Magnetic billiards & Symplectic quotients

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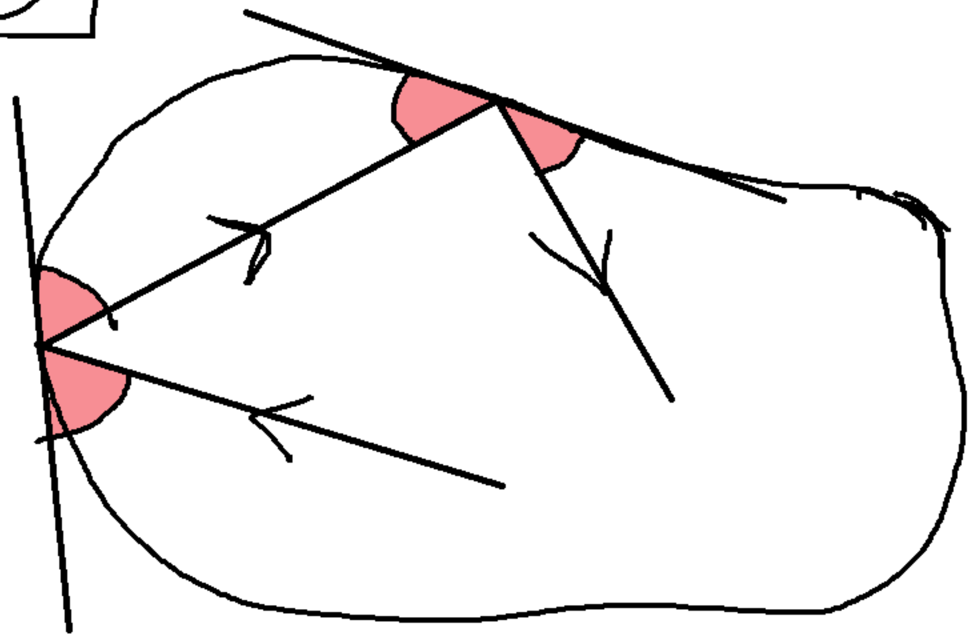
2D



"normal" billiards

"straight line flow" (SLF)

SLF is "limit case $B=0$"

$R \longrightarrow \infty$.

"magnetic" billiards

"circular flow" (CF)

constant radius R

$R = \frac{1}{|B|}$, $B > 0$: counter-clockwise.

$B \neq 0$ $\quad$ $B < 0$: clockwise.

(Work in progress)

2 Models: 1. Birkhoff billiards

2. Orbit dynamics
(using symplectic quotients)

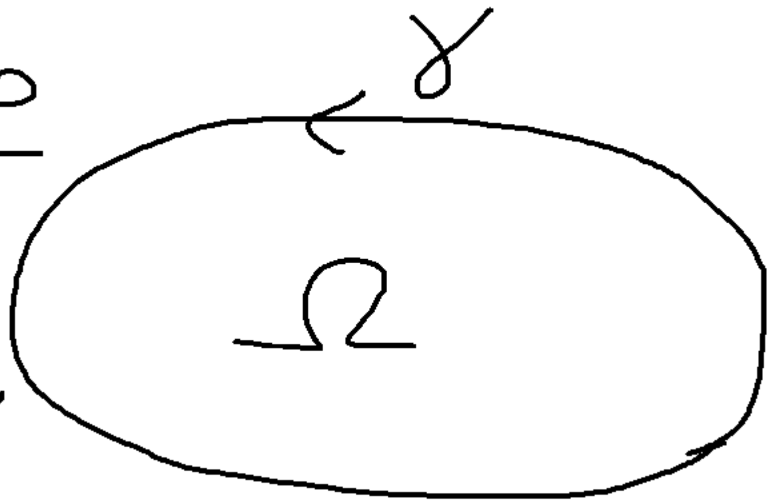
In both: Billiard flow as a discrete dynamics

$$T : PS \longrightarrow PS \quad , \; T \text{ is a symplectomorphism.}$$

billiard map.

phase space

1. Birkhoff billiards: Setup



$\gamma : \mathbb{R}/L\mathbb{Z} \longrightarrow \mathbb{R}^2$ counter-clockw. orientation, smooth, $\|\gamma'\| \equiv 1$.

Ω connected, compact, "no holes"

Regularity condition

SLF: Ω convex

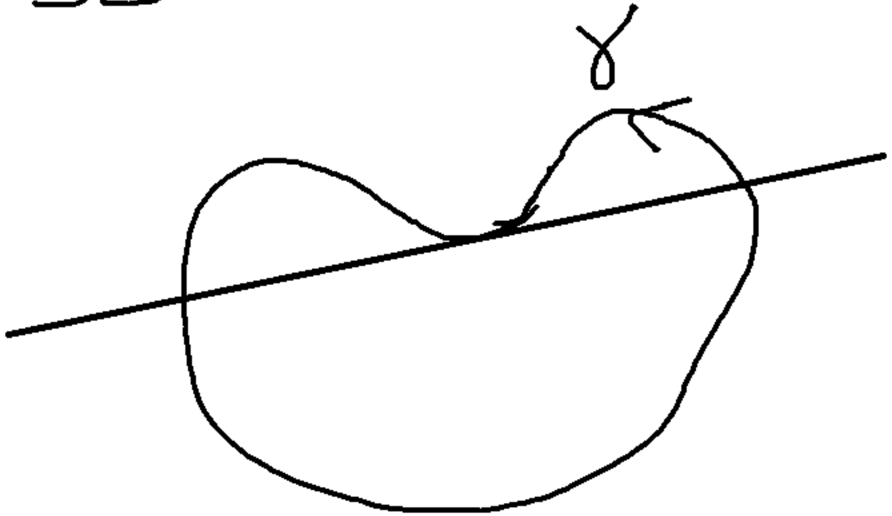
CF: $R > \max\limits_{\ell \in \mathbb{R}/L\mathbb{Z}} \rho(\ell)$ or $R < \min\limits_{\ell \in \mathbb{R}/L\mathbb{Z}} \rho(\ell)$.

$$\rho(\ell) := \frac{1}{\|\gamma''(\ell)\|}$$, radius of curvature.

