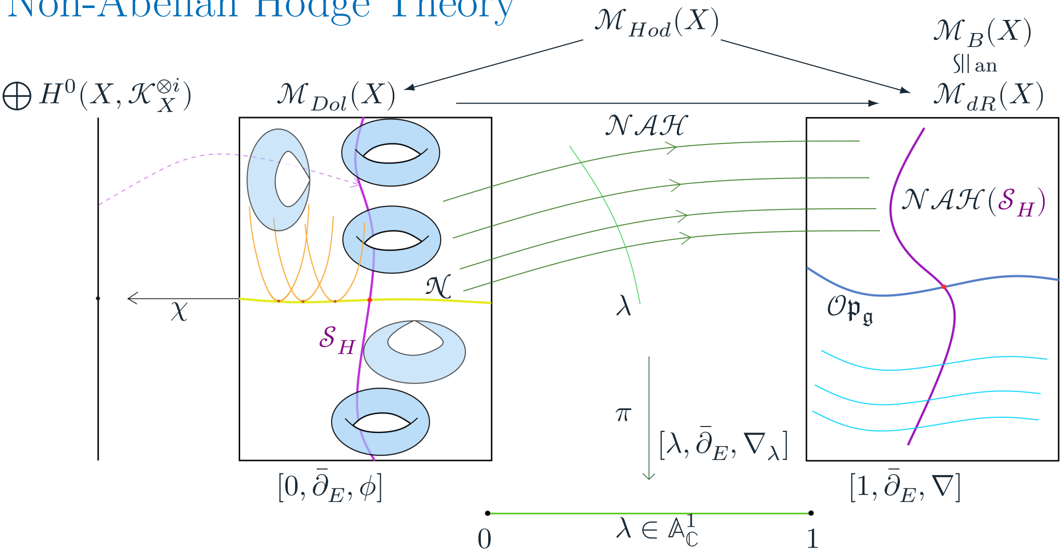
$$H^1(X,\mathbb{C}) \cong H^0(X,\mathcal{K}_X) \oplus H^1(X,\mathcal{O}_X)$$

and moreover:

$$H^1(X,\mathbb{C}^*) \underset{\text{not hol}}{\cong} H^0(X,\mathcal{K}_X) \times Pic^0(X)$$

We describe $GL_1(\mathbb{C})$- local systems, by a holomorphic line bundle and a holomorphic 1-form.
Remark: By GAGA, classifying holomorphic bundles on the Riemann surface is equivalent to the classification of algebraic bundles on the corresponding projective variety/$\mathbb{C}$.

Non-Abelian Hodge Theory



Moduli

Higgs Bundles and the Dolbeault Side

$\mathcal{H}(X) := \{(\bar{\partial}_E, \phi) |\ \bar{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E);\ \phi \in H^0(X, \mathcal{E}nd(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{K}_X)\}$

- The choice of $\bar{\partial}_E$ for a complex bundle E fixes the holomorphic structure on the bundle. Notation: $\mathcal{E} \leftrightarrow (\bar{\partial}_E, E) \leftrightarrow \bar{\partial}_E$

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- $\mathcal{H}(X) \subset \Omega^{0,1}(\mathcal{E}nd(\mathcal{E})) \oplus \Omega^{1,0}(\mathcal{E}nd(\mathcal{E}))$

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- Higgs field $\phi : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is a holomorphic ($\bar{\partial}_E \phi = 0$), $\mathcal{O}_X$-linear morphism.
- $\mathcal{H}(X) \subset \Omega^{0,1}(\mathcal{E}nd(\mathcal{E})) \oplus \Omega^{1,0}(\mathcal{E}nd(\mathcal{E}))$
- For simplicity we restrict to:

$$\bigwedge^n \mathcal{E} \cong \mathcal{O}_X$$

and $tr(\phi) = 0$, i.e. $\mathrm{SL}_n\mathbb{C}$- Higgs bundles.

