Seminar Lie algebras and representations
Problem sheet 3: Representations of $\mathfrak{sl}_2(\mathbb{C})$

Exercise 1 *Prove that, as representations of* $\mathfrak{sl}_2(\mathbb{C})$*,* $\bigwedge^3(\mathrm{Sym}^4(V)) \cong \mathrm{Sym}^3(\mathrm{Sym}^2(V))$*. Prove also that* $\mathrm{Sym}^3(\mathrm{Sym}^2(V)) \cong \mathrm{Sym}^2(\mathrm{Sym}^3(V))$*.*

Exercise 2 *A representation of a Lie group* G *is a homomorphism of Lie groups* $\rho : G \to \mathrm{GL}(V)$*, for some vector space* V*. Any such induces a representation of the respective Lie algebras*

$$d\rho : \mathfrak{g} \to \mathfrak{gl}(V).$$

On the other hand, given a finite dimensional representation of a Lie algebra $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$*, it lifts uniquely to a representation* $\rho : \widetilde{G} \to \mathrm{GL}(V)$*, where* $\widetilde{G}$ *is the only simply connected connected Lie group with Lie algebra* $\mathfrak{g}$*.*

Using the theory of representations of $\mathfrak{sl}_2(\mathbb{C})$*, simple connectedness of* $\mathrm{SL}_2(\mathbb{C})$ *and the above, prove that the universal cover of* $\mathrm{SL}_2(\mathbb{R})$ *has no finite dimensional representation which does not factor through* $\mathrm{SL}_2(\mathbb{R})$*. In other words, it has no faithful finite dimensional representation.*