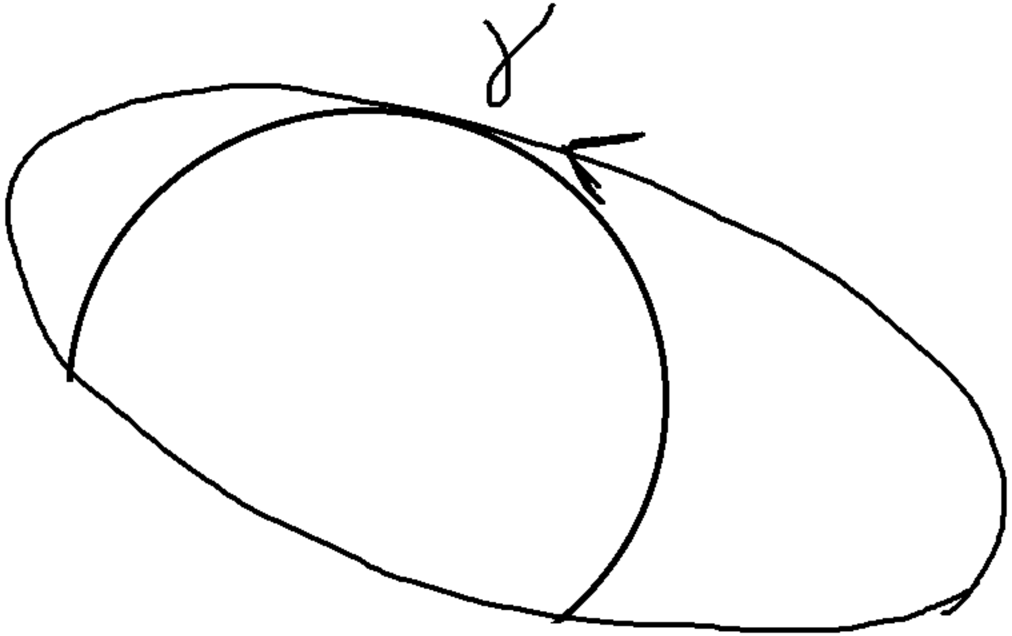
1. Birkhoff billiards: Setup

Regularity condition avoids "degenerate" cases:

SLF:



CF:



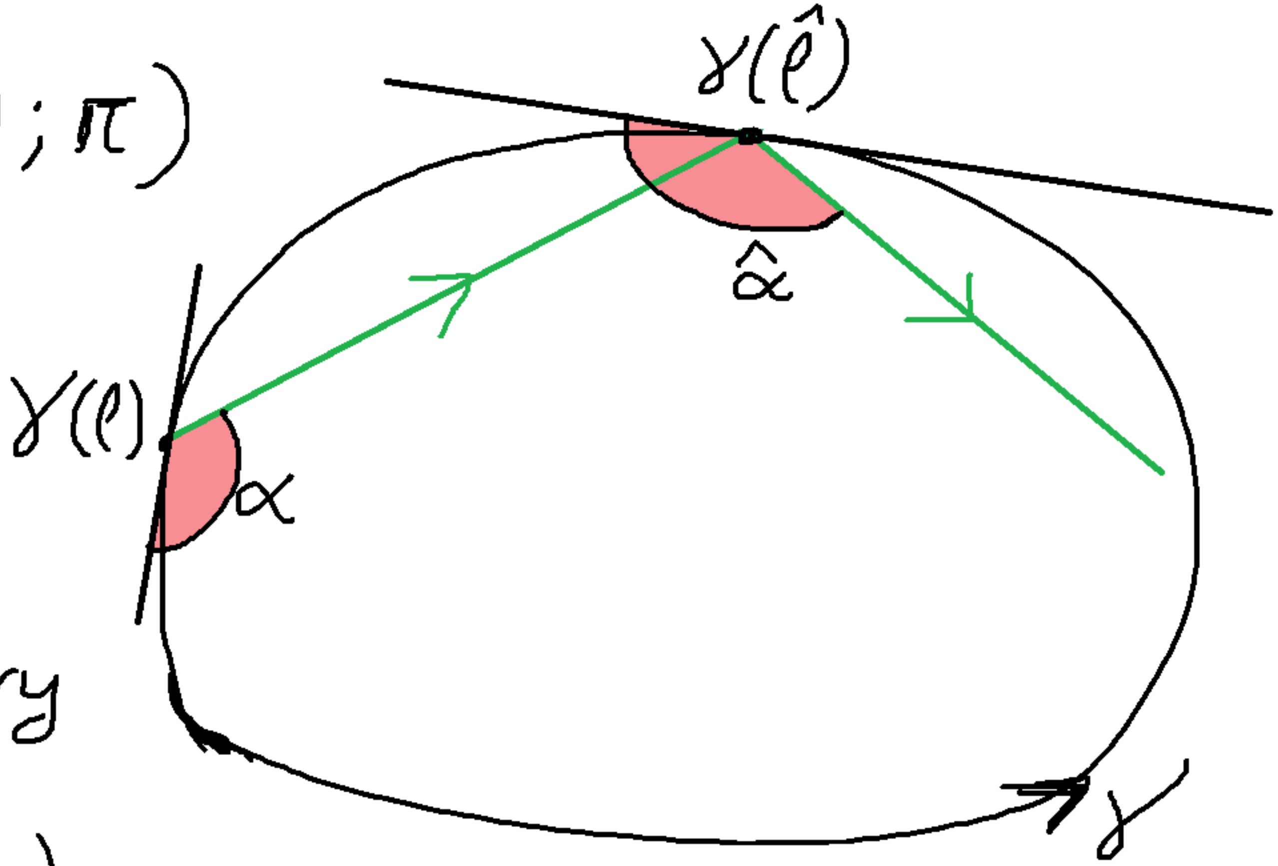
1. Birkhoff billiards: Phase space.

$$PS_{Birk} = \mathbb{R}/L\mathbb{Z} \times (0;\pi)$$

Coordinates (ℓ, α)

ℓ: Position on table boundary

α: Angle at which trajectory leaves boundary (between traj. & $\gamma'(\ell)$)



$$T : (\ell, \alpha) \longmapsto (\hat{\ell}(\ell,\alpha), \hat{\alpha}(\ell,\alpha)).$$