Higgs Bundles and the Dolbeaut Side

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- Reduce $\mathcal{H}$ to the subspace of polystable Higgs bundles. These have closed orbits under the action of the gauge group.

$$\text{Stability: } (\mathcal{E}, \phi) \text{ is stable} :\Longleftrightarrow \frac{deg(\mathcal{F})}{rank(\mathcal{F})} < \frac{deg(\mathcal{E})}{rank(\mathcal{E})};\ \forall \mathcal{F} <_{\phi\text{-inv.}} \mathcal{E}$$

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- The resulting quotient is the moduli space of Higgs Bundles:

$$\mathcal{M}_{Dol}(X) := \mathcal{H}^{ps}(X) \big/ \mathcal{G}(E) \supset \mathcal{H}^{st}(X) \big/ \mathcal{G}(E) =: \mathcal{M}^{sm}_{Dol}(X) \supset \mathcal{T}^*\mathcal{B}un^{st}(X)$$

$$\mathcal{M}_{Dol}(X) \underset{\substack{open\\dense}}{\supset} \mathcal{T}^*\mathcal{B}un^{st}(X)$$

Higgs Bundles and the Dolbeaut Side

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Higgs Bundles and the Dolbeaut Side

- $\mathcal{M}_{Dol}(X)$ is a complex analytic space/ normal quasi-projective variety$/_{\mathbb{C}}$. In addition to the $\mathbb{C}$- structure I, it carries a symplectic structure ω_I.
- There is a $\mathbb{C}^*$-action on $\mathcal{M}_{Dol}(X)$:

$$\lambda \cdot (\bar{\partial}_E, \phi) \longmapsto (\bar{\partial}_E, \lambda\phi);\ \lambda \in \mathbb{C}^*$$

Hitchin Map

- There is a proper holomorphic map:

$$\mathcal{M}_{Dol}(X) \longrightarrow \mathcal{B} := \bigoplus H^0(X, K^{\otimes i})$$
$$(\mathcal{E}, \phi) \longmapsto (tr(\phi^2), \dots, tr(\phi^n))$$

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- We will be interested is a special section of this map:

$$\mathcal{S}_H : \mathcal{B} \longrightarrow \mathcal{M}_{Dol}(X)$$
$$(q_2, \dots, q_n) \longmapsto (\mathcal{K}_X^{\frac{n-1}{2}} \oplus \mathcal{K}_X^{\frac{n-3}{2}} \oplus \dots \oplus \mathcal{K}_X^{\frac{1-n}{2}}, \begin{bmatrix} 0 & q_2 & \cdots & q_n \\ 1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & q_2 \\ & & 1 & 0 \end{bmatrix})$$

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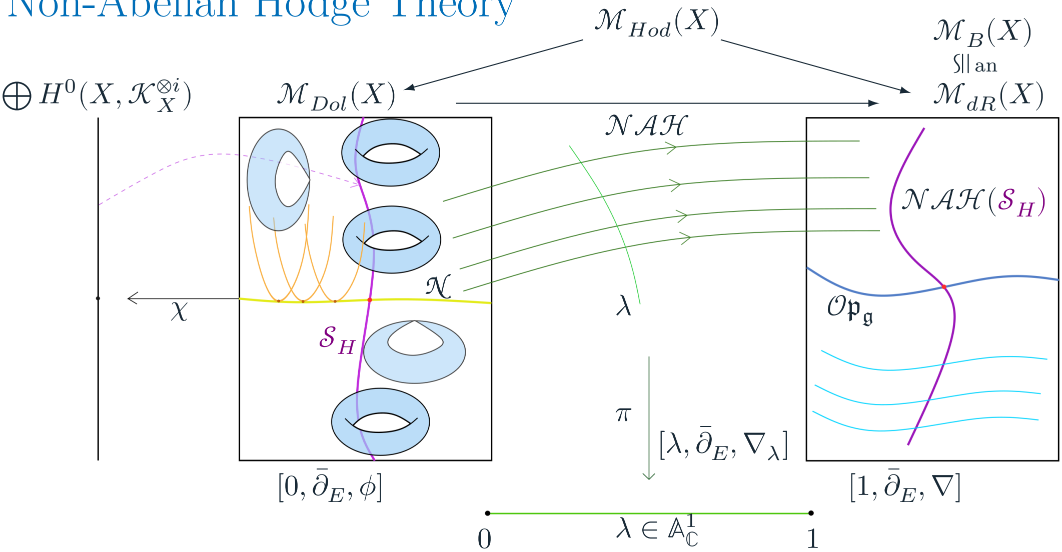
- Involves the choice of a $\mathcal{K}_X^{\frac{1}{2}} \in \mathcal{S}pin(X)$ where $\mathcal{S}pin(X)$ is affine over $\mathcal{J}ac(X)[2]$.