1. Birkhoff billiard: The billiard map.

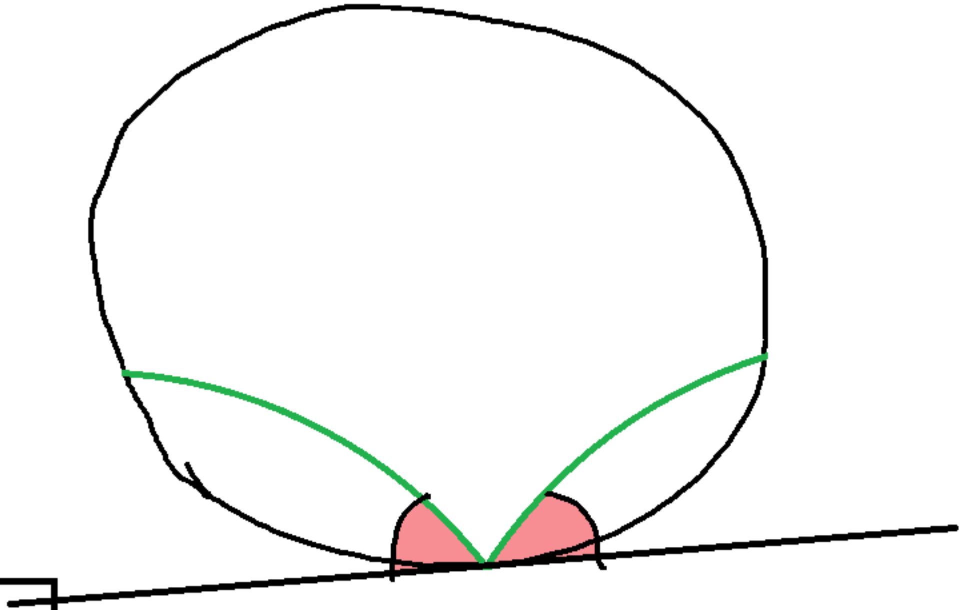
$$T_B : (\ell, \alpha) \longmapsto (\hat{\ell}(\ell,\alpha), \hat{\alpha}(\ell,\alpha)).$$

$$PS_{Birk} \longrightarrow PS_{Birk}.$$

Inverse map:

Let $(\tilde{\ell}, \tilde{\alpha}) := T_B(\ell, \pi - \alpha)$.

Then $T_B^{-1}(\ell,\alpha) = (\tilde{\ell}, \pi - \tilde{\alpha})$.



$\boxed{T_B \text{ is a diffeomorphism}}$.

1. Birkhoff billiards: Generating functions.

Proposition *Consider a smooth map* $T : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathrm{PS}_{\mathrm{Birk}}$ *and define* $\Delta := \{(l,l) | l \in {}^{\mathbb{R}}/_{L\mathbb{Z}}\} \subseteq {}^{\mathbb{R}}/_{L\mathbb{Z}} \times {}^{\mathbb{R}}/_{L\mathbb{Z}}$, *the "diagonal". Let* $G : ({}^{\mathbb{R}}/_{L\mathbb{Z}} \times {}^{\mathbb{R}}/_{L\mathbb{Z}}) \backslash \Delta \longrightarrow \mathbb{R}$ *be such that for* $l_0, l_1, l_2 \in \mathbb{R}/L\mathbb{Z}$ *and* $\alpha_0, \alpha_1, \alpha_2 \in (0;\pi)$ *following implication holds:*

$$(l_0, \alpha_0) \overset{T}{\mapsto} (l_1, \alpha_1) \overset{T}{\mapsto} (l_2, \alpha_2) \implies \left.\frac{\mathrm{d}}{\mathrm{d}l}\right|_{l=l_1} [G(l_0, l) + G(l, l_2)] = 0. \tag{1}$$

Then the 2-form $\omega := \frac{\partial}{\partial \alpha}\Big(\partial_1 G(l, \hat{l}(l,\alpha))\Big)\, \mathrm{d}\alpha \wedge \mathrm{d}l$ *on* $\mathrm{PS}_{\mathrm{Birk}}$ *is preserved by* T.

Consider $F : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathbb{R}, \quad (l,\alpha) \mapsto G(l, \hat{l}(l,\alpha))$. If

$dF = \underbrace{-\cos\alpha}_{=\partial_1 G(l,\hat{l}(l,\alpha))}\, dl + \underbrace{\cos\hat{\alpha}}_{=\partial_2 G(l,\hat{l}(l,\alpha))}\, d\hat{l}$, then it works.

$\boxed{\omega = \sin\alpha\, d\alpha \wedge dl}$

1. Birkhoff billiards: Generating functions.

Generating function condition: $(l_0, \alpha_0) \overset{T}{\mapsto} (l_1, \alpha_1) \overset{T}{\mapsto} (l_2, \alpha_2) \implies \left.\frac{\mathrm{d}}{\mathrm{d}l}\right|_{l=l_1} [G(l_0, l) + G(l, l_2)] = 0.$

For $F : \mathrm{PS}_{\mathrm{Birk}} \longrightarrow \mathbb{R}, \quad (l, \alpha) \mapsto G(l, \hat{l}(l, \alpha))$, why is

$dF = -\cos\alpha \, dl + \cos\hat{\alpha} \, d\hat{l}$ enough?

Because this implies

$$\left.\frac{d}{dl}\right|_{l=l_1} \big(G(l_0, l) + G(l, l_2)\big) = \cos\alpha_1 - \cos\alpha_1 = 0.$$

Proof in this case that $\omega := \sin\alpha \, d\alpha \wedge dl$ is preserved:

$$0 = d^2F = \sin\alpha \, d\alpha \wedge dl - \sin\hat{\alpha} \, d\hat{\alpha} \wedge d\hat{l} = \omega - T^*\omega. \qquad \square$$

1. Birkhoff billiards: Generating functions.

Subtlety:

$$dF = \underbrace{-\cos\alpha}_{} \, dl + \cos\hat{\alpha}\, d\hat{l}$$

$$= \partial_1 G(l, \hat{l}(l,\alpha))$$

So "$\cos\alpha$" is described depending on $l, \hat{l}$.

! Important for later. !

1- Birkhoff billiards: generating functions

Case $B=0$: $G:(\ell_1,\ell_2)\longmapsto \|\gamma(\ell_1)-\gamma(\ell_2)\|_2$.