Non-Abelian Hodge Theory



Theorem: Donaldson/ Narasimhan Seshadri

The Higgs bundle $(\mathcal{E}, \phi)$ is polystable as a Higgs bundle, i.e. $(\mathcal{E}, \phi) \in \mathcal{M}_{Dol}(X)$.

$$\Longleftrightarrow$$

There exists a hermitian metric h on E, with associated Chern connection $\mathcal{D}_h$ with $\mathcal{D}_h^{0,1} = \bar{\partial}_E$, that solves the following system of non-linear PDE's:

$$\begin{aligned} F_{\mathcal{D}_h} + [\phi, \phi^{\dagger_h}] &= 0 \\ \mathcal{D}_h^{0,1}\phi &= 0. \end{aligned} \qquad \text{Hitchin's Self-Duality Equations}$$

Moduli

Moduli Space of Flat connections/ the de Rham Side

- $\mathcal{D} := \{(\mathcal{V}, \nabla) | \nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathbb{C}} K_X \text{ holomorphic connection}\}$

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- $\mathcal{M}_{dR}(X) = \mathcal{D}^{cr} \big/ \mathcal{G}$

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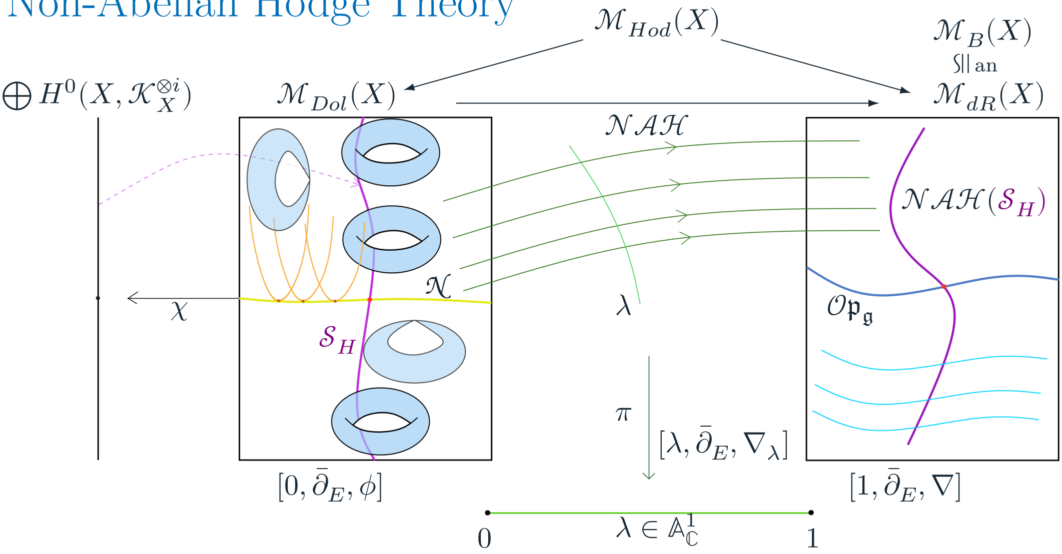
- $\mathcal{D}^{cr} := \{(\mathcal{V}, \nabla) \in \mathcal{D} | \ \mathcal{W} < \mathcal{V} \ \nabla\text{-inv. has a } \nabla\text{-inv. complement } \}$
- $\mathcal{M}_{dR}(X) = \mathcal{D}^{cr} \big/ \mathcal{G}$
- Is a Stein space and carries a symplectic structure, which is the pullback of the Atiyah-Bott-Goldman via the Riemann-Hilbert correspondence.

Non-Abelian Hodge Theory



λ-Connections/ the stuff in between, (in and around)/ 'The Hodge Span'

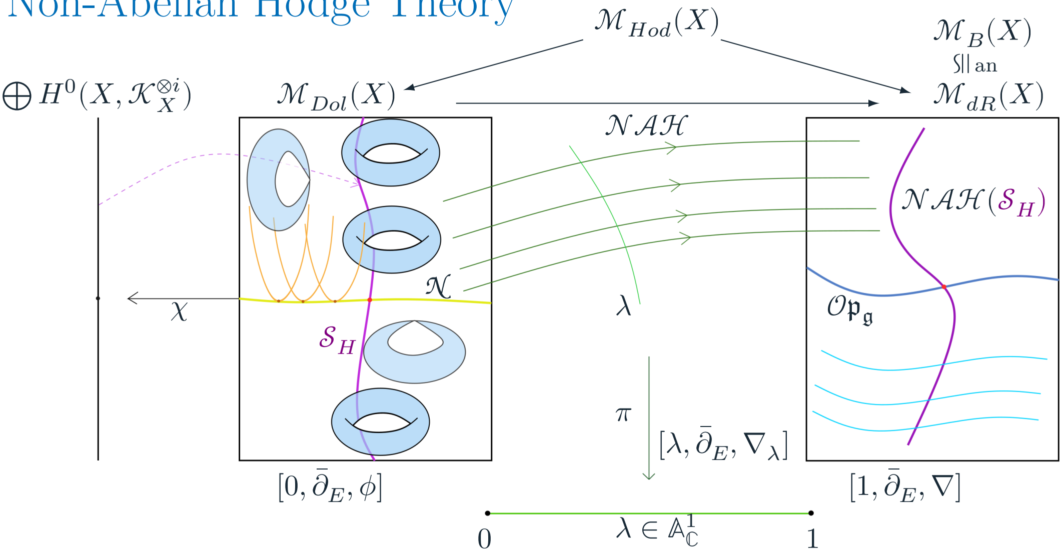
$\mathcal{M}_{Hod}(X) := \{(\lambda, \bar{\partial}_E, \nabla_\lambda) |\ \nabla_\lambda : \Omega^0(E) \to \Omega^{1,0}(E)\} \big/ \mathcal{G}$

- $(E, \bar{\partial}_E)$ is a homolorphic vector bundle on X,
- $\nabla_\lambda(fs) = f\nabla(s) + \lambda \cdot \partial f \otimes s;$ with $s \in \Omega^0(E),\ f \in C^\infty(E,)$.
- $[\nabla_\lambda, \bar{\partial}_E] = 0$.
- $\lambda \in \mathbb{C}$, defines a tautological map onto the affine line, also denoted by λ:

$$\lambda : \mathcal{M}_{Hod}(X) \longrightarrow \mathbb{A}^1_{\mathbb{C}},$$
$$[\lambda, \bar{\partial}_E, \nabla_\lambda] \longmapsto \lambda$$

  with fibers: $\lambda^{-1}(0) = \mathcal{M}_{Dol}(X)$ and $\lambda^{-1}(1) = \mathcal{M}_{dR}(X)$
  making $\mathcal{M}_{Hod}(X)$ into the NAH-span from $\mathcal{M}_{Dol}(X)$ to $\mathcal{M}_{dR}(X)$.
- the $\mathbb{C}^*$-action on $\mathcal{M}_{Dol}(X)$ extends on $\mathcal{M}_{Hod}(X)$, which preserves only the $\lambda^{-1}(0)$-fiber.

Non-Abelian Hodge Theory



Non-Abelian Hodge Correspondence

- Hitchin's equations are equivalent to the integrability of the following family of λ-connections:

$$\mathcal{D}_\lambda = \lambda^{-1}\phi + \mathcal{D}_h + \lambda\phi^{\dagger_h} \qquad \text{Deligne's twistor family}$$
$$\mathcal{D}_\lambda^2 = 0 \ \ \forall \lambda \in \mathbb{C}^*.$$

- Further, specializing the family to the fiber over $\lambda = 1$, we obtain a holomorphic flat connection, i.e. a point $(\mathcal{V}, \nabla) \in \mathcal{M}_{dR}(X)$:

$$(\mathcal{V} = (E, \mathcal{D}_1^{0,1}), \nabla = \mathcal{D}_1^{1,0}) \in \mathcal{M}_{dR}(X)$$

- Summarizing, the NAH correspondence is the map:

$$\mathcal{M}_{Dol}(X) \ni (\mathcal{E}, \phi) \mapsto (E, h, \mathcal{D}_h), \phi) \mapsto (E, \mathcal{D}_\lambda) \mapsto ((E, \mathcal{D}^{0,1}_{\lambda=1}), \mathcal{D}^{1,0}_{\lambda=1}) \in \mathcal{M}_{dR}(X)$$

Definition: $\mathrm{SL}_n\mathbb{C}$-Opers à la Beilinson-Drinfel'd

An a $\mathrm{SL}_n\mathbb{C}$-Oper on the curve X, is a triple $(\mathcal{E}, \nabla, \mathcal{F}^\bullet)$:

- $\mathcal{E}$ is a holomorphic bundle,
- ∇ is a holomorphic connection,
- $\mathcal{F}^\bullet$ is a complete holomorphic filtration of $\mathcal{E}$,

such that:

1. The filtration is Griffiths-transverse w.r.t. ∇:

$$\nabla_| : \mathcal{F}^n \hookrightarrow \mathcal{F}^{n+1} \otimes_{\mathcal{O}_X} \mathcal{K}_X, \quad (\mathbb{C}\text{-linear})$$

2. the induced map on the quotients:

$$\bar{\nabla} : \mathcal{F}^n / \mathcal{F}^{n-1} \longrightarrow \mathcal{F}^{n+1} / \mathcal{F}^n \otimes_{\mathcal{O}_X} \mathcal{K}_X, \quad (\mathcal{O}_X\text{-linear})$$

is an $\mathcal{O}_X$-linear isomorphism (i.e. the identity in $\mathcal{P}ic(X)$).

Baroque-Opers: R.C. Gunning was the first to take a shot at defining Opers:

An Oper on an projective algebraic curve is a globally defined differential operator of order-n, acting on $\mathcal{K}_X^{\frac{1-n}{2}}$.

I.e. define globally a higher order differential equation.
The higher order differential is not globally defined on a generic coordinate system (unlike the de Rham differential).

Remark

The Beilinson-Drinfel'd picture generalizes Gunning's, by interpreting Opers as cyclic $\mathcal{D}_X$-Modules.

The 2nd-order situation:

$$\hbar \frac{d^2}{dz^2}\psi(z) = q(z)\psi(z) \iff dz^2\left[\hbar\frac{d^2}{dz^2} - q(z)\right]\psi(z) = 0$$

Opers Vs Complex Projective Structures

- A simple computation shows that the condition for the last equation to be globally defined is that the coordinate change functions are Möbius transformations:

$$z_\alpha = \frac{a_{\alpha\beta} z_\beta + b_{\alpha\beta}}{c_{\alpha\beta} z_\beta + d_{\alpha\beta}}; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Aut}^\infty(\mathbb{H})$$

  as the obstruction to gluing/integr. takes the form of a Schwarzian derivative.
- The uniformization theorem for compact Riemann surfaces provides us with such a coordinate system.

Uniformizing $\mathbb{CP}^1$-Structure

- Uniformization: $\exists \rho : \pi_1(X) \to \mathrm{SL}_2\mathbb{R}$ such that: $X \cong \mathbb{H} \big/ \rho(\pi_1(X))$. Then the covering map $\mathbb{H} \to X$ endows X with a $\mathbb{P}\mathrm{SL}_2\mathbb{R}$- coordinate system.
- In this coordinate system we obtain 1-cocycles for the canonical sheaf and its square roots:

$$\mathcal{K}_X = \left\{ \frac{dz_\alpha}{dz_\beta} = (c_{\alpha\beta} z_\beta + d_{\alpha\beta})^2 \right\}_{\alpha\beta}$$

$$\mathcal{K}_X^{\frac{1}{2}} = \left\{ \xi_{\alpha\beta} := \sqrt{\frac{dz_\alpha}{dz_\beta}} \right\}_{\alpha\beta}$$

$$\mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}} = \left\{ \begin{bmatrix} \xi_{\alpha\beta} & 0 \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} \right\}_{\alpha\beta}$$

The associated connection to the equation

So we have: $dz^2\left[\hbar\frac{d^2}{dz^2}-q(z)\right]\psi(z)=0$ globally defined on $\{z_\alpha\}_\alpha$ with:

- $q_\alpha dz_\alpha^2 = q_\beta dz_\beta^2 \implies q \in H^0(X, \mathcal{K}_X^2)$
- $\psi_\alpha = \xi_{\alpha\beta}^{-1}\psi_\beta \implies \psi \in H^0(\mathcal{K}^{-\frac{1}{2}}) \implies d\psi \in H^0(\mathcal{K}^{\frac{1}{2}})$