Proof that if $(\ell_1,\alpha_1)\xrightarrow{T}(\ell_2,\alpha_2)$, then $\partial_1 G=-\cos\alpha_1$

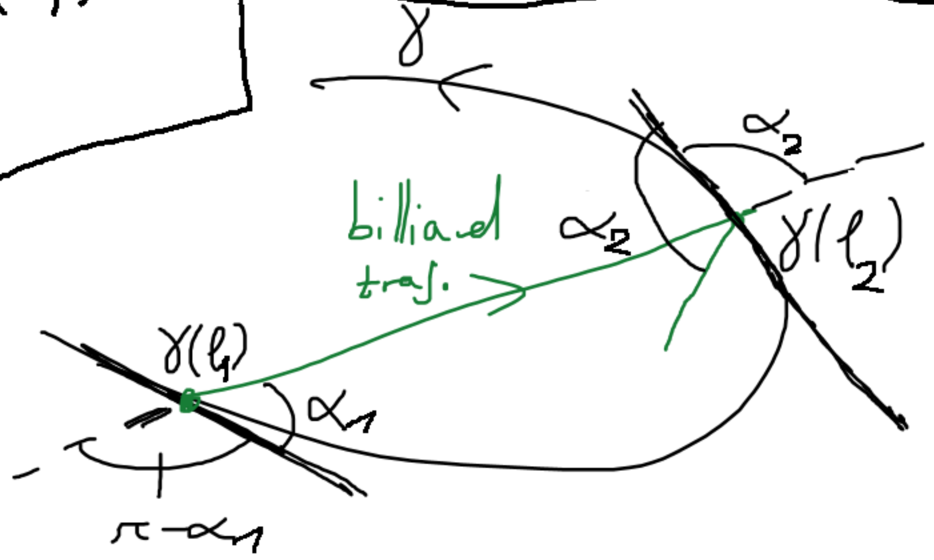
$\partial_2 G=\cos\alpha_2$:

$$\partial_1 G=\langle \underbrace{\partial_{\gamma(\ell_1)}\|\gamma(\ell_1)-\gamma(\ell_2)\|_2}_{\substack{=\text{unit vector from}\\ \gamma(\ell_2)\text{ to }\gamma(\ell_1)}},\gamma'(\ell_1)\rangle$$

$$=\cos(\pi-\alpha_1)=-\cos\alpha_1$$

$$\partial_2 G=\langle \underbrace{\partial_{\gamma(\ell_2)}\|\gamma(\ell_1)-\gamma(\ell_2)\|_2}_{=\text{unit vector from }\gamma(\ell_1)\text{ to }\gamma(\ell_2)},\gamma'(\ell_2)\rangle$$

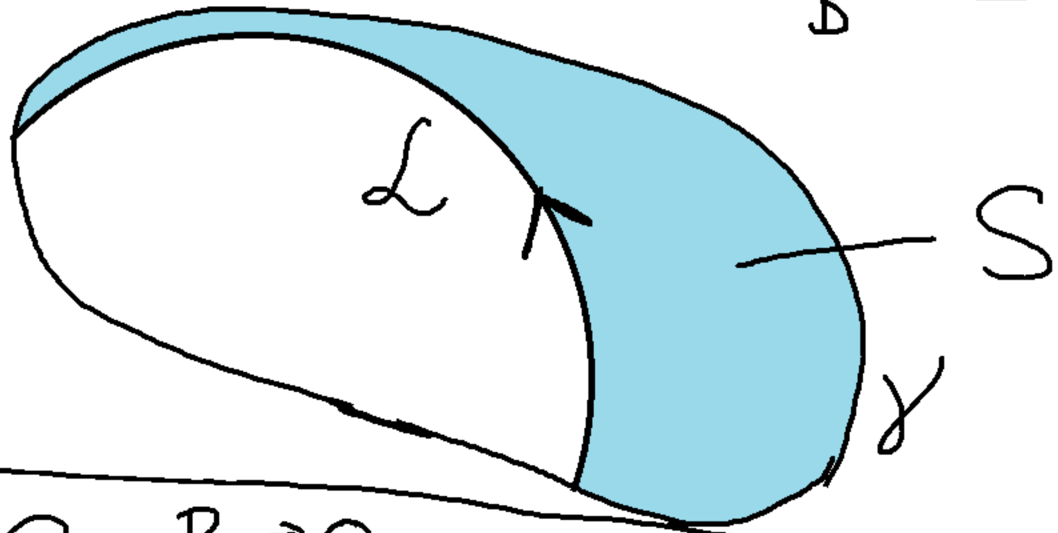
$$=\cos\alpha_2.$$



1. Birkhoff billiards: Generating functions

Case $B \neq 0$:

$$G_B = \mathcal{L} + B \cdot S$$



$\mathcal{L}$: Length of trajectory segment / circular arc

S: area to the right of trajectory, up to table boundary.

"$G_B \xrightarrow{B \to 0} G$ for $B = 0$"

[Source: J. Stat. Phys, 1996]