We can now rewrite our equation as follows:

$$\frac{d}{dz_\alpha}\begin{bmatrix}-\hbar\psi_\alpha' \\ \psi_\alpha\end{bmatrix} = \begin{bmatrix}-\hbar\psi_\alpha'' \\ \psi_\alpha'\end{bmatrix} = \begin{bmatrix}-q_\alpha\psi_\alpha \\ \psi_\alpha'\end{bmatrix} = \begin{bmatrix}0 & q_\alpha \\ -1 & 0\end{bmatrix}\begin{bmatrix}-\psi_\alpha' \\ \psi_\alpha\end{bmatrix}$$

$$\implies \left(d + \frac{1}{\hbar}\begin{bmatrix}0 & q_\alpha \\ -1 & 0\end{bmatrix}\cdot dz_\alpha\right)\begin{bmatrix}-\psi_\alpha' \\ \psi_\alpha\end{bmatrix} = 0$$

The associated connection to the equation

So we have:

- $\nabla_\hbar \begin{bmatrix} -\psi'_\alpha \\ \psi_\alpha \end{bmatrix} = 0$
- For $\nabla_\hbar := d + \frac{1}{\hbar}\phi_q$ and $\phi_q := \begin{bmatrix} 0 & q_\alpha \\ -1 & 0 \end{bmatrix} dz_a \in H^0(\mathcal{E}nd(\mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}) \otimes_{\mathcal{O}_X} \mathcal{K}_X)$.
- I.e. ϕ_q is a Higgs field, and the connection-matrix in $\nabla_\hbar$.
- BUT $\nabla_\hbar$ is not a connection on the holomorphic bundle: $\mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}$ as the connection matrix ϕ_q does not gauge properly:

$$\begin{bmatrix} \xi_{\alpha\beta} & 0 \\ 0 & \xi_{\alpha\beta}^{-1} \end{bmatrix} (\phi_q)_\beta \begin{bmatrix} \xi_{\alpha\beta}^{-1} & 0 \\ 0 & \xi_{\alpha\beta} \end{bmatrix} = (\phi_q)_\alpha$$

The associated Oper to the equation

On which bundle does $\nabla_\hbar$ live?

- The answer comes from interpreting $\hbar$ as an extension-class:

$$\hbar \in \mathbb{C} \cong Ext^1(\mathcal{K}_X^{-\frac{1}{2}}, \mathcal{K}_X^{\frac{1}{2}}) \cong H^1(X, \mathcal{K}_X) \cong H^0(X, \mathcal{O}_X)$$

- The holomorphic bundle on which $\nabla_\hbar$ live is:

$$0 \longrightarrow \mathcal{K}_X^{\frac{1}{2}} \longrightarrow \mathcal{V}_\hbar \longrightarrow \mathcal{K}_X^{-\frac{1}{2}} \longrightarrow 0$$

given explicitely by the cocyle:

$$g^\hbar_{\alpha\beta} := \begin{bmatrix} \xi_{\alpha\beta}^{-1} & \hbar \frac{d\xi_{\alpha\beta}}{dz_\beta} \\ 0 & \xi_{\alpha\beta} \end{bmatrix} ; \; g^\hbar_{\alpha\beta} \cdot \nabla^\hbar_\beta \cdot g^{\hbar-1}_{\alpha\beta} = \nabla^\hbar_\alpha .$$

The associated Oper to the equation

- $\mathcal{V}_{\hbar=0} \cong \mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}$
- $\mathcal{V}_{\hbar} \cong \mathcal{V}_1 \ \forall \hbar \neq 0$

Theorem: R.C.Gunning

For an point in the Hitchin section $(\mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}, \phi_q) \in \mathcal{S}_H^{\mathfrak{sl}_2}$

- The pairs $(\mathcal{V}_{\hbar}, \nabla^{\hbar}) \ \forall \hbar \in \mathbb{C}^*$ constructed above, are an 1-parameter family of SL_2- Opers,
- In particular :

$$\hbar = 0: \ (\mathcal{K}_X^{\frac{1}{2}} \oplus \mathcal{K}_X^{-\frac{1}{2}}, \phi_q) \in \mathcal{S}_H^{\mathfrak{sl}_2} \subset \mathcal{M}_{Dol}(X)$$
$$\hbar = 1: \ (\mathcal{V}_1, \nabla^1) \in \mathcal{O}\mathfrak{p}_{\mathfrak{sl}_2} \subset \mathcal{M}_{dR}(X)$$

Conformal Limit

Introduce a real parameter: $R \in \mathbb{R}^+$ in Hitchin's equations, by rescaling the Higgs pair/ rescaling the compactification circle :

$$(\mathcal{E}, \phi) \text{ stable } \to (\mathcal{E}, R\phi) \text{ stable.}$$

$$\Longrightarrow F_{\mathcal{D}_h} + R^2[\phi, \phi^{\dagger_h}] = 0$$
$$\mathcal{D}_h^{0,1}\phi = 0.$$

thereby 'twisting' the twistor family:

$$\mathcal{D}_{R,\lambda} = \frac{R}{\lambda}\phi + \mathcal{D}_h + R\lambda\phi^{\dagger_h}$$
$$\mathcal{D}_{R,\lambda}^2 = 0 \quad \forall \lambda \in \mathbb{C}^*.$$

The Gaiotto Conjecture

$$\mathcal{D}_{R,\lambda} = \frac{R}{\lambda}\phi + \mathcal{D}_h + R\lambda\phi^{\dagger_h} \underset{\hbar=\frac{\lambda}{R}}{\Longrightarrow} \mathcal{D}_{R,\hbar} = \hbar^{-1}\phi + \mathcal{D}_h + \hbar R^2\phi^{\dagger_h}$$

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1. $\lim\limits_{\substack{R,\lambda\to 0 \\ \hbar=\frac{\lambda}{R}\text{ fixed}}} \mathcal{D}_{R,\lambda}$ exists and is holomorphic.
2. If $(\mathcal{E}, \phi_q) = \mathcal{S}_H(q)$, then for $\mathcal{D}_{R,\lambda,q} = \frac{R}{\lambda}\phi_q + \mathcal{D}_h + R\lambda\phi_q^{\dagger_h}$ we have

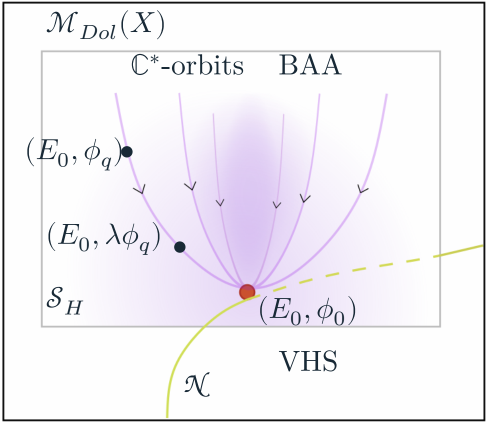
$$\lim_{\substack{R,\lambda\to 0 \\ \hbar=\frac{\lambda}{R}\text{ fixed}}} \mathcal{D}_{R,\lambda} = \lim_{\substack{R,\lambda\to 0 \\ \hbar=\frac{\lambda}{R}\text{ fixed}}} \frac{R}{\lambda}\phi_q + \mathcal{D}_h + R\lambda\phi_q^{\dagger_h}$$

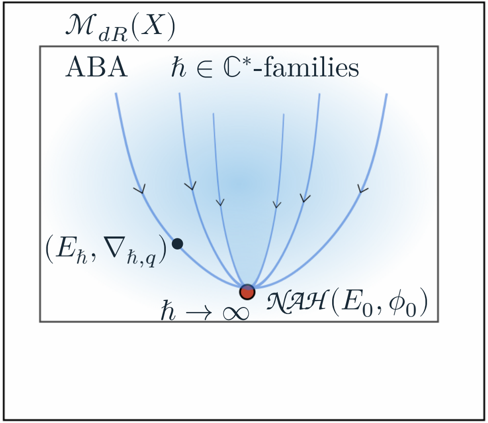
$$= \hbar^{-1}\phi_q + \mathcal{D}_{h\#} + \hbar\phi_0^{\dagger_{h\#}} = \nabla_{0,\hbar,q}$$

$$_{gauge} \sim d + \hbar^{-1}\phi_q$$

The resulting correspondence:

$$\mathcal{M}_{Dol}(X) \supset \mathbb{C}^*\text{-orbits in } \mathcal{S}_H \longrightarrow \mathbb{C}^*\text{-families in } \mathcal{O}\mathfrak{p}_{\mathfrak{sl}_n} \subset \mathcal{M}_{dR}(X)$$
$$[E_0, \lambda\phi_q]_{\lambda\in\mathbb{C}^*} \longmapsto [E_\hbar, \nabla_{\hbar,q}]_{\hbar\in\mathbb{C}^*}$$





Remarks on the Conformal Limit

- Interpretation of the conformal limit as a different (holomorphic) section of $\mathcal{M}_{Hod}(X)$
- The show must go on...but next time.
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