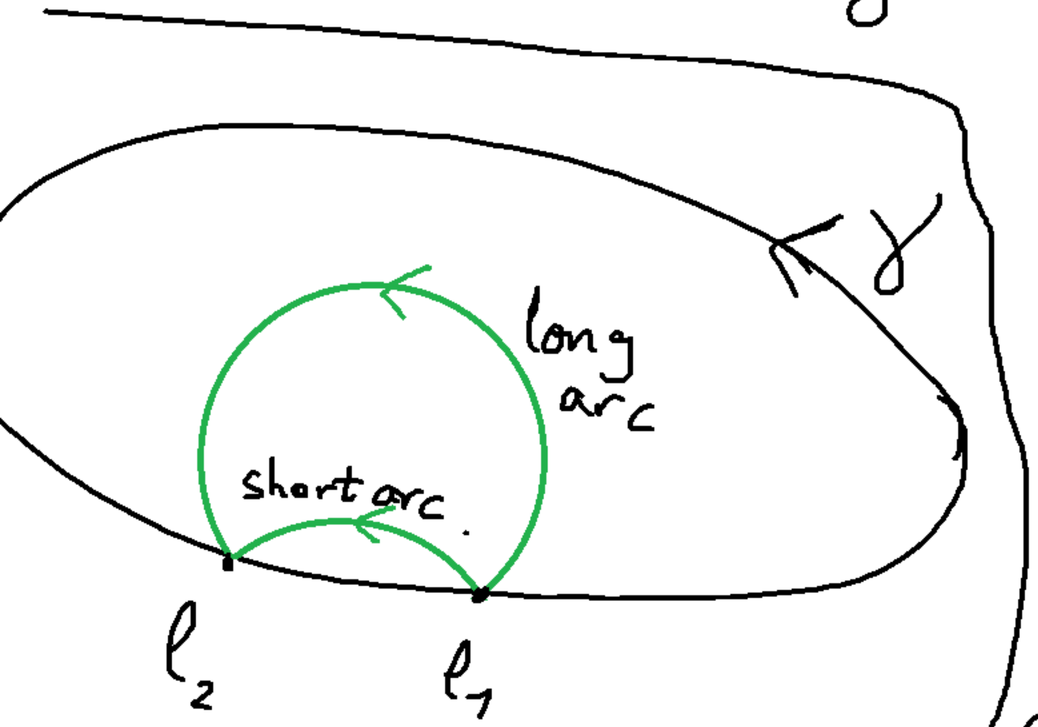
Source only considers $B > 0$, proof only for "small $B > 0$"

1. Birkhoff billiards: generating functions

$B \neq 0$: For small R / big $|B|$,

$\ell_1, \ell_2 \rightsquigarrow$ maybe two possibilities.

Solution: two generating functions



G_B^+: short arc, G_B^-: long arc

$$(\ell_1, \alpha_1^+) \xmapsto{T_B} (\ell_2, \alpha_2^+),$$

$$(\ell_1, \alpha_1^-) \xmapsto{T_B} (\ell_2, \alpha_2^-)$$

$$dG_B^\pm = -\cos\alpha_1^\pm \, d\ell_1 + \cos\alpha_2^\pm \, d\ell_2.$$

choose depending on α

$$\rightsquigarrow F : (\ell, \alpha) \longmapsto G_B^?(\ell, \hat{\ell}(\ell, \alpha))$$

$$dF = -\cos\alpha \, d\ell + \cos\hat{\alpha} \, d\hat{\ell} \qquad \square$$

1. Birkhoff-Billiards: Summary:

$PS_{Birk} = \mathbb{R}/L\mathbb{Z} \times (0;\pi)$, Coord (l, α)

position   direction of velocity

$T_B : PS_{Birk} \to PS_{Birk}$ symplectomorphisms

with $\omega = \sin\alpha \, d\alpha \wedge dl$, (proof by)

generating function $G_B^{\pm}(l_1, l_2)$ with

$F(l,\alpha) = G_B^{\pm}(l, \hat{l}(l,\alpha))$,

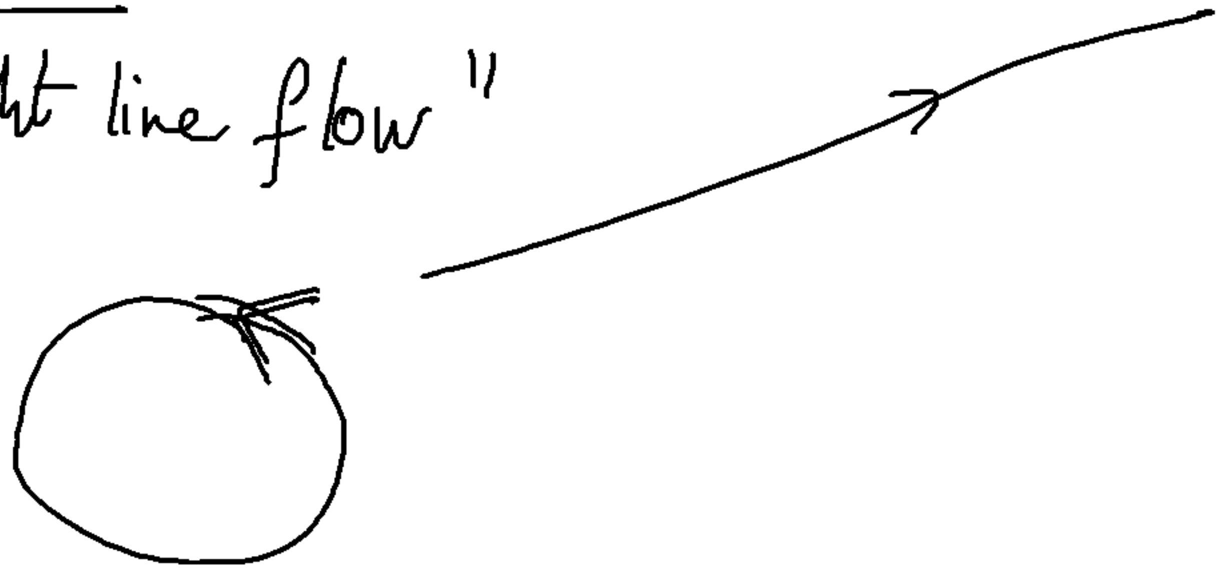
$dF = -\cos\alpha \, dl + \cos\hat{\alpha} \, d\hat{l} \Rightarrow 0 = d^2F = \omega - T_B^{*}\omega$.

2. Orbit dynamics

Step 1 | Want to describe "straight line flow"

and

"circular flow"



as Hamiltonian flow on

$\mathbb{R}^N \times \mathbb{R}^N$ with coordinates (x, v)

x: position of point mass

v: velocity of point mass.

Birkhoff: $N=2$.

Here, generalize constuction to higher dimensions.

Step 2 | Add the billiards.

2. Orbit dynamics: The physics.

$(\mathbb{R}^N \times \mathbb{R}^N, \omega)$ sympl. manifold.

$H : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ Hamiltonian.

X_H Ham. vector field, uniquely given by $\omega(X_H, \cdot) = dH$

ϕ_t Ham. flow (of X_H).

If $\omega = \sum dx_i \wedge dp_i$, then

$$\boxed{\begin{aligned} \omega(X_H, \cdot) = dH, \iff \quad & \frac{\partial H}{\partial p_i} = x_i \\ \frac{d}{dt}\phi^t = X_H \circ \phi_t \quad & \frac{\partial H}{\partial x_i} = -p_i \end{aligned}}$$

H: time-independant Hamiltonian.

p_i: generalized momenta.

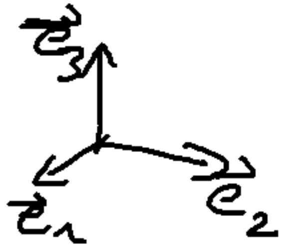
2. Orbit dynamics : the physics.

$m=1$
Charge $=-1$

$\underline{N=2}$ $M=\mathbb{R}^2\times\mathbb{R}^2$, coord. (x,p), $\omega = dx_1\wedge dp_1 + dx_2 \wedge dp_2$.

Straight line flow: No forces. $p_i = v_i$, $i=1,2$.

$$H(x,p)=\frac{1}{2}|p|^2 = \frac{1}{2}|v|^2.$$



Circular flow: Lorentz force (Magnetic field $\vec{B}=B\cdot\vec{e}_3$)

$$p_1 = v_1+\frac{B}{2}x_2,\ p_2=v_2-\frac{B}{2}x_1,\quad H(x,p)=\frac{1}{2}\left(\left(p_1-\frac{B}{2}x_2\right)^2+\left(p_2+\frac{B}{2}x_1\right)^2\right)$$

$\boxed{\text{Change of coordinates}}$ (x,p) to (x,v) results in

$$H(x,v)=\frac{1}{2}|v|^2,\quad \omega = dx_1\wedge dv_1 + dx_2\wedge dv_2 + B\,dx_1\wedge dx_2$$

$\Longrightarrow$ Hamiltonian is independent of B,
ω depends on B.

2. Orbit dynamics : Generalization to $N > 2$.

$M = \mathbb{R}^N \times \mathbb{R}^N$, $H(x,v) = \frac{1}{2}\|v\|_2^2$,

coord. (x,v)

SLF | $\omega = \sum_{j=1}^{N} dx_j \wedge dv_j$, $X_H|_{(x,v)} = (v,0)$, $\phi_t(x,v) = (x+tv, v)$.

CF | $N = 2n$, $\omega = \sum_{j=1}^{N} dx_j \wedge dv_j + B\sum_{j=1}^{n} dx_j \wedge dx_{n+j}$,

with complex coord. $z_j = x_j + i x_{n+j}$, $w_j = v_j + i v_{n+j}$,

$X_H|_{(z,w)} = (w, B \cdot i w)$, $\phi_t(z,w) = \left(z + \frac{iw}{B}(1 - e^{iBt}), w e^{iBt}\right)$

2nd component of flow is time-derivative of first component of flow: Because ω is of the form $\omega = \sum dx_j \wedge dv_j + \sum_{i,j} f_{ij} \cdot dx_i \wedge dx_j$

2. Orbit dynamics

SLF: Flow induces $\mathbb{R}$-action on $M=\mathbb{R}^N \times \mathbb{R}^N$.

CF: Flow is $\frac{2\pi}{|B|}$-periodic (since e^{iBt} is $\frac{2\pi}{|B|}$-periodic)

$\rightsquigarrow \mathbb{R}/P\mathbb{Z}$-action, $P=\frac{2\pi}{|B|}$.

Observe that $R=\frac{\|v\|}{|B|}$.

$\boxed{\text{Restrict to } \|v\|=1:}$ $Q:=H^{-1}\left(\frac{1}{2}\right)=\{(x,v) \mid \|v\|=1\}$.

$Q \subseteq M$ submanifold of codimension 1.

$\Rightarrow Q$ coisotropic, i.e. $T_qQ^{\omega} < T_qQ$

(Here $T_qQ^{\omega}=\{Y\in T_qM \mid \omega_q(Y,\tilde{Y})=0 \ \forall \tilde{Y}\in T_qQ\}$).

2. Orbit dynamics : Symplectic quotient.

(M,ω) sympl. manifold, $H: M \to \mathbb{R}$ Hamiltonian
with global Ham. flow, inducing either

$\mathbb{R}$-action $\mathbb{R} \circlearrowright M$, $t \cdot m := \phi_t(m)$, or

$\mathbb{R}/P\mathbb{Z}$-action $\mathbb{R}/P\mathbb{Z} \circlearrowright M$, $[t]m := \phi_t(m)$

Flow is periodic with period P everywhere.

Further assumptions :

- Action is free on $M \setminus \{\text{stationary points}\}$
- Fix $E \in \mathbb{R}$ regular value of H such that $Q := H^{-1}(E) \neq \emptyset$.

(It can be shown that the action restricts to Q.)

Action is proper on Q.

2. Orbit dynamics : symplectic quotient.

Then :

$$\begin{array}{c} Q \subseteq (M, \omega) \\ \big\downarrow \pi \\ \left(\overline{Q} := Q/G\,,\ \overline{\omega}\right) \end{array}$$

$G \curvearrowright M$, $G = \mathbb{R}$ or $G = \mathbb{R}/p\mathbb{Z}$

$G \curvearrowright Q \subseteq M$

free, proper action on Q.

by Ham. flow.

"Quotient manifold theorem + symplectic structure"

π smooth surjective submersion,

$\overline{\omega}$ uniquely defined by

$\pi^* \overline{\omega} = \omega|_Q$, and

$\overline{Q} = Q/G$ is the orbit space.

$(\overline{Q}, \overline{\omega})$ is a symplectic manifold.

$\dim \overline{Q} = \dim M - 2$.

2. Orbit dynamics: Imagining the orbit space

$\underline{N=2}$ | $\underline{\|V\|=1}$ | $\quad Q = \mathbb{R}^2 \times S^1.$

SLF | $\overline{Q} = Q/\mathbb{R}$: $\underbrace{\text{Every line in } \mathbb{R}^2}_{\text{every orbit of the Hamiltonian flow.}}$ is a point in $\overline{Q}$

CF | $\overline{Q} = Q/(\mathbb{R}/P\mathbb{Z})$: Every circle of radius $\frac{1}{|B|}$ is a point in $\overline{Q}$.

Let $q \in Q$, then $\pi(q) =: \overline{q}$ is the orbit of q in Q.

$$T_{\overline{q}}\overline{Q} \cong \frac{T_q Q}{T_q Q^{\omega}} = \frac{T_q Q}{\operatorname{span}\{X_H|_q\}}, \qquad d\pi|_q : T_q Q \longrightarrow \frac{T_q Q}{\operatorname{span}\{X_H|_q\}}$$

$$Y \longmapsto [Y].$$

2. Orbit dynamics: The billiard map

$N=2$

$$\Theta \subseteq \overline{Q} = Q/G$$

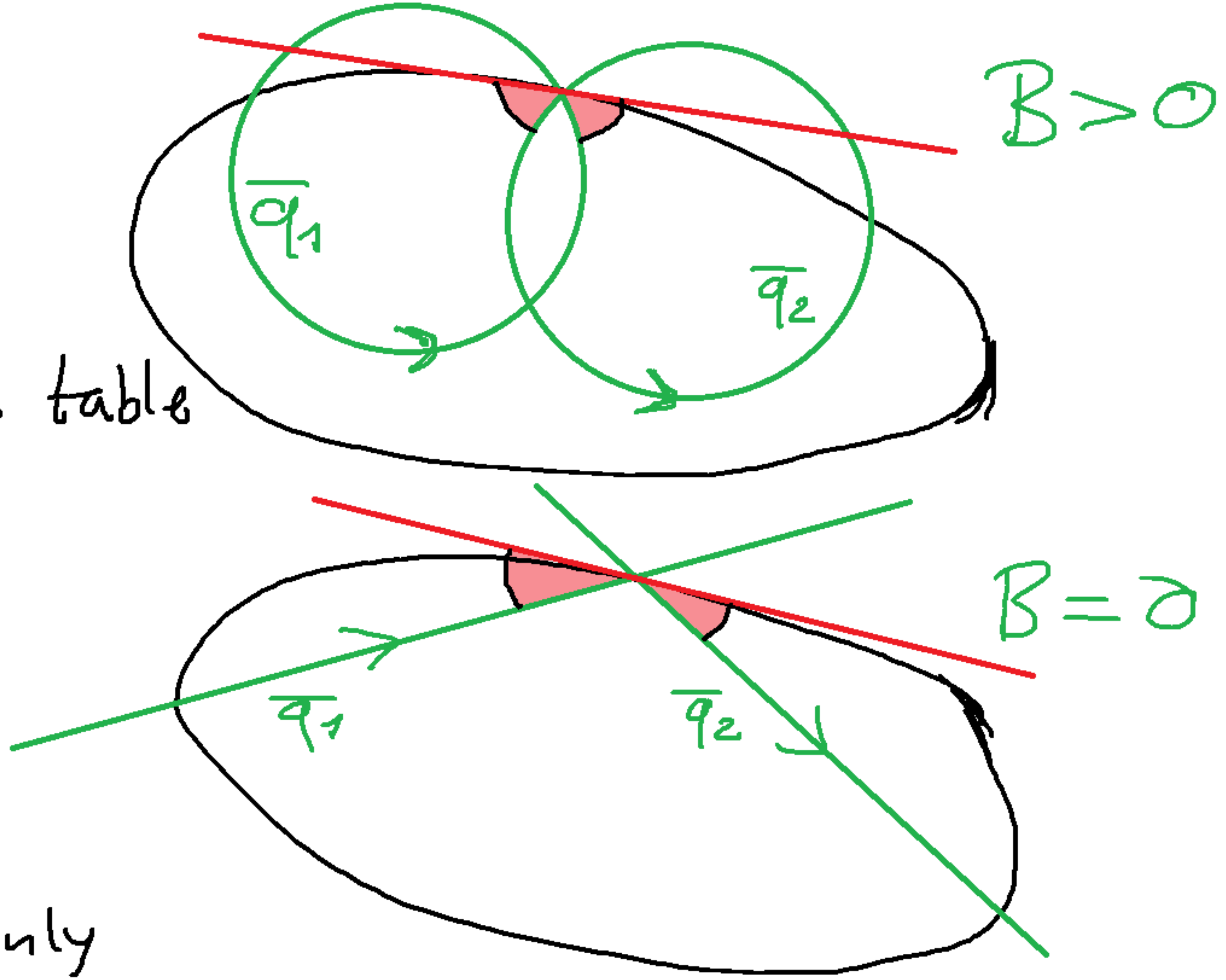
open subset of all orbits which transversally intersect billiards table boundary.

$$T_B : \Theta \longrightarrow \Theta .$$

Regularity conditions

$\Longrightarrow T_B$ is well-defined.

(Every orbit intersects table in only one trajectory segment)



$$\overline{q_1} \xmapsto{T} \overline{q_2}$$

2. Orbit dynamics: my results

- $T_B: \mathcal{O} \longrightarrow \mathcal{O}$ is a symplectomorphism.
- For $N=2$, choosing the natural map

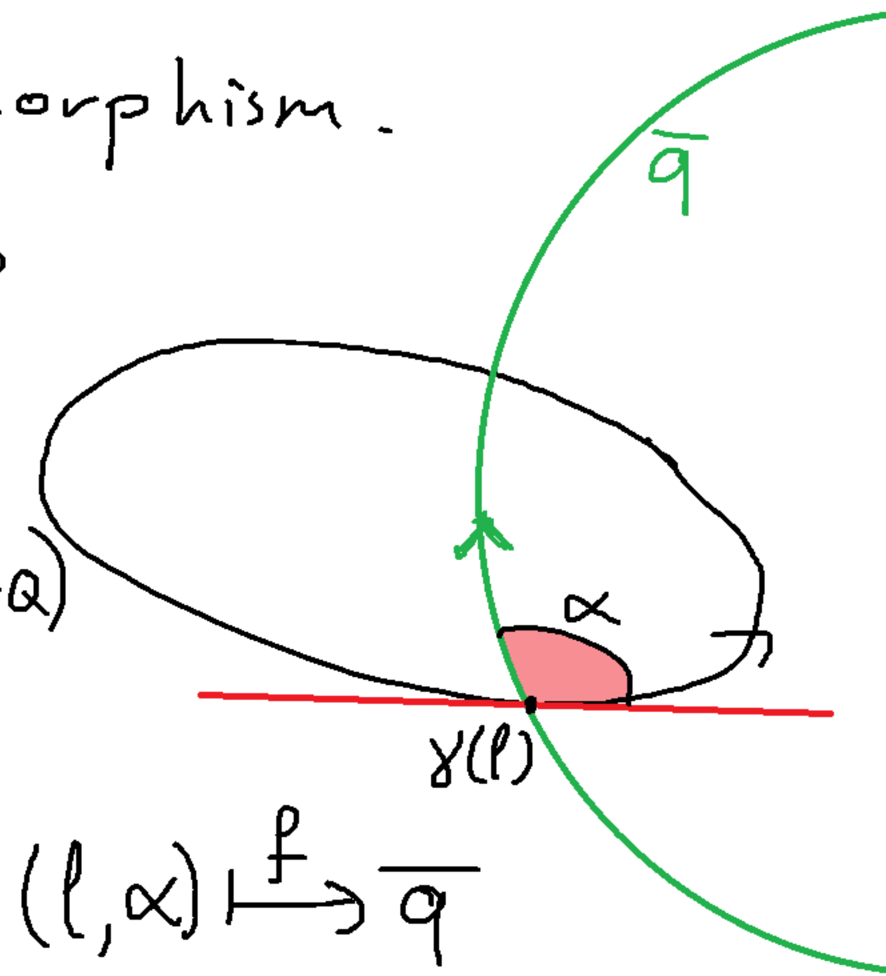
$f_B: PS_{Birk} \longrightarrow \mathcal{O}$, and with

$\overline{\omega}$ given by sympl. Quotient $(\pi^*\overline{\omega} = \omega|_Q)$

and $\omega_{Birk} = \sin\alpha \, d\alpha \wedge dl$,

$$\boxed{f_B^{*}\overline{\omega} = \omega_{Birk}.}$$

for $B \neq 0$
and $B = 0$.



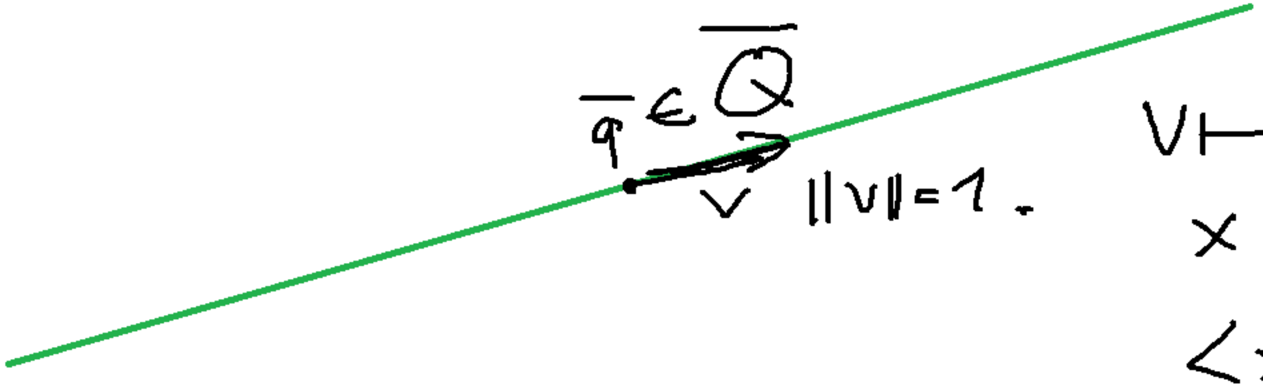
$(l,\alpha) \xmapsto{f} \overline{q}$

2. Orbit dynamics: my results.

For SLF: $(\overline{Q}, \overline{\omega}) \cong (TS^{N-1}, \omega_{std})$

(Pull back ω_{std} on T^*S^{N-1} via $TS^{N-1} \to T^*S^{N-1}$, $(v,x) \mapsto (v, \langle x, \cdot \rangle)$.

$\langle \cdot, \cdot \rangle$ on $T_v S^{N-1}$ given by $S^{N-1} \subseteq \mathbb{R}^N$ submanifold structure)



$v \mapsto (v, x) \in TS^{N-1}$, x unique point on line $\overline{q}$ such that $\langle x, v \rangle = 0$.

$\leadsto x \in T_v S^{N-1} \cong \{v\}^{\perp}$

2. Orbit dynamics: My results

For CF:

$$(\overline{Q}, \overline{\omega}) \cong \left(\mathbb{R}^{2n} \times \mathbb{CP}^{n-1}, B \cdot \omega_{\mathbb{R}^{2n}} - \frac{1}{B}\omega_{\mathbb{CP}^{n-1}}\right)$$

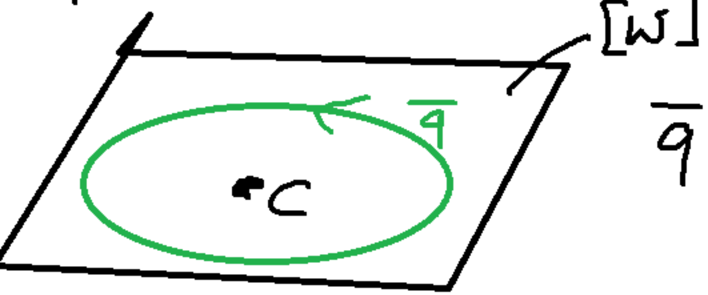
$\omega_{\mathbb{R}^{2n}} = dx \wedge dy = \sum_{i=1}^{n} dx_i \wedge dy_i$ standard sympl. form.

$\omega_{\mathbb{CP}^{n-1}}$ sympl. structure with $\mathbb{CP}^{n-1}$ as sympl. quotient:

$H: \mathbb{R}^{2n} \to \mathbb{R}$, $(x,y) \mapsto -\frac{1}{2}|(x,y)|^2$, Ham. flow induces $\mathbb{R}/2\pi\mathbb{Z}$-action.

Action: $z_j = x_j + i y_j$ complex coord, $\phi_t(z) = e^{it} z$.

$$\mathbb{CP}^{n-1} = H^{-1}(-\tfrac{1}{2}) / (\mathbb{R}/2\pi\mathbb{Z}) = S^{2n-1}/S^1$$



$\overline{q} \longmapsto (c, [w])$

c: center of orbitcircle

$[w]$: linear complex plane parallel to the affine plane in which orbit lies.

Thank You for your time!